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FLEURY'S SPANNING DIMENSION AND CHAIN CONDITIONS ON NON-ESSENTIAL ELEMENTS IN MODULAR LATTICES

BY

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Abstract. Based on a lattice-theoretic approach, we give a complete characterization of modules with Fleury's spanning dimension. An example of a non-Artinian, non-hollow module satisfying this finiteness condition is constructed. Furthermore we introduce and characterize the dual notion of Fleury's spanning dimension.

1. Introduction. Smith and Vedadi [SV] characterized modules which satisfy the ascending (resp. descending) chain condition for non-essential submodules. Modules that satisfy the descending chain condition for non-small submodules have been studied by Fleury in [F] in his search for a dual Goldie dimension. He termed modules with DCC on non-small modules *modules with finite spanning dimensions.* A conceptually cleaner dualization of Goldie's dimension than Fleury's had been carried out by Grzeszczuk and Puczylowski in [GP] by introducing a notion of Goldie dimension for modular lattice. The dual Goldie dimension of a module being the Goldie dimension of the dual of its lattice of submodules reassembles earlier dualization attempts made by Varadarajan [V], Takeuchi [T] and Reiter [Re]. Fleury's spanning dimension however remained a rather subtle module-theoretic condition between the Artinianness of a module and the finiteness of its dual Goldie dimension.

In this paper we will give a complete characterization of Fleury's notion and will also construct a non-Artinian, non-hollow example with finite spanning dimension. Following Grzeszczuk and Puczyłowski's idea we will prove Smith and Vedadi's results for modular lattices and apply them to the dual lattice of the lattice of submodules of a module to obtain this characterization. We close by considering modules with ascending chain conditions on non-small submodules.

Throughout this paper, R denotes an associative ring with unit and all modules are unitary left R-modules.

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2. Modular lattices. A *lattice* $(L, \wedge, \vee)$ (for short L) is a partial ordered set $(L, \leq)$ such that for any $a, b \in L$ there exist elements $a \wedge b$ and $a \vee b$ such that $a \wedge b$ is the largest element c in L with $c \leq a$ and $c \leq b$ while $a \vee b$ is the smallest element c in L with $a \leq c$ and $b \leq c$. For two elements $a \leq b$ we denote by $[a, b] = \{c \in L \mid a \leq c \leq b\}$ the interval of elements between a and b. A lattice $(L, \vee, \wedge)$ is *complete* if joins $\bigvee X$ and meets $\bigwedge X$ exist for any non-empty subsets $X \subseteq L$. In this case the smallest element of L is denoted by 0 and its largest element by 1. L is called *modular* if for all $a, b, c \in L$,

$$a \leq b \Rightarrow b \wedge (a \vee c) = a \vee (b \wedge c).$$

All lattices in this paper are supposed to be complete and modular. For a thorough introduction to lattice theory we refer the reader to Grätzer's book [G].

If M is a module over a ring R and $\mathcal{L}(M)$ is its set of submodules, then $(\mathcal{L}(M), +, \cap)$ is a complete modular lattice with inclusion as partial ordering.

The *dual lattice* L^o of a lattice $(L, \wedge, \vee)$ consists of the same underlying set L, but with reversed partial ordering $<^o$, i.e. for all $a, b \in L : a <^o b \Leftrightarrow a > b$. If L is complete (modular), then so is L^o.

An element a is a *complement* of an element b in a modular lattice L if $a \vee b = 1$ and $a \wedge b = 0$. We say that L is *decomposable* if there exist complements different from 0 and 1. An element $a \in L$ with be called *decomposable* if $[0, a]$ is decomposable. Note that in the case of $L = \mathcal{L}(M)$, a submodule A of M is a complement in $\mathcal{L}(M)$ if and only if it is a direct summand of M.

