COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 2

ABSOLUTE n-FOLD HYPERSPACE SUSPENSIONS

BY

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Abstract. The notion of an absolute n-fold hyperspace suspension is introduced. It is proved that these hyperspaces are unicoherent Peano continua and are dimensionally homogeneous. It is shown that the 2-sphere is the only finite-dimensional absolute 1-fold hyperspace suspension. Furthermore, it is shown that there are only two possible finite-dimensional absolute n-fold hyperspace suspensions for each $n \geq 3$ and none when $n = 2$. Finally, it is shown that infinite-dimensional absolute n-fold hyperspace suspensions must be unicoherent Hilbert cube manifolds.

1. Introduction. The notion of an absolute hyperspace was introduced recently [17]. The notion is a natural analogue of de Groot's concepts of absolute suspensions and absolute cones [3]. The continua that are absolute n-fold hyperspaces and the continua that are absolute hyperspaces of compacta were determined in [17].

The notion of the hyperspace suspension of a continuum was introduced in 1979 [15] and studied further in [2]. The notion was extended to n-fold hyperspace suspensions in [12].

Our purpose here is to study absolute hyperspaces for the case of n-fold hyperspace suspensions. Our main results are Theorem 3.6, Theorem 3.7, Theorem 4.2, Corollary 4.4 and Theorem 4.9 in the finite-dimensional case, and Theorem 5.2 in the infinite-dimensional case. Our other results, such as Theorems 4.6 and 4.8, are used in the proofs of our main results and are also of independent interest.

2. Definitions. If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$, the interior of A is denoted by $\mathrm{Int}_Z(A)$, and the closure of A is denoted by $\mathrm{Cl}(A)$. A *map* means a continuous function.

2000 *Mathematics Subject Classification*: Primary 54B20; Secondary 54F15.

Key words and phrases: absolute n-fold hyperspace suspension, continuum, dimensionally homogeneous, Hilbert cube, Hilbert cube manifolds, n-fold hyperspace suspension, Peano continua, 2-sphere, unicoherent continuum.

The research of the first named author was partially supported by the grant 42602 of CONACyT.

Let X and Y be homeomorphic spaces; we denote this by $X \approx Y$. If x_1 and x_2 are two points of X, and y_1 and y_2 are two points of Y, then $(X, x_1, x_2) \approx (Y, y_1, y_2)$ means that there is a homeomorphism of X onto Y that takes x_j onto y_j, $j \in \{1, 2\}$.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space Z. A *Peano continuum* is a locally connected continuum.

We let I denote the unit interval $[0, 1]$. An *arc* is any space homeomorphic to I. An *n-cell* is a space homeomorphic to I^n. The *Hilbert cube* is a space homeomorphic to I^∞, and it is denoted by $\mathcal{Q}$. A metric space Z is said to be a *$\mathcal{Q}$-manifold* provided that for each point $z \in Z$, there exists a neighborhood W of z in Z such that W is a Hilbert cube. An *n-sphere* is a space homeomorphic to the unit sphere $\mathcal{S}^n$ of the Euclidean space $\mathbb{R}^{n+1}$.

A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points (i.e., a 1-dimensional compact connected polyhedron). Given a graph X and a point $x \in X$, the *order of x in X*, denoted by $\mathrm{ord}_x(X)$, is the $\min\{n \in \mathbb{N} \mid x$ has a basis of open sets U in X such that $\mathrm{Cl}(U) \setminus U$ has exactly n points$\}$.

Given a continuum X and a positive integer n, we let $\mathcal{C}_n(X)$ denote the *n-fold hyperspace* of X; that is,

$$\mathcal{C}_n(X) = \{A \subset X \mid A \text{ is nonempty, closed and has at most } n \text{ components}\},$$

topologized with the Hausdorff metric, defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\};$$

$\mathcal{H}$ always denotes the Hausdorff metric. It is known that $\mathcal{C}_n(X)$ is an arcwise connected continuum (for $n = 1$ see [14, (1.12), p. 65], for $n \geq 2$ see [10, 3.1, p. 240]).

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of X; that is,

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has at most } n \text{ points}\}.$$

By the *n-fold hyperspace suspension* of a continuum X, denoted by $\mathrm{HS}_n(X)$, we mean the quotient space

$$\mathrm{HS}_n(X) = \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$$

with the quotient topology. The fact that $\mathrm{HS}_n(X)$ is a continuum follows from [16, 3.10, p. 40]. Notice that $\mathrm{HS}_1(X)$ corresponds to the hyperspace suspension $\mathrm{HS}(X)$ defined in [15].

