THE EULER AND HELMHOLTZ OPERATORS
ON FIBERED MANIFOLDS WITH ORIENTED BASES

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Abstract. We study naturality of the Euler and Helmholtz operators arising in the variational calculus in fibered manifolds with oriented bases.

Given two fibered manifolds $Z_1 \to M$ and $Z_2 \to M$ over the same base $M$, we denote the space of all base preserving fibered manifold morphisms of $Z_1$ into $Z_2$ by $C^\infty_M(Z_1, Z_2)$.

In [1], I. Kolář studied the Euler operator $E : C^\infty_M(J^s Y, \bigwedge^m T^* M) \to C^\infty_Y(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M)$ for fibered manifolds $p : Y \to M$. He deduced that all natural operators of this type are of the form $cE$, $c \in \mathbb{R}$, provided $m$ is sufficiently large.

In [3], Kolář and Vitolo studied the Helmholtz operator $H : C^\infty_Y(J^s Y, V^* Y \otimes \bigwedge^m T^* M) \to C^\infty_{J^s Y}(J^{2s} Y, V^* J^s Y \otimes V^* Y \otimes \bigwedge^m T^* M)$ for fibered manifolds $p : Y \to M$. They deduced that all natural operators of this type are of the form $cH$, $c \in \mathbb{R}$, provided $s = 1, 2$. In [4], we extended this result to all $s$.

In the present paper, for a fibered manifold $p : Y \to M$ with oriented basis, we study the naturality of the Euler operator

$$\tilde{E} : \mathcal{V}ol^+(M) \times C^\infty(J^s Y, \mathbb{R}) \to C^\infty(Y, J^{2s} Y, V^* Y)$$

given by $\tilde{E}(\eta, \lambda) \otimes \eta = E(\lambda \otimes \eta)$ for any $\eta \in \mathcal{V}ol^+(M)$ and $\lambda \in C^\infty(J^s Y, \mathbb{R})$, where $\mathcal{V}ol^+(M)$ is the set of all positive volume forms on $M$.

We also study, for fibered manifolds $p : Y \to M$ with oriented bases, the naturality of the Helmholtz operator

$$\tilde{H} : \mathcal{V}ol^+(M) \times C^\infty(Y, V^* Y) \to C^\infty_{J^s Y}(J^{2s} Y, V^* J^s Y \otimes V^* Y)$$

defined from $H$ just as $\tilde{E}$ from $E$.

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The first main result of the present paper is

**Theorem 1.** Let $m, n, s$ be natural numbers. Any $\mathcal{FM}_{m,n}^+$-natural $π_{2s}^*$-local, regular operator

$$D : \mathcal{Vol}^+(M) \times C^∞(J^sY, \mathbb{R}) \to C^∞_Y(J^{2s}Y, V^*Y),$$

$\mathbb{R}$-linear in the second factor and homogeneous of weight 0 in the first factor, is of the form $D = c\tilde{E}$, $c \in \mathbb{R}$.

**Remark 1.** $\mathcal{FM}_{m,n}^+$ denotes the category of all $(m, n)$-dimensional fibered manifolds with oriented bases and their fibered embeddings covering orientation preserving embeddings. The $\mathcal{FM}_{m,n}^+$-naturality of $D$ means that for any $\mathcal{FM}_{m,n}^+$-map $f : Y_1 \to Y_2$, any Lagrangians $λ_1 \in C^∞(J^sY_1, \mathbb{R})$ and $λ_2 \in C^∞(J^sY_2, \mathbb{R})$ and any positive volume forms $η_1 \in \mathcal{Vol}^+(M_1)$ and $η_2 \in \mathcal{Vol}^+(M_2)$ if $λ_1$ and $λ_2$ are $f$-related and $η_1$ and $η_2$ are $f$-related, then $D(η_1, λ_1)$ and $D(η_2, λ_2)$ are $f$-related. The regularity means that $D$ transforms smoothly parametrized families of Lagrangians and volume forms into smoothly parametrized families of respective morphisms. The locality means that $D(η, λ)_u$ depends on germ $π^\ast_{2s}(u)(λ)$ and germ $x(η)$ for any $u \in J^s_xY$, $x \in M$, where $π^\ast_{2s} : J^sY \to J^sY$ is the jet projection. The linearity in the second factor means that $D(η, λ)$ depends $\mathbb{R}$-linearly on $λ \in C^∞(J^sY, \mathbb{R})$ for any fixed $η \in \mathcal{Vol}^+(M)$. The homogeneity of weight 0 in the first factor means that $D(tη, λ) = D(η, λ)$ for $t > 0$.