A *pseudo-complement* of an element a in L is a maximal element of the set $\Omega_a = \{c \in L \mid a \wedge c = 0\}$, and L is called *pseudo-complemented* if every element of L has a pseudo-complement in L. Given a pseudo-complement b of a the element $a \vee b$ has the property that for any $c \in L$, $c \wedge (a \vee b) = 0 \Rightarrow c = 0$, since if $c \wedge (a \vee b) = 0$, then $(c \wedge (a \vee b)) \vee b = b$ and by modularity $b = (c \vee b) \wedge (a \vee b) = ((c \vee b) \wedge a) \vee b$. This implies $(c \vee b) \wedge a \leq b \wedge a = 0$. As b is maximal, $c \leq b$, and hence $c \leq c \wedge (b \vee a) = 0$, i.e. $c = 0$.

An element $x \in L$ such that $y \wedge x = 0 \Rightarrow y = 0$, for all $y \in L$, is called *essential*. The main object of this paper are chain conditions for non-zero elements that are not essential (called *non-essential*) with applications to the dual submodule lattice of a module. A lattice L is called *uniform* if every non-zero element of L is essential in L. An element $a \in L$ such that whenever $a \leq b \in L$ and a is essential in $[0, b]$ then $a = b$, is called *(essentially) closed* in L.

Goldie's dimension notion for modules is based on the notion of an *independent family of submodules* which generalizes the notion of a basis for vector spaces. Transferring Goldie's notion to modular lattices, Grzeszczuk and Puczyłowski called a subset $I \subseteq L \setminus \{0\}$ of a lattice L *independent* if for any finite subset X of I and $x \in I \setminus X$ one has $(\bigvee X) \wedge x = 0$. They proved the following theorem in [GP, Theorem 5]:

THEOREM 2.1. *For a complete modular lattice L the following conditions are equivalent:*

(a) *L does not contain infinite independent sets.*
(b) *L contains a finite independent set $\{a_1, \ldots, a_n\}$ such that $a_1 \vee \cdots \vee a_n$ is essential in L and the lattices $[0, a_i]$ are uniform for $1 \leq i \leq n$.*
(c) $\sup\{k \mid L$ *contains an independent subset of cardinality equal to* $k\} = n < \infty$.
(d) *For any ascending chain $a_1 < a_2 < \cdots$ of elements of L there exists j such that a_j is essential in $[0, a_k]$ for all $k \geq j$.*

We say that L has *finite Goldie dimension* if it satisfies one of the conditions above. The number n in (c) is called the *Goldie dimension* of L.

While complete modular lattices do not in general have to be pseudo-complemented, any submodule lattice $\mathcal{L}(M)$ of a module M is pseudo-complemented since $L = \mathcal{L}(M)$ has an even stronger property, namely for any element $a \in L$ and chain $C \subseteq L$,

$$a \wedge \bigvee C = \bigvee_{c \in C} (a \wedge c).$$

A complete modular lattice with this property is called *upper continuous*. If L is upper continuous, then, by Zorn's Lemma, Ω_a has a maximal element for each $a \in L$, i.e. L is pseudo-complemented. To characterize lattices that satisfy the ascending chain condition on non-essential elements, we need to weaken the above notions. We say that a complete modular lattice L is *weakly upper continuous* (or a **-lattice*) if for any $a \in L$ and chain $C \subseteq L$,

$$\bigvee_{c \in C} (a \wedge c) = 0 \;\Rightarrow\; a \wedge \bigvee C = 0.$$

Any weakly upper continuous lattice L is pseudo-complemented, because for any $a \in L$, the set $\Omega_a = \{b \in L \mid a \wedge b = 0\}$ is closed under joins of chains and hence has a maximal element by Zorn's Lemma. Call a lattice L *amply pseudo-complemented* if for any $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement a' of b with $a \leq a'$. Any weakly upper continuous lattice is amply pseudo-complemented, because the set $\Omega_{a,b} = \{a' \in L \mid a \leq a'$ and $a' \wedge b = 0\}$ is non-empty and closed under joins of chains. Hence by Zorn's Lemma it has a maximal element, which is the desired pseudo-complement of b.

Recall that a lattice L is called *compact* if $1 = \bigvee X$ for some subset X of L; then there exist $a_1, \ldots, a_n \in X$ with $1 = a_1 \vee \cdots \vee a_n$. Noetherian lattices are compact. We call an element a of a complete lattice *proper* if $a \neq 1$.