2.1. Notation. Given a continuum X, $q_X^n \colon \mathcal{C}_n(X) \twoheadrightarrow \mathrm{HS}_n(X)$ denotes the quotient map. We consider $q_X^n(X)$ as the top of $\mathrm{HS}_n(X)$ and denote it by T_X^n. We denote $q_X^n(\mathcal{F}_n(X))$ by F_X^n. For $n = 1$ we write q_X, T_X and F_X instead of q_X^1, T_X^1 and F_X^1.

2.2. Remark. Note that $\mathrm{HS}_n(X) \setminus \{F_X^n\}$ and $\mathrm{HS}_n(X) \setminus \{T_X^n, F_X^n\}$ are homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ and $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_n(X))$, respectively.

2.3. Example. It is easy to see that $\mathrm{HS}(I)$ is a 2-cell, and $\mathrm{HS}(\mathcal{S}^1)$ is a 2-sphere.

Given a continuum X, we define its *suspension* as the quotient space $(X \times I)/(X \times \{1\} \cup X \times \{0\})$. We denote by v^+ and v^- the images of $X \times \{1\}$ and $X \times \{0\}$ under the quotient map, respectively. The suspension of X is denoted by $\mathrm{Sus}(X, v^+, v^-)$.

De Groot [3] defined the notion of an absolute suspension. The general idea of his definition is to consider the vertices of a suspension as distinguished points of the suspension: A continuum X is an *absolute suspension* provided that for any two points p and q of X, there is a continuum $Y(p,q)$ such that $(X,p,q) \approx (\mathrm{Sus}(Y(p,q)), v^+, v^-)$.

In analogy with the definition of absolute suspension, we define absolute n-fold hyperspace suspensions by considering the points T_Y^n and F_Y^n as special points: A continuum X is said to be an *absolute n-fold hyperspace suspension* provided that for each pair of different points p and q of X there exists a continuum $Y(p,q)$, depending on p and q, such that

$$(X,p,q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)}).$$

When $n = 1$, we call X an *absolute hyperspace suspension* (omitting 1-fold).

Note that in our definition, the space $Y(p,q)$ may not be the same as the space $Y(q,p)$. On the other hand, in de Groot's definition of absolute suspension it can be assumed that the spaces $Y(p,q)$ and $Y(q,p)$ are the same space since there is a homeomorphism of any suspension onto itself that interchanges the vertices.

3. General theorems. We prove two theorems about the n-fold hyperspace of a graph. We then prove that absolute n-fold hyperspace suspensions are unicoherent Peano continua and that they are dimensionally homogeneous.

3.1. Theorem. *Let X be a continuum and let n be a positive integer. If* $\dim(\mathrm{HS}_n(X)) < \infty$, *then* $\dim(X) = 1$.

Proof. Let $k = \dim(\mathrm{HS}_n(X))$. Then since $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X) \approx \mathrm{HS}_n(X) \setminus \{F_X^n\}$ (Remark 2.2), $\dim(\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)) \le k$. Hence, $\dim(\mathcal{C}(X) \setminus \mathcal{F}_1(X)) \le k$. Thus, it follows from [9, 45.1, p. 269] that $\dim(\mathcal{C}(X)) \le k$. Therefore, since $k < \infty$, $\dim(X) = 1$ by [9, 72.5, p. 348, and 73.9, p. 354]. ∎

3.2. Theorem. *Let X be a graph, and let n be a positive integer. If Y is a continuum such that* $\mathrm{HS}_n(Y) \approx \mathrm{HS}_n(X)$, *then Y is a graph.*

Proof. Since X is a graph, it is a locally connected continuum. Hence, $\mathrm{HS}_n(X)$ is locally connected by [12, 5.2, p. 131]. Thus, $\mathrm{HS}_n(Y)$ is locally connected, and hence so is Y [12, 5.2, p. 131].

Since X is a graph, $\dim(\mathcal{C}_n(X)) = \dim_X(\mathcal{C}_n(X)) < \infty$ by the formula in [13]. Hence, $\dim(\mathrm{HS}_n(X) \setminus \{F_X^n\}) = \dim(\mathcal{C}_n(X))$. Thus, $\dim(\mathrm{HS}_n(X)) = \dim(\mathcal{C}_n(X))$ by [5, Corollary 2, p. 32]. Therefore, $\dim(\mathrm{HS}_n(Y)) < \infty$. Hence, $\dim(\mathcal{C}_n(Y) \setminus \mathcal{F}_n(Y)) < \infty$. Thus, $\dim_Y(\mathcal{C}_n(Y)) < \infty$. Therefore, since Y is a Peano continuum, Y is a graph by [17, 2.9]. ∎

3.3. QUESTION. Let X be a graph and let n be a positive integer. If Y is a continuum and $\mathrm{HS}_n(Y) \approx \mathrm{HS}_n(X)$, then is $Y \approx X$?

