COLLOQUIUM MATHEMATICUM

VOL. 124 2011 NO. 1

REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM AND THEIR ALMOST CONTACT METRIC STRUCTURES

BY

YOUNG HO KIM (Taegu) and SADAHIRO MAEDA (Saga)

Dedicated to Professor Naoto Abe
on the occasion of his sixtieth birthday

Abstract. We characterize homogeneous real hypersurfaces of types (A_0), (A_1) and (B) in a complex projective space or a complex hyperbolic space.

1. Introduction. We denote by $\widetilde{M}_n(c)$, $n \geq 2$, a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c ($\neq 0$). That is, $\widetilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according to whether c is positive or negative. $\widetilde{M}_n(c)$ is a so-called *nonflat complex space form* of constant holomorphic sectional curvature c.

In this paper, we study real hypersurfaces M^{2n-1} of $\widetilde{M}_n(c)$. It is known that every such hypersurface admits an almost contact metric structure (ϕ, ξ, η, g) induced from the ambient space. So it is natural to study the theory of real hypersurfaces from the viewpoint of contact geometry (for example, see [1, 2]). Motivated by a fundamental idea in contact geometry, for a real hypersurface M^{2n-1} of $\widetilde{M}_n(c)$ we shall investigate the equation

$$(1.1) \qquad d\eta(X,Y) = \pm k \cdot g(X, \phi Y) \quad \text{for all vectors } X, Y \in TM,$$

where k is a positive constant. Equation (1.1) means that the exterior derivative d of the contact form η of M satisfies either $d\eta(X,Y) = k \cdot g(X, \phi Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -k \cdot g(X, \phi Y)$ for all $X, Y \in TM$. Note that (1.1) can be rewritten as $\phi A + A\phi = \mp 2k\phi$, where A is the shape operator

2010 *Mathematics Subject Classification*: Primary 53C40; Secondary 53C22.
Key words and phrases: real hypersurfaces, nonflat complex space forms, real hypersurfaces of type (A_0), (A_1) and (B), holomorphic distribution, geodesics, circles, contact manifolds, Sasakian manifolds, Sasakian space forms.

DOI: 10.4064/cm124-1-8

of M in $\widetilde{M}_n(c)$. (Cf. the proof of Theorem 2.) This implies that every real hypersurface satisfying (1.1) must be a Hopf hypersurface.

We first classify the real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$ satisfying (1.1) (Theorems 1 and 2). From these classification theorems we can see that every such hypersurface is locally a homogeneous real hypersurface of $\widetilde{M}_n(c)$, namely it is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space. We next characterize the hypersurfaces M^{2n-1} among all real hypersurfaces in $\widetilde{M}_n(c)$ by observing some geodesics on M^{2n-1} (Theorem 3 and Proposition 2).

We here remark that there exist no real hypersurfaces M with $d\eta = 0$ on M in a nonflat complex space form (see Corollary 2.12 in [8]).

2. Fundamental notions in contact geometry. Let M be an odd-dimensional Riemannian manifold furnished with an almost contact metric structure (ϕ, ξ, η, g), which consists of a $(1,1)$-tensor ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi\xi = 0$ and $\eta(\phi(X)) = 0$. We say that such an odd-dimensional manifold is an *almost contact metric manifold*. When the exterior derivative $d\eta$ of the contact form η on an almost contact metric manifold M which is given by $d\eta(X, Y) := (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\}$ satisfies

$$d\eta(X, Y) = g(X, \phi Y) \quad \text{for all } X, Y \in TM, \tag{2.1}$$

the structure (ϕ, ξ, η, g) is said to be a *contact metric structure* on M. An almost contact metric manifold having a contact metric structure is called a *contact manifold*. Note that contact manifolds are analogues to Hermitian manifolds in Kähler geometry. An almost contact metric manifold M is said to be a *Sasakian manifold* if the structure tensor ϕ of M satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.2}$$

with the Riemannian connection ∇ on M associated with g for all $X, Y \in TM$. By an easy computation we find that the structure of a Sasakian manifold is a contact metric structure. However, in general a contact metric structure need not be Sasakian. For a unit tangent vector $u \in TM$ orthogonal to ξ in a Sasakian manifold M we call $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ its *ϕ-sectional curvature*, where R is the curvature tensor of M. A *Sasakian space form* is a Sasakian manifold whose ϕ-sectional curvatures do not depend on the choice of unit tangent vectors orthogonal to ξ. Sasakian manifolds and Sasakian space forms are analogues to Kähler manifolds and complex space forms in Kähler geometry, respectively. For more details on contact geometry see [5] for example.

3. Fundamental theory of real hypersurfaces in $\widetilde{M}_n(c)$. Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ in an n-dimensional nonflat complex space form $\widetilde{M}_n(c)$ with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{3.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX, \tag{3.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\widetilde{M}_n(c)$. We call $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ the *principal foliation* associated to the principal curvature λ.