Theorem 2.2. *The following statements are equivalent for a complete modular lattice L:*

(a) *L satisfies the ascending chain condition (ACC) on non-essential elements;*

(b) *L is weakly upper continuous and $[0, a]$ is Noetherian (resp. compact) for all $a \in S$, where S is one of the following sets:*

  (i) *the set of non-essential elements of L,*

  (ii) *the set of proper closed elements of L,*

  (iii) *the set of decomposable elements of L.*

In this case L is amply pseudo-complemented and has finite Goldie dimension.

Proof. (a)⇒(b.i). Let $0 \neq a \in L$ and $C \subseteq L$ a chain of elements with $\bigvee_{c\in C}(a \wedge c) = 0$. Then $a \wedge c = 0$ for all $c \in C$, which shows that the elements of C are non-essential. Thus C must be finite, since otherwise it would contain a countable infinite subchain $b_1 < b_2 < \cdots$ of non-essential elements that does not stop, which would contradict (a). Since C is finite, $\bigvee C$ is an element of C and we have $a \wedge \bigvee C = 0$. This shows that L is weakly upper continuous. If a is non-essential, then so is every non-zero element in $[0, a]$. In particular, any chain in $[0, a]$ stops, i.e. $[0, a]$ is Noetherian (resp. compact).

(b.i)⇒(b.ii) is trivial since any proper closed element is non-essential in L.

(b.ii)⇒(b.iii). Let a be a decomposable element with $a = b \vee c$ and $b \wedge c = 0$ where $0 < b, c < a$. By the remark preceding the theorem, a weakly upper continuous lattice is amply pseudo-complemented. Thus there are mutual pseudo-complements b' and c' such that $b \leq b'$ and $c \leq c'$ and b' is a pseudo-complement of c' and vice versa. Note that if $b' \leq x$ with b' being essential in $[0, x]$, then $(x \wedge c') \wedge b' = c' \wedge b' = 0$ implies $x \wedge c' = 0$. As b' is maximal with respect to $b' \wedge c' = 0$, we have $b' = x$, i.e. b' is closed. By assumption $[0, b']$ is Noetherian (resp. compact). Hence also $[0, b]$ is Noetherian (resp. compact). Analogously $[0, c]$ is Noetherian (resp. compact) and so is the interval $[0, b \vee c] = [0, a]$.

(b.iii)⇒(a). First note that L has to have finite Goldie dimension. Suppose that there exists a countably infinite independent family of elements $\{a_i\}_{i\in\mathbb{N}}$. Since $a_1 \wedge (a_2 \vee \cdots \vee a_n) = 0$ for any $n > 1$, by weak upper continuity we also have $a_1 \wedge (\bigvee_{i>1} a_i) = 0$. Thus $\bigvee_{i\geq 1} a_i$ is decomposable and by hypothesis $[0, \bigvee_{i\geq 1} a_i]$ is compact, i.e. there exists $n > 1$ such that $\bigvee_{i=1}^{n} a_i = \bigvee_{i\geq 1} a_i$, i.e. $a_{n+1} \leq \bigvee_{i=1}^{n} a_i$, which contradicts the assumption that the set $\{a_i\}_{i\in\mathbb{N}}$ is independent. Thus L has finite Goldie dimension.

Let $a_1 \leq a_2 \leq \cdots$ be a chain of non-essential elements. Since L has finite Goldie dimension, there exists by Theorem 2.1(d) an index i such that a_i

is essential in $[0,a_j]$ for any $j > i$. Given a pseudo-complement b of a_i we also have $b \wedge a_j = 0$ for $j > i$ since a_i is essential in $[0,a_j]$. By weak upper continuity, $b \wedge (\bigvee_{j \geq i} a_j) = 0$. Hence $b \vee \bigvee_{j \geq i} a_j = \bigvee_{j \geq i} b \vee a_j$ is decomposable and by assumption $[0, \bigvee_{j \geq i} b \vee a_j]$ is compact. Thus there exists $n \geq i$ such that $a_j \leq b \vee a_n$ and hence $a_j = a_n$ for all $j \geq n$. ■

Examples of lattices satisfying the conditions of Theorem 2.2 are obviously Noetherian lattices or uniform lattices. Before we apply our theorem to modules we note the following:

PROPOSITION 2.3. *Let L be a lattice such that every pseudo-complement is a complement. Then the following conditions are equivalent:*

(a) *L satisfies ACC on non-essential elements;*
(b) *L is pseudo-complemented and $[0,a]$ is Noetherian for any decomposable element $a \in L$;*
(c) *L is Noetherian or uniform.*

Proof. (a)⇒(b) is trivial by Theorem 2.2(b.iii).