3.4. REMARK. Regarding Question 3.3, note that [12, 5.5 and 5.6, p. 132] and Theorem 4.8 below give a positive answer for I and $\mathcal{S}^1$ for 1-fold and 2-fold hyperspace suspensions, respectively.

3.5. LEMMA. *If Y is a graph, then* $\dim_{F_Y^n}(\mathrm{HS}_n(Y)) \leq \dim_{T_Y^n}(\mathrm{HS}_n(Y))$.

Proof. First note that if for each point y of Y, $\mathrm{ord}_y(Y) \leq 2$, then Y is either an arc or a simple closed curve by [16, 8.40(b), p. 135]. Since $\mathrm{HS}_n(I)$ and $\mathrm{HS}_n(\mathcal{S}^1)$ are $2n$-dimensional Cantor manifolds [12, 3.10, p. 129], both spaces are dimensionally homogeneous by [5, A), p. 93]. Hence, in both cases we have $\dim_{F_Y^n}(\mathrm{HS}_n(Y)) = \dim_{T_Y^n}(\mathrm{HS}_n(Y))$.

Suppose there exists a point $y \in Y$ such that $\mathrm{ord}_y(Y) \geq 3$. By the formula in [13] for $\dim_A(\mathcal{C}_n(Y))$ for any A in $\mathcal{C}_n(Y)$, it follows immediately that $\dim_Y(\mathcal{C}_n(Y)) = \dim(\mathcal{C}_n(Y))$. Thus, $\dim_Y(\mathcal{C}_n(Y)) = \dim_{T_Y^n}(\mathrm{HS}_n(Y))$ by Remark 2.2; hence, $\dim_{T_Y^n}(\mathrm{HS}_n(Y)) = \dim(\mathrm{HS}_n(Y))$. Therefore, we have $\dim_{F_Y^n}(\mathrm{HS}_n(Y)) \leq \dim_{T_Y^n}(\mathrm{HS}_n(Y))$. ∎

3.6. THEOREM. *If a continuum X is an absolute n-fold hyperspace suspension, then X is a unicoherent Peano continuum.*

Proof. Note that for any continuum Y, using order arcs and [14, (1.8), p. 59], it is easy to see that for each $\varepsilon > 0$, $\mathcal{V}_\varepsilon^{\mathcal{H}}(Y) \cap \mathcal{C}_n(Y)$ is arcwise connected. Hence, $\mathcal{C}_n(Y)$ is locally connected at Y. Thus, $\mathrm{HS}_n(Y)$ is locally connected at T_Y^n.

Now, by definition, for any two different points p and q of X, there exists a continuum $Y(p,q)$ such that $(X,p,q) \approx (\mathrm{HS}_n(Y(p,q)), T_X^n, F_X^n)$. Hence, X is locally connected at p. Since p was arbitrary, X is a Peano continuum. Since n-fold hyperspace suspensions have property (b) [12, 4.1, p. 130], they are unicoherent [19, (7.3), p. 227]. Therefore, X is unicoherent. ∎

3.7. THEOREM. *If a continuum X is an absolute n-fold hyperspace suspension, then X is dimensionally homogeneous.*

Proof. Let X be an absolute n-fold hyperspace suspension. By Theorem 3.6, X is a Peano continuum. Let p and q be two points of X. Since

X is an absolute n-fold hyperspace suspension, there exists a continuum $Y(p,q)$ such that

$$(X,p,q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)}).$$

Thus, $\mathrm{HS}_n(Y(p,q))$ is a Peano continuum. Hence, by [12, 5.2, p. 131], $Y(p,q)$ is a Peano continuum. Thus, $\mathcal{C}_n(Y(p,q))$ is locally connected by [10, 3.2, p. 240].

If $\dim(X) < \infty$, then $Y(p,q)$ is a graph by [12, 3.6, p. 128] and [14, (1.109), p. 144]. Since $(X,p,q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)})$, we have $\dim_q(X) \le \dim_p(X)$, by Lemma 3.5. Interchanging the roles of p and q, we find that $\dim_p(X) \le \dim_q(X)$. Therefore, X is dimensionally homogeneous.

