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INDECOMPOSABLE REPRESENTATIONS FOR EXTENDED DYNKIN QUIVERS OF TYPE $\widetilde{\mathbb{E}}_8$

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Abstract. We discuss the problem of classification of indecomposable representations for extended Dynkin quivers of type $\widetilde{\mathbb{E}}_8$, with a fixed orientation. We describe a method for an explicit determination of all indecomposable preprojective and preinjective representations for those quivers over an arbitrary field and for all indecomposable representations in case the field is algebraically closed. This method uses tilting theory and results about indecomposable modules for a canonical algebra of type $(5, 3, 2)$ obtained by Kussin and Meltzer and by Komoda and Meltzer. Using these techniques we calculate all series of preprojective indecomposable representations of rank 6. The same method has been used by Kussin and Meltzer to determine indecomposable representations for extended Dynkin quivers of type $\widetilde{\mathbb{D}}_n$ and $\widetilde{\mathbb{E}}_6$. Moreover, our techniques can be applied to calculate indecomposable representations of extended Dynkin quivers of type $\widetilde{\mathbb{E}}_7$. The indecomposable representations for extended Dynkin quivers of type $\widetilde{\mathbb{A}}_n$ are known.

1. Introduction. Let K be an arbitrary field and $A = KQ/I$ be a finite-dimensional K-algebra of quiver type. It is well known that a finite-dimensional left A-module can be described by choosing a finite-dimensional vector space for each vertex and a linear map for each arrow of the quiver so that the relations of the ideal I are satisfied. We denote by A-mod the category of finitely generated left A-modules.

In general, an explicit description of indecomposable representations by vector spaces and matrices is very difficult. Often one only knows the dimension vectors of the indecomposable modules, but their explicit description is not known. One of the first results was given by Gabriel [4] in 1972. He proved that an algebra $A = KQ$ has only finitely many indecomposable representations if and only if Q is a Dynkin quiver $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. Moreover, Gabriel described all indecomposable representations in these cases explicitly. From the 1973 results by Donovan and Freislich [3] and Nazarova [11] we know that a path algebra $A = KQ$ is of tame type if and only if Q is one of

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the extended Dynkin quivers $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$. Moreover, Nazarova [12] described the indecomposable modules in the special case of the quiver $\widetilde{\mathbb{D}}_4$ with subspace orientation.

In this paper we discuss a method of description of indecomposable left modules over the path algebra of the extended Dynkin quiver $\widetilde{\mathbb{E}}_8$ with the following orientation:

$$Q: \quad \begin{tikzcd} & & \circ \arrow[d] & & & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] \end{tikzcd}$$

The method described here was used by Kussin and Meltzer [10] to compute all indecomposable preprojective representations in the case of $\widetilde{\mathbb{D}}_n$ and all indecomposable preprojective representations of rank 3 in the case of $\widetilde{\mathbb{E}}_6$, which are the most complicated in this situation. In our considerations it is essential that for each path algebra $A = KQ$ of an extended Dynkin quiver Q of type $\widetilde{\mathbb{E}}_8$ there is a tilting module T over the canonical algebra Λ of type $(5,3,2)$ such that $\mathrm{End}_\Lambda(T) \cong A^{\mathrm{op}}$. For definitions and results concerning tilting theory we refer to [1]. From the Brenner–Butler Theorem [2] we know that applying the functor $\mathrm{Hom}_\Lambda(T,-)$ to an indecomposable Λ-module M satisfying the condition $\mathrm{Ext}^1_\Lambda(T,M) = 0$ we get an indecomposable right A^{op}-module, hence a left indecomposable A-module. This module is of the form $\mathrm{Hom}_\Lambda(T,M)$. A description of indecomposable modules over a canonical algebra of type $(5,3,2)$ was given in [9] in the case when the field K is of characteristic different from 2 and in [8] in arbitrary characteristic. Applying the method above to the indecomposable preprojective Λ-modules satisfying the condition $\mathrm{Ext}^1_\Lambda(T,M) = 0$ we obtain all indecomposable preprojective left A-modules. We can also apply the functor $\mathrm{Hom}_\Lambda(T,-)$ to the indecomposable regular Λ-modules and obtain in this way all indecomposable regular left A-modules. Note that the structure of tubes in the Auslander–Reiten quiver does not change under tilting. Matrices for the indecomposable preinjective left A-modules can be obtained by dualizing those for the indecomposable preprojective representations of the opposite quiver. For this purpose a different tilting module for Λ has to be chosen. Also by changing the tilting module we can calculate matrices for the indecomposable modules for the quiver $\widetilde{\mathbb{E}}_8$ having a different orientation of the arrows.

2. Canonical algebras of type $(5,3,2)$ and series of indecomposable modules of rank 6. Canonical algebras, which are among the most important classes of algebras, were introduced by Ringel in 1984 [13]. Recall that a domestic canonical algebra of quiver type Λ is isomorphic to the path algebra of the quiver

$$
\begin{tikzcd}
 & i_1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{p-1}"] & i_{p-1} \arrow[dr, "\alpha_p"] & \\
0 \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & j_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{q-1}"] & j_{q-1} \arrow[r, "\beta_q"] & \infty \\
 & k_1 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{r-1}"] & k_{r-1} \arrow[ur, "\gamma_r"] &
\end{tikzcd}
$$

modulo the relation $\gamma_r \cdots \gamma_1 = \alpha_p \cdots \alpha_1 + \beta_q \cdots \beta_1$, where p, q, r are the lengths of the upper (middle, lower, respectively) arm, and the triples (p, q, r) are only of the form $(p, q, 1)$ with $p, q \geq 1$, $(p, 2, 2)$ with $p \geq 2$, $(3, 3, 2)$, $(4, 3, 2)$ or $(5, 3, 2)$.

It is clear that a finite-dimensional left Λ-module M consists of finite-dimensional linear spaces $M(i)$ for each vertex i and a linear map $M(\delta)$ for each arrow $\delta = \alpha_i$, β_n and γ_m, satisfying the relation

$$M(\gamma_r) \circ \cdots \circ M(\gamma_1) = M(\alpha_p) \circ \cdots \circ M(\alpha_1) + M(\beta_q) \circ \cdots \circ M(\beta_1).$$

We recall that a homomorphism $f : M \to N$ of Λ-modules is given by a set of linear maps $f_a : M(a) \to N(a)$ such that for each arrow $\alpha : a \to b$ of the quiver Λ we have $f_b M(\alpha) = N(\alpha) f_a$. It follows from [6, Lemma 4.3] that the linear maps $M(\alpha)$ for indecomposable preprojective modules are monomorphisms. Hence a homomorphism between two indecomposable preprojective modules M and N is uniquely determined by the map $f(\infty)$; further on we will identify this homomorphism with a matrix of $f(\infty)$.

The integer $\mathrm{rk}(M) = \dim M(\infty) - \dim M(0)$ is called the *rank* of the module M. An indecomposable module is of positive rank (negative rank, zero rank, respectively) if and only if it is preprojective (preinjective, regular, respectively). For all indecomposable modules M over a domestic canonical algebra we have $|\mathrm{rk}(M)| \leq 6$ and the case $|\mathrm{rk}(M)| = 6$ appears only for the type $(5, 3, 2)$. All indecomposable modules M with $|\mathrm{rk}(M)| \leq 5$ were described explicitly by matrices in [9]. Moreover, the case $|\mathrm{rk}(M)| = 6$ was described for a field of characteristic different from 2 in the same paper, and in [8] for an arbitrary field.

Now, we define some matrices necessary for the description of indecomposable modules over a canonical algebra of type $(5, 3, 2)$. Let n and i be natural numbers. Let I_n be the $n \times n$ identity matrix. Define

$$X_{n+i,n} = \left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right], Y_{n+i,n} = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline & I_n & \end{array}\right] \in \mathrm{M}_{n+i,n}(K),$$

both having i rows of zeros of length n. If H is some matrix, then we consider the matrix

$$\left[\begin{array}{c|ccc} \multirow{3}{*}{H} & 1 & & \\ & & \ddots & \\ & & & \\ \hline & 1 & & 1 \\ 0 & & \ddots & \\ & & & 1 \end{array}\right]$$

with entries 1 on two diagonals each of length $m \geq 0$, and call this matrix the *enlargement* of H. Furthermore, let us denote

$$Z' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$Z^{(3)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z^{(4)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad Z^{(5)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The notation is taken from [9]. Now, there are five series of indecomposable preprojective Λ-modules of rank 6. We denote them by $M_n^{(k)}$ for $n \in \mathbb{N}$ and $k = 1, 2, 3, 4, 5$. Those series are given as follows:

$$M_n : \quad \begin{tikzcd} & K^{n+i_1} \arrow[r, "X_{n+i_2,n+i_1}"] & K^{n+i_2} \arrow[r, "X_{n+i_3,n+i_2}"] & K^{n+i_3} \arrow[r, "X_{n+i_4,n+i_3}"] & K^{n+i_4} \arrow[dr, "X_{n+6,n+i_4}"] & \\ K^n \arrow[ur, "X_{n+i_1,n}"] \arrow[rr, "Y_{n+2,n}"] \arrow[drr, "Y_{n+3,n}"'] & & K^{n+2} \arrow[r, "Y_{n+4,n+2}"] & K^{n+4} \arrow[rr, "Y_{n+6,n+4}"] & & K^{n+6} \\ & & K^{n+3} \arrow[urrr, "Z_{n+6,n+3}"'] & & & \end{tikzcd}$$

where the numbers i_1, i_2, i_3, i_4 can be read off from the following table:

number╲type	$M_n^{(1)}$	$M_n^{(2)}$	$M_n^{(3)}$	$M_n^{(4)}$	$M_n^{(5)}$
i_1	2	1	1	1	1
i_2	3	3	2	2	2
i_3	4	4	4	3	3
i_4	5	5	5	5	4

and where $Z_{n+6,n+3}$ is the nth enlargement of the corresponding matrix Z. We can choose the matrix Z in two different ways. In the case when the characteristic of the field is different from 2, Z can be chosen to be Z'. In the other case we assume that the field has arbitrary characteristic; then Z depends on the type and the remainder of n divided by 6, as shown in the following table:

n (mod 6)$\diagdown$type	$M_n^{(1)}$	$M_n^{(2)}$	$M_n^{(3)}$	$M_n^{(4)}$	$M_n^{(5)}$
0	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(4)}$	$Z^{(5)}$
1	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(3)}$	$Z^{(2)}$
2	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$
3	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(4)}$	$Z^{(5)}$
4	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(1)}$
5	$Z^{(2)}$	$Z^{(1)}$	$Z^{(2)}$	$Z^{(3)}$	$Z^{(2)}$

Below we assume that K is of arbitrary characteristic and consider the series $M_n^{(k)}$, $n \in \mathbb{N}$ and $k = 1, 2, 3, 4, 5$. In the first case we can use the same method, but then we get a description of indecomposable representations of the extended Dynkin quiver of type $\widetilde{\mathbb{E}}_8$ by matrices with entries 0, 1, -1, 2. In our case we get matrices with entries 0, 1, -1 only, which is preferable.