**Remark 2.** Theorem 1 without the linearity assumption does not hold. For, let $h : \mathbb{R} \to \mathbb{R}$ be a non-constant function. Then the operator $\tilde{E}^{[h]}(η, λ) = (h \circ λ \circ π^\ast_{2s})\tilde{E}(η, λ)$ is not linear in the second factor.

**Remark 3.** If $C : C^∞_M(J^sY, \bigwedge^m T^*M) \to C^∞_Y(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M)$ is a natural $\mathbb{R}$-linear operator, then (similarly to $\tilde{E}$) one can define the corresponding natural operator $\tilde{C} : \mathcal{Vol}^+(M) \times C^∞(J^sY, \mathbb{R}) \to C^∞_Y(J^{2s}Y, V^*Y)$, $\mathbb{R}$-linear in the second factor and homogeneous of weight zero in the first factor. Using Theorem 1, we see that $\tilde{C} = c\tilde{E}$, and we recover the above mentioned result of [1] in the case of $\mathbb{R}$-linear operators. The inverse construction of $C$ from $\tilde{C}$ is impossible because we have no canonical surjection $C^∞_M(J^sY, \bigwedge^m T^*M) \to \mathcal{Vol}^+(M) \times C^∞(J^sY, \mathbb{R})$. So, Theorem 1 is not a consequence of the result of [1].

The second main result of the present paper is

**Theorem 2.** Let $m, n, s$ be natural numbers. Any $\mathcal{FM}_{m,n}^+$-natural, $π_{2s}^*$-local, regular operator

$$D : \mathcal{Vol}^+(M) \times C^∞_Y(J^sY, V^*Y) \to C^∞_Y(J^{2s}Y, V^*J^sY \otimes V^*Y),$$

$\mathbb{R}$-linear in the second factor and homogeneous with weight 0 in the first factor, is of the form $c\tilde{H}$, $c \in \mathbb{R}$. 
Remark 4. Theorem 2 without the assumption of linearity does not hold. For, we have a natural operator $\tilde{H}^0$ non-linear in the second factor given by $\langle \tilde{H}^0(\eta, B)j^2_\sigma, v \otimes w \rangle = \langle B_j^2 \sigma, T\pi_0^s(v) \rangle \langle B_j^2 \sigma, w \rangle$ for $j^2_\sigma \in J^2_\sigma Y$, $x \in M$, $v \in V_j^2 \sigma J^2_\sigma Y$, $w \in V_\sigma(x) Y$.

Proof of Theorem 1. From now on $\mathbb{R}^{m,n}$ is the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $x^1, \ldots, x^m, y^1, \ldots, y^n$ are the usual coordinates on $\mathbb{R}^{m,n}$.

Let $D$ be an operator in question.

Since an $\mathcal{F}M^{\pm}_{m,n}$-map $(x, y - \sigma(x))$ sends $j^2_0(\sigma)$ to $\theta = j^2_0(0) \in J^2_0(\mathbb{R}^m, \mathbb{R}^n) = J^2_0(\mathbb{R}^{m,n})$, $J^2(\mathbb{R}^{m,n})$ is the $\mathcal{F}M^+_{m,n}$-orbit of $\theta$. Therefore $D$ is uniquely determined by the evaluations

$$\langle D(\eta, \lambda) \theta, v \rangle \in \mathbb{R}$$

for all $\lambda \in C^\infty(J^s(\mathbb{R}^{m,n}), \mathbb{R})$, $\eta \in \text{Vol}^+(\mathbb{R}^m)$ and $v \in T_0 \mathbb{R}^n = V(0,0) \mathbb{R}^{m,n}$.