If $\dim(X) = \infty$, then $\dim(\mathrm{HS}_n(Y(p,q))) = \infty$. Hence, $\dim(\mathcal{C}_n(Y(p,q))) = \infty$. Since $Y(p,q)$ is locally connected and $\dim(\mathcal{C}_n(Y(p,q))) = \infty$, by [11, 5.1, p. 270], $Y(p,q)$ is not a graph. Thus, by [17, 2.9], $\dim_{Y(p,q)}(\mathcal{C}_n(Y(p,q))) = \infty$. Hence, by Remark 2.2, $\dim_{T^n_X}(\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)}) = \infty$. Therefore, $\dim_p(X) = \infty$. Since p was an arbitrary point of X, X is dimensionally homogeneous. ∎

4. Theorems when X is finite-dimensional. We prove that for each n, there are two possible finite-dimensional absolute n-fold hyperspace suspensions and they are themselves hyperspace suspensions, namely, $\mathrm{HS}_n(I)$ and $\mathrm{HS}_n(\mathcal{S}^1)$. We determine the situation exactly when X is finite-dimensional and $n \le 2$ (Corollaries 4.4 and 4.10); in particular, we prove that $\mathrm{HS}_2(I)$ is not an absolute 2-fold hyperspace suspension by proving that $\mathrm{HS}_2(I)$ is a 4-cell (Theorem 4.6) and proving that $\mathrm{HS}_2(\mathcal{S}^1)$ is unique with respect to 2-fold hyperspace suspensions (Theorem 4.8).

4.1. LEMMA. *If X is a continuum with a free arc, then there exists $\chi \in \mathrm{HS}_n(X)$ such that $\dim_\chi(\mathrm{HS}_n(X)) = 2n$.*

Proof. Let α be a free arc contained in X. Then, by [11, 5.3, p. 271], $\mathcal{C}_n(\alpha)$ is a $2n$-dimensional subspace of $\mathcal{C}_n(X)$, and there is $A \in \mathcal{C}_n(\alpha) \setminus \mathcal{F}_n(\alpha)$ such that $\dim_A(\mathcal{C}_n(X)) = 2n$. Hence, by Remark 2.2, $\dim_{q^n_X(A)}(\mathrm{HS}_n(X)) = 2n$. ∎

4.2. THEOREM. *Let X be a finite-dimensional continuum and let n be a positive integer. If X is an absolute n-fold hyperspace suspension, then either $X \approx \mathrm{HS}_n(I)$ or $X \approx \mathrm{HS}_n(\mathcal{S}^1)$.*

Proof. Suppose X is an absolute n-fold hyperspace suspension. Thus, by Theorem 3.6, X is a Peano continuum. Note that X is also dimensionally homogeneous by Theorem 3.7. Let p and q be two points of X. Hence, there exists a continuum $Y(p,q)$ such that $(X,p,q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)})$. By the proof of Theorem 3.7, $Y(p,q)$ is a graph. Thus, by Lemma 4.1, there exists $\chi \in \mathrm{HS}_n(Y(p,q))$ such that $\dim_\chi(\mathrm{HS}_n(Y(p,q))) = 2n$. Therefore, since X is dimensionally homogeneous, $\dim(\mathrm{HS}_n(Y(p,q))) = 2n$. Hence, by [12, 3.6, p. 128], [14, (1.109), p. 144], [7, 3.1, p. 182] and [16, 8.40(b), p. 135], $Y(p,q)$ is either an arc or a simple closed curve. Thus, either $X \approx \mathrm{HS}_n(I)$ or $X \approx \mathrm{HS}_n(\mathcal{S}^1)$. ■

Our next theorem shows that when $n = 1$, the converse of Theorem 4.2 is false for I but true for $\mathcal{S}^1$. As a corollary, we deduce that the 2-sphere is the only finite-dimensional absolute hyperspace suspension. We show in Theorem 4.9 that when $n = 2$, the converse of Theorem 4.2 is false for both I and $\mathcal{S}^1$.

4.3. THEOREM. $\mathrm{HS}(I)$ *is not an absolute hyperspace suspension, but* $\mathrm{HS}(\mathcal{S}^1)$ *is an absolute hyperspace suspension.*

Proof. By Example 2.3, $\mathrm{HS}(I)$ is a 2-cell. Let Y be a continuum such that $\mathrm{HS}(I) \approx \mathrm{HS}(Y)$. Then Y is a 2-cell [12, 5.5, p. 132]. Furthermore, it follows from [9, 5.1.1, p. 33] that T_Y and F_Y are points of the manifold boundary of $\mathrm{HS}(I)$. Thus, considering two points χ_1 and χ_2 in the manifold interior of $\mathrm{HS}(I)$, we see that $\mathrm{HS}(I)$ is not an absolute hyperspace suspension.