3. Tilting from a canonical algebra of type $(5, 3, 2)$ to a path algebra of extended Dynkin quiver of type $\widetilde{\mathbb{E}}_8$. Let A be a K-algebra. We denote by $\cdot_{A^{\mathrm{op}}}$ multiplication in the opposite algebra A^{op}. Further, for a right A-module M we denote by $*$ multiplication in the left A^{op}-module M. Moreover, for a quiver Q, we denote by Q^{op} the opposite quiver. We recall that the map $\omega \mapsto \omega^{\mathrm{op}}$ for each path ω in KQ induces an algebra isomorphism $(KQ)^{\mathrm{op}} \cong K(Q^{\mathrm{op}})$. Therefore each right $K(Q^{\mathrm{op}})$-module M is a left KQ-module with KQ-scalar multiplication defined by the formula

$$KQ \times M \to M, \qquad (\alpha, m) \mapsto \alpha * m := m\alpha^{\mathrm{op}}.$$

Let Λ denote a canonical algebra of type $(5, 3, 2)$ over an arbitrary field K. It is known that we can find a tilting module T in Λ-mod such that $\mathrm{End}_\Lambda(T) \cong K(Q^{\mathrm{op}})$, where Q^{op} is the extended Dynkin quiver with the following orientation:

$$Q^{\mathrm{op}}: \quad \begin{tikzcd} & & \circ & & & & & \\ \circ & \circ \arrow[l, "\alpha_1^{\mathrm{op}}"'] & \circ \arrow[l, "\alpha_2^{\mathrm{op}}"'] \arrow[u, "\alpha_3^{\mathrm{op}}"'] \arrow[r, "\alpha_4^{\mathrm{op}}"] & \circ \arrow[r, "\alpha_5^{\mathrm{op}}"] & \circ \arrow[r, "\alpha_6^{\mathrm{op}}"] & \circ \arrow[r, "\alpha_7^{\mathrm{op}}"] & \circ \arrow[r, "\alpha_8^{\mathrm{op}}"] & \circ \end{tikzcd}$$

The indecomposable direct summands T_k of T are given as follows:

$$T_0:\quad \begin{tikzcd}
 & K \arrow[r, "\mathrm{id}"] & K \arrow[dr, "X_{2,1}"] & \\
K \arrow[r, "Y_{2,1}"] & K^2 \arrow[rr, "\mathrm{id}"] & & K^2 \\
 & K \arrow[urr, "Z(T_0)"'] & &
\end{tikzcd}
\qquad
T_1:\quad \begin{tikzcd}
K \arrow[r, "\mathrm{id}"] & K \arrow[r, "X_{2,1}"] & K^2 \arrow[r, "X_{3,2}"] & K^3 \arrow[dr, "X_{4,3}"] & \\
 & K \arrow[r, "Y_{3,1}"] & K^3 \arrow[rr, "Y_{4,3}"] & & K^4 \\
 & & K^2 \arrow[urr, "Z(T_1)"'] & &
\end{tikzcd}$$

$$T_2:\quad \begin{tikzcd}
K \arrow[r, "X_{2,1}"] & K^2 \arrow[r, "X_{3,2}"] & K^3 \arrow[r, "X_{4,3}"] & K^4 \arrow[dr, "X_{6,4}"] & \\
 & K^2 \arrow[r, "Y_{4,2}"] & K^4 \arrow[rr, "Y_{6,4}"] & & K^6 \\
 & & K^3 \arrow[urr, "Z(T_2)"'] & &
\end{tikzcd}
\qquad
T_3:\quad \begin{tikzcd}
K \arrow[r, "\mathrm{id}"] & K \arrow[r, "X_{2,1}"] & K^2 \arrow[dr, "X_{3,2}"] & \\
K \arrow[r, "Y_{2,1}"] & K^2 \arrow[rr, "Y_{3,2}"] & & K^3 \\
 & K^2 \arrow[urr, "Z(T_3)"'] & &
\end{tikzcd}$$

$$T_4:\quad \begin{tikzcd}
K \arrow[r, "X_{2,1}"] & K^2 \arrow[r, "X_{3,2}"] & K^3 \arrow[r, "X_{4,3}"] & K^4 \arrow[dr, "X_{6,4}"] & \\
 & K^2 \arrow[r, "Y_{3,2}"] & K^3 \arrow[rr, "Y_{5,3}"] & & K^5 \\
 & & K^2 \arrow[urr, "Z(T_4)"'] & &
\end{tikzcd}
\qquad
T_5:\quad \begin{tikzcd}
K \arrow[r, "X_{2,1}"] & K^2 \arrow[r, "X_{3,2}"] & K^3 \arrow[r, "\mathrm{id}"] & K^3 \arrow[dr, "X_{4,3}"] & \\
 & K \arrow[r, "Y_{2,1}"] & K^2 \arrow[rr, "Y_{4,2}"] & & K^4 \\
 & & K^2 \arrow[urr, "Z(T_5)"'] & &
\end{tikzcd}$$

$$T_6:\quad \begin{tikzcd}
K \arrow[r, "X_{2,1}"] & K^2 \arrow[r, "\mathrm{id}"] & K^2 \arrow[r, "\mathrm{id}"] & K^2 \arrow[dr, "X_{3,2}"] & \\
 & K \arrow[r, "Y_{2,1}"] & K^2 \arrow[rr, "Y_{3,2}"] & & K^3 \\
 & & K \arrow[urr, "Z(T_6)"'] & &
\end{tikzcd}
\qquad
T_7:\quad \begin{tikzcd}
K \arrow[r, "\mathrm{id}"] & K \arrow[r, "\mathrm{id}"] & K \arrow[r, "\mathrm{id}"] & K \arrow[dr, "X_{2,1}"] & \\
 & K \arrow[r, "\mathrm{id}"] & K \arrow[rr, "Y_{2,1}"] & & K^2 \\
 & & K \arrow[urr, "Z(T_7)"'] & &
\end{tikzcd}$$

$$T_8:\quad \begin{tikzcd}
 & K \arrow[r, "\mathrm{id}"] & K \arrow[r, "\mathrm{id}"] & K \arrow[r, "\mathrm{id}"] & K \arrow[dr, "X_{2,1}"] & \\
K \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{id}"] \arrow[drrr, "\mathrm{id}"'] & & K \arrow[r, "\mathrm{id}"] & K \arrow[rr, "Y_{2,1}"] & & K^2 \\
 & & & K \arrow[urr, "Z(T_8)"'] & &
\end{tikzcd}$$

where

$$Z(T_0) = Z(T_7) = Z(T_8) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad Z(T_1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad Z(T_2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Z(T_3) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad Z(T_4) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Z(T_5) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad Z(T_6) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is easy to see that each T_i is preprojective and indecomposable, hence $\mathrm{Ext}^1_\Lambda(T_i, T_i) = 0$. Moreover, one easily calculates that $\mathrm{Ext}^1_\Lambda(T, T) = 0$. Because the number of indecomposable direct summands of T equals the number of simple Λ-modules, T is in fact a tilting module. Furthermore, it is easy to see that $\mathrm{End}_\Lambda(T) = K(Q^{\mathrm{op}})$.

It is clear that for each left Λ-module M the vector space $\mathrm{Hom}_\Lambda(T, M)$ is a right module over the endomorphism algebra of T, so it is a left KQ-module. Hence we can consider a functor $\mathrm{Hom}_\Lambda(T, -) : \Lambda\text{-mod} \to KQ\text{-mod}$, where

$$Q: \quad \begin{tikzcd} & & \overset{3}{\circ} \arrow[d, "\alpha_3"] & & & & & \\ \circ \arrow[r, "\alpha_1"] & \circ \arrow[r, "\alpha_2"] & \circ & \circ \arrow[l, "\alpha_4"'] & \circ \arrow[l, "\alpha_5"'] & \circ \arrow[l, "\alpha_6"'] & \circ \arrow[l, "\alpha_7"'] & \circ \arrow[l, "\alpha_8"'] \end{tikzcd}$$

The theorem of Brenner and Butler [2] implies that for any left indecomposable Λ-module M such that $\mathrm{Ext}^1_\Lambda(T, M) = 0$, the module $\mathrm{Hom}_\Lambda(T, M)$ is indecomposable. We are going to describe those modules.