It is well-known that M has an almost contact metric structure induced from the Kähler structure of the ambient space $\widetilde{M}_n(c)$. That is, we have a quadruple (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N}, \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

It follows from (3.1), (3.2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{3.3}$$

$$\nabla_X \xi = \phi AX. \tag{3.4}$$

We clarify here the meaning of the condition that a real hypersurface M is a contact manifold with respect to the almost contact metric structure induced from the ambient space $\widetilde{M}_n(c)$. On an orientable connected real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$, we have an almost contact metric structure (ϕ, ξ, η, g) associated with a unit normal vector $\mathcal{N}$ of M in $\widetilde{M}_n(c)$. Clearly the quadruple $(\phi, -\xi, -\eta, g)$ is also an almost contact metric structure on M which is associated with the unit normal $-\mathcal{N}$. We call a real hypersurface M *contact* if M satisfies either (2.1) or

$$d\eta(X, Y) = -g(X, \phi Y)$$

for all vectors $X, Y \in TM$. Similarly, a real hypersurface M is called *Sasakian* if M satisfies either (2.2) or

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for all vectors $X, Y \in TM$.

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned}(3.5)\quad & g(R(X,Y)Z,W) \\ &= (c/4)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) \\ &\quad - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ &\quad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W).\end{aligned}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. The following properties of principal curvatures of a Hopf hypersurface M in $\widetilde{M}_n(c)$ are well-known.

LEMMA 1.

(1) *The principal curvature δ associated with ξ is locally constant.*
(2) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + c/2)\phi v$. In particular, when $c > 0$, we have $A\phi v = ((\delta\lambda + c/2)/(2\lambda - \delta))\phi v$.*

REMARK 1. When $c < 0$, in Lemma 1(2) it can happen that both the equations $2\lambda - \delta = 0$ and $\delta\lambda + c/2 = 0$ hold. In fact, for example we may take a horosphere in $\mathbb{C}H^n(c)$. It is known that this real hypersurface has two distinct constant principal curvatures, either $\lambda = \sqrt{|c|}/2,\ \delta = \sqrt{|c|}$ or $\lambda = -\sqrt{|c|}/2,\ \delta = -\sqrt{|c|}$. Hence, when $c < 0$, we must consider two cases $2\lambda - \delta = 0$ and $2\lambda - \delta \neq 0$.

Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of a nonflat complex space form $\widetilde{M}_n(c)$ is a Hopf hypersurface. This means that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in a nonflat complex space form.

In $\mathbb{C}P^n(c)$ $(n \geq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally one of the following (cf. [8]):

(A_1) a geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) a tube of radius r around a totally geodesic $\mathbb{C}P^{\ell}(c)$ $(1 \leq \ell \leq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) a tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;
(C) a tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n (≥ 5) is odd;
(D) a tube of radius r around a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
(E) a tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Hypersurfaces of type (A_1) or (A_2) are called of type (A).

The number of distinct principal curvatures of the above real hypersurfaces is $2, 3, 3, 5, 5, 5$, respectively. Their principal curvatures are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

Notice that in $\mathbb{C}P^n(c)$ a tube of radius r $(0 < r < \pi/\sqrt{c})$ around a totally geodesic $\mathbb{C}P^\ell(c)$ $(0 \le \ell \le n-1)$ is a tube of radius $\pi/\sqrt{c} - r$ around a totally geodesic $\mathbb{C}P^{n-\ell-1}(c)$.

In $\mathbb{C}H^n(c)$ $(n \ge 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally one of the following (cf. [8]):

- (A_0) a horosphere in $\mathbb{C}H^n(c)$;
- ($A_{1,0}$) a geodesic sphere of radius r $(0 < r < \infty)$;
- ($A_{1,1}$) a tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
- (A_2) a tube of radius r around a totally geodesic $\mathbb{C}H^\ell(c)$ $(1 \le \ell \le n-2)$, where $0 < r < \infty$;
- (B) a tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either ($A_{1,0}$) or ($A_{1,1}$). Hypersurfaces of types (A_0), (A_1) or (A_2) are said to be of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log(2+\sqrt{3})$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3})$ (cf. [4]). Except for this real hypersurface, the number of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures is $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows:

	(A_0)	$(A_{1,0})$	$(A_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

In [8], the above two tables of principal curvatures are given in the case of $c = \pm 4$.