To prove the second part of our theorem, recall from Example 2.3 that $\mathrm{HS}(\mathcal{S}^1)$ is a 2-sphere. Hence, $\mathrm{HS}(\mathcal{S}^1)$ has the following property: If $\{\chi_1, \chi_2\}$ and $\{\chi_1', \chi_2'\}$ are two-point subsets of $\mathrm{HS}(\mathcal{S}^1)$, then there is a homeomorphism of $\mathrm{HS}(\mathcal{S}^1)$ onto $\mathrm{HS}(\mathcal{S}^1)$ taking χ_k to χ_k' for each $k \in \{1, 2\}$. It now follows immediately that $\mathrm{HS}(\mathcal{S}^1)$ is an absolute hyperspace suspension. ■

4.4. COROLLARY. *Let X be a finite-dimensional continuum. Then X is an absolute hyperspace suspension if and only if X is a 2-sphere.*

Proof. Assume that X is a finite-dimensional absolute hyperspace suspension. By Theorem 4.2, $X \approx \mathrm{HS}(I)$ or $X \approx \mathrm{HS}(\mathcal{S}^1)$. Hence, by Theorem 4.3, $X \approx \mathrm{HS}(\mathcal{S}^1)$. Therefore, X is a 2-sphere by Example 2.3.

Conversely, a 2-sphere is an absolute hyperspace suspension by Theorem 4.3 since $\mathrm{HS}(\mathcal{S}^1)$ is a 2-sphere (Example 2.3). ■

In Theorem 4.9, we show that the converse of Theorem 4.2 is false when $n = 2$. First, we obtain a model for $\mathrm{HS}_2(I)$ (Theorem 4.6), which is of interest independent of its use here.

4.5. LEMMA. *There is an embedding φ of $\mathcal{C}_2(I)$ in $\mathbb{R}^4$ such that $\varphi(\mathcal{F}_2(I))$ is convex.*

Proof. The proof of the lemma is a matter of making several observations about the proof that $\mathcal{C}_2(I)$ is a 4-cell as done in [6, 2.2, p. 349].

We let $\mathrm{Cone}(Y)$ denote the quotient space $(Y \times I)/(Y \times \{1\})$. We use the notation from the proof in [6, p. 349]:

- $D^1 = \{A \in \mathcal{C}_2(I) \mid 1 \in A\}$, $D^1_0 = \{A \in \mathcal{C}_2(I) \mid 0, 1 \in A\}$.

- f: $\text{Cone}(D^1) \to \mathcal{C}_2(I)$ is the homeomorphism given by

$$f(A,t) = \begin{cases} (1-t)A & \text{if } t < 1, \\ \{0\} & \text{if } t = 1. \end{cases}$$

- g: $\text{Cone}(D_0^1) \to D^1$ is the homeomorphism given by

$$g(A,t) = \begin{cases} t + (1-t)A & \text{if } t < 1, \\ \{1\} & \text{if } t = 1. \end{cases}$$

In the last paragraph of the above-mentioned proof of [6], it is shown that D_0^1 is a 2-cell S. The only element of $\mathcal{F}_2(I)$ in D_0^1 is $\{0,1\}$, which corresponds to the point $(0,1)$ under the homeomorphism in [6]. We consider $\{0,1\}$ to correspond to the point $(0,1,0,0)$ in $\mathbb{R}^4$.

The elements of $\mathcal{F}_2(I)$ in D^1 are those of the form $\{a,1\}$, $a \in I$. From the formula for g, we have

$$g^{-1}(\{a,1\}) = (\{0,1\}, a) \quad \text{for all } \{a,1\}.$$

Thus, letting $(0,1,1,0)$ be the vertex of $\text{Cone}(D_0^1)$, we can consider the elements of $\mathcal{F}_2(I)$ in D^1 as corresponding under g^{-1} to the line segment L in $\mathbb{R}^4$ from the point $(0,1,0,0)$ to the point $(0,1,1,0)$; specifically, $g^{-1}(\{a,1\}) = (0,1,a,0)$ for all $\{a,1\}$, $a \in I$.

From the formula for f, we have

$$f^{-1}(\{a,b\}) = \begin{cases} \left(\dfrac{1}{\max\{a,b\}}\{a,b\}, 1 - \max\{a,b\}\right) & \text{if } \{a,b\} \neq \{0\}, \\ \text{vertex } v \text{ of } \text{Cone}(D^1), & \text{if } \{a,b\} = \{0\}. \end{cases}$$

Hence, letting $(0,1,0,1)$ denote the vertex of $\text{Cone}(D^1)$, we can consider $\mathcal{F}_2(I)$ to correspond to the line segments in $\mathbb{R}^4$ joining all points of L to $(0,1,0,1)$; specifically, $f^{-1}(\{a,b\}) = (0,1,\min\{a,b\}, 1-\max\{a,b\})$ for all $\{a,b\} \in \mathcal{F}_2(I)$. Note that $f^{-1}(\{a,1\}) = g^{-1}(\{a,1\})$ for all $a \in I$.

Therefore, $\varphi = f^{-1}$ satisfies our lemma. ∎

4.6. THEOREM. $\text{HS}_2(I)$ *is a* 4*-cell.*

Proof. The hyperspace $\mathcal{C}_2(I)$ is a 4-cell (a result due to R. Schori; for a proof, see [6, 2.2, p. 349]). Let β denote the manifold boundary of $\mathcal{C}_2(I)$.