Now, if M is a Λ-module satisfying $\mathrm{Ext}^1_\Lambda(T, M) = 0$ then the left KQ-module $\mathrm{Hom}_\Lambda(T, M)$ is given as follows:

- To each vertex a of Q, there is associated the K-vector space
$$\mathrm{Hom}_\Lambda(T, M)_a = e_a * \mathrm{Hom}_\Lambda(T, M),$$
where e_a denotes the stationary path in KQ corresponding to the vertex a.
- To each arrow $\alpha : a \to b$ of Q, there is associated the K-linear map
$$f_\alpha : \mathrm{Hom}_\Lambda(T, M)_a \to \mathrm{Hom}_\Lambda(T, M)_b$$
defined by the formula
$$f_\alpha(e_a * m) = \alpha * e_a * m.$$

Now we show how the representation $\mathrm{Hom}_\Lambda(T, M)$ can be computed in practise. Using the definition of KQ-scalar multiplication in the module $\mathrm{Hom}_\Lambda(T, M)$ we obtain

- $\mathrm{Hom}_\Lambda(T, M)_a = e_a * \mathrm{Hom}_\Lambda(T, M) = \mathrm{Hom}_\Lambda(T, M)e_a$,
- $f_\alpha(e_a * m = me_a) = \alpha * e_a b * m = me_a\alpha^{\mathrm{op}}$, where $\alpha : a \to b$ is an arrow in Q.

Next, we know from [1] that there exist K-linear isomorphisms
$$\mathrm{Hom}_\Lambda(T, M)e_a \cong \mathrm{Hom}_\Lambda(T_a, M) \quad \text{and} \quad e_b KQ e_a \cong \mathrm{Hom}_\Lambda(T_a, T_b).$$
The first isomorphism gives vector spaces of the representation $\mathrm{Hom}_\Lambda(T, M)$. Using the second isomorphism we see that to each arrow $\alpha : a \to b$ of the quiver Q there corresponds a module homomorphism from T_b to T_a. Hence, if we choose a generator $h_\alpha : T_b \to T_a$ of the vector space $\mathrm{Hom}_\Lambda(T_b, T_a)$, the map $f_\alpha : \mathrm{Hom}_\Lambda(T_a, M) \to \mathrm{Hom}_\Lambda(T_b, M)$ is defined by the formula $f_\alpha = \mathrm{Hom}_\Lambda(h, M)$, because for each $g : T_a \to M$ we have the identity $f_\alpha(g) = gh = \mathrm{Hom}_\Lambda(h, M)(g)$.

Thus we obtain

Property 3.1. *Let M be an indecomposable Λ-module satisfying* $\mathrm{Ext}^1_\Lambda(T,M)=0$. *Then the left KQ-module* $\mathrm{Hom}_\Lambda(T,M)$ *is formed by*

- *vector spaces* $\mathrm{Hom}_\Lambda(T,M)_a = \mathrm{Hom}_\Lambda(T_a,M)$ *for each vertex a of Q,*
- *maps* $f_\alpha = \mathrm{Hom}_\Lambda(h_\alpha, M)$, *where h_α are generators of* $\mathrm{Hom}_\Lambda(T_b,T_a)$, *for each arrow $\alpha : a \to b$ of the quiver Q.*

From Property 3.1 it follows that, in order to find a desired representation, we need to choose a generator of each linear space $\mathrm{Hom}_\Lambda(T_b,T_a)$ for each arrow $\alpha : a \to b$ in the quiver Q. The diagram below shows a choice of generators of the vector spaces $\mathrm{Hom}_\Lambda(T_b,T_a)$, which we will use later:

$$\begin{tikzcd} & & T_3 & & & & & \\ T_0 & T_1 \arrow[l, "T_{1,0}"'] & T_2 \arrow[l, "T_{2,1}"'] \arrow[u, "T_{2,3}"] \arrow[r, "T_{2,4}"] & T_4 \arrow[r, "T_{4,5}"] & T_5 \arrow[r, "T_{5,6}"] & T_6 \arrow[r, "T_{6,7}"] & T_7 \arrow[r, "T_{7,8}"] & T_8 \end{tikzcd}$$

where

$$T_{1,0} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{2,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{2,3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad T_{2,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$T_{4,5} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \qquad T_{5,6} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{6,7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_{7,8} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall that a homomorphism f is identified with the matrix $f(\infty)$. The proof of the fact above is straightforward and is left to the reader.

4. Explicit computation of matrices. Let $\{M_n^{(5)} \mid n \in \mathbb{N}\}$ be one of the series of indecomposable preprojective modules over the canonical algebra Λ of type $(5,3,2)$ introduced in Section 2. It is easy to verify that in the preprojective component of the Auslander–Reiten quiver the modules $M_n^{(5)}$ are positioned to the right of the direct summands T_k of the tilting module T. Then applying the Auslander–Reiten formula and the fact that there do not exist non-zero homomorphisms from right to left [13], we obtain the relation $\mathrm{Ext}^1_\Lambda(T, M_n^{(5)}) = 0$. Because each module $M_n^{(5)}$ is indecomposable and $\mathrm{Ext}^1_\Lambda(T, M_n^{(5)}) = 0$, we know that all KQ-modules $\mathrm{Hom}_\Lambda(T, M_n^{(5)})$ are indecomposable.

Now, we will compute the modules $\mathrm{Hom}_\Lambda(T, M_n)$. From Property 3.1 we have to find bases of the linear spaces $\mathrm{Hom}_\Lambda(T_i, M_n^{(5)})$, and matrices of the homomorphisms $\mathrm{Hom}_\Lambda(h_\alpha, M)$.

First, let us consider the case of $n \equiv 0$ or $n \equiv 3 \bmod 6$. Let $Z_{n+6,n+3} \in \mathrm{M}_{n+6,n+3}(K)$ be the nth enlargement of the matrix $Z^{(5)}$ introduced in Section 2 and let $H = (h_{ij}) \in \mathrm{M}_{n+3,t}(K)$ be arbitrary. Then for the matrix

$$Z_{n+6,n+3}H = \begin{bmatrix} h_{3i} + h_{4i} \\ h_{1i} + h_{3i} + h_{5i} \\ h_{3i} + h_{6i} \\ h_{2i} + h_{3i} + h_{7i} \\ h_{1i} + h_{8i} \\ h_{2i} + h_{9i} \\ h_{4i} + h_{10i} \\ h_{5i} + h_{11i} \\ h_{6i} + h_{12i} \\ \vdots \end{bmatrix}_{i=1,\dots,t}$$

we have the following identities:

$$\begin{aligned}
h_{1i} + h_{3i} + h_{5i} &= (h_{3i} + h_{4i}) + (h_{1i} + h_{8i}) + \sum_{k=0}^{\infty}\{(h_{12k+5,i} + h_{12k+11,i}) \\
&\quad + (h_{12k+14,i} + h_{12k+20,i}) + (h_{12k+10,i} + h_{12k+16,i}) \\
&\quad - (h_{12k+4,i} + h_{12k+10,i}) - (h_{12k+8,i} + h_{12k+14,i}) \\
&\quad - (h_{12k+11,i} + h_{12k+17,i})\}, \\
h_{3i} + h_{6i} &= (h_{3i} + h_{4i}) + \sum_{k=0}^{\infty}\{(h_{12k+6,i} + h_{12k+12,i}) \\
&\quad + (h_{12k+10,i} + h_{12k+16,i}) - (h_{12k+4,i} + h_{12k+10,i}) \\
&\quad - (h_{12k+12,i} + h_{12k+18,i})\}, \\
h_{2i} + h_{3i} + h_{7i} &= (h_{3i} + h_{4i}) + (h_{2i} + h_{9i}) + \sum_{k=0}^{\infty}\{(h_{12k+7,i} + h_{12k+13,i}) \\
&\quad + (h_{12k+15,i} + h_{12k+21,i}) + (h_{12k+10,i} + h_{12k+16,i}) \\
&\quad - (h_{12k+9,i} + h_{12k+15,i}) - (h_{12k+4,i} + h_{12k+10,i}) \\
&\quad - (h_{12k+13,i} + h_{12k+19,i})\},
\end{aligned} \tag{4.1}$$

where we put, for simplicity, $h_{ij} = 0$ for each $i > n + 3$ and $j = 1, \dots, t$.

We use the above three identities to compute bases of the linear spaces $\mathrm{Hom}_\Lambda(T_i, M_n^{(5)})$, $i = 0, 1, \dots, 8$.

4.1. Computation of $\mathrm{Hom}_\Lambda(T_0, M_n^{(5)})$**.** A homomorphism $g : T_0 \to M_n^{(5)}$ is given by matrices $D = (d_{ij}) \in \mathrm{M}_{n+3,1}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,1}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,2}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,1}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,2}(K)$ such that $E = X_{n+4,n+3}D$, $SX_{2,1} = X_{n+6,n+4}E$, $GY_{2,1} = X_{n+4,n+2}F$, $S = Y_{n+6,n+4}G$, $SZ(T_3) = Z_{n+6,n+3}H$. From the first four conditions we get

$$s_{11} = s_{12} = s_{21} = s_{22} = s_{32} = s_{42} = s_{n+4,1} = s_{n+5,1} = s_{n+6,1} = 0, \tag{4.2}$$

where the equality $SZ(T_3) = Z_{n+6,n+3}H$ yields

$$\begin{bmatrix} 0 \\ 0 \\ s_{31} \\ s_{41} \\ s_{51}+s_{52} \\ s_{61}+s_{62} \\ s_{71}+s_{72} \\ s_{81}+s_{82} \\ \vdots \end{bmatrix} = \begin{bmatrix} h_{31}+h_{41} \\ h_{11}+h_{31}+h_{51} \\ h_{31}+h_{61} \\ h_{21}+h_{31}+h_{71} \\ h_{11}+h_{81} \\ h_{21}+h_{91} \\ h_{41}+h_{10,1} \\ h_{51}+h_{11,1} \\ \vdots \end{bmatrix}. \tag{4.3}$$

Applying (4.1) to the right-hand side of (4.3) we get

$$\begin{aligned} s_{31} = \sum_{k=0}^{\infty}\{&(s_{12k+9,1}+s_{12k+9,2}) + (s_{12k+13,1}+s_{12k+13,2}) \\ &- (s_{12k+7,1}+s_{12k+7,2}) - (s_{12k+15,1}+s_{12k+15,2})\}, \\ s_{41} = \sum_{k=0}^{\infty}\{&(s_{12k+10,1}+s_{12k+10,2}) + (s_{12k+6,1}+s_{12k+6,2}) \\ &+ (s_{12k+13,1}+s_{12k+13,2}) - (s_{12k+12,1}+s_{12k+12,2}) \\ &- (s_{12k+7,1}+s_{12k+7,2}) - (s_{12k+16,1}+s_{12k+16,2})\}, \\ s_{51} = -s_{52} - \sum_{k=0}^{\infty}\{&(s_{12k+8,1}+s_{12k+8,2}) + (s_{12k+13,1}+s_{12k+13,2}) \\ &+ (s_{12k+17,1}+s_{12k+17,2}) - (s_{12k+7,1}+s_{12k+7,2}) \\ &- (s_{12k+11,1}+s_{12k+11,2}) - (s_{12k+14,1}+s_{12k+14,2})\}, \end{aligned} \tag{4.4}$$

where we put $s_{ij} = 0$ for $t > n+6$ and $j = 1, 2, 3, 4$.