It is well-known that our ambient manifold $\widetilde{M}_n(c)$ admits no totally umbilic real hypersurfaces. In this context, we recall that a real hypersurface M of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$, is called *totally η-umbilic* if its shape operator A is of the form $A = \alpha I + \beta\eta\otimes\xi$ for some smooth functions α and β on M. This is equivalent to saying that $Au = \alpha u$ for each vector u on M which is orthogonal to the characteristic vector ξ of M, where α is a smooth function on M. It is known that every totally η-umbilic hypersurface is a Hopf hypersurface with two distinct constant principal curvatures α and $\alpha+\beta$.

A totally η-umbilic hypersurface M^{2n-1}, $n \geq 2$, with shape operator $A = \alpha I + \beta\eta\otimes\xi$ in a nonflat complex space form $\widetilde{M}_n(c)$ is locally one of the following:

- (P) a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$, where $\alpha = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\beta = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$;
- (H_i) a horosphere in $\mathbb{C}H^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$;
- (H_{ii}) a geodesic sphere of radius r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$, where $\alpha = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$ and $\beta = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$;
- (H_{iii}) a tube of radius r $(0 < r < \infty)$ around a totally geodesic complex hyperplane $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$, where

$$\alpha = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2), \qquad \beta = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2).$$

Totally η-umbilic hypersurfaces are interesting examples of Riemannian manifolds. The length spectrum of such a hypersurface was studied in detail (see [3]). Moreover, it is well-known that every geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$ is a Berger sphere ([9]).

We recall here characterizations of real hypersurfaces of type (A) and type (B) in a nonflat complex space form. It is known that a real hypersurface M of a nonflat complex space form is of type (A) if and only if $\phi A = A\phi$ on M (see [8]). The following characterization of real hypersurfaces of type (B) was established in [6].

LEMMA 2. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$. Then the following two conditions are equivalent:*

(1) *M is a real hypersurface of type* (B).
(2) *The holomorphic distribution $T^0M = \{X \in TM \mid X \perp \xi\}$ of M decomposes into the direct sum of restricted principal foliations $V^0_{\lambda_i} = \{X \in T^0M \mid AX = \lambda_i X\}$. Moreover, every restricted principal foliation $V^0_{\lambda_i}$ is integrable and each of its leaves is a totally geodesic submanifold of M.*

In contrast with the conclusion of Lemma 2, for every Hopf hypersurface M in a nonflat complex space form, the holomorphic distribution T^0M is *not* integrable (see Proposition 2 in [6]).

In this paper, real hypersurfaces of types (A), (B), (C), (D) and (E) in $\widetilde{M}_n(c)$ are said to be *standard real hypersurfaces*. It is well-known that every standard real hypersurface M is a homogeneous real hypersurface of $\widetilde{M}_n(c)$.

4. Statements of results

THEOREM 1. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of $\mathbb{C}P^n(c)$. If* (1.1) *holds on M, then M is locally one of the following homogeneous real hypersurfaces:*

(1) *a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c}\,)\tan^{-1}\bigl(\sqrt{c}/(2k)\bigr)$, $0 < r < \pi/\sqrt{c}$,*
(2) *a tube of radius $r = (2/\sqrt{c}\,)\tan^{-1}\bigl((\sqrt{c+4k^2}-\sqrt{c}\,)/(2k)\bigr)$, $0 < r < \pi/(2\sqrt{c}\,)$, around a complex hyperquadric $\mathbb{C}Q^{n-1}$.*

THEOREM 2. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of $\mathbb{C}H^n(c)$. If* (1.1) *holds on M, then M is locally one of the following homogeneous real hypersurfaces:*

(1) *a horosphere in $\mathbb{C}H^n(c)\,(c = -4k^2)$,*
(2) *either a geodesic sphere $G(r)$ of radius $r = (1/\sqrt{|c|}\,)\{\log(2k+\sqrt{|c|}\,)-\log(2k-\sqrt{|c|}\,)\}$ or a tube of radius $r = (1/(2\sqrt{|c|}\,))\{\log(2k+\sqrt{|c|}\,)-\log(2k-\sqrt{|c|}\,)\}$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)$ $(-4k^2 < c < 0)$,*
(3) *a tube of radius $r = (1/\sqrt{|c|}\,)\{\log(\sqrt{|c|}+2k)-\log(\sqrt{|c|}-2k)\}$ around a totally geodesic $\mathbb{C}H^{n-1}(c)\,(c < -4k^2)$.*

Proof of Theorem 1. It follows from (1.1) and (3.4) that

$$0 = g(\phi AX, Y) - g(\phi AY, X) \mp 2kg(X, \phi Y) = g((\phi A + A\phi \pm 2k\phi)X, Y)$$

for each $X, Y \in TM$. This implies that a real hypersurface M of $\mathbb{C}P^n(c)$

satisfies (1.1) if and only if

$$\phi A + A\phi = \mp 2k\phi. \tag{4.1}$$

So we shall determine real hypersurfaces M satisfying (4.1). We then have $\phi A\xi = 0$, which shows that ξ is principal. We denote by δ its principal curvature. We study principal curvatures λ associated with principal curvature vectors orthogonal to ξ. We remark here that (4.1) shows that $A\phi X = (\mp 2k - \lambda)\phi X$ for each vector X perpendicular to ξ. This, together with Lemma 1(2) and (4.1), means that the principal curvature λ satisfies one of the following quadratic equations:

$$4\lambda^2 + 8k\lambda + c - 4k\delta = 0 \quad \text{or} \quad 4\lambda^2 - 8k\lambda + c + 4k\delta = 0. \tag{4.2}$$

