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THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE OF A STACKED MONOMIAL ALGEBRA

BY

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Dedicated to the memory of Sheila Brenner

Abstract. This paper studies the Hochschild cohomology of finite-dimensional monomial algebras. If $\Lambda = K\mathcal{Q}/I$ with I an admissible monomial ideal, then we give sufficient conditions for the existence of an embedding of $K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$ into the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$. We also introduce stacked algebras, a new class of monomial algebras which includes Koszul and D-Koszul monomial algebras. If Λ is a stacked algebra, we prove that $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$, where $\mathcal{N}$ is the ideal in $\mathrm{HH}^*(\Lambda)$ generated by the homogeneous nilpotent elements. In particular, this shows that the Hochschild cohomology ring of Λ modulo nilpotence is finitely generated as an algebra.

Introduction. Let K be a field and let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional K-algebra where $\mathcal{Q}$ is a quiver and I is an admissible ideal. We assume that Λ is a monomial algebra, that is, the ideal I is generated by a finite set of paths ϱ. We take the set ϱ to be a minimal generating set for I. The Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ is given by

$$\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$$

with the Yoneda product, where Λ^{e} is the enveloping algebra $\Lambda^{\mathrm{op}} \otimes_K \Lambda$ of Λ. Let $\mathcal{N}$ be the ideal in $\mathrm{HH}^*(\Lambda)$ generated by the homogeneous nilpotent elements.

The first part of the paper studies subrings of the Hochschild cohomology ring of a monomial algebra. In particular, in Theorem 2.13 we give sufficient conditions for the existence of non-nilpotent elements $x_1, \ldots, x_r$ in $\mathrm{HH}^*(\Lambda)$ with $x_a x_b = 0$ for $a \neq b$ so that $K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$ is a subalgebra of $\mathrm{HH}^*(\Lambda)$. We then introduce a new class of monomial algebras

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which we call (D, A)-stacked monomial algebras (Definition 3.1). The class of (D, A)-stacked monomial algebras includes the Koszul monomial algebras as well as the D-Koszul monomial algebras of [2, 6]. The second part of the paper determines, for a finite-dimensional (D, A)-stacked monomial algebra with $\operatorname{char} K \neq 2$, the quotient $\mathrm{HH}^*(\Lambda)/\mathcal{N}$, and shows in Theorem 3.4 that $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$. In particular, we show that this quotient is finitely generated as a K-algebra and of Krull dimension at most 1.

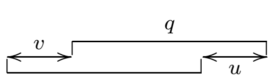
It was conjectured in [11] that $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is finitely generated as a ring for any artin algebra Λ over a commutative artinian ring. Some evidence for this conjecture came from [11] where it was shown for a finite-dimensional Nakayama algebra with one relation, and from [8] where it was shown for a finite-dimensional selfinjective indecomposable algebra of finite representation type over an algebraically closed field. The conjecture is also known to be true for any block of a group ring of a finite group [3, 12], and any block of a finite-dimensional cocommutative Hopf algebra [4] (and see [11]). In [9], the conjecture is shown for all monomial algebras. The results of this paper differ from those in [9] in that we obtain a complete description of the ring $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ for a (D, A)-stacked monomial algebra Λ.

1. Background. One of the main tools we use in this paper is the minimal projective resolution of a monomial algebra Λ over Λ^{e} as given in [1]. This requires the concept of overlaps of [5] and [10]. We recall the relevant definitions here, once we have introduced some basic notation.

An arrow α starts at the vertex $\mathfrak{o}(\alpha)$ and ends at the vertex $\mathfrak{t}(\alpha)$; arrows in a path are read from left to right. If $p = \alpha_1 \cdots \alpha_n$ is a path with arrows $\alpha_1, \ldots, \alpha_n$ then $\mathfrak{o}(p) = \mathfrak{o}(\alpha_1)$ and $\mathfrak{t}(p) = \mathfrak{t}(\alpha_n)$. (Note that if $n = 0$ so that the path is a vertex v, then $\mathfrak{o}(v) = v = \mathfrak{t}(v)$.) We denote the *length* of a path p by $\ell(p)$. We fix ϱ as a minimal generating set for the ideal I, and refer to an element of ϱ as a *relation.* An arrow α *begins* (respectively *ends*) a relation r in ϱ if $r = \alpha p$ (respectively $r = p\alpha$) for some path p. A path p is a *prefix* of a path q if there is some path p' such that $q = pp'$. A path p is a *suffix* of a path q if there is some path p' such that $q = p'p$.

Definition 1.1.

(1) A path q *overlaps* a path p *with overlap* pu if there are paths u and v such that $pu = vq$ and $1 \leq \ell(u) < \ell(q)$. We may illustrate the definition with the following diagram:



Note that we allow $\ell(v) = 0$ here.

(2) A path q *properly overlaps* a path p *with overlap* pu if q overlaps p and $\ell(v) \geq 1$.
(3) A path p *has no overlaps* with a path q if p does not properly overlap q and q does not properly overlap p.

To describe a minimal projective resolution of Λ over Λ^{e}, we use sets $\mathcal{R}^n$ which we now define recursively. Let

$$\mathcal{R}^0 = \text{the set of vertices of } \mathcal{Q},$$
$$\mathcal{R}^1 = \text{the set of arrows of } \mathcal{Q},$$
$$\mathcal{R}^2 = \varrho, \text{ the minimal set of paths in the generating set of } I.$$

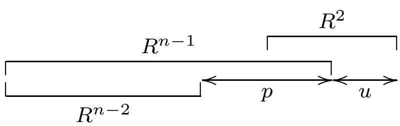
For $n \geq 3$, we say $R^2 \in \mathcal{R}^2$ *maximally overlaps* $R^{n-1} \in \mathcal{R}^{n-1}$ *with overlap* $R^n = R^{n-1}u$ if

(1) $R^{n-1} = R^{n-2}p$ for some path p;
(2) R^2 overlaps p with overlap pu;
(3) there is no element of $\mathcal{R}^2$ which overlaps p with overlap being a proper prefix of pu.

We may also say that R^n is a *maximal overlap* of $R^2 \in \mathcal{R}^2$ with $R^{n-1} \in \mathcal{R}^{n-1}$.

The set $\mathcal{R}^n$ is defined to be the set of all overlaps R^n formed in this way.

The construction of the paths in $\mathcal{R}^n$ may be illustrated with the following diagram of R^n:



We also recall from [10] that if $R_1^n p = R_2^n q$ for $R_1^n, R_2^n \in \mathcal{R}^n$ and paths p, q, then $R_1^n = R_2^n$ and $p = q$.

Let (P^*, δ^*) be the minimal projective Λ^{e}-resolution of Λ from [1]. Then

$$P^n = \coprod_{R^n \in \mathcal{R}^n} \Lambda\mathfrak{o}(R^n) \otimes \mathfrak{t}(R^n)\Lambda$$

where we write $\otimes$ for $\otimes_K$ throughout.

Any element R^n in $\mathcal{R}^n$ may be expressed uniquely as $R_j^{n-1}a_j$ and as $b_k R_k^{n-1}$ for some $R_j^{n-1}, R_k^{n-1} \in \mathcal{R}^{n-1}$ and paths a_j, b_k. We say that the elements R_j^{n-1} and R_k^{n-1} *occur in* R^n.

The map

$$\delta^{2n+1} \colon P^{2n+1} \to P^{2n}$$

is given as follows. If $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n} \in \mathcal{R}^{2n+1}$ then

$$\mathfrak{o}(R^{2n+1}) \otimes \mathfrak{t}(R^{2n+1}) \mapsto \mathfrak{o}(R_j^{2n}) \otimes a_j - b_k \otimes \mathfrak{t}(R_k^{2n})$$

where the first tensor lies in the summand corresponding to R_j^{2n} and the second tensor lies in the summand corresponding to R_k^{2n}.

For even degree elements, any element R^{2n} in $\mathcal{R}^{2n}$ may be expressed in the form $p_j R_j^{2n-1} q_j$ for some $R_j^{2n-1} \in \mathcal{R}^{2n-1}$ and paths p_j, q_j with $n \geq 1$. Let $R^{2n} = p_1 R_1^{2n-1} q_1 = \cdots = p_r R_r^{2n-1} q_r$ be all expressions of R^{2n} which contain some element of $\mathcal{R}^{2n-1}$ as a subpath. Then the map

$$\delta^{2n} \colon P^{2n} \to P^{2n-1}$$

is given as follows. If $R^{2n} \in \mathcal{R}^{2n}$ then, with the above notation,

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \sum_{j=1}^{r} p_j \otimes q_j$$

where the tensor $p_j \otimes q_j$ lies in the summand of P^{2n-1} corresponding to R_j^{2n-1}.

If not specified, then it will always be clear from the context in which summand of a projective module our tensors lie.

We now recall the product structure of $\mathrm{HH}^*(\Lambda)$. An element η of $\mathrm{HH}^n(\Lambda)$ is represented by a map $P^n \to \Lambda$ of Λ^{e}-modules; by abuse of notation we also denote our chosen representative map by η. The *liftings* of η are choices of maps $\Omega^m\eta\colon P^{n+m} \to P^m$, for $m \geq 0$, such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+m} \arrow[r, "\delta^{n+m}"] \arrow[d, "\Omega^m\eta"'] & \cdots \arrow[r, "\delta^{n+2}"] & P^{n+1} \arrow[r, "\delta^{n+1}"] \arrow[d, "\Omega\eta"'] & P^n \arrow[d, "\Omega^0\eta"'] \arrow[dr, "\eta"] & \\
\cdots \arrow[r] & P^m \arrow[r, "\delta^m"] & \cdots \arrow[r, "\delta^2"] & P^1 \arrow[r, "\delta^1"] & P^0 \arrow[r, "\delta^0"] & \Lambda
\end{tikzcd}$$

For homogeneous elements $\theta \in \mathrm{HH}^m(\Lambda)$ and $\eta \in \mathrm{HH}^n(\Lambda)$ represented by maps $P^m \to \Lambda$ and $P^n \to \Lambda$ respectively, the product $\theta\eta$ in Hochschild cohomology is the element of $\mathrm{HH}^{n+m}(\Lambda)$ which is represented by the map $\theta \circ \Omega^m\eta$, where $\circ$ denotes the usual composition of maps. Recall that this agrees with the Yoneda product and is independent of the choice of representatives and liftings for η and θ.

2. Subalgebras of $\mathrm{HH}^*(\Lambda)$**.** In this section we give sufficient conditions on a finite-dimensional monomial algebra $\Lambda = K\mathcal{Q}/I$ for $\mathrm{HH}^*(\Lambda)$ to have a subalgebra of the form

$$K[x_1, \ldots, x_r]/\langle x_i x_j \text{ for } i \neq j\rangle.$$

The first result looks at elements of $\mathrm{HH}^*(\Lambda)$ which are determined by certain closed paths in the quiver, and for which we need some preliminary lemmas. The proofs of these lemmas are straightforward, but we include them for completeness.

DEFINITION 2.1. A *closed path* C in $\mathcal{Q}$ is a non-trivial path C in $K\mathcal{Q}$ such that $C = vCv$ for some vertex v. We may say that C is a closed path at the vertex v. We do not make any assumptions with this terminology as to whether or not C is a non-zero element in the algebra Λ.

If C is a closed path at the vertex v then we say that v is *not internal* to C if $C = v\sigma_1 v\sigma_2 v$ for paths σ_1, σ_2 implies that $\sigma_1 = v$ or $\sigma_2 = v$.

