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ON STABILITY PROPERTIES OF POSITIVE CONTRACTIONS OF L^1-SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

BY

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Abstract. We extend the notion of Dobrushin coefficient of ergodicity to positive contractions defined on the L^1-space associated with a finite von Neumann algebra, and in terms of this coefficient we prove stability results for L^1-contractions.

1. Introduction. The importance of investigating asymptotic behavior of Markov operators on commutative L^1-spaces is well known (see [K]). On the other hand, these investigations involve several notions of mixing (weak mixing, mixing, complete mixing etc.) of L^1-contractions of a measure space. Relations between these notions are of great interest (see for example [BLRT], [BKLM]). However, in those investigations the lattice property of L^1-spaces is essentially used. Therefore it is natural to consider Markov operators on partially ordered Banach spaces which are not lattices. One class of such spaces consists of L^1-spaces associated with von Neumann algebras. Note that these Banach spaces are ordered by strongly normal cones (see [EW1]). In [EW1], [EW2], [S] certain asymptotic properties of Markov semigroups on non-commutative L^1-spaces were studied.

In this paper we study uniformly (resp. strongly) asymptotically stable contractions of L^1-spaces associated with finite von Neumann algebras in terms of the Dobrushin coefficients. The paper is organized as follows. Section 2 contains some preliminary facts and definitions. In Section 3 we introduce the Dobrushin coefficient of ergodicity of an L^1-contraction. Using this notion we prove a uniform asymptotic stability criterion for stochastic operators, which is a non-commutative analog of Bartoszek's result (see [B]). Further in Section 4 we give an analog of the Akcoglu–Sucheston theorem (see [AS]) for non-commutative L^1-spaces. We hope that this result will lead to subsequential ergodic theorems in a non-commutative setting (see

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[CL], [LM]). In the final Section 5 using the results of the previous section we prove a strong asymptotic stability criterion for positive L^1-contractions. We note that our results are not valid when the von Neumann algebra is only semi-finite.

2. Preliminaries. Throughout the paper, M will be a von Neumann algebra with unit $\mathbb{1}$, and τ a faithful normal finite trace on M. Recall that $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa}. We denote by M_* a pre-dual space to M. An element $p \in M_{\mathrm{sa}}$ is called a *projector* if $p^2 = p$. Let ∇ be the set of all projectors; ∇ forms a logic. For $p \in \nabla$ we set $p^\perp = \mathbb{1} - p$ (for more definitions see [BR], [T]).

The map $\|\cdot\|_1 : M \to [0, \infty)$ defined by the formula $\|x\|_1 = \tau(|x|)$ is a norm (see [N]). The completion of M with respect to this norm is denoted by $L^1(M, \tau)$. It is known [N] that the spaces $L^1(M, \tau)$ and M_* are isometrically isomorphic, so they can be identified. We will use this fact without explicit mention.

THEOREM 2.1 ([N]). *The space $L^1(M, \tau)$ coincides with the set*

$$L^1 = \Big\{x = \int_{-\infty}^{\infty} \lambda\, de_\lambda : \int_{-\infty}^{\infty} |\lambda|\, d\tau(e_\lambda) < \infty\Big\}.$$

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda|\, d\tau(e_\lambda).$$

Furthermore, if $x, y \in L^1(M, \tau)$, $x, y \ge 0$ and $x \cdot y = 0$ then $\|x + y\|_1 = \|x\|_1 + \|y\|_1$.

It is known [N] that

$$L^1(M, \tau) = L^1(M_{\mathrm{sa}}, \tau) + iL^1(M_{\mathrm{sa}}, \tau). \tag{1}$$

Note that $L^1(M_{\mathrm{sa}}, \tau)$ is a pre-dual to M_{sa}.

Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a bounded linear operator. We say that T is *positive* if $Tx \ge 0$ whenever $x \ge 0$, and a *contraction* if $\|Tx\|_1 \le \|x\|_1$ for all $x \in L^1(M_{\mathrm{sa}}, \tau)$. A positive operator T is called *stochastic* if $\tau(Tx) = \tau(x)$ for all $x \ge 0$. It is clear that any stochastic operator is a contraction. For given $y \in L^1(M_{\mathrm{sa}}, \tau)$ and $z \in M_{\mathrm{sa}}$ define a linear operator $T_{y,z} : L^1(M_{\mathrm{sa}}, \tau) \to L^1(M_{\mathrm{sa}}, \tau)$ as follows:

$$T_{y,z}x = \tau(xz)y$$

and extend it to $L^1(M, \tau)$ as $T_{y,z}x = T_{y,z}x_1 + iT_{y,z}x_2$ for $x = x_1 + ix_2$, $x_1, x_2 \in L^1(M_{\mathrm{sa}}, \tau)$.

Put $T_y := T_{y,\mathbb{1}}$. A linear operator $T : L^1(M,\tau) \to L^1(M,\tau)$ is called *uniformly* (resp. *strongly*) *asymptotically stable* if there exist elements $y \in L^1(M_{\mathrm{sa}},\tau)$ and $z \in M_{\mathrm{sa}}$ such that

$$\lim_{n\to\infty} \|T^n - T_{y,z}\| = 0$$

(resp.

$$\lim_{n\to\infty} \|T^n x - T_{y,z}x\|_1 = 0$$

for every $x \in L^1(M,\tau)$.)