Since k and δ are constant, this implies that λ is also constant on the connected real hypersurface M. Thus we can see that our real hypersurface is a Hopf hypersurface with at most three distinct constant principal curvatures. In view of the list of principal curvatures in Section 3 we find that M is of type either (A_1), (A_2) or (B). But real hypersurfaces of type (A_2) do not satisfy (4.1). Thus we only have to check (4.1) in detail for real hypersurfaces of type (A_1) or (B).

When M is of type (A_1), since all nonzero vectors orthogonal to ξ are principal curvature vectors associated with the principal curvature $(\sqrt{c}/2)\times \cot(\sqrt{c}\,r/2)$, (4.1) yields $\cot(\sqrt{c}\,r/2) = \mp 2k/\sqrt{c}$ $(0 < r < \pi/\sqrt{c})$. Thus the sign must be positive and $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/(2k))$.

When M is of type (B), (4.1) turns into $\lambda_1 + \lambda_2 = \mp 2k$ with principal curvatures $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 - \pi/4)$ and $\lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 + \pi/4)$. Since $0 < r < \pi/(2\sqrt{c})$, we have $\lambda_1 < -\sqrt{c}/2$ and $0 < \lambda_2 < \sqrt{c}/2$. Therefore, the sign must be negative. As $\lambda_1 + \lambda_2 = -2k$ is equivalent to the equality

$$\frac{\tan(\sqrt{c}\,r/2)+1}{\tan(\sqrt{c}\,r/2)-1} - \frac{\tan(\sqrt{c}\,r/2)-1}{\tan(\sqrt{c}\,r/2)+1} = -\frac{4k}{\sqrt{c}},$$

we obtain $\tan(\sqrt{c}\,r/2) = (\sqrt{c+4k^2} - \sqrt{c}\,)/(2k)$ because $0 < r < \pi/(2\sqrt{c}\,)$. We hence get the conclusion. ∎

Proof of Theorem 2. By the proof of Theorem 1 we only have to determine Hopf hypersurfaces M with $A\xi = \delta\xi$ satisfying (4.1). Since $c < 0$, we must consider the case that $2\lambda - \delta = 0$ at some point x of M (see Lemma 1). Towards a contradiction suppose that the function $2\lambda - \delta$ vanishes identically on no neighborhood of x. Then there exists a sequence $\{x_n\}$ in M with $\lim_{n\to\infty} x_n = x$ and $(2\lambda - \delta)(x_n) \neq 0$ for each n. The discussion in the proof of Theorem 1 means that for each n the function $2\lambda - \delta$ is a nonzero constant on some sufficiently small neighborhood of x_n. This, together with the continuity of $2\lambda - \delta$ on M, shows that $2\lambda - \delta \neq 0$ at x, which is a contradiction. Hence the principal curvature λ is also constant locally if $2\lambda - \delta = 0$

at some point x of M. Thus our real hypersurface is a Hopf hypersurface with at most four distinct constant principal curvatures. By considering the list of principal curvatures in Section 3 we see that M is of type either (A_0), (A_1), (A_2) or (B). But real hypersurfaces of type (A_2) do not satisfy (4.1). So we only have to investigate (4.1) for real hypersurfaces of type (A_0), (A_1) or (B).

When M is of type (A_0), (4.1) turns into $\sqrt{|c|} = \mp 2k$. Hence the sign must be positive and $c = -4k^2$. When M is of type ($\mathrm{A}_{1,0}$), (4.1) can be written as $\coth(\sqrt{|c|}\, r/2) = \mp 2k/\sqrt{|c|}$. Then the sign must be positive and $-4k^2 < c < 0$. Solving this, we obtain $r = (1/\sqrt{|c|})\{\log(2k + \sqrt{|c|}\,) - \log(2k - \sqrt{|c|}\,)\}$. When M is of type ($\mathrm{A}_{1,1}$), (4.1) turns into $\tanh(\sqrt{|c|}\, r/2) = \mp 2k/\sqrt{|c|}$. Hence the sign must be positive and $c < -4k^2$. Solving this, we obtain $r = (1/\sqrt{|c|})\{\log(\sqrt{|c|} + 2k) - \log(\sqrt{|c|} - 2k)\}$.