Next, $\mathcal{F}_2(I) \subset \beta$. This can be seen from the proof of Lemma 4.5 or by considering the maps f_ε: $\mathcal{C}_2(I) \to \mathcal{C}_2(I) \setminus \mathcal{F}_2(I)$, $\varepsilon > 0$, given by

$$f_\varepsilon(A) = \text{Cl}(\mathcal{V}_\varepsilon(A)) \quad \text{for each } A \in \mathcal{C}_2(I);$$

the maps f_ε show that each element of $\mathcal{F}_2(I)$ is an unstable value of the identity map on $\mathcal{C}_2(I)$ (as defined in [5, VI 1, p. 74]). Thus, since $\mathcal{C}_2(I)$ is a 4-cell, $\mathcal{F}_2(I) \subset \beta$ [5, Example VI 2, p. 75].

By Lemma 4.5, we can assume that $\mathcal{C}_2(I)$ is in $\mathbb{R}^4$ with $\mathcal{F}_2(I)$ being convex. Thus, since $\mathcal{F}_2(I) \subset \beta$ (as just proved) and since β is a 3-sphere, it follows that the quotient space $\beta/\mathcal{F}_2(I)$ is a 3-sphere by the Theorem in [1, p. 21].

Finally, we prove that $\mathrm{HS}_2(I)$ is a 4-cell. Let $q\colon \beta \to \beta/\mathcal{F}_2(I)$ denote the quotient map. Since β and $\beta/\mathcal{F}_2(I)$ are 3-spheres and $\mathcal{F}_2(I)$ is compact, q is a cellular map [18, 1.8.1, p. 44]. Hence, q can be extended to a map f from the 4-cell $\mathcal{C}_2(I)$ onto a 4-cell I^4 such that f is a homeomorphism on the manifold interior of $\mathcal{C}_2(I)$ onto the manifold interior of I^4 [4, Theorem, p. 1279]. Note that the only nondegenerate fiber of f is $q^{-1}(\{\mathcal{F}_2(I)\}) = \mathcal{F}_2(I)$, which is also the only nondegenerate fiber of the quotient map

$$q_I^2\colon \mathcal{C}_2(I) \to \mathcal{C}_2(I)/\mathcal{F}_2(I) = \mathrm{HS}_2(I).$$

Therefore, $q_I^2 \circ f^{-1}$ is a homeomorphism of I^4 onto $\mathrm{HS}_2(I)$ (continuity is by [16, 3.22, p. 45]). This proves $\mathrm{HS}_2(I)$ is a 4-cell. ∎

4.7. LEMMA. *Each $A \in \mathcal{C}_2(\mathcal{S}^1)\setminus\{\mathcal{S}^1\}$ has a 4-cell neighborhood in $\mathcal{C}_2(\mathcal{S}^1)$ and the point $\mathcal{S}^1$ does not have a 4-cell neighborhood in $\mathcal{C}_2(\mathcal{S}^1)$. Hence, the point $T^2_{\mathcal{S}^1}$ of $\mathrm{HS}_2(\mathcal{S}^1)$ does not have a 4-cell neighborhood in $\mathrm{HS}_2(\mathcal{S}^1)$.*

Proof. Each $A \in \mathcal{C}_2(\mathcal{S}^1) \setminus \{\mathcal{S}^1\}$ is contained in the interior of an arc $B \subset \mathcal{S}^1$; clearly, $\mathcal{C}_2(B)$ is a neighborhood of A in $\mathcal{C}_2(\mathcal{S}^1)$, and $\mathcal{C}_2(B)$ is a 4-cell [6, 2.2, p. 349].

We now prove that $\mathcal{S}^1$ does not have a 4-cell neighborhood in $\mathcal{C}_2(\mathcal{S}^1)$. A. Illanes has proved that $\mathcal{C}_2(\mathcal{S}^1)$ is the cone $\mathcal{K}$ over a solid (3-dimensional) torus [8, p. 118]. Clearly, the vertex v of $\mathcal{K}$ is the only point of $\mathcal{K}$ that could fail to have a 4-cell neighborhood in $\mathcal{K}$. In fact, v does not have such a neighborhood, which we show as follows: If $\mathcal{N}$ were a 4-cell neighborhood of v in $\mathcal{K}$, then $\mathcal{N}\setminus\{v\}$ would retract onto a solid torus that is a level of $\mathcal{K}$ in $\mathcal{N}$. Thus, $\mathcal{N} \setminus \{v\}$ would not be simply connected. However, $\mathcal{N} \setminus \{v\}$ is simply connected since a 4-cell minus any point is simply connected. Therefore, v does not have a 4-cell neighborhood in $\mathcal{K}$.