LEMMA 2.2. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra where I is an admissible ideal with minimal set of generators ϱ. Suppose that there is a closed path C in the quiver $\mathcal{Q}$ at the vertex v such that $C \neq p^r$ for any path p with $r \geq 2$ and that $C^s \in \varrho$ for some $s \geq 2$. Suppose also that there are no overlaps of C^s with any relation in $\varrho \setminus \{C^s\}$. Then the vertex v is not internal to C.*

Proof. Suppose that v is internal to C; then $C = v\sigma_1 v\sigma_2 v$ for distinct non-trivial closed paths σ_1, σ_2. Since Λ is finite-dimensional, there are natural numbers N_1, N_2 with $\sigma_i^{N_i} \in I$ (for $i = 1, 2$) and hence there is a subword w_i of $\sigma_i^{N_i}$ in ϱ, the set of generators for I. But these subwords are in $\varrho \setminus \{C^s\}$. Moreover no subword of σ_i is in ϱ since $C^s \in \varrho$. Thus each w_i must be of the form $a_i\sigma_i^{t_i}b_i$ with a_i a suffix of σ_i, b_i a prefix of σ_i and $t_i \geq 0$. We see that C properly overlaps w_1, and w_2 properly overlaps C (noting that if a_i and b_i are both vertices then $t_i \geq 2$). So there is a proper overlap of C^s with some element of $\varrho \setminus \{C^s\}$, which contradicts the hypothesis. ∎

LEMMA 2.3. *Let a and b be paths of length at least 1 and suppose that $ab = ba$. Then there is a path p and integers $r, s \geq 1$ such that $a = p^r$ and $b = p^s$.*

Proof. The proof is by induction on $\ell(ab)$. For the initial case, if $\ell(ab) = 2$ then we may take $a = b = p$ and we are done. Now assume the assertion is true for paths z, z' with $\ell(zz') < n$ and $zz' = z'z$. Suppose that $\ell(ab) = n$ and that $ab = ba$. If $\ell(a) = \ell(b)$ then $a = b$ and we may take $p = a = b$. So, without loss of generality, suppose that $\ell(a) > \ell(b)$. Then we may write $a = bq = q'b$ for some paths q, q' with $\ell(q) = \ell(q') \geq 1$. So $bqb = ab = ba = bq'b$ and hence $q = q'$. Thus $bq = qb$ with $\ell(b), \ell(q) \geq 1$ and $\ell(bq) < n$. By the induction hypothesis, there is a path p and integers $r, s \geq 1$ such that $b = p^r$ and $q = p^s$. Hence $a = p^{r+s}$. This completes the proof. ∎

LEMMA 2.4. *Let a, b and c be paths such that $1 \leq \ell(a) < \ell(c)$ and $c^s a = bc^s$ for some $s \geq 1$. Then there is a path p and integer $t \geq 2$ such that $c = p^t$.*

Proof. Since $\ell(a) = \ell(b) < \ell(c)$ we may write $c = bq = q'a$ for some paths q, q' with $\ell(q) = \ell(q') \geq 1$. Then

$$bqbq \cdots bqa = c^s a = bc^s = bq'aq'a \cdots q'a.$$

Using $\ell(q) = \ell(q')$ and $\ell(a) = \ell(b)$, it follows that $q = q'$ and $a = b$. Hence $c = bq = qb$. Thus we may apply Lemma 2.3 to obtain a path p and integers $r, r' \geq 1$ such that $b = p^r$ and $q = p^{r'}$. Hence $c = p^t$ where $t = r + r' \geq 2$. ∎

We now come to our first result. Recall that ϱ is a fixed minimal generating set for I and that we refer to an element of ϱ as a relation.

PROPOSITION 2.5. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra where I is an admissible ideal with minimal set of generators ϱ. Suppose that there is a closed path C in the quiver $\mathcal{Q}$ at the vertex v such that $C \neq p^r$ for any path p with $r \geq 2$ and that $C^s \in \varrho$ for some $s \geq 2$. Suppose also that there are no overlaps of C^s with any relation in $\varrho \setminus \{C^s\}$.*

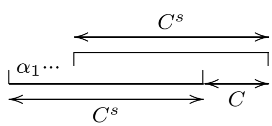
Then there is a subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$ where x is in degree 2 and is represented by the map $P^2 \to \Lambda$ where, for $R^2 \in \mathcal{R}^2$,

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v & \textit{if } R^2 = C^s, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. From Lemma 2.2 we know that the vertex v is not internal to C. Let $C = \alpha_1 \cdots \alpha_r$ where the α_i are arrows.

The element C^s in $\mathcal{R}^2$ properly overlaps itself with overlap C^{s+1}. If C^{s+1} is not in $\mathcal{R}^3$ then there is some path u with $\ell(u) < \ell(C)$ such that $C^s u \in \mathcal{R}^3$. Hence $C^s u = vR^2$ for some path v and $R^2 \in \mathcal{R}^2$. But then R^2 overlaps C^s and so by hypothesis $R^2 = C^s$. Now Lemma 2.4 contradicts the hypothesis that $C \neq p^r$ for any path p with $r \geq 2$. Hence $C^{s+1} \in \mathcal{R}^3$. Since there are no overlaps of C^s with any other relation in ϱ, this is also the only element of $\mathcal{R}^3$ in which C^s occurs.

Since $\alpha_1 \in \mathcal{R}^1$, we may illustrate the element $C^{s+1} \in \mathcal{R}^3$ by the following diagram:

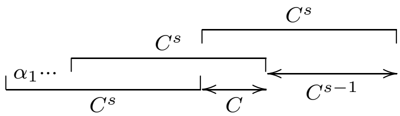


The image of $\mathfrak{o}(C^{s+1}) \otimes \mathfrak{t}(C^{s+1})$ under the map $P^3 \xrightarrow{\delta^3} P^2$ is $v \otimes C - C \otimes v$ in the summand corresponding to the element C^s of $\mathcal{R}^2$. For $R^2 \in \mathcal{R}^2$, define $x \colon P^2 \to \Lambda$ by

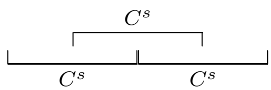
$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v & \text{if } R^2 = C^s, \\ 0 & \text{otherwise.} \end{cases}$$

Then the composition $P^3 \xrightarrow{\delta^3} P^2 \xrightarrow{x} \Lambda$ is zero. Moreover, since the image of any composition $P^2 \xrightarrow{\delta^2} P^1 \to \Lambda$ is in the Jacobson radical of Λ, it is clear that x does not lie in $\operatorname{Im} \delta^{2*}$, where δ^{2*} is the induced map $\operatorname{Hom}_{\Lambda^e}(P^1, \Lambda) \to \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$. Thus x represents a non-zero element of $\mathrm{HH}^2(\Lambda)$.

Now, the element $C^s \in \mathcal{R}^2$ overlaps the path C with overlap C^s. So we have the situation:



Thus it is clear from Definition 1.1 that the element C^s in $\mathcal{R}^2$ maximally overlaps the element C^{s+1} in $\mathcal{R}^3$ with overlap C^{2s} in $\mathcal{R}^4$. We may illustrate this element $C^{2s} \in \mathcal{R}^4$ more simply by:



Moreover, since there are no overlaps of C^s with any element of $\mathcal{R}^2$ except C^s it follows that there are no overlaps of C with any element of $\mathcal{R}^2$ except C^s. The construction of elements of $\mathcal{R}^4$ now gives that the only element of $\mathcal{R}^4$ which has C^{s+1} as a subpath is C^{2s}.

The map x lifts to the map $\Omega x\colon P^3 \to P^1$ given by

$$\mathfrak{o}(R^3) \otimes \mathfrak{t}(R^3) \mapsto \begin{cases} \sum_{j=0}^{r-1} \alpha_1 \cdots \alpha_j \otimes \alpha_{j+2} \cdots \alpha_r & \text{if } R^3 = C^{s+1}, \\ 0 & \text{otherwise.} \end{cases}$$

The projective P^4 has summand $\Lambda\mathfrak{o}(C^{2s}) \otimes \mathfrak{t}(C^{2s})\Lambda$. From Lemma 2.4 and using $C \neq p^r$ for any path p with $r \geq 2$, the expressions of C^{2s} which contain some element of $\mathcal{R}^3$ as a subpath are precisely those of the form $C^j C^{s+1} C^{s-j-1}$ for $j = 0, 1, \ldots, s-1$. So, the image of $\mathfrak{o}(C^{2s}) \otimes \mathfrak{t}(C^{2s})$ under the map $P^4 \xrightarrow{\delta^4} P^3$ is $\sum_{j=0}^{s-1} C^j \otimes C^{s-j-1}$, which lies in the summand of P^3 corresponding to the element C^{s+1} of $\mathcal{R}^3$. A simple computation shows that x may be lifted to the map $\Omega^2 x\colon P^4 \to P^2$ given by

$$\mathfrak{o}(R^4) \otimes \mathfrak{t}(R^4) \mapsto \begin{cases} v \otimes v \text{ in the } C^s\text{-component} & \text{if } R^4 = C^{2s}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus x^2 in $\mathrm{HH}^4(\Lambda)$ is represented by the element $P^4 \to \Lambda$ with

$$\mathfrak{o}(R^4) \otimes \mathfrak{t}(R^4) \mapsto \begin{cases} v & \text{if } R^4 = C^{2s}, \\ 0 & \text{otherwise.} \end{cases}$$

In order to construct x^n for $n > 2$ as a map $P^{2n} \to \Lambda$, we consider the sets $\mathcal{R}^{2n}$ and $\mathcal{R}^{2n+1}$. Inductively, it may be verified that $C^{(n-1)s+1} \in \mathcal{R}^{2n-1}$, C^{ns} is the only element in $\mathcal{R}^{2n}$ which has $C^{(n-1)s+1}$ as a subpath, and C^{ns+1} is the only element in $\mathcal{R}^{2n+1}$ which has C^{ns} as a subpath. Thus, for $n \geq 2$ and from Section 1, the map $\delta^{2n+1}\colon P^{2n+1} \to P^{2n}$ is given, on the component corresponding to $C^{ns+1} \in \mathcal{R}^{2n+1}$, by

$$\mathfrak{o}(C^{ns+1}) \otimes \mathfrak{t}(C^{ns+1}) \mapsto v \otimes C - C \otimes v$$

with image lying in the component corresponding to $C^{ns} \in \mathcal{R}^{2n}$, and the map $\delta^{2n}\colon P^{2n} \to P^{2n-1}$ is given, on the component corresponding to $C^{ns} \in \mathcal{R}^{2n}$, by

$$\mathfrak{o}(C^{ns}) \otimes \mathfrak{t}(C^{ns}) \mapsto \sum_{j=0}^{s-1} C^j \otimes C^{s-j-1}$$

with image lying in the component corresponding to $C^{(n-1)s+1} \in \mathcal{R}^{2n-1}$. It is now straightforward to show that the map $\Omega^{2(n-1)}x\colon P^{2n} \to P^{2n-2}$ given by

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \begin{cases} v \otimes v \text{ in the } C^{(n-1)s}\text{-component} & \text{if } R^{2n} = C^{ns}, \\ 0 & \text{otherwise,} \end{cases}$$

is indeed a lifting of x, and hence that x^n is represented by $P^{2n} \to \Lambda$ with

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \begin{cases} v & \text{if } R^{2n} = C^{ns}, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, x^n is a non-zero element of $\mathrm{HH}^{2n}(\Lambda)$ for all $n \geq 1$, since any composition of the form $P^{2n} \xrightarrow{\delta^{2n}} P^{2n-1} \to \Lambda$ has image in the Jacobson radical of Λ.