When M is of type (B), (4.1) turns into $\lambda_1 + \lambda_2 = \mp 2k$ with principal curvatures $\lambda_1 = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\, r/2)$ and $\lambda_2 = (\sqrt{|c|}/2)\cdot\tanh(\sqrt{|c|}\, r/2)$. Hence the sign must be positive. Rewriting the relation $\lambda_1 + \lambda_2 = 2k$, we have

$$\frac{\exp(\sqrt{|c|}\, r) + 1}{\exp(\sqrt{|c|}\, r) - 1} + \frac{\exp(\sqrt{|c|}\, r) - 1}{\exp(\sqrt{|c|}\, r) + 1} = \frac{4k}{\sqrt{|c|}};$$

we therefore obtain $-4k^2 < c < 0$ and $r = (1/(2\sqrt{|c|}))\{\log(2k + \sqrt{|c|}\,) - \log(2k - \sqrt{|c|}\,)\}$. ■

As an immediate consequence of statements (1) and (2) in Theorem 2 we obtain the following characterization of a horosphere and the homogeneous real hypersurface of type (B) with two distinct constant principal curvatures in $\mathbb{C}H^n(c)$.

COROLLARY 1. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of $\mathbb{C}H^n(c)$. Then:*

(1) *M is locally a horosphere in $\mathbb{C}H^n(c)$ if and only if* (1.1) *holds on M with $k = \sqrt{|c|}/2$.*
(2) *M is locally either a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{|c|}\,) \times \log(2 + \sqrt{3}\,)$ or a tube of radius $r = (1/\sqrt{|c|}\,)\log(2 + \sqrt{3}\,)$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)$ if and only if* (1.1) *holds on M with $k = \sqrt{|c|/3}$.*

When $k = 1$, Theorems 1 and 2 give the following classification theorems of real hypersurfaces which are contact in a nonflat complex space form.

COROLLARY 2. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of $\mathbb{C}P^n(c)$. If it is contact, then it is locally one of the following homogeneous real hypersurfaces:*

(1) *a geodesic sphere* $G(r)$ *of radius* $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)$, $0 < r < \pi/\sqrt{c}$,
(2) *a tube of radius* $r = (2/\sqrt{c})\tan^{-1}((\sqrt{c+4} - \sqrt{c})/2)$ *around a complex hyperquadric* $\mathbb{C}Q^{n-1}$, $0 < r < \pi/(2\sqrt{c})$.

Corollary 3. *Let* M^{2n-1} $(n \geq 2)$ *be a connected real hypersurface of* $\mathbb{C}H^n(c)$. *If it is contact, then it is locally one of the following homogeneous real hypersurfaces:*

(1) *a horosphere in* $\mathbb{C}H^n(c)$ $(c = -4)$,
(2) *either a geodesic sphere* $G(r)$ *of radius* $r = (1/\sqrt{|c|})\{\log(2+\sqrt{|c|}) - \log(2 - \sqrt{|c|})\}$ *or a tube of radius* $r = (1/(2\sqrt{|c|}))\{\log(2 + \sqrt{|c|}) - \log(2 - \sqrt{|c|})\}$ *around a totally real totally geodesic* $\mathbb{R}H^n(c/4)$ $(-4 < c < 0)$,
(3) *a tube of radius* $r = (1/\sqrt{|c|})\big\{\log(\sqrt{|c|}+2) - \log(\sqrt{|c|}-2)\big\}$ *around a totally geodesic* $\mathbb{C}H^{n-1}(c)$ $(c < -4)$.

Motivated by Corollaries 2 and 3, we establish the following classification theorem of real hypersurfaces which are Sasakian in a nonflat complex space form (cf. [4]).

Proposition 1. *Let* M^{2n-1} $(n \geq 2)$ *be a connected Sasakian real hypersurface of a nonflat complex space form* $\widetilde{M}_n(c)$. *Then* M *is locally one of the following homogeneous real hypersurfaces of the ambient space* $\widetilde{M}_n(c)$:

(i) *a geodesic sphere* $G(r)$ *of radius* r *with* $\tan(\sqrt{c}\,r/2) = \sqrt{c}/2$, *i.e.* $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)$ $(0 < r < \pi/\sqrt{c})$ *in* $\mathbb{C}P^n(c)$;
(ii) *a horosphere in* $\mathbb{C}H^n(c)$ $(c = -4)$;
(iii) *a geodesic sphere* $G(r)$ *of radius* r *with* $\tanh(\sqrt{|c|}\,r/2) = \sqrt{|c|}/2$ $(0 < r < \infty)$, *i.e.* $r = (1/\sqrt{|c|})\{\log(2 + \sqrt{|c|}) - \log(2 - \sqrt{|c|})\}$ *in* $\mathbb{C}H^n(c)$ $(-4 < c < 0)$;
(iv) *a tube of radius* r *around a totally geodesic* $\mathbb{C}H^{n-1}(c)$ *with*
$$\tanh(\sqrt{|c|}\,r/2) = 2/\sqrt{|c|}\ (0 < r < \infty),$$
i.e. $r = (1/\sqrt{|c|})\{\log(\sqrt{|c|}+2) - \log(\sqrt{|c|}-2)\}$ *in* $\mathbb{C}H^n(c)$ $(c < -4)$.