We see that the vector space $\mathrm{Hom}_\Lambda(T_0, M_n^{(5)})$ is given by the matrices $S = (s_{ij}) \in \mathrm{M}_{n+6,2}(K)$, where the nine entries from (4.2) equal zero, and the entries s_{31}, s_{41}, s_{51} are linear combinations of other s_{ij}'s. Therefore $\dim_K \mathrm{Hom}_\Lambda(T_0, M_n^{(5)}) = 2n$ and a basis is given by the $(n+6) \times 2$-matrices $q_{61}, q_{71}, \dots, q_{n+3,1}, q_{52}, q_{62}, \dots, q_{n+6,2}$ where q_{ij} is the matrix with the (i, j) entry 1, the $(3, 1)$, $(4, 1)$, $(5, 1)$ entries calculated using the above identities, and all the other entries zero. The following table shows these coefficients for the initial i, j's:

matrix	q_{61}	q_{71}	q_{81}	q_{91}	$q_{10,1}$	$q_{11,1}$	$q_{12,1}$	$q_{13,1}$	$q_{14,1}$	$q_{15,1}$	$q_{16,1}$	$q_{17,1}$
s_{31}	0	-1	0	1	0	0	0	1	0	-1	0	0
s_{41}	1	-1	0	0	1	0	-1	1	0	0	-1	0
s_{51}	0	1	-1	0	0	1	0	-1	1	0	0	-1

matrix	q_{52}	q_{62}	q_{72}	q_{82}	$q_{9,2}$	$q_{10,2}$	$q_{11,2}$	$q_{12,2}$	$q_{13,2}$	$q_{14,2}$	$q_{15,2}$	$q_{16,2}$
s_{31}	0	0	-1	0	1	0	0	0	1	0	-1	0
s_{41}	0	1	-1	0	0	1	0	-1	1	0	0	-1
s_{51}	-1	0	1	-1	0	0	1	0	-1	1	0	0

4.2. Computation of $\mathrm{Hom}_\Lambda(T_1, M_n^{(5)})$**.** A homomorphism $g : T_1 \to M_n^{(5)}$ is given by matrices $B = (b_{ij}) \in \mathrm{M}_{n+1,1}(K)$, $C = (c_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $D = (d_{ij}) \in \mathrm{M}_{n+3,2}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,3}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,3}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,2}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,4}(K)$ such that $C = X_{n+2,n+1}B$, $DX_{2,1} = X_{n+3,n+2}C$, $X_{n+4,n+3}D = EX_{3,2}$, $SX_{4,3} = X_{n+6,n+4}E$, $GY_{3,1} = X_{n+4,n+2}F$, $SY_{4,3} = Y_{n+6,n+4}G$, $SZ(T_2) = Z_{n+6,n+3}H$.

From the first four conditions we get $s_{n+2,1} = s_{n+3,1} = s_{n+4,1} = s_{n+4,2} = s_{n+5,1} = s_{n+5,2} = s_{n+5,3} = s_{n+6,1} = s_{n+6,2} = s_{n+6,3} = 0$. Moreover, the next two conditions imply $s_{12} = s_{13} = s_{14} = s_{22} = s_{23} = s_{24} = s_{34} = s_{44} = 0$. Applying (4.1) to the last matrix equality we obtain a linear dependence of the entries s_{21}, s_{31}, s_{41}, s_{33}, s_{43}, s_{53}. Hence $\dim_K \mathrm{Hom}_\Lambda(T_1, M_n^{(5)}) = 4n$. Let w_{ij} be the $(n+6)\times 4$-matrix with the (i, j) entry 1, the $(2, 1)$, $(3, 1)$, $(4, 1)$, $(3, 3)$, $(4, 3)$, $(5, 3)$ entries linearly dependent on other entries, and all other entries zero. Thus the matrices $w_{11}, w_{51}, \ldots, w_{n+1,1}, w_{32}, w_{42}, \ldots, w_{n+3,2}, w_{63}, w_{73}, \ldots, w_{n+4,3}, w_{54}, w_{64}, \ldots, w_{n+6,4}$ form a basis of $\mathrm{Hom}_\Lambda(T_1, M_n^{(5)})$.

4.3. Computation of $\mathrm{Hom}_\Lambda(T_2, M_n^{(5)})$**.** A homomorphism $g : T_2 \to M_n^{(5)}$ is given by matrices $B = (b_{ij}) \in \mathrm{M}_{n+1,1}(K)$, $C = (c_{ij}) \in \mathrm{M}_{n+2,2}(K)$, $D = (d_{ij}) \in \mathrm{M}_{n+3,3}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,4}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,2}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,4}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,3}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,6}(K)$ such that $CX_{2,1} = X_{n+2,n+1}B$, $DX_{3,2} = X_{n+3,n+2}C$, $EX_{4,3} = X_{n+4,n+3}D$, $SX_{6,4} = X_{n+6,n+4}E$, $GY_{4,2} = Y_{n+4,n+2}F$, $SY_{6,4} = Y_{n+6,n+4}G$, $SZ(T_0) = Z_{n+6,n+3}H$.

It is easy to see that then $s_{13} = s_{14} = s_{15} = s_{16} = s_{23} = s_{24} = s_{25} = s_{26} = s_{35} = s_{36} = s_{45} = s_{46} = s_{n+2,1} = s_{n+3,1} = s_{n+3,2} = s_{n+4,1} = s_{n+4,2} = s_{n+4,3} = s_{n+5,1} = s_{n+5,2} = s_{n+5,3} = s_{n+5,4} = s_{n+6,1} = s_{n+6,2} = s_{n+6,3} = s_{n+6,4} = s_{n+6,5} = 0$. Furthermore, the entries s_{21}, s_{31}, s_{41}, s_{22}, s_{32}, s_{42}, s_{34}, s_{44}, s_{54} are linearly dependent on other s_{ij}'s. Thus $\dim_K \mathrm{Hom}_\Lambda(T_2, M) =$

$6n+1$ and a basis is given by the $(n+6)\times 6$-matrices $v_{11}, v_{51}, \dots, v_{n+1,1}, v_{12}, v_{52}, \dots, v_{n+2,2}, v_{33}, v_{43}, \dots, v_{n+3,3}, v_{64}, \dots, v_{n+4,4}, v_{55}, \dots, v_{n+6,5}, v_{56}, \dots, v_{n+6,6}$ where v_{ij} is the matrix with the (i,j) entry 1 and all other entries 0 except for the $(2,1)$, $(3,1)$, $(4,1)$, $(2,2)$, $(3,2)$, $(4,2)$, $(3,4)$, $(4,4)$, $(5,4)$ entries, which are dependent on other ones.

4.4. Computation of $\mathrm{Hom}_{\Lambda}(T_3, M_n^{(5)})$**.** A homomorphism $g : T_3 \to M_n^{(5)}$ is given by matrices $C = (c_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $D = (d_{ij}) \in \mathrm{M}_{n+3,1}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,2}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,2}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,2}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,3}(K)$ such that $D = X_{n+3,n+2}C$, $EX_{2,1} = X_{n+4,n+3}D$, $SX_{3,2} = X_{n+6,n+4}E$, $GY_{2,1} = Y_{n+4,n+2}F$, $SY_{3,2} = Y_{n+6,n+4}G$, $SZ(T_1) = Z_{n+6,n+3}H$.

The first six equations yield $s_{12} = s_{13} = s_{22} = s_{23} = s_{33} = s_{43} = s_{n+3,1} = s_{n+4,1} = s_{n+5,1} = s_{n+5,2} = s_{n+6,1} = s_{n+6,2} = 0$. Moreover, from the equation $SZ(T_1) = Z_{n+6,n+3}H$ we find that s_{21}, s_{31}, s_{41}, s_{32}, s_{42}, s_{52} are linearly dependent on other s_{ij}'s. Hence $\dim_K \mathrm{Hom}_{\Lambda}(T_3, M_n^{(5)}) = 3n$. A basis can be constructed as follows. Let u_{ij} be the $(n+6)\times 3$-matrix with the (i,j) entry 1 and other entries zero, except for the $(2,1)$, $(3,1)$, $(4,1)$, $(3,2)$, $(4,2)$, $(5,2)$ entries that depend on other ones. Then the matrices $u_{11}, u_{51}, \dots, u_{n+2,1}, u_{62}, u_{72}, \dots, u_{n+4,2}, u_{53}, u_{63}, \dots, u_{n+6,3}$ form a basis of $\mathrm{Hom}_{\Lambda}(T_3, M_n^{(5)})$.