Using the invariance of $D$ with respect to $\mathcal{F}M^+_{m,n}$-morphisms of the form $id_{\mathbb{R}^m} \times \psi$ for linear $\psi$ we see that $D$ is uniquely determined by the evaluations

$$\left( D(\eta, \lambda) \theta, \frac{\partial}{\partial y^1_0} \right) \in \mathbb{R}$$

for all $\lambda \in C^\infty(J^s(\mathbb{R}^{m,n}), \mathbb{R})$ and $\eta \in \text{Vol}^+(\mathbb{R}^m)$.

Consider an arbitrary positive volume form $\eta = f(x^1, \ldots, x^m)dx^1 \wedge \cdots \wedge dx^m$ on $\mathbb{R}^m$. There is a map $F : \mathbb{R}^m \to \mathbb{R}^m$ such that $\frac{\partial}{\partial x^i}F = f$ and $F(0) = 0$. Then the locally defined $\mathcal{F}M^+_{m,n}$-map $(F, x^2, \ldots, x^m, y^1, \ldots, y^n)$ preserves $\theta$, $\frac{\partial}{\partial y^1_0} \theta$ and sends germ$_0(d^m x)$ into germ$_0(\eta)$, where $d^m x = dx^1 \wedge \cdots \wedge dx^m$. Then by naturality $D$ is uniquely determined by the evaluations

$$\left( D(d^m x, \lambda) \theta, \frac{\partial}{\partial y^1_0} \right) \in \mathbb{R}$$

for all $\lambda \in C^\infty(J^s(\mathbb{R}^{m,n}), \mathbb{R})$.

By the $\mathbb{R}$-linearity in the second factor of $D$ and by Corollary 19.8 in [1] we see that $D$ is determined by the values

$$\langle D(d^m x, x^\beta M(y^j_\alpha)) \theta, \frac{\partial}{\partial y^1_0} \rangle,$$

where $(x^i, y^j_\alpha)$ is the induced coordinate system on $J^s(\mathbb{R}^{m,n})$ and $M$ is an arbitrary monomial in the $y^j_\alpha$'s. (Here and below, $\alpha$ and $\beta$ are arbitrary $m$-tuples with $|\alpha| \leq s$ and $j = 1, \ldots, n$.)

Now, using the invariance of $D$ with respect to the $\mathcal{F}M^+_{m,n}$-maps

$$(x^1, \ldots, x^m, \tau_1 y^1, \ldots, \tau^n y^n)$$

for $\tau^j > 0$, we get the homogeneity condition which gives that (1) is zero if
$M(y^1_{\alpha})$ is not of the form $y^1_{\alpha}$. So, $D$ is determined by the values
\[
\left\langle D(d^mx,x^{\beta}y^1_{\alpha})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle
\]
for $\alpha$ and $\beta$ as above.

Next, using the invariance of $D$ with respect to the $\mathcal{F}\mathcal{M}^+_{m,n}$-maps
\[
(x^1, \ldots, \tau^i x^i, \ldots, x^m, y^1, \ldots, y^n)
\]
for $\tau^i > 0$ and using the $\mathbb{R}$-linearity in the second factor and the homogeneity of weight 0 in the first factor of $D$ we get
\[
(2) \quad \left\langle D(d^mx,x^{\beta}y^1_{\alpha})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle = 0
\]
if only $\beta_i - \alpha_i \neq 0$ for some $i = 1, \ldots, m$ (i.e. if $\alpha \neq \beta$).

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an $m$-tuple with $|\alpha| \leq s$.

Suppose $\alpha_{i_1} > 0$ for some $i_1 = 1, \ldots, m$.