In these cases, M *has constant* ϕ*-sectional curvature* $c+1$. *Conversely, each of the hypersurfaces* (i)–(iv) *is Sasakian.*

Proof. Assume that our real hypersurface M is a Sasakian manifold. Then it follows from (2.2) and (3.3) that

$$g(X,Y)\xi - \eta(Y)X = \eta(Y)AX - g(AX,Y)\xi \tag{4.3}$$

for all $X, Y \in TM$. Setting $X = Y = \xi$ in (4.3), we see that ξ is principal. Hence we can choose a principal curvature vector u orthogonal to ξ. Then, setting $Y = \xi$ in (4.3), we find that $Au = -u$, so that the tangent bundle TM of M decomposes as $TM = \{\xi\}_{\mathbb{R}} \oplus V_{-1}$, where $V_{-1} = \{X \in TM \mid$

$AX = -X\}$. Thus a Sasakian real hypersurface M is a totally η-umbilic hypersurface with coefficients $\alpha = -1$ and $\beta = c/4$ in $\widetilde{M}_n(c)$. Here, we change the unit normal vector $\mathcal{N}$ into $-\mathcal{N}$ for each member in the list of totally η-umbilic hypersurfaces in Section 3. Then we know that M is locally one of (i)–(iv). Next, for each unit vector u perpendicular to ξ, we compute the ϕ-sectional curvature $K(u, \phi u)$ of M. It follows from (3.5) and the equality $A = -I + (c/4)\eta \otimes \xi$ that $K(u, \phi u) = c + 1$.

Conversely, assume that a real hypersurface M is locally one of (i)–(iv). Then the shape operator A of M is of the form $A = -I + (c/4)\eta \otimes \xi$ by changing $\mathcal{N}$ into $-\mathcal{N}$ for each member in the list of totally η-umbilic hypersurfaces in Section 3. This, combined with (3.3), yields (2.2), so that M is a Sasakian manifold. ∎

Theorems 1 and 2 show that real hypersurfaces satisfying (1.1) in a nonflat complex space form are of type (A) or (B). We shall characterize real hypersurfaces of type (A) satisfying (1.1).

THEOREM 3. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of a nonflat complex space form. Then for each positive constant k, the following conditions* (1) *and* (2) *are equivalent:*

(1) *M is locally one of the following:*
  - (1_a) *a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c}\,)\tan^{-1}(\sqrt{c}/(2k))$, $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$,*
  - (1_b) *a horosphere in $\mathbb{C}H^n(c)\,(c = -4k^2)$,*
  - (1_c) *a geodesic sphere $G(r)$ of radius $r = (1/\sqrt{|c|}\,)\{\log(2k+\sqrt{|c|}\,) - \log(2k - \sqrt{|c|}\,)\}$ in $\mathbb{C}H^n(c)\,(-4k^2 < c < 0)$,*
  - (1_d) *a tube of radius $r = (1/\sqrt{|c|}\,)\{\log(\sqrt{|c|} + 2k) - \log(\sqrt{|c|} - 2k)\}$ around a totally geodesic $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)\,(c < -4k^2)$.*

(2) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2} \in T_xM$ which are orthogonal to the characteristic vector ξ_x and satisfy:*
  - (2_a) *All geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ on M with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ are mapped to a circle of the same curvature k in $\widetilde{M}_n(c)$.*
  - (2_b) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)\,(1 \leq i < j \leq 2n-2)$ on M with $\gamma_{ij}(0) = x$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ are mapped to a circle of the same curvature k in $\widetilde{M}_n(c)$.*

Before proving Theorem 3 we review the definition of circles in Riemannian geometry. A real smooth curve $\gamma = \gamma(s)$ parameterized by its arclength s in a Riemannian manifold M with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies the ordinary differential equations $\nabla_{\dot{\gamma}}\dot{\gamma} = kY_s$, $\nabla_{\dot{\gamma}}Y_s = -k\dot{\gamma}$ with a field Y_s of unit vectors along γ.