Hence x is a non-nilpotent element of $\mathrm{HH}^*(\Lambda)$ and generates a subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$. This completes the proof. ■

Example 2.6. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} & \cdot \arrow[dd, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] & & \cdot \arrow[ul, "\delta"'] \\ & \cdot \arrow[ul, "\varepsilon"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Let $\Lambda = K\mathcal{Q}/I$ where $C = \alpha\beta\gamma\delta\beta\varepsilon$ and I is the ideal $\langle C^2, \alpha\beta\varepsilon, \delta\beta\gamma\rangle$. Note that C is a closed path which satisfies the conditions of Proposition 2.5 and moreover has repeated arrows. From Proposition 2.5, there is a subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$ where x is in degree 2.

In the next result we consider elements of $\mathrm{HH}^*(\Lambda)$ which come from closed trails or generalizations of such paths. We start with some definitions.

Definition 2.7.

(1) A *closed trail* T in $\mathcal{Q}$ is a non-trivial closed path $T = \alpha_1 \cdots \alpha_m$ in $K\mathcal{Q}$ such that $\alpha_1, \ldots, \alpha_m$ are all distinct arrows.
(2) For $A \geq 1$, a *closed A-trail* T in $\mathcal{Q}$ is a non-trivial closed path $T = \alpha_1 \cdots \alpha_m$ in $K\mathcal{Q}$ such that $\alpha_1, \ldots, \alpha_m$ are all distinct paths of length A.

Thus if $A = 1$ then a closed 1-trail is a closed trail. Since $\mathcal{Q}$ is a finite quiver, there are a finite number of closed A-trails for each $A \geq 1$. If $A > 1$, then an A-trail may have repeated arrows.

DEFINITION 2.8. Let p be any path and let q be a closed path in $\mathcal{Q}$. Then p *lies on* q if p is a subpath of q^s for some $s \geq 1$.

Fix $A \geq 1$. Let T be a closed A-trail in $\mathcal{Q}$ at the vertex v; for ease of notation write $T = \alpha_0\alpha_1 \cdots \alpha_{m-1}$ for $m \geq 1$, where the α_i are distinct paths of length A. Let $e_i = \mathfrak{o}(\alpha_i)$ for $i = 0, \dots, m-1$, so $e_0 = v$. Let

$$\begin{aligned} T_1 &= \alpha_1 \cdots \alpha_{m-1}\alpha_0, \\ T_2 &= \alpha_2 \cdots \alpha_0\alpha_1, \\ &\vdots \\ T_{m-1} &= \alpha_{m-1}\alpha_0 \cdots \alpha_{m-2} \end{aligned}$$

and set $T_0 = T$. Then the paths $T_0, T_1, \dots, T_{m-1}$ are all of length Am and lie on the closed path $\alpha_0\alpha_1 \cdots \alpha_{m-1}$. We say that $\{T_0, T_1, \dots, T_{m-1}\}$ is a *complete set of closed A-trails* on $\alpha_0\alpha_1 \cdots \alpha_{m-1}$.

Fix $L \geq 2$ and write $L = Nm + l$ where $0 \leq l \leq m-1$ and $N \geq 0$. For $t \in \mathbb{N}$, let $[t] \in \{0, 1, \dots, m-1\}$ denote the residue of t modulo m. Let $W = T_0^N \alpha_0\alpha_1 \cdots \alpha_{l-1}$ with the conventions that if $N = 0$ then $T_0^N = e_0$ and if $l = 0$ then $W = T_0^N$. More generally, for $k = 0, 1, \dots, m-1$, define

$$\sigma^k(W) = T_k^N \alpha_k\alpha_{k+1} \cdots \alpha_{k+l-1}$$

with the conventions that

(i) if $t \geq m$ then $\alpha_t = \alpha_{[t]}$,
(ii) if $N = 0$ then $T_k^N = e_k$,
(iii) if $l = 0$ then $\sigma^k(W) = T_k^N$.

Note that, for all k, $\sigma^k(W)$ lies on the A-trails $T_0, T_1, \dots, T_{m-1}$. Define ϱ_T to be the set

$$\varrho_T = \{W, \sigma(W), \dots, \sigma^{m-1}(W)\}.$$

We say that ϱ_T is the *set of paths of length AL that are associated to the A-trail T*. Note that $\{W, \sigma(W), \dots, \sigma^{m-1}(W)\}$ is also the set of paths of length AL that are associated to each A-trail T_k for $k = 0, \dots, m-1$.

We keep this notation throughout the rest of the paper.

PROPOSITION 2.9. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ϱ. Let $A \geq 1$. Let T be a closed A-trail in the quiver $\mathcal{Q}$ at the vertex v; write $T = \alpha_0\alpha_1 \cdots \alpha_{m-1}$ where the α_i are distinct paths of length A in $\mathcal{Q}$. Suppose that there is some integer $L \geq 2$ such that ϱ contains the set $\varrho_T = \{W, \sigma(W), \dots, \sigma^{m-1}(W)\}$ of paths of length AL that are associated* *to the A-trail T. Let $L = Nm + l$ with $0 \leq l \leq m-1$ and $N \geq 0$. Suppose also that, for each i, the path α_i has no overlaps with any relation in $\varrho \setminus \varrho_T$.*

Then there exists a subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$ such that x is in degree $2m/\gcd(L,m)$ and is represented by the map $P^{2m/\gcd(L,m)} \to \Lambda$, where, for $R^{2m/\gcd(L,m)} \in \mathcal{R}^{2m/\gcd(L,m)}$,

$$\mathfrak{o}(R^{2m/\gcd(L,m)}) \otimes \mathfrak{t}(R^{2m/\gcd(L,m)}) \mapsto \begin{cases} \mathfrak{o}(T_k) & \text{if } R^{2m/\gcd(L,m)} = T_k^{L/\gcd(L,m)} \\ & \qquad \text{for } k = 0, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. With the above notation we have $T = T_0, W = T_0^N \alpha_0 \cdots \alpha_{l-1}$, α_i is the prefix of length A of $\sigma^i(W)$, and $\alpha_{[l+i-1]}$ is the suffix of length A of $\sigma^i(W)$.

Since no path α_i has overlaps with any relation in $\varrho \setminus \varrho_T$, the path W has no overlaps with any relation in $\varrho \setminus \varrho_T$. Thus if a relation R^2 overlaps W then $R^2 \in \varrho_T$. The element $\sigma(W)$ in $\mathcal{R}^2$ maximally overlaps the element $W \in \mathcal{R}^2$ with overlap $W\alpha_{[l]} \in \mathcal{R}^3$. The relation $\sigma^l(W)$ maximally overlaps $W\alpha_{[l]} \in \mathcal{R}^3$ with overlap $W\sigma^l(W)$ in $\mathcal{R}^4$. We continue in this way with maximal overlaps to obtain the element $W\sigma^l(W) \cdots \sigma^{l(\mu-1)}(W)$ in $\mathcal{R}^{2\mu}$ where $\mu = m/\gcd(L,m)$ and, moreover, μ is minimal such that we obtain an element of $\mathcal{R}^n$ with n even which is also a closed path in $\mathcal{Q}$. Note that $W\sigma^l(W) \cdots \sigma^{l(\mu-1)}(W) = T_0^{L/\gcd(L,m)}$. Similarly we may use a sequence of maximal overlaps to give elements $T_k^{L/\gcd(L,m)}$ in $\mathcal{R}^{2\mu}$, for $k = 1, \dots, m-1$. Let $R_k^{2\mu} = T_k^{L/\gcd(L,m)}$ for $k = 0, \dots, m-1$, and observe that $\mathfrak{o}(R_k^{2\mu}) = \mathfrak{t}(R_k^{2\mu}) = e_k$.

Noting that $\mathfrak{o}(T_k) = e_k$, for $R^{2\mu} \in \mathcal{R}^{2\mu}$, define $x \colon P^{2\mu} \to \Lambda$ by

$$\mathfrak{o}(R^{2\mu}) \otimes \mathfrak{t}(R^{2\mu}) \mapsto \begin{cases} e_k & \text{if } R^{2\mu} = R_k^{2\mu} \text{ for } k = 0, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the set $\mathcal{R}^{2\mu+1}$. The relation $\sigma^{[l(\mu-1)+1]}(W)$ maximally overlaps $R_0^{2\mu}$ to give $R_0^{2\mu}\alpha_0$ in $\mathcal{R}^{2\mu+1}$ (since $R_0^{2\mu}$ is a closed path in $\mathcal{Q}$). We may write this element of $\mathcal{R}^{2\mu+1}$ as $R_0^{2\mu+1} = R_0^{2\mu}\alpha_0 = \alpha_0 R_1^{2\mu}$. Similarly, we may define elements $R_k^{2\mu+1} = R_k^{2\mu}\alpha_k = \alpha_k R_{k+1}^{2\mu} \in \mathcal{R}^{2\mu+1}$ for $k = 0, \dots, m-1$. Moreover none of the $\mathcal{R}_k^{2\mu}$ occur in any other elements of $\mathcal{R}^{2\mu+1}$, since if a relation R^2 overlaps an element of ϱ_T then $R^2 \in \varrho_T$, and if an element of ϱ_T overlaps a relation R^2 then also $R^2 \in \varrho_T$.

The image of $\mathfrak{o}(R_k^{2\mu+1}) \otimes \mathfrak{t}(R_k^{2\mu+1})$ under the map $P^{2\mu+1} \xrightarrow{\delta^{2\mu+1}} P^{2\mu}$ is $e_k \otimes \alpha_k - \alpha_k \otimes e_{k+1}$ with the first tensor in the summand of $P^{2\mu}$ corresponding to the element $R_k^{2\mu}$ and the second tensor in the summand corresponding to the element $R_{k+1}^{2\mu}$. Hence the composition $P^{2\mu+1} \xrightarrow{\delta^{2\mu+1}} P^{2\mu} \xrightarrow{x} \Lambda$ is zero.

Moreover, since the image of any composition $P^{2\mu} \xrightarrow{\delta^{2\mu}} P^{2\mu-1} \to \Lambda$ is in the Jacobson radical of Λ, it is clear that x does not lie in $\operatorname{Im}\delta^{2\mu*}$. Thus x represents a non-zero element of $\mathrm{HH}^{2\mu}(\Lambda)$.

By considering maximal overlaps it can be verified in a similar way to that shown in the proof of Proposition 2.5 that there is a lifting of x to the map $P^{4\mu} \to P^{2\mu}$ given by

$$\mathfrak{o}(R^{4\mu}) \otimes \mathfrak{t}(R^{4\mu}) \mapsto \begin{cases} e_k \otimes e_k \text{ in the } T_k^{L/\gcd(L,m)}\text{-component} \\ \qquad \text{if } R^{4\mu} = T_k^{2L/\gcd(L,m)} \text{ for } k = 0, \ldots, m-1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Note that $T_k^{2L/\gcd(L,m)} = (R_k^{2\mu})^2$ for $k = 0, \ldots, m-1$. Then x^2 in $\mathrm{HH}^{4\mu}(\Lambda)$ is represented by the element $P^{4\mu} \to \Lambda$ with

$$\mathfrak{o}(R^{4\mu}) \otimes \mathfrak{t}(R^{4\mu}) \mapsto \begin{cases} e_k & \text{if } R^{4\mu} = T_k^{2L/\gcd(L,m)} \text{ for } k = 0, \ldots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Using similar computations to those in the proof of Proposition 2.5, we deduce, more generally, that x^n is represented by $P^{2\mu n} \to \Lambda$ with

$$\mathfrak{o}(R^{2\mu n}) \otimes \mathfrak{t}(R^{2\mu n}) \mapsto \begin{cases} e_k & \text{if } R^{2\mu n} = T_k^{nL/\gcd(L,m)} \text{ for } k = 0, \ldots, m-1, \\ 0 & \text{otherwise,} \end{cases}$$

and x^n is a non-zero element of $\mathrm{HH}^{2\mu n}(\Lambda)$ for all $n \geq 1$.