(b)⇒(c). If L is not uniform, then there exists a non-zero non-essential element $a \in L$. Since L is pseudo-complemented, a has a pseudo-complement in L, which by assumption is a complement. Thus 1 is a decomposable element and hence Noetherian by assumption.

(c)⇒(a) is trivial. ■

3. Modules with descending chain condition on non-small submodules. Let M be a left R-module over a (unital associative) ring R. As mentioned before, the submodule lattice $\mathcal{L}(M)$ is upper continuous, hence Theorem 2.2 becomes [SV, Theorem 1.8]. It is also interesting to apply Theorem 2.2 to the dual of $\mathcal{L}(M)$.

The dual Goldie dimension of a module is the Goldie dimension of $\mathcal{L}(M)^o$. A submodule N of M is called *small* if it is essential in $\mathcal{L}(M)^o$, and *coclosed* if it is essentially closed in $\mathcal{L}(M)^o$. Given two submodules N and L of M, N is called a *supplement* of L in M if N is a pseudo-complement of L in $\mathcal{L}(M)^o$. If $\mathcal{L}(M)^o$ is (amply) pseudo-complemented, then M is called *(amply) supplemented.* The existence of supplements in a module is not secured, since in general, the lattice $\mathcal{L}(M)^o$ is not upper continuous. According to Grothendieck, an object in an abelian category is said to satisfy (AB5) if its lattice of subobjects is upper continuous. Modules M whose dual submodule lattice $\mathcal{L}(M)^o$ is upper continuous are said to satisfy (AB5*), i.e. for any chain of submodules $\{B_i\}_i$ and submodule A of M one has $A + (\bigcap_i B_i) = \bigcap_i (A + B_i)$. We say that a module M satisfies *weak* (AB5*) if $\mathcal{L}(M)^o$ is weakly upper continuous, i.e. for any chain of submodules $\{B_i\}_i$ and submodule A of M with $A + B_i = M$ for all i, also $A + (\bigcap_i B_i) = M$. With this terminology, the dual version of Theorem 2.2 yields the following characterization of modules with finite spanning dimension. Recall that in his attempt to dualize the Goldie dimension for modules, P. Fleury said that a module has *finite spanning dimension* if for any descending chain of submodules $N_1 \supseteq N_2 \supseteq \cdots$ there exists an $i \in \mathbb{N}$ such that N_j is small in M for all $j \geq i$ (see [F]). This condition is obviously equivalent to M satisfying the descending chain condition for non-small submodules. For more information on modules with spanning dimension we refer the reader to [Ra, S1, S2, S3].

THEOREM 3.1. *The following statements are equivalent for a left R-module M:*

(1) *M has finite spanning dimension, i.e. M satisfies the descending chain condition (DCC) on non-small submodules;*
(2) *M satisfies weak* (AB5*) *and every factor M/N by a non-small submodule N of M is Artinian (resp. finitely cogenerated);*
(3) *M satisfies weak* (AB5*) *and every factor M/N by a non-zero coclosed submodule N of M is Artinian (resp. finitely cogenerated);*
(4) *M satisfies weak* (AB5*) *and every decomposable factor module of M is Artinian (resp. finitely cogenerated).*

In this case M is amply supplemented and has finite dual Goldie dimension.

The *radical* $\mathrm{Rad}(M)$ of a module M is the sum of all small submodules. In general $\mathrm{Rad}(M)$ does not need to be small. In [AS] Al-Khazzi and Smith proved that $\mathrm{Rad}(M)$ is Artinian if and only if M has DCC on small submodules. Note that $M/\mathrm{Rad}(M)$ is Artinian for any module with dual Goldie dimension. Thus M is Artinian if and only if M has DCC on small submodules and on non-small submodules.