Here k (≥ 0) is constant and Y_s is called the unit principal normal vector of γ. A circle of null curvature is nothing but a geodesic. A circle can be equivalently defined to be a curve $\gamma = \gamma(s)$ on M with Riemannian metric g satisfying the ordinary differential equation

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{4.4}$$

Proof of Theorem 3. We assume (1). Then the above discussion implies that M satisfies both (4.1) and $\phi A = A\phi$. So, we can choose the normal vector $\mathcal{N}$ of M in the ambient space $\widetilde{M}_n(c)$ in such a way that

$$AX = kX + \beta\eta(X)\xi \quad \text{for each } X \in TM \text{ with some constant } \beta. \tag{4.5}$$

We take an arbitrary geodesic $\gamma = \gamma(s)$ on M with $\langle\dot\gamma(0), \xi_{\gamma(0)}\rangle = 0$ and consider the function $\rho_\gamma(s) := \langle\dot\gamma(s), \xi_{\gamma(s)}\rangle$ along γ, called the *structure torsion* of γ (cf. [3]). Then ρ_γ is constant along γ. Indeed, from (3.4) and (4.5) we have

$$\nabla_{\dot\gamma}\rho_\gamma = \dot\gamma\langle\dot\gamma, \xi\rangle = \langle\dot\gamma, \nabla_{\dot\gamma}\xi\rangle = \langle\dot\gamma, \phi A\dot\gamma\rangle = k\langle\dot\gamma, \phi\dot\gamma\rangle = 0.$$

This, combined with $\langle\dot\gamma(0), \xi_{\gamma(0)}\rangle = 0$, implies that $\dot\gamma(s)$ is perpendicular to $\xi_{\gamma(s)}$ for each s, so that γ satisfies $A\dot\gamma(s) = k\dot\gamma(s)$ for any s. Hence, from (3.1) and (3.2) we find that the geodesic γ is mapped to a circle of positive curvature k in the ambient space $\widetilde{M}_n(c)$, proving (2).

Conversely, assume (2) holds. Then, from (4.4) and (2_a),

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = -k^2\dot\gamma_i. \tag{4.6}$$

On the other hand, from (3.1) and (3.2) we have

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = g((\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i)\mathcal{N} - g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i. \tag{4.7}$$

Comparing the tangential components of (4.6) and (4.7), we see that

$$g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i = k^2\dot\gamma_i,$$

so that at $s = 0$ we get

$$g(Av_i, v_i)Av_i = k^2 v_i \quad \text{for } 1 \leq i \leq 2n-2,$$

which yields

$$Av_i = kv_i \quad \text{or} \quad Av_i = -kv_i \quad \text{for } 1 \leq i \leq 2n-2. \tag{4.8}$$

This implies that ξ is a principal curvature vector, because $\langle A\xi, v_i\rangle = \langle\xi, Av_i\rangle = 0$ for $1 \leq i \leq 2n-2$. Therefore M is a Hopf hypersurface with at most three distinct constant principal curvatures, $k, -k$ and $\delta = g(A\xi, \xi)$ at each its points. On the other hand, applying the same discussion as above to condition (2_b), we get the following corresponding to (4.8):

$$\begin{aligned} &A((v_i + v_j)/\sqrt{2}\,) = k(v_i + v_j)/\sqrt{2} \quad \text{or} \\ &A((v_i + v_j)/\sqrt{2}\,) = -k(v_i + v_j)/\sqrt{2} \end{aligned} \tag{4.9}$$

for $1 \leq i < j \leq 2n-2$. Thus, from (4.8) and (4.9) we can see that either $Av_i = kv_i$ $(1 \leq i \leq 2n-2)$ or $Av_i = -kv_i$ $(1 \leq i \leq 2n-2)$. This implies that M is totally η-umbilic with coefficient $\alpha = \pm k$ in the ambient space $\widetilde{M}_n(c)$, which yields (1). ∎

REMARK 2. Condition (2_b) in Theorem 3 cannot be omitted. In fact, consider a real hypersurface M which is a tube of radius $\pi/(2\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)\,(1 \leq \ell \leq n-2)$ in the ambient space $\mathbb{C}P^n(c), n \geq 3$. Note that this hypersurface is of type (A_2) in $\mathbb{C}P^n(c)$. The tangent bundle TM decomposes as $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\sqrt{c}/2} \oplus V_{-\sqrt{c}/2}$ with $A\xi = 0$ (see the table of principal curvatures in Section 3). At an arbitrary fixed point $x \in M$, we take orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ_x in such a way that $\{v_1, \ldots, v_{2n-2\ell-2}\}$ and $\{v_{2n-2\ell-1}, \ldots, v_{2n-2}\}$ are orthonormal bases of $V_{\sqrt{c}/2}$ and $V_{-\sqrt{c}/2}$, respectively. Then all geodesics $\gamma_i = \gamma_i(s)\,(1 \leq i \leq 2n-2)$ on M with $\dot{\gamma}_i(0) = v_i$ are mapped to the circle of the same curvature $\sqrt{c}/2$ lying on the totally real totally geodesic $\mathbb{R}P^2(c/4)$ in $\mathbb{C}P^n(c)$ (for details, see [7]).