The locally defined $\mathcal{F}\mathcal{M}^+_{m,n}$-map $\psi = (x^1, \ldots, x^m, y^1 + x^{i_1}y^1, \ldots, y^n)^{-1}$ preserves $x^1, \ldots, x^m$, $\Theta$ and $\frac{\partial}{\partial y^1_0}$ and sends $y^1_{\alpha}$ to $y^1_{\alpha} + x^{i_1}y^1_{\alpha} + y^1_{\alpha-1_{i_1}}$ (as $y^1_{\alpha} \circ J^s\psi^{-1}(j^s_{x^1\sigma}) = \partial_{\alpha}(\sigma^1 + x^{i_1}\sigma^1)(x_0) = \partial_{\alpha}\sigma^1(x_0) + x^{i_1}\partial_{\alpha}\sigma^1(x_0) + \partial_{\alpha-1_{i_1}}\sigma^1(x_0) = (y^1_{\alpha} + x^{i_1}y^1_{\alpha} + y^1_{\alpha-1_{i_1}})(j^s_{x^1\sigma})$ for $j^s_{x^1\sigma} \in J^s\mathbb{R}^{m,n}$, where $\partial_{\alpha}$ is the iterated partial derivative with respect to the index $\alpha$ multiplied by $1/|\alpha|$!). Then using the invariance of $D$ with respect to $\psi$, from
\[
\left\langle D(d^mx,x^{\alpha-1_{i_1}}y^1_{\alpha})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle = 0
\]
(see (2)) we see that
\[
\left\langle D(d^mx,x^{\alpha}y^1_{\alpha})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle = -\left\langle D(d^mx,x^{\alpha-1_{i_1}}y^1_{\alpha-1_{i_1}})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle.
\]
Continuing this process we see that
\[
\left\langle D(d^mx,x^{\alpha}y^1_{\alpha})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle = (-1)^{|\alpha|}\left\langle D(d^mx,y^1_{(0)})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle.
\]
Summing up, $D$ is determined by the value
\[
\left\langle D(d^mx,y^1_{(0)})\Theta, \frac{\partial}{\partial y^1_0} \right\rangle \in \mathbb{R}.
\]
Thus the vector space of all $D$ in question is of dimension less than or equal to 1. Hence $D = c\tilde{E}$ for some $c \in \mathbb{R}$.

**Proof of Theorem 2.** Let $D$ be an operator in question. Let $\Theta$ be as in the proof of Theorem 1.
As in that proof, $D$ is uniquely determined by
\[
\left\langle D(\eta, B)\Theta, \frac{d}{dt_0} \left( t j_0^s(g(x), 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle \in \mathbb{R}
\]
for all $B \in C_{\mathbb{R}^m}^\infty(J^s(\mathbb{R}^{m,n}), V^* \mathbb{R}^{m,n})$, $\eta \in Vol^+(\mathbb{R}^m)$ and $g : \mathbb{R}^m \to \mathbb{R}$.

Using the invariance of $D$ with respect to the $\mathcal{F}M_{m,n}^+$-maps $(x^1, \ldots, x^m, y^1 + g(x)y^1, y^2, \ldots, y^n)$ preserving $\Theta$ we find that $D$ is uniquely determined by
\[
\left\langle D(\eta, B)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle \in \mathbb{R}
\]
for all $B \in C_{\mathbb{R}^m}^\infty(J^s(\mathbb{R}^{m,n}), V^* \mathbb{R}^{m,n})$.

Then similarly to the proof of Theorem 1 (using $\mathcal{F}M_{m,n}^+$-naturality), $D$ is uniquely determined by
\[
\left\langle D(d^m x, B)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle \in \mathbb{R}
\]
for all $B$ as above.

Let $B \in C_{\mathbb{R}^m}^\infty(J^s(\mathbb{R}^{m,n}), V^* \mathbb{R}^{m,n})$. Using the invariance of $D$ with respect to the $\mathcal{F}M_{m,n}^+$-maps $\psi_\tau = (x^1, \ldots, x^m, (1/\tau^1)y^1, \ldots, (1/\tau^n)y^n)$ for $\tau^j \neq 0$ we get the homogeneity condition
\[
\left\langle D(d^m x, (\psi_\tau)_* B)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = \tau^1 \tau^2 \left\langle D(d^m x, B)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle
\]
for $\tau = (\tau^1, \ldots, \tau^n)$. Then by the second factor linearity of $D$ and by Corollary 19.8 in [2] of the Peetre theorem,
\[
\left\langle D(d^m x, B)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle
\]
is determined by the values
\[
\left\langle D(d^m x, x^\beta y^2_\alpha dy^1)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle,
\]
\[
\left\langle D(d^m x, x^\beta y^2_\alpha dy^2)\Theta, \frac{d}{dt_0} \left( t j_0^s(1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial y_0^2} \right\rangle
\]
for all $m$-tuples $\alpha$ and $\beta$ with $|\alpha| \leq s$.