Hence x is a non-nilpotent element of $\mathrm{HH}^*(\Lambda)$ and generates the required subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$. ■

The following corollary is the special case when $A = 1$.

COROLLARY 2.10. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ϱ. Let T be a closed trail in the quiver $\mathcal{Q}$ at the vertex v; write $T = \alpha_0\alpha_1 \cdots \alpha_{m-1}$ where the α_i are distinct arrows in $\mathcal{Q}$. Suppose that there is some integer $L \geq 2$ such that ϱ contains the set $\varrho_T = \{W, \sigma(W), \ldots, \sigma^{m-1}(W)\}$ of paths of length L that lie on the trail T. Let $L = Nm + l$ with $0 \leq l \leq m-1$ and $N \geq 0$. Suppose also that, for each i, the arrow α_i does not begin or end any relation in $\varrho \setminus \varrho_T$.*

Then there exists a subalgebra $K[x]$ of $\mathrm{HH}^*(\Lambda)$ where x is in degree $2m/\gcd(L,m)$ and is represented by the map $P^{2m/\gcd(L,m)} \to \Lambda$ where, for $R^{2m/\gcd(L,m)} \in \mathcal{R}^{2m/\gcd(L,m)}$,

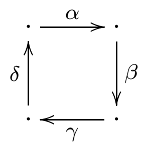
$$\mathfrak{o}(R^{2m/\gcd(L,m)}) \otimes \mathfrak{t}(R^{2m/\gcd(L,m)}) \mapsto \begin{cases} \mathfrak{o}(T_k) & \textit{if } R^{2m/\gcd(L,m)} = T_k^{L/\gcd(L,m)} \\ & \qquad \textit{for } k = 0, \ldots, m-1, \\ 0 & \qquad \textit{otherwise.} \end{cases}$$

REMARK. (1) Suppose that a trail T satisfies the conditions of Corollary 2.10 and has a repeated vertex. Then $T = p_1p_2p_3$ for paths p_i with

$p_1 \in vK\mathcal{Q}w$, $p_2 \in wK\mathcal{Q}w$ and $p_3 \in wK\mathcal{Q}v$ for some vertices v, w, and $1 \leq \ell(p_2) < \ell(T)$. Since Λ is a finite-dimensional monomial algebra, there is some positive integer N with $p_2^N \in I$ and hence there is a subpath q of p_2^N which lies in ϱ. Now the first arrow of q is an arrow on the trail T and so q must be in ϱ_T since the conditions of Corollary 2.10 hold. Hence $N = 1$ and thus $p_2 \in I$.

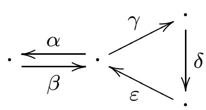
(2) In the case $A = 1$ and $m = 1$, the closed trail T is simply a loop α and $\varrho_T = \{\alpha^L\}$ for some $L \geq 2$. Furthermore if $\alpha^L \in \varrho$ and α neither begins nor ends any relation in $\varrho \setminus \{\alpha^L\}$ then both Propositions 2.5 and 2.9 apply.

EXAMPLE 2.11. This is an example of an A-trail with $A = 2$. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver



and $I = \langle \alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta \rangle$. We may apply Proposition 2.9 with 2-trail $\alpha\beta\gamma\delta$ to show that $K[x]$ is a subalgebra of $\mathrm{HH}^*(\Lambda)$ where x is in degree 4.

EXAMPLE 2.12. In this example $A = 1$ and the trail has a repeated vertex. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver



and $I = \langle \alpha\beta, \beta\gamma, \gamma\delta, \delta\varepsilon, \varepsilon\alpha \rangle$. Then we may apply Corollary 2.10 with trail $\alpha\beta\gamma\delta\varepsilon$ to show that $K[x]$ is a subalgebra of $\mathrm{HH}^*(\Lambda)$ where x is in degree 10.

Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ϱ. Then ϱ is a finite set. Thus there are a finite number of closed paths $C_1, \ldots, C_u$ in $\mathcal{Q}$ such that for each C_i with $1 \leq i \leq u$, we have $C_i \neq p_i^{r_i}$ for any path p_i with $r_i \geq 2$, $C_i^{s_i} \in \varrho$ for some $s_i \geq 2$ and there are no overlaps of $C_i^{s_i}$ with any relation in $\varrho \setminus \{C_i^{s_i}\}$. By Proposition 2.5, for each C_i there is a map x_i of degree 2 from P^2 to Λ. We call x_i the element of $\mathrm{HH}^2(\Lambda)$ *corresponding to the closed path* C_i.

Also, for each $A \geq 2$, there are a finite number of closed A-trails $T_{u+1}, \ldots$ $\ldots, T_r$ in $\mathcal{Q}$ such that for each T_i with $u + 1 \leq i \leq r$, there is some integer $L_i \geq 2$ so that the set ϱ_{T_i} of paths of length AL_i that are associated to the A-trail T_i is contained in ϱ but that no path $\alpha_{i\,j}$ of length A on the A-trail T_i has overlaps with a relation in $\varrho \setminus \varrho_{T_i}$, where $T_i = \alpha_{i\,0}\alpha_{i\,1} \cdots \alpha_{i\,m_i-1}$. We say that two A-trails are *distinct* if neither lies on the other.

By Proposition 2.9, for each A_i-trail T_i there is a map x_i of degree $2m_i/\gcd(L_i, m_i)$ from $P^{2m_i/\gcd(L_i,m_i)}$ to Λ. We say that x_i is the element of $\mathrm{HH}^{2m_i/\gcd(L_i,m_i)}(\Lambda)$ *corresponding to the A_i-trail T_i.*

Keeping the above notation, we now combine Propositions 2.5 and 2.9 in the following theorem.

THEOREM 2.13. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ϱ.*

Let $C_1, \dots, C_u$ be closed paths in the quiver $\mathcal{Q}$ at the vertices $v_1, \dots, v_u$ respectively, such that for each C_i with $1 \le i \le u$, we have $C_i \neq p_i^{r_i}$ for any path p_i with $r_i \ge 2$, $C_i^{s_i} \in \varrho$ for some $s_i \ge 2$ and there are no overlaps of $C_i^{s_i}$ with any relation in $\varrho \setminus \{C_i^{s_i}\}$.

Let $T_{u+1}, \dots, T_r$ be closed paths in the quiver $\mathcal{Q}$ such that the T_i are distinct closed A_i-trails with $A_i \ge 1$. For each $u+1 \le i \le r$, write $T_i = \alpha_{i\,0}\alpha_{i\,1}\cdots\alpha_{i\,m_i-1}$ where each $\alpha_{i\,j}$ is a path of length A_i. Suppose that there are integers $L_i \ge 2$ so that the set ϱ_{T_i} of paths of length A_iL_i which are associated to the trail T_i is contained in ϱ but no path $\alpha_{i\,j}$ has overlaps with any relation in $\varrho \setminus \varrho_{T_i}$.

Then

$$K[x_1, \dots, x_r]/\langle x_a x_b \textit{ for } a \neq b\rangle$$

is a subalgebra of $\mathrm{HH}^*(\Lambda)$ where x_j corresponds to the closed path C_j for $j = 1, \dots, u$ and to the closed trail T_j for $j = u+1, \dots, r$.

For $j = 1, \dots, u$, the vertices $v_1, \dots, v_u$ are distinct, and the element x_j corresponding to the closed path C_j is in degree 2 and is represented by the map $P^2 \to \Lambda$ where, for $R^2 \in \mathcal{R}^2$,

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v_j & \textit{if } R^2 = C_j^{s_j}, \\ 0 & \textit{otherwise.} \end{cases}$$

For $j = u+1, \dots, r$, let $T_{j,0}, \dots, T_{j,m_j-1}$ denote the complete set of closed A_j-trails on the closed path T_j. The element x_j corresponding to the closed A_j-trail T_j is, in the above notation, in degree $2\mu_j$ where $\mu_j = m_j/\gcd(L_j, m_j)$ and is represented by the map $P^{2\mu_j} \to \Lambda$ where, for $R^{2\mu_j} \in \mathcal{R}^{2\mu_j}$,

$$\mathfrak{o}(R^{2\mu_j}) \otimes \mathfrak{t}(R^{2\mu_j}) \mapsto \begin{cases} \mathfrak{o}(T_{j,k}) & \textit{if } R^{2\mu_j} = T_{j,k}^{L_j/\gcd(L_j,m_j)} \\ & \qquad \textit{for } k = 0, \dots, m_j - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We start by showing that the vertices $v_1, \dots, v_u$ are distinct. Suppose that there are distinct closed paths C_a, C_b at the vertex $v_a = v_b$ such that $C_a^{s_a}, C_b^{s_b} \in \varrho$ for some $s_a, s_b \ge 2$ and there are no overlaps of $C_a^{s_a}$ with any relation in $\varrho \setminus \{C_a^{s_a}\}$ or of $C_b^{s_b}$ with any relation in $\varrho \setminus \{C_b^{s_b}\}$. The algebra Λ is finite-dimensional so there is some $N \ge 1$ with $(C_aC_b)^N \in I$. Since no subword of C_a or of C_b is in ϱ, there is some relation in ϱ of the form $w(C_aC_b)^{N_0}w'$ for some paths w, w' and $N_0 \geq 0$.

If w is trivial, that is, $w = v_a$, then $w(C_aC_b)^{N_0}w'$ overlaps $C_a^{s_a}$, which contradicts the hypothesis on C_a. If w is non-trivial and is a subword of C_b, so that there is a path p with $C_b = pw$, then $w(C_aC_b)^{N_0}w'$ overlaps $C_b^{s_b}$, which contradicts the hypothesis on C_b. Finally, if C_b is a subword of w with $w = pC_b$ and p non-trivial, then $w(C_aC_b)^{N_0}w'$ overlaps $C_a^{s_a}$, which contradicts the hypothesis on C_a. Hence the vertices $v_1, \ldots, v_u$ are distinct.

For $j = 1, \ldots, u$, define $x_j \colon P^2 \to \Lambda$ by

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v_j & \text{if } R^2 = C_j^{s_j}, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.5, each of the elements $x_1, \ldots, x_u$ is in degree 2 and generates a subalgebra of $\mathrm{HH}^*(\Lambda)$ isomorphic to $K[x]$.

Now there is a lifting of x_a to the map $P^4 \to P^2$ given by

$$\mathfrak{o}(R^4) \otimes \mathfrak{t}(R^4) \mapsto \begin{cases} v_a \otimes v_a \text{ in the } C_a^{s_a}\text{-component} & \text{if } R^4 = C_a^{2s_a}, \\ 0 & \text{otherwise,} \end{cases}$$

and hence, for $a \neq b$, the composition x_ax_b is zero since the vertices v_a and v_b are distinct.

For $j = 1, \ldots, u$, each x_j is non-nilpotent, so if for some $t \geq 1$ and scalars $c_i \in K$ we have $\sum_{i=1}^u c_ix_i^t = 0$ then $0 = x_j \sum_{i=1}^u c_ix_i^t = c_jx_j^{t+1}$ and hence $c_j = 0$. Thus

$$K[x_1, \ldots, x_u]/\langle x_ax_b \text{ for } a \neq b\rangle$$

is a subalgebra of $\mathrm{HH}^*(\Lambda)$.