4.5. Computation of $\mathrm{Hom}_{\Lambda}(T_4, M_n^{(5)})$**.** A homomorphism $g : T_4 \to M_n^{(5)}$ is given by matrices $B = (b_{ij}) \in \mathrm{M}_{n+1,1}(K)$, $C = (c_{ij}) \in \mathrm{M}_{n+2,2}(K)$, $D = (d_{ij}) \in \mathrm{M}_{n+3,3}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,4}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,2}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,3}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,2}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,5}(K)$ such that $CX_{2,1} = X_{n+2,n+1}B$, $DX_{3,2} = X_{n+3,n+2}C$, $EX_{4,3} = X_{n+4,n+3}D$, $SX_{5,4} = X_{n+6,n+4}E$, $GY_{3,2} = Y_{n+4,n+2}F$, $SY_{5,3} = Y_{n+6,n+4}G$, $SZ(T_4) = Z_{n+6,n+3}H$.

It is easy to verify that we have $s_{13} = s_{14} = s_{15} = s_{23} = s_{24} = s_{25} = s_{34} = s_{35} = s_{44} = s_{45} = s_{n+2,1} = s_{n+3,1} = s_{n+3,2} = s_{n+4,1} = s_{n+4,2} = s_{n+4,3} = s_{n+5,1} = s_{n+5,2} = s_{n+5,3} = s_{n+5,4} = s_{n+6,1} = s_{n+6,2} = s_{n+6,3} = s_{n+6,4} = 0$. Moreover s_{21}, s_{31}, s_{41}, s_{22}, s_{32}, s_{42} are linearly dependent on other s_{ij}'s. Hence $\dim_K \mathrm{Hom}_{\Lambda}(T_4, M_n^{(5)}) = 5n$ and a basis is given by the $(n+6)\times 5$-matrices $x_{11}, x_{51}, \dots, x_{n+1,1}, x_{12}, x_{52}, \dots, x_{n+2,2}, x_{33}, x_{43}, \dots, x_{n+3,3}, x_{54}, x_{64}, \dots, x_{n+4,4}, x_{55}, x_{65}, \dots, x_{n+6,5}$, where x_{ij} is the matrix with the (i,j) entry 1 and all other entries zero, except the $(2,1)$, $(3,1)$, $(4,1)$, $(2,2)$, $(3,2)$, $(4,2)$ entries which are dependent on other ones.

4.6. Computation of $\mathrm{Hom}_{\Lambda}(T_5, M_n^{(5)})$**.** A homomorphism $g : T_5 \to M_n^{(5)}$ is given by matrices $B = (b_{ij}) \in \mathrm{M}_{n+1,1}(K)$, $C = (c_{ij}) \in \mathrm{M}_{n+2,2}(K)$,

$D=(d_{ij})\in \mathrm{M}_{n+3,3}(K)$, $E=(e_{ij})\in \mathrm{M}_{n+4,3}(K)$, $F=(f_{ij})\in \mathrm{M}_{n+2,1}(K)$, $G=(g_{ij})\in \mathrm{M}_{n+4,2}(K)$, $H=(h_{ij})\in \mathrm{M}_{n+3,2}(K)$ and $S=(s_{ij})\in \mathrm{M}_{n+6,4}(K)$ such that $CX_{2,1}=X_{n+2,n+1}B$, $DX_{3,2}=X_{n+3,n+2}C$, $X_{n+4,n+3}D=E$, $SX_{4,3}=X_{n+6,n+4}E$, $GY_{2,1}=Y_{n+4,n+2}F$, $SY_{4,2}=Y_{n+6,n+4}G$, $SZ(T_5)=Z_{n+6,n+3}H$.

Again it is easy to see that $s_{13}=s_{14}=s_{23}=s_{24}=s_{34}=s_{44}=s_{n+2,1}=s_{n+3,1}=s_{n+3,2}=s_{n+4,1}=s_{n+4,2}=s_{n+4,3}=s_{n+5,1}=s_{n+5,2}=s_{n+5,3}=s_{n+6,1}=s_{n+6,2}=s_{n+6,3}=0$ and the entries s_{21}, s_{31}, s_{41}, s_{22}, s_{32}, s_{42} are linearly dependent on other s_{ij}'s. Hence $\dim_K \mathrm{Hom}_\Lambda(T_5, M_n^{(5)})=4n$ and the matrices $y_{11}, y_{51}, \ldots, y_{n+1,1}, y_{12}, y_{52}, \ldots, y_{n+2,2}, y_{33}, y_{43}, \ldots, y_{n+3,3}, y_{54}, y_{64}, \ldots, y_{n+6,4}$ form a basis, where y_{ij} is the $(n+6)\times 4$-matrix with the (i,j) entry 1, the $(3,1)$, $(4,1)$, $(5,1)$ entries depending on other ones, and all other entries zero.

4.7. Computation of $\mathrm{Hom}_\Lambda(T_6, M_n^{(5)})$**.** A homomorphism $g: T_6\to M_n^{(5)}$ is given by matrices $B=(b_{ij})\in \mathrm{M}_{n+1,1}(K)$, $C=(c_{ij})\in \mathrm{M}_{n+2,2}(K)$, $D=(d_{ij})\in \mathrm{M}_{n+3,2}(K)$, $E=(e_{ij})\in \mathrm{M}_{n+4,2}(K)$, $F=(f_{ij})\in \mathrm{M}_{n+2,1}(K)$, $G=(g_{ij})\in \mathrm{M}_{n+4,2}(K)$, $H=(h_{ij})\in \mathrm{M}_{n+3,1}(K)$ and $S=(s_{ij})\in \mathrm{M}_{n+6,3}(K)$ such that $CX_{2,1}=X_{n+2,n+1}B$, $D=X_{n+3,n+2}C$, $E=X_{n+4,n+3}D$, $SX_{3,2}=X_{n+6,n+4}E$, $GY_{2,1}=Y_{n+4,n+2}F$, $SY_{3,2}=Y_{n+6,n+4}G$, $SZ(T_6)=Z_{n+6,n+3}H$.

The first six equations imply the conditions $s_{12}=s_{13}=s_{22}=s_{23}=s_{33}=s_{43}=s_{n+2,1}=s_{n+3,1}=s_{n+3,2}=s_{n+4,1}=s_{n+4,2}=s_{n+5,1}=s_{n+5,2}=s_{n+6,1}=s_{n+6,2}=0$. The last matrix identity implies that the entries s_{21}, s_{31}, s_{41} are linearly dependent on other s_{ij}'s. Therefore $\dim_K \mathrm{Hom}_\Lambda(T_6, M_n^{(5)})=3n$. Moreover, a basis is given by the $(n+6)\times 3$-matrices $z_{11}, z_{51}, \ldots, z_{n+1,1}$, $z_{32}, z_{42}, \ldots, z_{n+2,2}, z_{53}, z_{63}, \ldots, z_{n+6,3}$, where z_{ij} is the matrix with 1 at place (i,j), with the $(2,1)$, $(3,1)$ and $(4,1)$ entries depending on other ones, and all other entries zero.

4.8. Computation of $\mathrm{Hom}_\Lambda(T_7, M_n^{(5)})$**.** A homomorphism $g: T_7\to M_n^{(5)}$ is given by matrices $B=(b_{ij})\in \mathrm{M}_{n+1,1}(K)$, $C=(c_{ij})\in \mathrm{M}_{n+2,1}(K)$, $D=(d_{ij})\in \mathrm{M}_{n+3,1}(K)$, $E=(e_{ij})\in \mathrm{M}_{n+4,1}(K)$, $F=(f_{ij})\in \mathrm{M}_{n+2,1}(K)$, $G=(g_{ij})\in \mathrm{M}_{n+4,1}(K)$, $H=(h_{ij})\in \mathrm{M}_{n+3,1}(K)$ and $S=(s_{ij})\in \mathrm{M}_{n+6,2}(K)$ such that $C=X_{n+2,n+1}B$, $D=X_{n+3,n+2}C$, $E=X_{n+4,n+3}D$, $SX_{2,1}=X_{n+6,n+4}E$, $G=Y_{n+4,n+2}F$, $SY_{2,1}=Y_{n+6,n+4}G$, $Z_{n+6,n+3}H=SZ(T_7)$.

Then we obtain $s_{12}=s_{22}=s_{32}=s_{42}=s_{n+2,1}=s_{n+3,1}=s_{n+4,1}=s_{n+5,1}=s_{n+6,1}=0$. Furthermore the entries s_{21}, s_{31}, s_{41} are linearly dependent on other s_{ij}'s. Therefore $\dim_K \mathrm{Hom}_\Lambda(T_7, M_n^{(5)})=2n$ and a basis is given by the $(m+6)\times 2$-matrices $r_{11}, r_{51}, \ldots, r_{n+1,1}, r_{52}, r_{62}, \ldots, r_{n+6,2}$, where r_{ij} is the matrix with the (i,j) entry 1 and all other entries 0, except the $(2,1)$, $(3,1)$, $(4,1)$ entries which are linearly dependent on other ones.

4.9. Computation of $\mathrm{Hom}_\Lambda(T_8, M_n^{(5)})$**.** A homomorphism $g : T_8 \to M_n^{(5)}$ is given by matrices $A = (a_{ij}) \in \mathrm{M}_{n,1}(K)$, $B = (b_{ij}) \in \mathrm{M}_{n+1,1}(K)$, $C = (c_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $D = (d_{ij}) \in \mathrm{M}_{n+3,1}(K)$, $E = (e_{ij}) \in \mathrm{M}_{n+4,1}(K)$, $F = (f_{ij}) \in \mathrm{M}_{n+2,1}(K)$, $G = (g_{ij}) \in \mathrm{M}_{n+4,1}(K)$, $H = (h_{ij}) \in \mathrm{M}_{n+3,1}(K)$ and $S = (s_{ij}) \in \mathrm{M}_{n+6,2}(K)$ such that $B = X_{n+1,n}A$, $C = X_{n+2,n+1}B$, $D = X_{n+3,n+2}C$, $E = X_{n+4,n+3}D$, $SX_{2,1} = X_{n+6,n+4}E$, $F = Y_{n+2,n}A$, $G = Y_{n+4,n+2}F$, $SY_{2,1} = Y_{n+6,n+4}G$, $SZ(T_8) = Z_{n+6,n+3}H$.