Then by the invariance of $D$ with respect to the $\mathcal{F}M_{m,n}^+$-maps
\[
(\tau^1 x^1, \ldots, \tau^m x^m, y^1, \ldots, y^n)
\]
for $\tau^i > 0$ and the first factor 0-weight homogeneity of $D$ we get

$$(3) \quad \left\langle D(d^m x, x^\beta y_\alpha^2 dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle$$

if only $\beta \neq \alpha$.

Suppose $\alpha = (\alpha_1, \ldots, \alpha_m)$ is an $m$-tuple with $|\alpha| \leq s$ and $\alpha_i \neq 0$ for some $i$. Then using the invariance of $D$ with respect to the locally defined $\mathcal{M}_{m,n}^+$-map $\psi = (x^1, \ldots, x^m, y^1, y^2 + x^i y^2, \ldots, y^n)^{-1}$ preserving $x^1, \ldots, x^m, y^1, \Theta, j_0^s(1,0,\ldots,0)$ and $\frac{\partial}{\partial y_0^2}$ and sending $y_\alpha^2$ to $y_\alpha^2 + x^i y_\alpha^2 + y_{\alpha-1,i}^2$, from

$$\left\langle D(d^m x, x^{\alpha-1} y_\alpha^2 dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = 0$$

(see (3)) we deduce that

$$\left\langle D(d^m x, x^\alpha y_\alpha^2 dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = - \left\langle D(d^m x, x^{\alpha-1} y_{\alpha-1}^2, dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle.$$

Then for any $m$-tuple $\alpha$ with $|\alpha| \leq s$ we have

$$\left\langle D(d^m x, x^\alpha y_\alpha^2 dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = (-1)^{|\alpha|} \left\langle D(d^m x, y_0^2 dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle.$$

By the same arguments (since $\psi$ sends $dy_2$ to $dy_2 + x^i dy_2$), from

$$\left\langle D(d^m x, x^{\alpha-1} y_\alpha^1 dy^2) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = 0$$

we obtain

$$\left\langle D(d^m x, x^\alpha y_\alpha^1 dy^2) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = 0$$

if $\alpha \neq (0)$.

Using the invariance of $D$ with respect to the local $\mathcal{M}_{m,n}^+$-map

$$(x^1, \ldots, x^m, y^1 + y^1 y^2, \ldots, y^n)^{-1}$$

preserving $\Theta, j_0^s(1,0,\ldots,0)$ and $\frac{\partial}{\partial y_0^2}$, from

$$\left\langle D(d^m x, dy^1) \xi, \frac{d}{dt_0} (t j_0^s(1,0,\ldots,0)) \otimes \frac{\partial}{\partial y_0^2} \right\rangle = 0$$
we deduce that
\[ \left\langle D(d^m x, y^2_{(0)} dy^1_0) \Theta, \frac{d}{dt_0} (t_{j_0}^s (1,0,\ldots,0)) \otimes \frac{\partial}{\partial y^2_0} \right\rangle = -\left\langle D(d^m x, y^1_{(0)} dy^2_0) \Theta, \frac{d}{dt_0} (t_{j_0}^s (1,0,\ldots,0)) \otimes \frac{\partial}{\partial y^2_0} \right\rangle. \]

Thus \( D \) is uniquely determined by
\[ \left\langle D(d^m x, y^2_{(0)} dy^1_0) \Theta, \frac{d}{dt_0} (t_{j_0}^s (1,0,\ldots,0)) \otimes \frac{\partial}{\partial y^2_0} \right\rangle \in \mathbb{R}. \]

Therefore the vector space of all \( D \) in question is of dimension less than or equal to 1. Hence \( D = cH \) for some \( c \in \mathbb{R} \).

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