Hence, from the first part of our lemma, any homeomorphism of $\mathcal{C}_2(\mathcal{S}^1)$ onto $\mathcal{K}$ must take $\mathcal{S}^1$ to v. Therefore, $\mathcal{S}^1$ does not have a 4-cell neighborhood in $\mathcal{C}_2(\mathcal{S}^1)$.

The last part of our lemma now follows from Remark 2.2 since, by the definition of hyperspace suspension, $\mathrm{HS}_2(\mathcal{S}^1)$ is locally the same at the point $T^2_{\mathcal{S}^1}$ as $\mathcal{C}_2(\mathcal{S}^1)$ is at the point $\mathcal{S}^1$. ∎

The following theorem gives the definitive version of Theorem 5.7 of [12, p. 132] for the case when $n = 2$.

4.8. THEOREM. *Let Y be a continuum. If $\mathrm{HS}_2(I) \approx \mathrm{HS}_2(Y)$, then $Y \approx I$; if $\mathrm{HS}_2(\mathcal{S}^1) \approx \mathrm{HS}_2(Y)$, then $Y \approx \mathcal{S}^1$.*

Proof. Assume that $\mathrm{HS}_2(I) \approx \mathrm{HS}_2(Y)$. Then, by Theorem 4.6, $\mathrm{HS}_2(Y)$ is a 4-cell. Also, by Theorem 3.2, Y is a graph. Thus, it follows easily that

$\mathrm{ord}_y(Y) \leq 2$ for all $y \in Y$ (otherwise, $\dim_\chi(\mathrm{HS}_2(Y)) = 5$ when $(q_Y^2)^{-1}(\chi)$ is the disjoint union of a simple triod and an arc contained in the manifold interior of a free arc in Y). Therefore, $Y \approx I$ or $Y \approx \mathcal{S}^1$. However, $\mathrm{HS}_2(\mathcal{S}^1)$ is not a 4-cell by Lemma 4.7. Therefore, $Y \approx I$. This proves the first part of the theorem.

To prove the second part, assume that $\mathrm{HS}_2(\mathcal{S}^1) \approx \mathrm{HS}_2(Y)$. Then $\mathrm{HS}_2(Y)$ is a 4-dimensional Cantor manifold [12, 3.10, p. 129]. Thus, $\dim_\chi(\mathrm{HS}_2(Y)) = 4$ for all $\chi \in \mathrm{HS}_2(Y)$ [5, A), p. 93]. Also, Y is a graph by Theorem 3.2. Hence, $\mathrm{ord}_y(Y) \leq 2$ for all $y \in Y$ (as at the beginning of the proof). Thus, $Y \approx I$ or $Y \approx \mathcal{S}^1$. Since $\mathrm{HS}_2(\mathcal{S}^1)$ is not a 4-cell by Lemma 4.7, we know from Theorem 4.6 that $Y \not\approx I$. Therefore, $Y \approx \mathcal{S}^1$. ∎

4.9. Theorem. *$\mathrm{HS}_2(I)$ and $\mathrm{HS}_2(\mathcal{S}^1)$ are not absolute 2-fold hyperspace suspensions.*

Proof. By Theorem 4.6, $\mathrm{HS}_2(I)$ is a 4-cell. It is actually the case that T_I^2 and F_I^2 are both in the manifold boundary of $\mathrm{HS}_2(I)$. However, we need not verify this. Instead, we simply choose χ_1 and χ_2 as follows: Let χ_1 and χ_2 be points in the manifold interior of $\mathrm{HS}_2(I)$ if at least one of T_I^2 and F_I^2 is in the manifold boundary of $\mathrm{HS}_2(I)$; let χ_1 and χ_2 be points in the manifold boundary of $\mathrm{HS}_2(I)$ if T_I^2 and F_I^2 are both in the manifold interior of $\mathrm{HS}_2(I)$. Then we see that

$$(\mathrm{HS}_2(I), \chi_1, \chi_2) \not\approx (\mathrm{HS}_2(I), T_I^2, F_I^2).$$

Therefore, $\mathrm{HS}_2(I)$ is not an absolute 2-fold hyperspace suspension (since the continuum $Y(\chi_1, \chi_2)$ in the definition of absolute 2-fold hyperspace suspension must be I by Theorem 4.8).