The following is a characterization of real hypersurfaces of type (B) satisfying (1.1).

PROPOSITION 2. *Let M^{2n-1} $(n \geq 2)$ be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$. Then for each positive constant k, M is locally either a tube of radius $r = (2/\sqrt{c}\,)\tan^{-1}((\sqrt{c+4k^2} - \sqrt{c}\,)/(2k))$, $0 < r < \pi/(2\sqrt{c}\,)$, around a complex hyperquadric $\mathbb{C}Q^{n-1}$ in $\mathbb{C}P^n(c)$ or a tube of radius $r = (1/(2\sqrt{|c|}\,))\{\log(2k + \sqrt{|c|}\,) - \log(2k - \sqrt{|c|}\,)\}$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)\,(-4k^2 < c < 0)$ in $\mathbb{C}H^n(c)$ if and only if M satisfies the following two conditions.*

(i) *The holomorphic distribution $T^0M = \{X \in TM \mid X \perp \xi\}$ decomposes into the direct sum of restricted principal foliations $V^0_{\lambda_i} = \{X \in T^0M \mid AX = \lambda_i X\}$. Moreover, every restricted principal foliation $V^0_{\lambda_i}$ is integrable and each of its leaves is a totally geodesic submanifold of M.*
(ii) *There exists an integral curve of ξ on M which is mapped to a circle of positive curvature $|c|/(2k)$ in the ambient space $\widetilde{M}_n(c)$.*

Proof. By Lemma 2 we only need to show that a homogeneous real hypersurface M of type (B) satisfies (4.1) if and only if it satisfies (ii). When $c > 0$, M has three distinct constant principal curvatures

$$\lambda_1 = \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right), \qquad \lambda_2 = \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right), \qquad \delta = \sqrt{c}\cot(\sqrt{c}\,r).$$

On the other hand, we have

$$\begin{aligned}\lambda_1+\lambda_2&=\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)\\&=\sqrt{c}\cot\left(\sqrt{c}\,r-\frac{\pi}{2}\right)\\&=-\sqrt{c}\tan(\sqrt{c}\,r).\end{aligned}$$

Hence M satisfies (1.1) if and only if $A\xi=(c/(2k))\xi$, i.e. $\delta=c/(2k)$. Note that in this case every integral curve of ξ, considered as a curve in the ambient space $\mathbb{C}P^n(c)$, is a circle of positive curvature $c/(2k)$ (see (3.1), (3.2) and (3.4)). This, together with the constancy of the principal curvature δ, implies that a homogeneous real hypersurface M of type (B) satisfies (1.1) if and only if it satisfies (ii).

When $c<0$, we have

$$\lambda_1+\lambda_2=\frac{\sqrt{|c|}}{2}\left\{\coth\left(\frac{\sqrt{|c|}}{2}\,r\right)+\tanh\left(\frac{\sqrt{|c|}}{2}\,r\right)\right\}=\sqrt{|c|}\,\coth(\sqrt{|c|}\,r).$$

By the same discussion as in the case of $c>0$, we also obtain the desired conclusion. ∎

Acknowledgments. The first author is partially supported by Grant from KOSEF R01-2007-000-20014-0 (2007).

REFERENCES

[1] T. Adachi, M. Kameda and S. Maeda, *Geometric meaning of Sasakian space forms from the viewpoint of submanifold theory*, Kodai Math. J. 33 (2010), 383–397.

[2] —, —, —, *Real hypersurfaces which are contact in a nonflat complex space form*, Hokkaido Math. J. 40 (2011), 207–219.

[3] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form*, J. Math. Soc. Japan 54 (2002), 373–408.

[4] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. 395 (1989), 132–141.

[5] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser, 2002.

[6] S. Maeda, *A characterization of the homogeneous real hypersurface of type (B) with two distinct constant principal curvatures in a complex hyperbolic space*, Sci. Math. Jpn. 68 (2008), 1–10.

[7] S. Maeda and K. Ogiue, *Characterizations of geodesic hyperspheres in a complex projective space by observing the extrinsic shape of geodesics*, Math. Z. 225 (1997), 537–542.

[8] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*, in: Tight and Taut Submanifolds, T. E. Cecil and S. S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.

[9] A. Weinstein, *Distance spheres in complex projective spaces*, Proc. Amer. Math. Soc. 39 (1973), 649–650.

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail: yhkim@knu.ac.kr

Sadahiro Maeda
Department of Mathematics
Saga University
1 Honzyo, Saga 840-8502, Japan
E-mail: smaeda@ms.saga-u.ac.jp

Received 22 October 2010;
revised 16 June 2011 (5438)