For $j = u+1, \ldots, r$, let $T_{j,0}, \ldots, T_{j,m_j-1}$ denote the complete set of closed A_j-trails on the closed path T_j. Define $x_j \colon P^{2\mu_j} \to \Lambda$ by

$$\mathfrak{o}(R^{2\mu_j}) \otimes \mathfrak{t}(R^{2\mu_j}) \mapsto \begin{cases} \mathfrak{o}(T_{j,k}) & \text{if } R^{2\mu_j} = T_{j,k}^{L_j/\gcd(L_j,m_j)} \\ & \qquad \text{for } k = 0, \ldots, m_j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.9, each of the elements $x_{u+1}, \ldots, x_r$ generates a subalgebra of $\mathrm{HH}^*(\Lambda)$ isomorphic to $K[x]$ and, with the above notation, x_j is in degree $2m_j/\gcd(L_j, m_j) = 2\mu_j$.

Next we show that $x_ax_b = 0$ for $1 \leq a \leq r$ and $u + 1 \leq b \leq r$.

Let $T_b = \alpha_0\alpha_1 \cdots \alpha_{m_b-1}$ with $\mathfrak{o}(\alpha_i) = e_i$ for $k = 0, \ldots, m_b - 1$. Keeping the notation of Proposition 2.9, let $\varrho_{T_b} = \{W_b, \sigma(W_b), \ldots, \sigma^{m_b-1}(W_b)\}$ be the set of paths of length A_bL_b that are associated to the trail T_b and let $\mu_b = m_b/\gcd(L_b, m_b)$. For $k = 0, \ldots, m_b-1$, let $R_k^{2\mu_b} \in \mathcal{R}^{2\mu_b}$, $R_k^{2\mu_b+1} \in \mathcal{R}^{2\mu_b+1}$ be defined as in the proof of Proposition 2.9. Recall that $R_k^{2\mu_b} = T_{b,k}^{L_b/\gcd(L_b,m_b)}$, so that $\mathfrak{o}(R_k^{2\mu_b}) = \mathfrak{t}(R_k^{2\mu_b})$, and also that $R_k^{2\mu_b+1} = R_k^{2\mu_b}\alpha_k$. Again from the proof of Proposition 2.9, the map x_b lifts to the map $P^{2\mu_b+1} \to P^1$ given by

$$\mathfrak{o}(R^{2\mu_b+1}) \otimes \mathfrak{t}(R^{2\mu_b+1}) \mapsto \begin{cases} \sum_{i=1}^{A_b} \beta_{k\,1} \cdots \beta_{k\,i-1} \otimes \beta_{k\,i+1} \cdots \beta_{k\,A_b} \\ \quad \text{if } R^{2\mu_b+1} = R_k^{2\mu_b+1} \text{ for } k = 0, \dots, m_b - 1, \\ 0 \quad \text{otherwise,} \end{cases}$$

where the path $\alpha_k = \beta_{k\,1}\beta_{k\,2} \cdots \beta_{k\,A_b}$ with arrows $\beta_{k\,1}, \beta_{k\,2}, \dots, \beta_{k\,A_b}$, and where $\beta_{k\,1} \cdots \beta_{k\,i-1} \otimes \beta_{k\,i+1} \cdots \beta_{k\,A_b}$ lies in the component $\Lambda\mathfrak{o}(\beta_{k\,i}) \otimes \mathfrak{t}(\beta_{k\,i})\Lambda$ of P^1.

For each $k = 0, \dots, m_b - 1$, the only element of $\mathcal{R}^{2\mu_b+2}$ which contains $R_k^{2\mu_b+1}$ as a subpath is obtained from $\sigma^k(W_b)$ in $\mathcal{R}^2$ maximally overlapping $R_k^{2\mu_b+1}$, with overlap $R_k^{2\mu_b}\sigma^k(W_b)$. Let $R_k^{2\mu_b+2} = R_k^{2\mu_b}\sigma^k(W_b)$ for $k = 0, \dots, m_b - 1$. Then the map x_b lifts to the map $P^{2\mu_b+2} \to P^2$ given by

$$\mathfrak{o}(R^{2\mu_b+2}) \otimes \mathfrak{t}(R^{2\mu_b+2}) \mapsto \begin{cases} \mathfrak{o}(\sigma^k(W_b)) \otimes \mathfrak{t}(\sigma^k(W_b)) & \text{if } R^{2\mu_b+2} = R_k^{2\mu_b+2} \\ & \quad \text{for } k = 0, \dots, m_b - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $1 \le a \le u$, the composition of x_b with x_a is zero since the relation $C_a^{s_a}$ is not in ϱ_{T_b}.

For $u + 1 \le a \le r$, we now specify further liftings of the map x_b in order to compute $x_b x_a$.

Let $R^n \in \mathcal{R}^n$ for some $n \ge 2\mu_b$ be such that R^n contains some element of ϱ_{T_b} as a subpath. Then R^n is formed from elements of ϱ_{T_b} in the sense that R^n lies on T_b. Moreover there are precisely m_b elements of $\mathcal{R}^n$ which are formed in this way. We may label these elements as R_k^n for $k = 0, \dots, m_b - 1$ in such a way that the element R_k^n starts at the vertex e_k. Since $R_k^{2\mu_b}$ is a path in $e_k K\mathcal{Q} e_k$, we also have $R_k^{2\mu_b+q} = R_k^{2\mu_b} R_k^q$ for $k = 0, \dots, m_b - 1$, $R_k^q \in \mathcal{R}^q$ and $q \ge 2$. Thus for each $q \ge 2$, x_b lifts to the map $P^{2\mu_b+q} \to P^q$ given by

$$\mathfrak{o}(R^{2\mu_b+q}) \otimes \mathfrak{t}(R^{2\mu_b+q}) \mapsto \begin{cases} \mathfrak{o}(R_k^q) \otimes \mathfrak{t}(R_k^q) & \text{if } R^{2\mu_b+q} = R_k^{2\mu_b+q} \\ & \quad \text{for } k = 0, \dots, m_b - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is now clear that the composition $x_b x_a$ is zero for $a \neq b$ and $u+1 \le a \le r$, since for each n, the elements of the sets $\mathcal{R}^n$ formed from the A_a-trail T_a are distinct from the elements of the sets $\mathcal{R}^n$ formed from the A_b-trail T_b.

As noted earlier, for each $i = 1, \dots, r$, the elements x_i being non-nilpotent implies that if $\sum_{i=1}^r c_i x_i^{t_i}$ is a homogeneous element in $\mathrm{HH}^*(\Lambda)$ for some $t_i \ge 1$ and scalars $c_i \in K$ and if $\sum_{i=1}^r c_i x_i^{t_i} = 0$, then each $c_i = 0$. Hence

$$K[x_1, \dots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$$

is a subalgebra of $\mathrm{HH}^*(\Lambda)$ and the proof is complete. ∎

In Section 3, we apply this theorem to the class of (D, A)-stacked monomial algebras (see Definition 3.1), which includes the Koszul monomial algebras, but we first present two examples illustrating Theorem 2.13.

Example 2.14. Distinct A_i-trails considered in Theorem 2.13 may have vertices in common as the following example illustrates. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} \cdot \arrow[dd, "\beta"'] & & \cdot \arrow[dd, "\eta"] \\ & \cdot \arrow[ul, "\alpha"'] \arrow[ur, "\zeta"] & \\ \cdot \arrow[ur, "\gamma"'] & & \cdot \arrow[ul, "\theta"] \end{tikzcd}$$

and $I = \langle \alpha\beta, \beta\gamma, \gamma\alpha, \zeta\eta, \eta\theta, \theta\zeta \rangle$. Then we may apply Theorem 2.13 with trails $\alpha\beta\gamma$ and $\zeta\eta\theta$ (so $A_i = 1$) to show $\mathrm{HH}^*(\Lambda)$ has $K[x, y]/(xy)$ as a subalgebra where x and y are both in degree 6.

Example 2.15. This example shows that distinct closed paths considered in Theorem 2.13 may have arrows in common. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] & \cdot \arrow[d, "\beta"] & \cdot \arrow[l, "\eta"'] \\ \cdot \arrow[u, "\delta"] & \cdot \arrow[l, "\gamma"] \arrow[r, "\varepsilon"'] & \cdot \arrow[u, "\zeta"'] \end{tikzcd}$$

$C_1 = \alpha\beta\gamma\delta$, $C_2 = \zeta\eta\beta\varepsilon$ and I is the ideal $\langle C_1^2, C_2^2, \alpha\beta\varepsilon \rangle$. Then, from Theorem 2.13, $K[x, y]/(xy)$ is a subalgebra of $\mathrm{HH}^*(\Lambda)$ where x and y are in degree 2.

Since $\mathrm{HH}^*(\Lambda)$ is a graded commutative ring, we have the following corollary to Theorem 2.13.

Corollary 2.16. *With the hypotheses and notation of Theorem* 2.13, *let* $S = K[x_1, \dots, x_r]/\langle x_a x_b \text{ for } a \neq b \rangle$. *Then* $S \cap \mathcal{N} = \{0\}$ *and hence there is an embedding of* S *into* $\mathrm{HH}^*(\Lambda)/\mathcal{N}$.

3. (D, A)-stacked algebras. In this section we determine the quotient $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ for all (D, A)-stacked monomial algebras Λ when $\operatorname{char} K \neq 2$, showing that the subalgebra of Corollary 2.16 is isomorphic to the ring $\mathrm{HH}^*(\Lambda)/\mathcal{N}$. This class includes all Koszul monomial algebras. Moreover, we show that $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is a finitely generated K-algebra of Krull dimension at most 1, giving an affirmative answer to the conjecture of [11] for these algebras.

Definition 3.1. Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ϱ. Then Λ is said to be a (D, A)-*stacked monomial algebra* if there is some $D \geq 2$

and $A \geq 1$ such that, for all $n \geq 2$ and $R^n \in \mathcal{R}^n$,

$$\ell(R^n) = \begin{cases} \frac{n}{2}D & \text{if } n \text{ is even,} \\ \frac{n-1}{2}D + A & \text{if } n \text{ is odd.} \end{cases}$$

In particular all relations in ϱ are of length D.

REMARK. By [5], a monomial algebra Λ is (D, A)-stacked if and only if $\varrho = \mathcal{R}^2$ has the following properties:

(1) every path in ϱ is of length D;
(2) if $R_2^2 \in \mathcal{R}^2$ properly overlaps $R_1^2 \in \mathcal{R}^2$ with overlap $R_1^2 u$ then $\ell(u) \geq A$ and there exists $R_3^2 \in \mathcal{R}^2$ which properly overlaps R_1^2 with overlap $R_1^2 u'$, $\ell(u') = A$ and u' is a prefix of u.

For $D \geq 2$, $A \geq 1$ with A dividing D we now give an algebra Λ which is a (D, A)-stacked monomial algebra. We denote the global dimension of an algebra Λ by $\operatorname{gldim} \Lambda$.

EXAMPLE 3.2. Let $D \geq 2$, $A \geq 1$ with $D = dA$ for some $d \geq 2$. Let $\mathcal{Q}$ be the oriented cycle with D vertices $v_0, v_1, \ldots, v_{D-1}$ and D arrows $\alpha_0, \alpha_1, \ldots, \alpha_{D-1}$ with $\mathfrak{o}(\alpha_i) = v_i$ for all i. Let I be the ideal

$$\langle \alpha_{kA}\alpha_{kA+1} \cdots \alpha_{D-1}\alpha_0 \cdots \alpha_{kA-1} : 0 \leq k \leq d-1 \rangle.$$

Then $\Lambda = K\mathcal{Q}/I$ is a (D, A)-stacked monomial algebra of infinite global dimension.