It remains to prove that $\mathrm{HS}_2(\mathcal{S}^1)$ is not an absolute 2-fold hyperspace suspension. Let $A, B \in \mathcal{C}_2(\mathcal{S}^1) \setminus \{\mathcal{S}^1\}$ such that $A \notin \mathcal{F}_2(\mathcal{S}^1)$ and $B \notin \mathcal{F}_2(\mathcal{S}^1)$. By Lemma 4.7, A and B have 4-cell neighborhoods in $\mathcal{C}_2(\mathcal{S}^1)$. Hence, $q_{\mathcal{S}^1}^2(A)$ and $q_{\mathcal{S}^1}^2(B)$ have 4-cell neighborhoods in $\mathrm{HS}_2(\mathcal{S}^1)$. By the second part of Lemma 4.7, $T_{\mathcal{S}^1}^2$ does not have a 4-cell neighborhood in $\mathrm{HS}_2(\mathcal{S}^1)$. Thus,

$$(\mathrm{HS}_2(\mathcal{S}^1), q_{\mathcal{S}^1}^2(A), q_{\mathcal{S}^1}^2(B)) \not\approx (\mathrm{HS}_2(\mathcal{S}^1), T_{\mathcal{S}^1}^2, F_{\mathcal{S}^1}^2).$$

Therefore, $\mathrm{HS}_2(\mathcal{S}^1)$ is not an absolute 2-fold hyperspace suspension (by Theorem 4.8). ∎

4.10. Corollary. *No finite-dimensional continuum is an absolute 2-fold hyperspace suspension.*

Proof. Apply Theorems 4.2 and 4.9. ∎

We do not know if Theorem 4.9 extends to $n \geq 3$:

4.11. Question. Is $\mathrm{HS}_n(I)$ or $\mathrm{HS}_n(\mathcal{S}^1)$ an absolute n-fold hyperspace suspension for each $n \geq 3$?

5. A theorem when X is infinite-dimensional. We show that the Hilbert cube is an inifinite-dimensional absolute n-fold hyperspace suspension. Then we prove that any such absolute n-fold hyperspace suspension must be a unicoherent $\mathcal{Q}$-manifold.

5.1. Example. The Hilbert cube is an absolute n-fold hyperspace suspension. To see this, note that by [12, 5.4, p. 132], $\mathrm{HS}_n(\mathcal{Q}) \approx \mathcal{Q}$. Hence, for any two points p and q of $\mathcal{Q}$, $(\mathcal{Q}, p, q) \approx (\mathrm{HS}_n(\mathcal{Q}), T^n_{\mathcal{Q}}, F^n_{\mathcal{Q}})$ by Anderson's theorem on homogeneity (see [9, 11.9.1, p. 93]).

5.2. Theorem. *Let X be an inifinte-dimensional continuum and let n be a positive integer. If X is an absolute n-fold hyperspace suspension, then X is a unicoherent $\mathcal{Q}$-manifold.*

Proof. Suppose X is an infinite-dimensional absolute n-fold hyperspace suspension. Thus, by Theorem 3.6, X is a unicoherent Peano continuum. Let p and q be two points of X. By hypothesis, there exists a continuum $Y(p,q)$ such that $(X, p, q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)})$.

Since X is a Peano continuum, $\mathrm{HS}_n(Y(p,q))$ is locally connected. Hence, by [12, 5.2, p. 131], $Y(p,q)$ is a Peano continuum. Thus, $\mathcal{C}_n(Y(p,q))$ is locally connected by [10, 3.2, p. 240].

Since $\dim(X) = \infty$ and X is dimensionally homogeneous (Theorem 3.7), we see from Lemma 4.1 that $Y(p,q)$ does not have a free arc. Hence, by [10, 7.1, p. 250], $\mathcal{C}_n(Y(p,q))$ is a Hilbert cube. This implies that $Y(p,q)$ has a Hilbert cube neighborhood $\mathcal{W}$ in $\mathcal{C}_n(Y(p,q))$ such that $\mathcal{W} \cap \mathcal{F}_n(Y(p,q)) = \emptyset$. Hence, by Remark 2.2 and since $(X, p, q) \approx (\mathrm{HS}_n(Y(p,q)), T^n_{Y(p,q)}, F^n_{Y(p,q)})$, p has a Hilbert cube neighborhood W in X. Therefore, X is a $\mathcal{Q}$-manifold. ∎

5.3. Question. Does there exist a compact $\mathcal{Q}$-manifold other than $\mathcal{Q}$ that is an absolute n-fold hyperspace suspension?

5.4. Remark. Note that $\mathcal{S}^1 \times \mathcal{Q}$ is a $\mathcal{Q}$-manifold which is not an n-fold hyperspace suspension since it is not contractible with respect to $\mathcal{S}^1$, equivalently, it does not have property (b) [19, p. 226], and it is known that n-fold hyperspace suspensions have property (b) [12, 4.1, p. 130].

Acknowledgements. We thank Professor Robert Daverman for a stimulating series of communications concerning the proof of Theorem 4.6 and for references [1] and [4].

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Received 22 June 2005;
revised 10 November 2005 (4625)