A module M is called *hollow* if $\mathcal{L}(M)^o$ is uniform. Artinian and hollow modules have finite spanning dimension. For $M = R$, these are the only possibilities, as we will see. Recall that a module M is called *π-projective* if whenever $M = N + K$, then $\mathrm{End}(M) = \mathrm{Hom}(M, N) + \mathrm{Hom}(M, K)$. Projective modules are π-projective. Proposition 2.3 yields the following corollary:

COROLLARY 3.2. *Let M be a module such that every supplement is a direct summand. Then the following statements are equivalent:*

(a) *M has finite spanning dimension;*
(b) *M is supplemented and decomposable factor modules of M are Artinian;*
(c) *M is Artinian or hollow.*

In particular a π-projective module M with finite spanning dimension is Artinian or hollow.

Proof. The first statement is a direct translation of Proposition 2.3. A supplemented π-projective module has the property that supplements are direct summands since those are precisely the *quasi-discrete* modules (see [CLVW]). ∎

In order to find a module with finite spanning dimension which is neither Artinian nor hollow we might think of finding a module whose radical $\mathrm{Rad}(M)$ is a *waist*, i.e. either $N \subseteq \mathrm{Rad}(M)$ or $\mathrm{Rad}(M) \subseteq N$ for any submodule N of M. In this case $\mathrm{Rad}(M)$ is the largest small submodule of M and M has finite spanning dimension if and only if $M/\mathrm{Rad}(M)$ has. Thus we need to find a module with finite dual Goldie dimension greater than 1, whose radical is non-Artinian and a waist.

EXAMPLE 3.3. Let K be a field and V a vector space over K. Denote by $R = [K, V \oplus V]$ the trivial extension of K by $V \oplus V$, i.e. as K-vector space $R = K \oplus V \oplus V$ and multiplication is defined by $(\lambda, v, w).(\lambda', v', w') = (\lambda\lambda', \lambda v' + v\lambda', \lambda w' + w\lambda')$ for all $\lambda, \lambda' \in K$ and $v, v', w, w' \in V$. Let $M = V \oplus K \oplus K$ and define a left R-module structure on it by

$$(\lambda, v, w) \cdot (u, \alpha, \beta) = \lambda(u, \alpha, \beta) + (\alpha v + \beta w, 0, 0)$$

where $\lambda, \alpha, \beta \in K$ and $u, v, w \in V$.

CLAIM. $(V, 0, 0)$ *is a waist of* M.

Let $(u, \alpha, \beta) \in M$. If $(u, \alpha, \beta) \notin (V, 0, 0)$, then we can assume that $\alpha \neq 0$ or $\beta \neq 0$. Without loss of generality, suppose $\alpha \neq 0$. For any $v \in V$, we have

$$(0, \alpha^{-1}v, 0) \cdot (u, \alpha, \beta) = 0 \cdot (u, \alpha, \beta) + (\alpha\alpha^{-1}v, 0, 0) = (v, 0, 0).$$

Thus $(V, 0, 0) \subseteq R(u, \alpha, \beta)$.

As a consequence, $(V, 0, 0)$ is small in M, as any waist is. Moreover $M/(V, 0, 0) \cong K^2$ is semisimple and so $\mathrm{Rad}(M) = (V, 0, 0)$ and every non-small submodule of M contains $(V, 0, 0)$. Thus any chain of non-small submodules of M can be considered a chain in $M/(V, 0, 0)$, which is finite-dimensional. Hence every such chain stops.

M is not hollow as it has the two maximal submodules $R \cdot (0, 1, 0)$ and $R \cdot (0, 0, 1)$.

Also note that M is Artinian if and only if $\mathrm{Rad}(M)$ is Artinian if and only if $\dim(V) < \infty$.

So we see that for any infinite-dimensional vector space V over K, $M = V \oplus K \oplus K$ is a non-Artinian non-hollow left R-module with finite spanning dimension.