In Proposition 3.3 we will show that if Λ is a (D, A)-stacked monomial algebra with $\operatorname{gldim} \Lambda \geq 4$ then necessarily A divides D.

Let $\mathfrak{r}$ denote the Jacobson radical of a finite-dimensional algebra Λ. The Ext algebra $E(\Lambda)$ of Λ is defined by

$$E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}).$$

REMARK. (1) From [7], the (D, A)-stacked monomial algebras are precisely the monomial algebras for which every projective module in the minimal projective resolution of $\Lambda/\mathfrak{r}$ over Λ is generated in a single degree and for which the Ext algebra of Λ is finitely generated as a K-algebra. Moreover, from [7], $E(\Lambda)$ is generated in degrees 0, 1, 2 and 3.

(2) The $(2, 1)$-stacked monomial algebras are precisely the quadratic monomial algebras, or equivalently, the Koszul monomial algebras. In this case, $E(\Lambda)$ is generated in degrees 0 and 1.

(3) The $(D, 1)$-stacked monomial algebras for $D \geq 2$ are also known as D-Koszul monomial algebras ([2, 6]). In this case, $E(\Lambda)$ is generated in degrees 0, 1 and 2.

(4) The algebra defined in Example 2.11 is a $(6,2)$-stacked monomial algebra and, using [10], one may check that $E(\Lambda)$ is generated in degrees 0, 1, 2 and 3, but not in degrees 0, 1 and 2.

We now give some properties of these algebras.

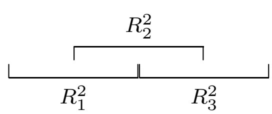
PROPOSITION 3.3. *Let Λ be a (D,A)-stacked monomial algebra. Then*

(1) *for $n \geq 2$, each path in $\mathcal{R}^{2n}$ can be written as $R_1^2 R_2^2 \cdots R_n^2$ with $R_i^2 \in \varrho$;*
(2) *if* $\operatorname{gldim} \Lambda \geq 3$ *then $D > A$;*
(3) *if* $\operatorname{gldim} \Lambda \geq 4$ *then $D = dA$ for some $d \geq 2$.*

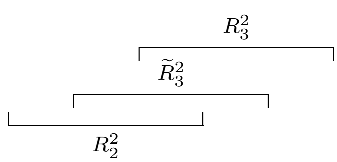
Proof. (1) The proof is by induction on n. The result is clear when $n = 1$. Assume true for $n-1$ and consider a path $R^{2n} \in \mathcal{R}^{2n}$. Write $R^{2n} = R^{2n-1}q$ and $R^{2n-1} = R^{2n-2}q'$ for some paths q, q', so that $R^{2n} = R^{2n-2}q'q$. By induction, $R^{2n-2} = R_1^2 \cdots R_{n-1}^2$ with $R_i^2 \in \varrho$. We know that $\ell(R^{2n-2}) = (n-1)D$ and $\ell(R^{2n}) = nD$ so that $\ell(q'q) = D$. Suppose that R^{2n} is the overlap formed from $R_n^2 \in \mathcal{R}^2$ maximally overlapping R^{2n-1}. Then R_n^2 is a suffix of $q'q$. Since R_n^2 has length D it follows that $R_n^2 = q'q$ and so $R^{2n} = R_1^2 \cdots R_{n-1}^2 R_n^2$, and the proof is complete.

(2) Suppose $\operatorname{gldim} \Lambda \geq 3$. Then $\mathcal{R}^3 \neq \emptyset$ so there is some $R^3 \in \mathcal{R}^3$, and $\ell(R^3) = D + A$. The element R^3 is constructed from an element of $\mathcal{R}^2$ maximally overlapping an element of $\mathcal{R}^2$, and both these elements have length D. Hence $D + A < 2D$ so that $A < D$.

(3) Suppose $\operatorname{gldim} \Lambda \geq 4$. Then there is some $R^4 \in \mathcal{R}^4$. From (1) we may write $R^4 = R_1^2 R_3^2$ with $R_1^2, R_3^2 \in \varrho$ and suppose R_2^2 is the relation which maximally overlaps R_1^2; we illustrate R^4 with the following diagram:



The path $R_3^2 \in \mathcal{R}^2$ overlaps $R_2^2 \in \mathcal{R}^2$ with overlap of length $2D - A$. Then there is some relation, $\widetilde{R}_3^2$ say, which maximally overlaps R_2^2 with overlap of length $D + A$. The paths R_2^2, R_3^2 and $\widetilde{R}_3^2$ are placed as follows:



By maximality, $D + A \leq 2D - A$. If $R_3^2 = \widetilde{R}_3^2$ then $D = 2A$ and we are done. So suppose $R_3^2 \neq \widetilde{R}_3^2$ and $D > 2A$.

Now the path $R_3^2 \in \mathcal{R}^2$ overlaps $\widetilde{R}_3^2 \in \mathcal{R}^2$ with overlap of length $2D - 2A$. Then there is some relation, $\widetilde{R}_4^2$ say, which maximally overlaps $\widetilde{R}_3^2$ with overlap of length $D+A$. By maximality, $D+A \leq 2D-2A$. If $R_3^2 = \widetilde{R}_4^2$ then $D = 3A$ and we are done. So suppose $R_3^2 \neq \widetilde{R}_4^2$ and $D > 3A$.

Continuing in this way, at the $(d-1)$-st stage we get $D+A \leq 2D-(d-1)A$ and so $D \geq dA$. This process must terminate eventually with equality and hence $D = dA$ for some $d \geq 2$. ∎

With the notation of Proposition 3.3, if $R^{2n} \in \mathcal{R}^{2n}$ is written as $R_1^2 \cdots R_n^2$, then we say that R_1^2 is the *first relation* in R^{2n}, and that R_n^2 is the *last relation* in R^{2n}.

We now prove our main theorem, and include in our hypotheses the requirement that $\operatorname{gldim} \Lambda \geq 4$. Note that if $\operatorname{gldim} \Lambda$ is finite then $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K$ and there are no closed paths C or A-trails T which satisfy the hypotheses of Theorem 3.4.

THEOREM 3.4. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional (D,A)-stacked monomial algebra, where I is an admissible ideal with minimal set of generators ϱ. Suppose* $\operatorname{char} K \neq 2$ *and* $\operatorname{gldim} \Lambda \geq 4$.

Let $C_1, \ldots, C_u$ be all the closed paths in the quiver $\mathcal{Q}$ at the vertices $v_1, \ldots, v_u$ respectively, such that for each C_i with $1 \leq i \leq u$, we have $C_i \neq p_i^{r_i}$ for any path p_i with $r_i \geq 2$, $C_i^d \in \varrho$ where $d = D/A$, and there are no overlaps of C_i^d with any relation in $\varrho \setminus \{C_i^d\}$.

Let $T_{u+1}, \ldots, T_r$ be all the distinct closed A-trails in the quiver $\mathcal{Q}$ such that for each T_i with $u+1 \leq i \leq r$, the set ϱ_{T_i} of paths of length D which are associated to the trail T_i is contained in ϱ but, if $T_i = \alpha_{i\,0}\alpha_{i\,1}\cdots\alpha_{i\,m_i-1}$, then no path $\alpha_{i\,j}$ of length A has overlaps with any relation in $\varrho \setminus \varrho_{T_i}$.

Then
$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$$
where

(a) *for $j = 1, \ldots, u$, the vertices $v_1, \ldots, v_u$ are distinct, and the element x_j corresponding to the closed path C_j is in degree 2 and is represented by the map $P^2 \to \Lambda$ where, for $R^2 \in \mathcal{R}^2$,*
$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v_j & \text{if } R^2 = C_j^d, \\ 0 & \text{otherwise}, \end{cases}$$

(b) *for $j = u+1, \ldots, r$, let $T_{j,0}, \ldots, T_{j,m_j-1}$ denote the complete set of A-trails on the closed path T_j. Then the element x_j corresponding to the closed A-trail T_j is, in the above notation, in degree $2\mu_j$ where $\mu_j = m_j/\gcd(d, m_j)$ and is represented by the map $P^{2\mu_j} \to \Lambda$ where, for $R^{2\mu_j} \in \mathcal{R}^{2\mu_j}$,*
$$\mathfrak{o}(R^{2\mu_j}) \otimes \mathfrak{t}(R^{2\mu_j}) \mapsto \begin{cases} \mathfrak{o}(T_{j,k}) & \text{if } R^{2\mu_j} = T_{j,k}^{d/\gcd(d,m_j)} \\ & \qquad \text{for } k = 0, \ldots, m_j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider Theorem 2.13 and its notation. We begin by showing that $s = d$ and each $A_i = A$ and $L_i = d$.

Let C be a closed path in the quiver $\mathcal{Q}$ at the vertex v such that $C \neq p^r$ for any path p with $r \geq 2$, $C^s \in \varrho$ for some $s \geq 2$ and there are no overlaps of C^s with any relation in $\varrho \setminus \{C^s\}$. Then, from the proof of Proposition 2.5, $C^s \in \mathcal{R}^2$ and $C^{s+1} \in \mathcal{R}^3$. Hence $\ell(C^s) = D$ and $\ell(C^{s+1}) = D + A$. Thus $\ell(C) = A$ and hence $As = D$. Thus $s = d$.

In the notation of Theorem 2.13, let T_i be an A_i-trail with ϱ_{T_i} containing paths of length $A_i L_i$. Since all paths in ϱ are necessarily of length D, we have $A_i L_i = D$. Now, from the proof of Proposition 2.9, we have $W_i \in \mathcal{R}^2$ and $W_i \alpha_{[l]} \in \mathcal{R}^3$ with $\alpha_{[l]}$ a path of length A_i. Since Λ is (D, A)-stacked, we know that $\ell(W_i) = D$ and $\ell(W_i \alpha_{[l]}) = D + A$. Hence $\ell(\alpha_{[l]}) = A$ and $A_i = A$. Now we have $AL_i = D$ and so $L_i = d$.

Thus, applying Theorem 2.13 and Corollary 2.16, and with the above notation, we see that $K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$ is isomorphic to a subalgebra of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$.

Since $\operatorname{char} K \neq 2$ and $\mathrm{HH}^*(\Lambda)$ is graded commutative, every homogeneous element of odd degree in $\mathrm{HH}^*(\Lambda)$ is nilpotent and thus is in $\mathcal{N}$. So, in order to determine $\mathrm{HH}^*(\Lambda)/\mathcal{N}$, it is enough to consider an arbitrary element $\chi \in \mathrm{HH}^{2n}(\Lambda)$ represented by the map $\chi\colon P^{2n} \to \Lambda$. We claim that $\chi = \sum_{j=1}^{u} c_j x_j^n + \sum_{j=u+1}^{r} c_j x_j^{q_j} + \eta$ for some $q_j \geq 0$ with $2n = 2\mu_j q_j$, $c_j \in K$ and η nilpotent. Hence $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$, thus proving the result. Our proof proceeds by altering χ by a nilpotent element η_1 and then by elements of the form $\sum_{j=1}^{u} c_j x_j^n$ and $\sum_{j=u+1}^{r} c_j x_j^{q_j}$, resulting in an element of $\mathrm{HH}^{2n}(\Lambda)$ whose image is in $\mathfrak{r}$. Since such elements are nilpotent from [11, Proposition 4.4], we will be done.