From these conditions we get $s_{12} = s_{22} = s_{32} = s_{42} = s_{52} = s_{62} = s_{n+1,1} = s_{n+2,1} = s_{n+3,1} = s_{n+4,1} = s_{n+5,1} = s_{n+6,1} = 0$, and we have the identities $s_{k,1} = s_{k+6,2}$ for $k = 1, \ldots, n$. Therefore $\dim_K \mathrm{Hom}_\Lambda(T_8, M_n^{(5)}) = n$ and a basis is given by $s_1, \ldots, s_n$ where s_i is the matrix with the $(i, 1)$ and $(i + 6, 2)$ entries 1 and all other entries zero.

4.10. Computation of matrices for $\mathrm{Hom}_\Lambda(h_{\alpha_i}, M_n^{(5)})$**,** $i = 0, 1, \ldots, 8$**.** We computed bases of the vector spaces $\mathrm{Hom}_\Lambda(T_i, M_n^{(5)})$ for $i = 0, 1, \ldots, 8$. Now we describe the matrices of the representation $\mathrm{Hom}_\Lambda(T, M_n^{(5)})$ for the extended Dynkin quiver Q. We need to find the matrices of the linear maps $\mathrm{Hom}_\Lambda(h_i, M_n^{(5)})$ in the bases obtained above. The linear maps $\mathrm{Hom}_\Lambda(h_i, M_n^{(5)})$ can be treated as multiplication from the right. In particular, the K-homomorphism

$$\mathrm{Hom}_\Lambda(h_{\alpha_1}, M_n^{(5)}) : \mathrm{Hom}_\Lambda(T_0, M_n^{(5)}) \to \mathrm{Hom}_\Lambda(T_1, M_n^{(5)})$$

is given by the formula

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s_{31} & 0 \\ s_{41} & 0 \\ s_{51} & s_{52} \\ \vdots & \vdots \\ s_{n+3,1} & s_{n+3,2} \\ 0 & s_{n+4,2} \\ 0 & s_{n+5,2} \\ 0 & s_{n+6,2} \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -s_{31} & s_{31} & 0 \\ 0 & -s_{41} & s_{41} & 0 \\ 0 & -s_{51} & s_{51} & s_{52} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -s_{n+3,1} & s_{n+3,1} & s_{n+3,2} \\ 0 & 0 & 0 & s_{n+4,2} \\ 0 & 0 & 0 & s_{n+5,2} \\ 0 & 0 & 0 & s_{n+6,2} \end{bmatrix}.$$

This linear map in the bases $q_{61}, q_{71}, \ldots, q_{n+3,1}, q_{52}, q_{62}, \ldots, q_{n+6,2}$ and $w_{11}, w_{51}, \ldots, w_{n+1,1}, w_{32}, w_{34}, \ldots, w_{n+3,2}, w_{63}, w_{73}, \ldots, w_{n+4,3}, w_{54}, w_{64}, \ldots, w_{n+6,4}$ has the form

$$A_5^{(5)} = \begin{array}{|c|c|} \hline & \\ \hline P_{n-2}^{(5)} & \begin{matrix} 0 & \\ 0 & P_{n+1}^{(5)} \\ 1 & \end{matrix} \\ \hline -I_{n-2} & \\ \hline \begin{matrix} & I_{n-2} & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array},$$

where

$$P_k^{(5)} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix}$$

is the $k \times 3$-matrix with three periodic rows of period 12 and the empty matrix-blocks are zero matrices of appropriate size.

By calculating the other linear maps in the same way we obtain a series of indecomposable representations of $\mathrm{Hom}_\Lambda(T, M_n^{(5)})$ for the quiver Q in the case $n \equiv 0$ or $n \equiv 3 \bmod 6$, namely

(4.5)
$$\begin{tikzcd} & & K^{3n} \arrow[d, "C_5"] & & & & & \\ K^{2n} \arrow[r, "A_5^{(5)}"] & K^{4n} \arrow[r, "B_5"] & K^{6n+1} & K^{5n} \arrow[l, "D_5"'] & K^{4n} \arrow[l, "E_5"'] & K^{3n} \arrow[l, "F_5^{(5)}"'] & K^{2n} \arrow[l, "G_5"'] & K^{n} \arrow[l, "H_5^{(5)}"'] \end{tikzcd}$$

where

$$B_5 = \begin{array}{|c|c|c|c|} \hline I_{n-2} & & & \\ \hline & & & \\ \hline & I_{n+1} & & \\ \hline & & I_{n-1} & \\ \hline & & & \\ \hline & & & I_{n+2} \\ \hline \end{array}, \quad C_5 = \begin{array}{|c|c|c|} \hline & & \\ \hline I_{n-1} & & \\ \hline & & \\ \hline & I_{n-1} & \\ \hline & & -I_{n+1} \\ \hline & & I_{n+1} \\ \hline \end{array}, \quad G_5 = \begin{array}{|c|c|} \hline I_{n-2} & \\ \hline & \\ \hline & I_{n+2} \\ \hline \end{array},$$

$$D_5 = \begin{array}{|c|c|c|c|c|} \hline I_{n-2} & & & & \\ \hline & I_{n-1} & & & \\ \hline & & I_{n+1} & & \\ \hline & & & \begin{matrix} 0 & \\ \vdots & I_n \\ 0 & \end{matrix} & \\ \hline & & & & I_{n+2} \\ \hline & & & \begin{matrix} & -I_n & \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \\ \hline \end{array}, \quad E_5 = \begin{array}{|c|c|c|c|} \hline I_{n-2} & & & \\ \hline \begin{matrix} & -I_{n-2} & \\ 0 & \cdots & 0 \end{matrix} & I_{n-1} & & \\ \hline & & I_{n+1} & \\ \hline & & & \\ \hline & & & -I_{n+2} \\ \hline \end{array},$$

$$F_5^{(5)} = \begin{array}{|c|c|c|} \hline I_{n-2} & & \\ \hline & \begin{matrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & -I_{n-2} & \\ 0 & 0 & & \end{matrix} & \\ \hline & \begin{matrix} & & I_n & \\ 0 & 0 & \cdots & 0 \end{matrix} & \\ \hline & & I_{n+2} \\ \hline \end{array}, \quad H_5^{(5)} = \begin{array}{|cccccc|} \hline 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & I_{n-4} & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ & & & I_n & & \\ \hline \end{array}.$$

In the case of $n \equiv 1$ or $n \equiv 5 \bmod 6$, we obtain the representations of the form

(4.6)
$$\begin{tikzcd} & & K^{3n} \arrow[d, "C_5"] & & & & & \\ K^{2n} \arrow[r, "A_5^{(2)}"] & K^{4n} \arrow[r, "B_5"] & K^{6n+1} & K^{5n} \arrow[l, "D_5"'] & K^{4n} \arrow[l, "E_5"'] & K^{3n} \arrow[l, "F_5^{(2)}"'] & K^{2n} \arrow[l, "G_5"'] & K^{n} \arrow[l, "H_5^{(2)}"'] \end{tikzcd}$$

with

$$A_5^{(2)}=\begin{array}{|c|c|}\hline & \\ \hline \begin{matrix}1 & \\ -1 & P^{(2)}_{n-3} \\ 1 & \end{matrix} & \begin{matrix}0 & 1 & \\ 0 & -1 & P^{(2)}_{n} \\ 1 & 1 & \end{matrix} \\ \hline -I_{n-2} & \\ \hline \begin{matrix} & I_{n-2} & \\ 0 & \cdots & 0\end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array},$$

where

$$P_k^{(2)}=\begin{bmatrix}1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & \cdots\end{bmatrix}$$

is the $k\times 3$-matrix with three periodic rows of period 12 and

$$F_5^{(2)}=\begin{array}{|c|c|c|}\hline I_{n-2} & & \\ \hline & \begin{matrix}0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & -I_{n-3} & \\ 0 & 0 & 0 & & \end{matrix} & \\ \hline & \begin{matrix} & I_n & \\ 0 & \cdots & 0\end{matrix} & \\ \hline & & I_{n+2} \\ \hline \end{array},\qquad H_5^{(2)}=\begin{array}{|c|}\hline \begin{matrix}1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & I_{n-6} & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\end{matrix} \\ \hline \begin{matrix}0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\end{matrix} \\ I_n \\ \hline \end{array}.$$

and B_5, C_5, D_5, E_5, G_5 have the same form as in the case of $n\equiv 0$ or $n\equiv 3$ mod 6.

In the last case $n\equiv 2$ or $n\equiv 4$ mod 6, we get the representations of the form

(4.7)
$$\begin{tikzcd} & & K^{3n} \arrow[d, "C_5"] & & & & & \\ K^{2n} \arrow[r, "A_5^{(1)}"] & K^{4n} \arrow[r, "B_5"] & K^{6n+1} & K^{5n} \arrow[l, "D_5"'] & K^{4n} \arrow[l, "E_5"'] & K^{3n} \arrow[l, "F_5^{(2)}"'] & K^{2n} \arrow[l, "G_5"'] & K^{n} \arrow[l, "H_5^{(1)}"'] \end{tikzcd}$$

with

$$H_5^{(1)}=\begin{array}{|c|}\hline \begin{matrix}1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & I_{n-5} & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\end{matrix} \\ \hline \begin{matrix}0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\end{matrix} \\ I_n \\ \hline \end{array},\qquad A_5^{(1)}=\begin{array}{|c|c|}\hline & \\ \hline P^{(1)}_{n-2} & \begin{matrix}0 & \\ 0 & P^{(1)}_{n+1} \\ 1 & \end{matrix} \\ \hline -I_{n-2} & \\ \hline \begin{matrix} & I_{n-2} & \\ 0 & \cdots & 0\end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array},$$

where

$$P_k^{(1)}=\begin{bmatrix}0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\ -1 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & \cdots\end{bmatrix}$$

is the $k\times 3$-matrix with three periodic rows of period 12 and B_5, C_5, D_5, E_5, G_5, $F_5^{(2)}$ have the same form as in the other cases.