We can show a kind of Fitting's Lemma for modules with chain conditions on non-small submodules. A ring S is called *strongly π-regular* if for any $f \in S$ the chain $fS \supseteq f^2S \supseteq \cdots$ stops, while a module M is called *strongly co-Hopfian* if for every endomorphism f of M, the chain $\mathrm{Im}(f) \supseteq \mathrm{Im}(f^2) \supseteq \cdots$

stops (see [HKC]). Recall from [CLVW] that the *small ideal* of a module M is defined as $\nabla(M) = \{f \in \mathrm{End}_R(M) \mid \mathrm{Im}(f) \ll M\}$. We see that any module M with finite spanning dimension and $\nabla(M) = 0$ is strongly co-Hopfian. On the other hand, if a module M has finite dual Goldie dimension, then any epimorphism $f : M \to M$ has a small kernel (see [CLVW]). These modules are called *generalized Hopfian* in [GH].

Recall that a module M is called *semi-projective* if $fS = \mathrm{Hom}(M, \mathrm{Im}(f))$ for all $f \in S = \mathrm{End}(M)$ (see [CLVW, 4.20]).

Proposition 3.4. *Let M be a module with finite spanning dimension and let $S = \mathrm{End}(M)$. For any $f \in S$ there exists $n > 0$ such that $f^n \in \nabla(M)$ or $\mathrm{Im}(f^n)$ is a supplement of $\mathrm{Ker}(f^n)$ in M. In particular if M is semi-projective, then $S/\nabla(M)$ is strongly π-regular.*

Proof. If $f^n \notin \nabla(M)$ for all $n > 0$, then $\mathrm{Im}(f) \supseteq \mathrm{Im}(f^2) \supseteq \cdots$ is a descending chain of non-small submodules and must stop. Thus there exists $n > 0$ such that $\mathrm{Im}(f^n) = \mathrm{Im}(f^m)$ for all $m > n$. Let $x \in M$. Then there exists $y \in M$ such that $(x)f^n = (y)f^{2n}$ as $\mathrm{Im}(f^n) = \mathrm{Im}(f^{2n})$. Hence $x = (y)f^n + (x - (y)f^n) \in \mathrm{Im}(f^n) + \mathrm{Ker}(f^n)$.

Note that $\mathrm{Im}(f^n)$, as a factor module of M, has finite dual Goldie dimension and that any epimorphism of a module with finite dual Goldie dimension has small kernel. Since $f^n : \mathrm{Im}(f^n) \to \mathrm{Im}(f^{2n}) = \mathrm{Im}(f^n)$ is an epimorphism, its kernel $\mathrm{Ker}(f^n) \cap \mathrm{Im}(f^n)$ is small in $\mathrm{Im}(f^n)$, i.e. $\mathrm{Im}(f^n)$ is a supplement of $\mathrm{Ker}(f^n)$ in M.

Assume that M is semi-projective. Let $f \in S$. If there exists $n > 0$ such that $f^n \in \nabla(M)$, then f^n is nilpotent in $S' = S/\nabla(M)$ and so $f^nS' = f^mS'$ for all $m > n$. If $f^n \notin \nabla(M)$ for all $n > 0$, then there exists $n > 0$ such that $\mathrm{Im}(f^n) = \mathrm{Im}(f^m)$ for all $m > n$. Thus $f^nS = \mathrm{Hom}(M, \mathrm{Im}(f^m)) = f^mS$ and also $f^nS' = f^mS'$, i.e. $S' = S/\nabla(M)$ is strongly π-regular. ∎

4. Ascending chain conditions on non-small submodules. We close this paper by dualizing Smith and Vedadi's result on modules with DCC on non-essential submodules.

Theorem 4.1. *The following statements are equivalent for a lattice L:*

(a) *L satisfies DCC on non-essential elements;*
(b) *$[0, a]$ is Artinian for any non-essential element $a \in L$;*
(c) *$[0, a]$ is Artinian for any decomposable element $a \in L$.*

If L is amply pseudo-complemented, then the following statement is equivalent to (a)–(c)*:*

(d) *$[0, a]$ is Artinian for any proper closed element $a \in L$.*

Proof. (a)$\Rightarrow$(b) is clear, since any descending chain in $[0,a]$ with a non-essential is a descending chain of non-essential elements.