First suppose that $\widetilde{R}^{2n} \in \mathcal{R}^{2n}$ does not occur in any element of $\mathcal{R}^{2n+1}$. We modify χ so that $\chi(\mathfrak{o}(\widetilde{R}^{2n}) \otimes \mathfrak{t}(\widetilde{R}^{2n})) = 0$. Define $\eta_1 : P^{2n} \to \Lambda$ as follows. If $R^{2n} \in \mathcal{R}^{2n}$ then

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \begin{cases} \chi(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) \\ \quad \text{if } R^{2n} \text{ does not occur in any element of } \mathcal{R}^{2n+1}, \\ 0 \quad \text{otherwise.} \end{cases}$$

Any element R^{2n+1} in $\mathcal{R}^{2n+1}$ may be written in the form $R^{2n+1} = R^{2n} p_1 = p_2 \widetilde{R}^{2n}$ for some $R^{2n}, \widetilde{R}^{2n} \in \mathcal{R}^{2n}$ and paths p_1, p_2. Then

$$\begin{aligned} &\eta_1 \delta^{2n+1}(\mathfrak{o}(R^{2n+1}) \otimes \mathfrak{t}(R^{2n+1})) \\ &\qquad = \eta_1(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}))p_1 - p_2 \eta_1(\mathfrak{o}(\widetilde{R}^{2n}) \otimes \mathfrak{t}(\widetilde{R}^{2n})) = 0 \end{aligned}$$

since both R^{2n} and $\widetilde{R}^{2n}$ occur in R^{2n+1}. Thus $\eta_1 \in \mathrm{HH}^{2n}(\Lambda)$. Moreover there is a lifting of η_1 to the zero map $P^{2n+1} \to P^1$ and hence $\eta_1^2 = 0$. Thus η_1 is nilpotent. So, without loss of generality, we may replace χ by $\chi - \eta_1$, and the new χ has the desired property.

We now proceed with our next modification. For $j = 1, \dots, u$, from the proof of Proposition 2.5, the element x_j^n is represented by the map $P^{2n} \to \Lambda$ where, for $R^{2n} \in \mathcal{R}^{2n}$,

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \begin{cases} v_j & \text{if } R^{2n} = C_j^{nd}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we may subtract a suitable scalar multiple c_j of each x_j^n from χ so that $\chi(\mathfrak{o}(C_j^{nd}) \otimes \mathfrak{t}(C_j^{nd})) \in \mathfrak{r}$ for each $1 \leq j \leq u$. We also denote this resulting element in $\mathrm{HH}^{2n}(\Lambda)$ by χ.

It is therefore enough to show that this new element χ is of the form $\sum_{j=u+1}^{r} c_j x_j^{q_j} + \eta$ with η nilpotent. Note that if $R^{2n} \in \mathcal{R}^{2n}$ and $\mathfrak{o}(R^{2n}) \neq \mathfrak{t}(R^{2n})$, then $\chi(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) \in \mathfrak{r}$. Thus $\chi \colon P^{2n} \to \Lambda$ satisfies, for $R^{2n} \in \mathcal{R}^{2n}$,

$$\chi(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) \in \begin{cases} \{0\} & \text{if } R^{2n} \text{ does not occur in any element of } \mathcal{R}^{2n+1}, \\ \mathfrak{r} & \text{if } \mathfrak{o}(R^{2n}) \neq \mathfrak{t}(R^{2n}), \\ \mathfrak{r} & \text{if } R^{2n} = C_j^{nd} \text{ for } j = 1, \dots, u. \end{cases}$$

For $j = u+1, \dots, r$, if $2\mu_j$ divides $2n$ with $2n = 2\mu_j q_j$ then $x_j^{q_j}$ is in degree $2n$ and, from Proposition 2.9, $x_j^{q_j}$ is represented by the map $P^{2n} \to \Lambda$ where, for $R^{2n} \in \mathcal{R}^{2n}$,

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \begin{cases} \mathfrak{o}(T_{j,k}) & \text{if } R^{2n} = T_{j,k}^{q_j d/\gcd(d,m_j)} \text{ for } k = 0, \dots, m_j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We now show that we may subtract suitable scalar multiples of the elements $x_{u+1}^{q_{u+1}}, \dots, x_r^{q_r}$ from χ to give an element η in $\mathrm{HH}^{2n}(\Lambda)$ represented by $P^{2n} \to \Lambda$ where $\eta(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) \in \mathfrak{r}$ for all $R^{2n} \in \mathcal{R}^{2n}$.

Consider all $R^{2n} \in \mathcal{R}^{2n}$ such that $R^{2n} \in vK\mathcal{Q}v$ for some vertex v, $R^{2n} \neq C_j^{nd}$ for any j, and R^{2n} occurs in some element of $\mathcal{R}^{2n+1}$. Then $v = \mathfrak{o}(R^{2n}) = \mathfrak{t}(R^{2n})$ and, for such R^{2n}, we may write $\chi(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) = \kappa\mathfrak{o}(R^{2n}) + y$ with $\kappa \in K, y \in \mathfrak{r}$. Let

$$Z = \{R^{2n} \in \mathcal{R}^{2n} : \chi(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) = \kappa\mathfrak{o}(R^{2n}) + y \text{ with } \kappa \neq 0\}.$$

Note that if $R^{2n} \in Z$ then $R^{2n} \in vK\mathcal{Q}v$ for some vertex v, $R^{2n} \neq C_j^{nd}$ for any j, and R^{2n} occurs in some element of $\mathcal{R}^{2n+1}$. Hence if $Z = \emptyset$ then we are done.

Assume $Z \neq \emptyset$. Let $R_1^{2n} \in Z$ so that $\chi(\mathfrak{o}(R_1^{2n}) \otimes \mathfrak{t}(R_1^{2n})) = \kappa_1\mathfrak{o}(R_1^{2n}) + y_1$ with $\kappa_1 \neq 0$ and $y_1 \in \mathfrak{r}$. Since R_1^{2n} must occur in some element of $\mathcal{R}^{2n+1}$ we have $R_1^{2n+1} \in \mathcal{R}^{2n+1}$ and $R_2^{2n} \in \mathcal{R}^{2n}$ with either $R_1^{2n+1} = R_1^{2n} a_1 = a_2 R_2^{2n}$ or $R_1^{2n+1} = a_1 R_1^{2n} = R_2^{2n} a_2$ for some paths a_1, a_2. Without loss of generality, suppose $R_1^{2n+1} = R_1^{2n} a_1 = a_2 R_2^{2n}$.

Since $\chi \in \mathrm{HH}^{2n}(\Lambda)$, we have

$$\begin{aligned} 0 &= \chi\delta^{2n+1}(\mathfrak{o}(R_1^{2n+1}) \otimes \mathfrak{t}(R_1^{2n+1})) \\ &= \chi(\mathfrak{o}(R_1^{2n}) \otimes \mathfrak{t}(R_1^{2n}))a_1 - a_2\chi(\mathfrak{o}(R_2^{2n}) \otimes \mathfrak{t}(R_2^{2n})) \\ &= \kappa_1 a_1 + y_1 a_1 - a_2\chi(\mathfrak{o}(R_2^{2n}) \otimes \mathfrak{t}(R_2^{2n})). \end{aligned}$$

Write $\chi(\mathfrak{o}(R_2^{2n}) \otimes \mathfrak{t}(R_2^{2n})) = \kappa_2\mathfrak{o}(R_2^{2n}) + y_2$ with $\kappa_2 \in K$, $y_2 \in \mathfrak{r}$. Then $0 = \kappa_1 a_1 + y_1 a_1 - \kappa_2 a_2 - a_2 y_2$. By hypothesis $\ell(R_1^{2n+1}) = nD + A$ and $\ell(R_i^{2n}) = nD$ so the paths a_1, a_2 are of length A. Then by length arguments, noting that $\ell(y_1 a_1), \ell(a_2 y_2) \geq A + 1$, it follows that $0 = \kappa_1 a_1 - \kappa_2 a_2$. But $\kappa_1 \neq 0$ and so $a_1 = a_2$ and $\kappa_1 = \kappa_2$. Hence $\kappa_2 \neq 0$ and $R_2^{2n} \in Z$. Thus we have $R_1^{2n+1} = R_1^{2n} a_1 = a_1 R_2^{2n}$ with $R_1^{2n}, R_2^{2n} \in Z$ and $\kappa_1 = \kappa_2$.

From $R_1^{2n} a_1 = a_1 R_2^{2n}$, the first relation R^2 of R_1^{2n} has prefix the path a_1 and the last relation of R_2^{2n} has suffix the path a_1. Now we may write $R_2^{2n} = R^{2n-1} p$ for some $R^{2n-1} \in \mathcal{R}^{2n-1}$ and path p; moreover the path p has length $D - A$. From Proposition 3.3, $D - A \geq A$, and so the path p has suffix a_1, and R^2 overlaps p. Thus there is a maximal overlap of a relation with R_2^{2n} giving the element $R_2^{2n+1} = R_2^{2n} b_2 \in \mathcal{R}^{2n+1}$ for some path b_2. Write $R_2^{2n+1} = R_2^{2n} b_2 = b_3 R_3^{2n}$ for some path b_3 and $R_3^{2n} \in \mathcal{R}^{2n}$, noting that $\ell(b_2) = \ell(b_3) = A$. Repeating the above argument shows that $b_2 = b_3$, $R_3^{2n} \in Z$ and $\kappa_2 = \kappa_3$.

Continuing in this way gives a sequence $R_1^{2n}, \ldots, R_h^{2n}$ of elements of Z and $R_1^{2n+1}, \ldots, R_{h-1}^{2n+1} \in \mathcal{R}^{2n+1}$ with $\kappa_1 = \kappa_2 = \cdots = \kappa_h \neq 0$ and $R_i^{2n+1} = R_i^{2n}\alpha_i = \alpha_i R_{i+1}^{2n}$ for $i = 1, \ldots, h-1$ and paths $\alpha_1, \ldots, \alpha_h$ of length A. Since Z is a finite set there is some h minimal such that $R_h^{2n} = R_t^{2n}$ with $t < h$. Suppose for contradiction that $t \neq 1$. Now $R_{t-1}^{2n}\alpha_{t-1} = \alpha_{t-1}R_t^{2n}$ and $R_{h-1}^{2n}\alpha_{h-1} = \alpha_{h-1}R_h^{2n}$. Thus R_t^{2n} has suffix α_{t-1} and R_h^{2n} has suffix α_{h-1}. But $R_h^{2n} = R_t^{2n}$ so $\alpha_{t-1} = \alpha_{h-1}$. Thus $R_{t-1}^{2n}\alpha_{t-1} = \alpha_{t-1}R_t^{2n} = \alpha_{h-1}R_h^{2n} = R_{h-1}^{2n}\alpha_{h-1}$ and hence it follows that $R_{t-1}^{2n} = R_{h-1}^{2n}$, contradicting the minimality of h. Hence $t = 1$. Thus there is some h minimal such that $R_h^{2n} = R_1^{2n}$, and $h \geq 2$.

So we have constructed $R_1^{2n}, \ldots, R_{h-1}^{2n} \in Z$ giving the following equalities:

$$\begin{aligned} R_1^{2n}\alpha_1\alpha_2\cdots\alpha_{h-1} &= \alpha_1 R_2^{2n}\alpha_2\cdots\alpha_{h-1} \\ &= \alpha_1\alpha_2 R_3^{2n}\cdots\alpha_{h-1} \\ &\;\;\vdots \\ &= \alpha_1\alpha_2\cdots R_{h-1}^{2n}\alpha_{h-1} \\ &= \alpha_1\alpha_2\cdots\alpha_{h-1}R_1^{2n}. \end{aligned}$$

Similarly,

$$R_i^{2n}\alpha_i\cdots\alpha_{h-1}\alpha_1\cdots\alpha_{i-1} = \alpha_i\cdots\alpha_{h-1}\alpha_1\cdots\alpha_{i-1}R_i^{2n}$$

for $i = 1, \ldots, h-1$.