5. Other examples of indecomposable preprojective KQ-modules. In this section we will present the other indecomposable rank 6 representations over the path algebra of the extended Dynkin quiver $\widetilde{\mathbb{E}}_8$. We say that an indecomposable preprojective module N over the path algebra KQ has rank n if there exists a Λ-module M of rank n such that $\operatorname{Ext}^1_\Lambda(T, M) = 0$ and $N \cong \operatorname{Hom}_\Lambda(T, M)$. In Section 4 we computed one series of indecomposable representations of rank 6. Now, we will present the other indecomposable representations of rank 6. Note that the matrices $P_k^{(1)}$, $P_k^{(2)}$, $P_k^{(5)}$ have the same form as in the last section.

a) Type I: $\operatorname{Hom}_\Lambda(T, M_n^{(1)})$

• *Case $n \equiv 0$ or $n \equiv 2$ or $n \equiv 4$* mod 6:

$$\tag{5.1} \begin{array}{ccccccccccccccc} & & & & K^{3n+2} & & & & & & & & & & \\ & & & & \big\downarrow{\scriptstyle C_1} & & & & & & & & & & \\ K^{2n+1} & \xrightarrow{A_1^{(1)}} & K^{4n+3} & \xrightarrow{B_1} & K^{6n+5} & \xleftarrow{D_1} & K^{5n+4} & \xleftarrow{E_1} & K^{4n+3} & \xleftarrow{F_1} & K^{3n+2} & \xleftarrow{G_1} & K^{2n+1} & \xleftarrow{H_1^{(1)}} & K^n \end{array}$$

where

$$A_1^{(1)} = \begin{array}{|c|c|}\hline & \\ \hline P_{n-2}^{(1)} & \begin{matrix}0\\0\\1\end{matrix}\ P_{n+1}^{(1)} \\ \hline -I_{n-1} & \\ \hline \begin{matrix} I_{n-1} \\ 0 \ \cdots \ 0\end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array}, \qquad B_1 = \begin{array}{|c|c|c|c|}\hline I_{n-1} & & & \\ \hline & & & \\ \hline & I_{n+2} & & \\ \hline & & I_n & \\ \hline & & & \\ \hline & & & I_{n+2} \\ \hline \end{array},$$

$$D_1 = \begin{array}{|c|c|c|c|c|}\hline I_{n-1} & & & & \\ \hline & I_n & & & \\ \hline & & I_{n+2} & & \\ \hline & & & \begin{matrix}0 & \\ \vdots & I_n \\ 0 & \end{matrix} & \\ \hline & & & & I_{n+2} \\ \hline & & & \begin{matrix} -I_{n+1} \\ 0 \ \cdots \ 0 \end{matrix} & \\ \hline \end{array}, \qquad E_1 = \begin{array}{|c|c|c|c|}\hline I_{n-1} & & & \\ \hline \begin{matrix} -I_{n-1} \\ 0 \ \cdots \ 0\end{matrix} & I_n & & \\ \hline & & I_{n+2} & \\ \hline & & & \\ \hline & & & -I_{n+2} \\ \hline \end{array},$$

$$F_1 = \begin{array}{|c|c|c|}\hline I_{n-1} & & \\ \hline & \begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ & & & -I_{n-2} & \\ 0 & 0 & 0 & & \end{matrix} & \\ \hline & \begin{matrix} I_{n+1} \\ 0 \ \cdots \ 0 \end{matrix} & \\ \hline & & I_{n+2} \\ \hline \end{array}, \qquad H_1^{(1)} = \begin{array}{|c|}\hline \begin{matrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & I_{n-5} & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ & & & I_n & & & \end{matrix} \\ \hline \end{array},$$

$$C_1 = \begin{array}{|c|c|c|}\hline & & \\ \hline I_n & & \\ \hline & & \\ \hline & I_n & \\ \hline & & -I_{n+2} \\ \hline & & I_{n+2} \\ \hline \end{array}, \qquad G_1 = \begin{array}{|c|c|}\hline I_{n-1} & \\ \hline & \\ \hline & I_{n+2} \\ \hline \end{array}.$$

• *Case $n \equiv 1$ or $n \equiv 3$ or $n \equiv 5$* mod 6:

$$\tag{5.2} \begin{array}{ccccccccccccccc} & & & & K^{3n+2} & & & & & & & & & & \\ & & & & \big\downarrow{\scriptstyle C_1} & & & & & & & & & & \\ K^{2n+1} & \xrightarrow{A_1^{(2)}} & K^{4n+3} & \xrightarrow{B_1} & K^{6n+5} & \xleftarrow{D_1} & K^{5n+4} & \xleftarrow{E_1} & K^{4n+3} & \xleftarrow{F_1} & K^{3n+2} & \xleftarrow{G_1} & K^{2n+1} & \xleftarrow{H_1^{(2)}} & K^n \end{array}$$

where

$$A_1^{(2)} = \begin{array}{|c|c|} \hline & \\ \hline \begin{matrix} 1 & \\ -1 & P_{n-2}^{(2)} \\ 1 & \end{matrix} & \begin{matrix} 0 & 1 & \\ 0 & -1 & P_n^{(2)} \\ 1 & 1 & \end{matrix} \\ \hline -I_{n-1} & \\ \hline \begin{matrix} & I_{n-1} & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array}, \qquad H_1^{(2)} = \begin{array}{|cccccccc|} \hline 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & I_{n-6} & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \multicolumn{8}{|c|}{I_n} \\ \hline \end{array},$$

and B_1, C_1, D_1, E_1, F_1, G_1 have the same form as in the above case.

b) Type II: $\mathrm{Hom}_\Lambda(T, M_n^{(2)})$

• *Case* $n \equiv 1$ *or* $n \equiv 3$ *or* $n \equiv 5 \bmod 6$:

(5.3)
$$\begin{tikzcd} & & K^{3n+2} \arrow[d, "C_2"] & & & & & \\ K^{2n+1} \arrow[r, "A_1^{(1)}"] & K^{4n+2} \arrow[r, "B_2"] & K^{6n+4} & K^{5n+3} \arrow[l, "D_2"'] & K^{4n+2} \arrow[l, "E_2"'] & K^{3n+1} \arrow[l, "F_2"'] & K^{2n} \arrow[l, "G_2"'] & K^n \arrow[l, "H_2^{(1)}"'] \end{tikzcd}$$

where

$$B_2 = \begin{array}{|c|c|c|c|} \hline I_{n-2} & & & \\ \hline & & & \\ \hline & I_{n+2} & & \\ \hline & & I_n & \\ \hline & & & \\ \hline & & & I_{n+2} \\ \hline \end{array}, \qquad C_2 = \begin{array}{|c|c|c|} \hline & & \\ \hline I_n & & \\ \hline & & \\ \hline & I_n & \\ \hline & & -I_{n+2} \\ \hline & & I_{n+2} \\ \hline \end{array}, \qquad G_2 = \begin{array}{|c|c|} \hline I_{n-2} & \\ \hline & \\ \hline & I_{n+2} \\ \hline \end{array},$$

$$D_2 = \begin{array}{|c|c|c|c|c|} \hline I_{n-2} & & & & \\ \hline & I_n & & & \\ \hline & & I_{n+2} & & \\ \hline & & & \begin{matrix} 0 & \\ \vdots & I_n \\ 0 & \end{matrix} & \\ \hline & & & & I_{n+2} \\ \hline & & & \begin{matrix} & -I_{n+1} & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline \end{array}, \qquad E_2 = \begin{array}{|c|c|c|c|} \hline I_{n-2} & & & \\ \hline \begin{matrix} & -I_{n-2} & \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & I_n & & \\ \hline & & I_{n+2} & \\ \hline & & & \\ \hline & & & -I_{n+2} \\ \hline \end{array},$$

$$F_2 = \begin{array}{|c|c|c|} \hline I_{n-2} & & \\ \hline & \begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & -I_{n-2} & \\ 0 & 0 & 0 & & \end{matrix} & \\ \hline & \begin{matrix} & I_{n+1} & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & & I_{n+2} \\ \hline \end{array}, \qquad H_2^{(1)} = \begin{array}{|ccccccc|} \hline 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & I_{n-5} & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{|c|}{I_n} \\ \hline \end{array}.$$

• *Case* $n \equiv 0$ *or* $n \equiv 2$ *or* $n \equiv 4 \bmod 6$:

(5.4)
$$\begin{tikzcd} & & K^{3n+2} \arrow[d, "C_2"] & & & & & \\ K^{2n+1} \arrow[r, "A_1^{(2)}"] & K^{4n+2} \arrow[r, "B_2"] & K^{6n+4} & K^{5n+3} \arrow[l, "D_2"'] & K^{4n+2} \arrow[l, "E_2"'] & K^{3n+1} \arrow[l, "F_2"'] & K^{2n} \arrow[l, "G_2"'] & K^n \arrow[l, "H_2^{(2)}"'] \end{tikzcd}$$

where

$$H_2^{(2)} = \left[\begin{array}{cccccccc} 1&0&0&0&0&0&\cdots&0\\ 0&1&0&0&0&0&\cdots&0\\ 0&0&0&0&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&I_{n-6}&\\ 0&0&0&0&0&0&\cdots&0\\ \hline 0&0&0&0&0&0&\cdots&0\\ 0&0&0&0&0&0&\cdots&0\\ \multicolumn{8}{c}{I_n} \end{array}\right],$$

and $A_1^{(2)}$, B_2, C_2, D_2, E_2, F_2, G_2 have the same form as in the other cases.

c) Type III: $\mathrm{Hom}_\Lambda(T, M_n^{(3)})$

• *Case* $n \equiv 0$ *or* $n \equiv 2$ *or* $n \equiv 4$ mod 6:

(5.5)

$$\begin{tikzcd} & & K^{3n+1} \arrow[d, "C_3"] & & & & & \\ K^{2n+1} \arrow[r, "A_1^{(1)}"] & K^{4n+2} \arrow[r, "B_2"] & K^{6n+3} & K^{5n+2} \arrow[l, "D_3"'] & K^{4n+1} \arrow[l, "E_3"'] & K^{3n} \arrow[l, "F_3"'] & K^{2n} \arrow[l, "G_2"'] & K^n, \arrow[l, "H_2^{(1)}"'] \end{tikzcd}$$

where

$$D_3 = \left[\begin{array}{c|c|c|c|c} I_{n-2} & & & & \\ \hline & I_{n-1} & & & \\ \hline & & I_{n+2} & & \\ \hline & & & \begin{array}{cc} 0 & \\ \vdots & I_n \\ 0 & \end{array} & \\ \hline & & & & I_{n+2} \\ \hline & & & \begin{array}{ccc} & -I_{n+1} & \\ 0 & \cdots & 0 \end{array} & \end{array}\right], \quad F_3 = \left[\begin{array}{c|c|c} I_{n-2} & & \\ \hline & \begin{array}{ccccc} 0&0&0&\cdots&0\\ 0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&-I_{n-3}&\\ 0&0&0&& \end{array} & \\ \hline & \begin{array}{ccc} & I_n & \\ 0&\cdots&0\\ 0&\cdots&0 \end{array} & \\ \hline & & I_{n+2} \end{array}\right],$$

$$C_3 = \left[\begin{array}{c|c|c} & & \\ \hline I_{n-1} & & \\ \hline & & \\ \hline & I_n & \\ \hline & & -I_{n+2} \\ \hline & & I_{n+2} \end{array}\right], \quad E_3 = \left[\begin{array}{c|c|c|c} I_{n-2} & & & \\ \hline \begin{array}{ccc} & -I_{n-2} & \\ 0 & \cdots & 0 \end{array} & I_{n-1} & & \\ \hline & & I_{n+2} & \\ \hline & & & \\ \hline & & & -I_{n+2} \end{array}\right].$$

• *Case* $n \equiv 1$ *or* $n \equiv 3$ *or* $n \equiv 5$ mod 6:

(5.6)

$$\begin{tikzcd} & & K^{3n+1} \arrow[d, "C_3"] & & & & & \\ K^{2n+1} \arrow[r, "A_1^{(2)}"] & K^{4n+2} \arrow[r, "B_2"] & K^{6n+3} & K^{5n+2} \arrow[l, "D_3"'] & K^{4n+1} \arrow[l, "E_3"'] & K^{3n} \arrow[l, "F_3"'] & K^{2n} \arrow[l, "G_2"'] & K^n \arrow[l, "H_2^{(2)}"'] \end{tikzcd}$$

where $A_1^{(1)}$, $A_1^{(2)}$, B_2, G_2, $H_2^{(1)}$, $H_2^{(2)}$ have the same form as in the other cases.

d) Type IV: $\mathrm{Hom}_\Lambda(T, M_n^{(4)})$

• *Case* $n \equiv 0$ *or* $n \equiv 3$ mod 6:

(5.7)

$$\begin{tikzcd} & & K^{3n+1} \arrow[d, "C_3"] & & & & & \\ K^{2n+1} \arrow[r, "A_4^{(4)}"] & K^{4n+1} \arrow[r, "B_4"] & K^{6n+2} & K^{5n+1} \arrow[l, "D_4"'] & K^{4n} \arrow[l, "E_4"'] & K^{3n} \arrow[l, "F_4"'] & K^{2n} \arrow[l, "G_2"'] & K^n \arrow[l, "H_4"'] \end{tikzcd}$$

where

$$B_4 = \begin{array}{|c|c|c|c|}\hline I_{n-2} & & & \\ \hline & & & \\ \hline & I_{n+1} & & \\ \hline & & I_n & \\ \hline & & & \\ \hline & & & \\ \hline & & & I_{n+2} \\ \hline \end{array}, \qquad A_4^{(4)} = \begin{array}{|c|c|}\hline & \\ \hline P^{(4)}_{n-2} & \begin{matrix} 0 & \\ 1 & P^{(4)}_{n+1} \\ 1 & \end{matrix} \\ \hline -I_{n-2} & \\ \hline \begin{matrix} & I_{n-2} & \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array},$$

where

$$P_k^{(4)} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & \cdots \end{bmatrix}$$

is the $k \times 3$-matrix with three periodic rows of period 12 and

$$D_4 = \begin{array}{|c|c|c|c|c|}\hline I_{n-2} & & & & \\ \hline & I_{n-1} & & & \\ \hline & & I_{n+1} & & \\ \hline & & & \begin{matrix} 0 & \\ \vdots & I_n \\ 0 & \end{matrix} & \\ \hline & & & & I_{n+2} \\ \hline & & & \begin{matrix} & -I_{n+1} & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline \end{array}, \qquad F_4 = \begin{array}{|c|c|c|}\hline I_{n-2} & & \\ \hline & \begin{matrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & -I_{n-2} & \\ 0 & 0 & & \end{matrix} & \\ \hline & \begin{matrix} & I_n & \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & & I_{n+2} \\ \hline \end{array},$$

$$E_4 = \begin{array}{|c|c|c|c|}\hline I_{n-2} & & & \\ \hline \begin{matrix} & -I_{n-2} & \\ 0 & \cdots & 0 \end{matrix} & I_{n-1} & & \\ \hline & & I_{n+1} & \\ \hline & & & \\ \hline & & & -I_{n+2} \\ \hline \end{array}, \qquad H_4 = \begin{array}{|c|}\hline \begin{matrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & I_{n-4} & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ & & & I_n & & \end{matrix} \\ \hline \end{array}$$

and C_3, G_2 have the same form as in the other cases.

- *Case* $n \equiv 1$ *or* $n \equiv 5 \bmod 6$:

(5.8)

$$\begin{array}{c} \qquad\qquad\qquad\qquad K^{3n+1} \\ \qquad\qquad\qquad\qquad \downarrow C_3 \\ K^{2n+1} \xrightarrow{A_4^{(3)}} K^{4n+1} \xrightarrow{B_4} K^{6n+2} \xleftarrow{D_4} K^{5n+1} \xleftarrow{E_4} K^{4n} \xleftarrow{F_4} K^{3n} \xleftarrow{G_2} K^{2n} \xleftarrow{H_4} K^n, \end{array}$$

with

$$A_4^{(3)} = \begin{array}{|c|c|}\hline & \\ \hline P^{(3)}_{n-2} & \begin{matrix} 0 & \\ 0 & P^{(3)}_{n+1} \\ 1 & \end{matrix} \\ \hline -I_{n-2} & \\ \hline \begin{matrix} & I_{n-2} & \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \\ \hline & I_{n+2} \\ \hline \end{array},$$

where

$$P_k^{(3)} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & \cdots \end{bmatrix}$$

is the $k \times 3$-matrix with three periodic rows of period 12 and C_3, G_2 have the same form as in the other cases.

- *Case $n \equiv 2$ or $n \equiv 4$* mod 6:

(5.9)
$$\begin{tikzcd}
 & & K^{3n+1} \arrow[d, "C_3"] & & & & & \\
K^{2n+1} \arrow[r, "A_4^{(2)}"] & K^{4n+1} \arrow[r, "B_4"] & K^{6n+2} & K^{5n+1} \arrow[l, "D_4"'] & K^{4n} \arrow[l, "E_4"'] & K^{3n} \arrow[l, "F_4"'] & K^{2n} \arrow[l, "G_2"'] & K^{n} \arrow[l, "H_2"']
\end{tikzcd}$$

where

$$A_4^{(2)} = \left[\begin{array}{ccc|cccc}
 & & & & & & \\
\hline
1 & & & 0 & 1 & & \\
-1 & P_{n-2}^{(2)} & & 0 & -1 & P_n^{(2)} & \\
1 & & & 1 & 1 & & \\
\hline
 & -I_{n-2} & & & & & \\
\hline
 & I_{n-2} & & & & & \\
0 & \cdots & 0 & & & & \\
\hline
 & & & & I_{n+2} & &
\end{array}\right]$$

and C_3, G_2, H_2 have the same form as in the other cases.

One can also calculate explicit matrices for the indecomposable preprojective modules of smaller rank. In the case of an algebraically closed field the indecomposable regular modules are calculated in the same way. For this the shape of the indecomposable Λ-modules obtained in [9] can be used. Note that we get all indecomposable preprojective and regular KQ-modules. This follows from the fact that the functor $\mathrm{Hom}_\Lambda(T, -)$ preserves Auslander–Reiten sequences. As stated in the introduction, matrices for the indecomposable preinjective KQ-modules can be obtained by choosing a tilting module over Λ with endomorphism ring isomorphic to KQ^{op} and considering its indecomposable preprojective modules. In particular we have proven

THEOREM 5.1. *Let K be a field and Q an extended Dynkin quiver of type $\widetilde{\mathbb{E}}_8$ with subspace orientation. The formulas* (4.5)–(4.7) *and* (5.1)–(5.9) *give a description of all indecomposable preprojective representations of rank* 6 *for this quiver by vector spaces and matrices.*

The indecomposability of the modules $N_n = \mathrm{Hom}_\Lambda(T, M_n)$ above follows from the Brenner–Butler theorem. Alternatively, one can calculate their endomorphism rings. It turns out that $\mathrm{End}_{KQ}(N_n) = K$, which also proves the indecomposability.

Thus we have obtained a practical method of describing the preprojective indecomposable KQ-modules by vector spaces and matrices. We emphasize that our method can also be applied to concealed-canonical algebras in the sense of [7] of domestic representation type, since these algebras are defined as endomorphism algebras of tilting bundles over weighted projective lines in the sense of [6]. Thus they can be obtained by tilting from domestic canonical algebras.

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