(b)$\Rightarrow$(c). If a is decomposable with $a = b \vee c$ and $b \wedge c = 0$, then b and c are non-essential, hence $[0,b]$ and $[0,c]$ are Artinian and so is their direct sum $[0,a]$.

(c)$\Rightarrow$(a). If $a_1 \geq a_2 \geq \cdots$ is a descending chain of non-essential elements, then there exists $b \in L$ with $b \wedge a_1 = 0$. Thus $a_i \in [0, b \vee a_1]$, which is Artinian by hypothesis, and hence the chain has to stop.

(b)$\Rightarrow$(d) is clear since proper closed elements are non-essential.

Suppose that L is amply pseudo-complemented and $[0,a]$ is Artinian for all closed elements $a \in L$. Let $a \in L$ be a non-essential element and b an element with $a \wedge b = 0$. There exists a pseudo-complement a' of b such that $a \leq a'$, as L is amply pseudo-complemented. Hence $[0,a']$ is Artinian as pseudo-complements are closed, and also $[0,a]$ is Artinian as $a \leq a'$. ∎

As mentioned before, the submodule lattice $\mathcal{L}(M)$ is upper continuous and hence amply pseudo-complemented. Thus Theorem 4.1 becomes [SV, Theorem 1.4]. It is more interesting to apply Theorem 4.1 to the dual of $\mathcal{L}(M)$, which leads to the following theorem.

THEOREM 4.2. *The following conditions are equivalent for a module M:*

(a) *M satisfies ACC on non-small submodules;*
(b) *M/N is Noetherian for every non-small submodule N of M;*
(c) *every decomposable factor module of M is Noetherian.*

If M is amply supplemented, then (a)–(c) *are also equivalent to:*

(d) *M/N is Noetherian for every non-zero coclosed submodule N of M.*

Note that condition (d) does not necessarily imply (a)–(c) if M is not amply supplemented, since (d) is trivially fulfilled for modules M whose only coclosed submodule is M. Recall that a coclosed ideal in a commutative ring is idempotent (see [CLVW, 4.17]). Hence any commutative ring R whose only idempotent ideals are 0 and R, has only one coclosed ideal, namely $I = R$, which fulfills condition (d) trivially. On the other hand if $\mathrm{Rad}(M) = 0$ for a module M, then any submodule is non-small and "ACC on non-smalls" means "Noetherianness". Thus any commutative non-Noetherian ring R with $\mathrm{Jac}(R) = 0$ and without non-trivial idempotent ideals is an example of a module satisfying 4.2(d) but not having ACC on non-smalls. We shall give such an example now:

LEMMA 4.3. *If F is a field of characteristic zero and X is any infinite set of variables, then the polynomial ring $R = F[X]$ in the variables $x \in X$ over F is a non-Noetherian integral domain with* $\mathrm{Jac}(R) = 0$ *and without non-trivial idempotent ideals.*

Proof. Surely R is a commutative non-Noetherian domain with $\mathrm{Jac}(R) = 0$. Any element $0 \neq f \in R$ can be uniquely written as a finite linear combination of monomials in variables $x \in X$ and the constant polynomial 1. Denote by $\mathrm{supp}(f)$ the support of f, which is the finite set of variables that appear in the monomials which span f. Let I be an idempotent ideal of R and consider the set $\{|\mathrm{supp}(f)| \mid 0 \neq f \in I\}$, which is non-empty subset of $\mathbb{N}$, since $I \neq 0$. Let $0 \neq f \in I$ with $|\mathrm{supp}(f)|$ minimal. Suppose that $\mathrm{supp}(f)$ is not empty and let $x \in \mathrm{supp}(f)$. We can write $f = \sum_{l=0}^{n} g_l x^l$ for some polynomials g_l, with $g_n \neq 0$, whose support is contained in $\mathrm{supp}(f) \setminus \{x\}$. Applying n times the partial derivative $\frac{\partial}{\partial x}$ to f yields $\frac{\partial^n}{\partial x^n}(f) = (n!)g_n$, which is non-zero since $g_n \neq 0$ and F has characteristic zero. Note that I is closed under the action of any derivation D, i.e. $D(I) \subseteq I$, because $D(I) = D(II) \subseteq D(I)I + ID(I) \subseteq I$. Hence $0 \neq g_n \in I$ having smaller support than f contradicts the minimality of the support of f. Thus $\mathrm{supp}(f)$ must be empty, which makes f a non-zero, hence invertible, constant. Thus $I = R$. ■