Our next goal is to show, for $i = 1, \ldots, h-1$, that each R_i^{2n} is a power of a closed A-trail. For this we start by showing that if $\alpha_f = \alpha_g$ then $\alpha_{f+1} = \alpha_{g+1}$.

Suppose that there is some f such that $\alpha_f = \alpha_g$ with $f < g$. Write $R_{f+1}^{2n} = R^{2n-1}p$ with $R^{2n-1} \in \mathcal{R}^{2n-1}$, p a path of length $D - A$. Since $R_f^{2n}\alpha_f = \alpha_f R_{f+1}^{2n}$ and $D - A \geq A$ we see that p has suffix α_f of length A. For each $i = 1, \ldots, h-1$, the first D arrows of R_i^{2n} form a relation in ϱ. Since each α_i is of length A and $D = dA$, each $\alpha_i\alpha_{i+1}\cdots\alpha_{d+i-1}$ is in ϱ. Thus, since $\alpha_f = \alpha_g$, the relation $\alpha_g\alpha_{g+1}\cdots\alpha_{d+g-1}$ overlaps p. So there is a maximal overlap $R^{2n+1} = R_{f+1}^{2n}U \in \mathcal{R}^{2n+1}$ of a relation in $\mathcal{R}^2$ with $R_{f+1}^{2n} \in \mathcal{R}^{2n}$ such that R^{2n+1} is a prefix of the path $R_{f+1}^{2n}\alpha_{g+1}\cdots\alpha_{d+g-1}$. By hypothesis $\ell(U) = A$ so that $U = \alpha_{g+1}$ and $R^{2n+1} = R_{f+1}^{2n}\alpha_{g+1}$. Then $R^{2n+1} = R_{f+1}^{2n}\alpha_{g+1} = \gamma R^{2n}$ with $R^{2n} \in \mathcal{R}^{2n}$ and γ a path of length A. Since $\chi \in \mathrm{HH}^{2n}(\Lambda)$, by the argument immediately following the definition of Z, we have $\alpha_{g+1} = \gamma$. Hence $\alpha_{f+1} = \alpha_{g+1}$.

Now we show that $\alpha_1, \ldots, \alpha_{h-1}$ are all distinct paths. Suppose for contradiction that $\alpha_f = \alpha_g$ with $1 \leq f < g \leq h-1$. From the above considerations, $\alpha_{f+1} = \alpha_{g+1}$, $\alpha_{f+2} = \alpha_{g+2}$, $\ldots$. Hence $R_f^{2n} = \alpha_f\alpha_{f+1}\cdots\alpha_{f+nd-1} = \alpha_g\alpha_{g+1}\cdots\alpha_{g+nd-1} = R_g^{2n}$. This contradicts the choice of h. Thus $\alpha_1\cdots\alpha_{h-1}$ is a closed A-trail.

Let T denote the closed A-trail $\alpha_1\cdots\alpha_{h-1}$. Then $R_1^{2n}T = TR_1^{2n}$. From Lemma 2.3, there is a path p and integers $r, s \geq 1$ such that $R_1^{2n} = p^r$ and $T = p^s$. We show that $s = 1$ so that $T = p$ and $R_1^{2n} = T^r$ for some $r \geq 1$.

Suppose for contradiction that $s \geq 2$. If A does not divide $\ell(p)$ then write $\ell(p) = \lambda A + \widetilde{s}$ where $1 \leq \widetilde{s} < A$. Then $p = \alpha_1\cdots\alpha_\lambda q$ with $\ell(q) = \widetilde{s}$ and $\alpha_{\lambda+1} = q\widetilde{q}$ for some path $\widetilde{q}$. Let R^2 be the first relation in R_1^{2n} and let $\widetilde{R}^2$ be the first relation in $R_{\lambda+1}^{2n}$. Since $T = p^s$ with $s \geq 2$ we know that $\widetilde{q}$ is a prefix of α_1 and hence of R^2. Therefore R^2 overlaps $\widetilde{R}^2$ with overlap of length $D + \ell(q)$. But $\ell(q) < A$. Hence there is an element of $\mathcal{R}^3$ of length strictly less than $D + A$, which is a contradiction. Now if A divides $\ell(p)$ then write $\ell(p) = \lambda A$ so that $p = \alpha_1\cdots\alpha_\lambda$. Since $s \geq 2$ and $\lambda \leq h-2$, we have $\alpha_{\lambda+1} = \alpha_1$, which contradicts the choice of h. Hence $s = 1$.

Thus $R_1^{2n} = T^{nD/(h-1)A} = T^{nd/(h-1)}$. Similarly, each R_i^{2n} equals $T_i^{nd/(h-1)}$ where $T = T_1$ and $T_1, \ldots, T_{h-1}$ is the complete set of closed A-trails on the closed path T.

Finally, we show that, for each i, the path α_i has no overlaps with any relation in $\varrho \setminus \varrho_T$. Suppose for contradiction that some relation R^2 in $\varrho \setminus \varrho_T$ overlaps α_1. Write $R^2 = \alpha_1'\alpha_2\cdots\alpha_{g-1}\alpha_g'b\beta$ where $\alpha_1 = \alpha_1''\alpha_1'$ with $\ell(\alpha_1') \geq 1$, $\alpha_g = \alpha_g'\alpha_g''$ with $\ell(\alpha_g'') \geq 1$, b is an arrow, and the first arrow of α_g'' is not equal to b. Note that $2 \leq g \leq d+1$.

First suppose that $g = d+1$. Since $\ell(R^2) = D$ we have $\ell(\alpha'_1) < A$ and $1 \le \ell(\alpha'_{d+1} b\beta) < A$. We see that R^2 overlaps the relation $\alpha_1 \cdots \alpha_d$ with overlap $\alpha_1 \cdots \alpha_d \alpha'_{d+1} b\beta$. So there is a maximal overlap in $\mathcal{R}^3$ of length strictly less than $D + A$, which is a contradiction.

Thus we may assume that $2 \le g \le d$. Consider

$$R_g^{2n} = (\alpha_g \cdots \alpha_{h-1} \alpha_1 \cdots \alpha_{g-1})^{nd/(h-1)}.$$

Write $R_g^{2n} = R^{2n-1} q$ for some $R^{2n-1} \in \mathcal{R}^{2n-1}$ and path q with $\ell(q) = D - A$. Since $g \le d$, we have $\ell(\alpha'_1 \alpha_2 \cdots \alpha_{g-1}) \le (g-1)A \le (d-1)A = D - A$ so R^2 overlaps q. Hence there is a maximal overlap $R^{2n+1} = R_g^{2n} U \in \mathcal{R}^{2n+1}$ of a relation in $\mathcal{R}^2$ with $R_g^{2n} \in \mathcal{R}^{2n}$, such that R^{2n+1} is a prefix of the path $R_g^{2n} \alpha'_g b\beta$. Since $\ell(\alpha''_g) \ge 1$ we see that $\ell(\alpha'_g) < A$. Noting also that $\ell(U) = A$, we deduce that $\alpha'_g b$ is a prefix of U. Then $R^{2n+1} = R_g^{2n} U = \gamma R^{2n}$ with $R^{2n} \in \mathcal{R}^{2n}$ and γ a path of length A. Since $\chi \in \mathrm{HH}^{2n}(\Lambda)$, by the argument immediately following the definition of Z, it follows that $U = \gamma$. Hence $U = \alpha_g$, which is a contradiction. Thus, for each i, the path α_i has no overlaps with any relation in $\varrho \setminus \varrho_T$.

Hence T is a closed A-trail in $\mathcal{Q}$ such that the set ϱ_T of paths of length D associated to the A-trail T is contained in ϱ, but no path α_i of length A has overlaps with any relation in $\varrho \setminus \varrho_T$.

By hypothesis, it follows that $T \in \{T_{u+1}, \ldots, T_r\}$. Thus we may write $T = T_j$ for some j with $u+1 \le j \le r$. Moreover, for $i = 1, \ldots, h-1$, since $\kappa_1 = \cdots = \kappa_{h-1}$, we have $\chi(\mathfrak{o}(R_i^{2n}) \otimes \mathfrak{t}(R_i^{2n})) = \kappa_1 \mathfrak{o}(R_i^{2n}) + y_i$ with $y_i \in \mathfrak{r}$. Thus $\chi - \kappa_1 x_j^{q_j}$ is such that $(\chi - \kappa_1 x_j^{q_j})(\mathfrak{o}(R_i^{2n}) \otimes \mathfrak{t}(R_i^{2n})) \in \mathfrak{r}$ for $i = 1, \ldots, h-1$.

Since Z is a finite set, we may continue subtracting suitable scalar multiples of the $x_j^{q_j}$ from χ. Thus there are scalars $c_j \in K$ such that $\chi - \sum_{j=u+1}^{r} c_j x_j^{q_j}$ is represented by a map $\eta\colon P^{2n} \to \Lambda$ where $\eta(\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n})) \in \mathfrak{r}$ for all $R^{2n} \in \mathcal{R}^{2n}$. This completes the proof. ■

Example 3.5. Theorem 3.4 applies to Examples 2.11, 2.12 and 2.14 with $\operatorname{char} K \ne 2$ to show that the subalgebra described there is indeed isomorphic to the quotient $\mathrm{HH}^*(\Lambda)/\mathcal{N}$.

The final theorem follows from Theorem 3.4 and gives an affirmative answer to the conjecture of [11] for (D, A)-stacked monomial algebras.

Theorem 3.6. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional (D, A)-stacked monomial algebra, where I is an admissible ideal. Suppose* $\operatorname{char} K \ne 2$. *Then $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is finitely generated as an algebra and the number of generators is bounded above by the number of paths of length A in the quiver. Moreover $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is of Krull dimension at most* 1.

Proof. Note first that if $\operatorname{gldim} \Lambda \leq 3$ then $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K$ and the theorem is immediate. Thus suppose $\operatorname{gldim} \Lambda \geq 4$. The ideal I has minimal set of generators ϱ where ϱ is a finite set since Λ is finite-dimensional. So, using Theorem 3.4 with the notation thereof,

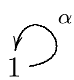
$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$$

where r is finite. Hence $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is finitely generated as an algebra and of Krull dimension at most 1.

Moreover, the generators $x_1, \ldots, x_r$ are defined precisely in terms of the paths $\{C_1, \ldots, C_u\} \cup \{\alpha_{i\,j} : j = 0, \ldots, m_i - 1,\ i = u+1, \ldots, r\}$, each of which is of length A in the quiver $\mathcal{Q}$. It is easy to verify that these paths are all distinct, since there are no overlaps of C_i^d with any relation in $\varrho \setminus \{C_i^d\}$, and there are no overlaps of any $\alpha_{i\,j}$ with any relation in $\varrho \setminus \varrho_{T_i}$. Thus the number r of generators of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is bounded above by the number of paths of length A in the quiver $\mathcal{Q}$. ∎

The final example is of a $(D, 1)$-stacked monomial algebra where the number of generators of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is equal to the number of arrows in $\mathcal{Q}$.

EXAMPLE 3.7. Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver



and $I = \langle \alpha^2 \rangle$ with $\operatorname{char} K \neq 2$. Then Λ is a Koszul algebra. From Theorem 3.4, $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x]$ with x in degree 2.

If $A > 1$, it is open as to whether or not the number of paths of length A in $\mathcal{Q}$ is the best possible upper bound for the number of generators of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$.

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