Example 3.3 is an example of a module with ACC on non-small submodules which is neither Noetherian nor hollow. Another example of such a module is the following:

EXAMPLE 4.4. Let R be the trivial extension of $\mathbb{Z}$ by $\mathbb{Q}$, i.e. $R = \mathbb{Z} \times \mathbb{Q}$ with componentwise addition and multiplication defined by

$$(n,q)(m,p) = (nm, np + qm), \qquad \forall n,m \in \mathbb{Z}, q,p \in \mathbb{Q}.$$

Note that $\mathrm{Jac}(R) = 0 \times \mathbb{Q}$ is a waist. To see this, take $(n,q) \in R$ with $n \neq 0$. Then for all $p \in \mathbb{Q}$ we have $(0, \frac{1}{n}p)(n,q) = (0,p)$, i.e. $R(n,q) \supset 0 \times \mathbb{Q}$. Thus each non-small submodule contains $\mathrm{Jac}(R)$ and as $R/\mathrm{Jac}(R) \simeq \mathbb{Z}$ is Noetherian, R has ACC on non-small submodules, but is neither Noetherian nor hollow (as $\mathbb{Z}$ is not hollow).

Any Noetherian, non-Artinian, non-local ring is an example of a module with ACC, but not DCC, on non-small submodules, while for any Artinian, non-Noetherian module M over some ring R, the module $M \oplus M$ satisfies DCC, but not ACC, on non-small submodules.

Since $\mathrm{Rad}(M)$ contains all small submodules, all submodules that properly contain $\mathrm{Rad}(M)$ are non-small. Thus if M has ACC on non-small submodules, then any chain of submodules containing $\mathrm{Rad}(M)$ stops and we have:

PROPOSITION 4.5. *$M/\mathrm{Rad}(M)$ is Noetherian if M satisfies ACC on non-small submodules.*

The last observation and a result by Al-Khazzi and Smith in [AS] which says that $\mathrm{Rad}(M)$ is Noetherian if and only if M has ACC on small submodules allows one to conclude that a module is Noetherian if and only if it satisfies ACC on small submodules and on non-small submodules.

Analogously to Propositon 3.4 we finish the paper with a statement on endomorphisms of modules with ACC on non-small submodules.

PROPOSITION 4.6. *Let M be a module with ACC on non-small submodules. Then for any $f \in S$ there exists $n > 0$ such that* $\mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n) = 0$ *or* $\mathrm{Ker}(f)$ *is small in M. In particular M is generalized Hopfian* (see [GH]).

Proof. If $\mathrm{Ker}(f)$ is not small in M, then $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(f^2) \subseteq \cdots$ is an ascending chain of non-small submodules that must stop. Thus there exists $n > 0$ such that $\mathrm{Ker}(f^n) = \mathrm{Ker}(f^m)$ for all $m > n$. Let $x \in \mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n)$ and $y \in M$ with $x = (y)f^n$. Since $0 = (x)f^n = (y)f^{2n}$, we have $y \in \mathrm{Ker}(f^{2n}) = \mathrm{Ker}(f^n)$, i.e. $x = (y)f^n = 0$.

If f is an epimorphism of M, so is every power of f. Hence if there exists $n > 0$ with $0 = \mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n) = \mathrm{Ker}(f^n)$, then $\mathrm{Ker}(f^n) = 0 = \mathrm{Ker}(f)$ and f is an isomorphism. Otherwise $\mathrm{Ker}(f)$ is small in M. ■

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