3. Uniformly asymptotically stable contractions. Let M be a von Neumann algebra with a faithful normal finite trace τ. Let $L^1(M,\tau)$ be the associated L^1-space.

Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a bounded linear operator. Define

$$X = \{x \in L^1(M_{\mathrm{sa}},\tau) : \tau(x) = 0\},$$

$$\overline{\alpha}(T) = \sup_{x\in X,\, x\neq 0} \frac{\|Tx\|_1}{\|x\|_1}, \qquad \alpha(T) = \|T\| - \overline{\alpha}(T). \tag{2}$$

The quantity $\alpha(T)$ is called the *Dobrushin coefficient of ergodicity* of T.

REMARK 3.1. In the commutative case, the Dobrushin coefficient of ergodicity was introduced in [C], [D], [ZZ].

We have the following theorem which extends the results of [C], [ZZ].

THEOREM 3.1. *Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a bounded linear operator. Then*

$$\|Tx\|_1 \le \overline{\alpha}(T)\|x\|_1 + \alpha(T)|\tau(x)| \tag{3}$$

for every $x \in L^1(M_{\mathrm{sa}},\tau)$.

Proof. Assume that x is positive. Then $\|x\|_1 = \tau(x)$ and

$$\overline{\alpha}(T)\|x\|_1 + \alpha(T)|\tau(x)| = \overline{\alpha}(T)\tau(x) + (\|T\| - \overline{\alpha}(T))\tau(x) = \|T\|\|x\|_1 \ge \|Tx\|_1.$$

So (3) is valid. If $x \le 0$ the same argument works. If $x \in X$ then (3) follows easily from (2).

Suppose now that none of the above three cases holds. Then $x = x^+ - x^-$, $\|x^+\|_1 \neq 0$, $\|x^-\|_1 \neq 0$, $\|x^+\|_1 \neq \|x^-\|_1$ (see [T]). Let $\|x^+\|_1 > \|x^-\|_1$. Put

$$y = \frac{\|x^-\|_1}{\|x^+\|_1} x^+ - x^-, \qquad z = \frac{\|x^+\|_1 - \|x^-\|_1}{\|x^+\|_1} x^+.$$

Then $x = y + z$ and $\|x\|_1 = \|y\|_1 + \|z\|_1$; here Theorem 2.1 has been used. It is clear that $y \in X$ and $z \ge 0$, so (3) is valid for y and z. Hence, we get

$$\begin{aligned}\|Tx\|_1 \le \|Ty\|_1 + \|Tz\|_1 &\le \overline{\alpha}(T)\|y\|_1 + \overline{\alpha}(T)\|z\|_1 + \alpha(T)\tau(z)\\ &= \overline{\alpha}(T)\|x\|_1 + \alpha(T)|\tau(x)|.\end{aligned}$$

Before formulating the main result of this section we need some lemmas.

LEMMA 3.2. *For every $x, y \in L^1(M_{\mathrm{sa}}, \tau)$ such that $x - y \in X$ there exist $u, v \in L^1(M_{\mathrm{sa}}, \tau)$ with $u, v \geq 0$ and $\|u\|_1 = \|v\|_1 = 1$ such that*

$$x - y = \frac{\|x - y\|_1}{2}\,(u - v).$$

Proof. We have $x - y = (x - y)^+ - (x - y)^-$. Define

$$u = \frac{(x-y)^+}{\|(x-y)^+\|_1}, \qquad v = \frac{(x-y)^-}{\|(x-y)^-\|_1}.$$

It is clear that $u, v \geq 0$ and $\|u\|_1 = \|v\|_1 = 1$. Since $x - y \in X$, we have

$$\tau(x-y) = \tau((x-y)^+) - \tau((x-y)^-) = \|(x-y)^+\|_1 - \|(x-y)^-\|_1 = 0,$$

that is, $\|(x-y)^+\|_1 = \|(x-y)^-\|_1$. As $\|x-y\|_1 = \|(x-y)^+\|_1 + \|(x-y)^-\|_1$ we get $\|(x-y)^+\|_1 = \|x-y\|_1/2$. Consequently,

$$u - v = \frac{(x-y)^+}{\|x-y\|_1/2} - \frac{(x-y)^-}{\|x-y\|_1/2} = \frac{2}{\|x-y\|_1}\,(x-y).$$

LEMMA 3.3. *Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a stochastic operator. Then*

$$\overline{\alpha}(T) = \sup\{\|Tu - Tv\|_1/2 : u, v \in L^1(M_{\mathrm{sa}}, \tau),\ u, v \geq 0,\ \|u\|_1 = \|v\|_1 = 1\}. \tag{4}$$

Proof. For $x \in X$, $x \neq 0$, using Lemma 3.2 we have

$$\frac{\|Tx\|_1}{\|x\|_1} = \frac{\|T(x^+ - x^-)\|_1}{\|x^+ - x^-\|_1} = \frac{\frac{\|x^+ - x^-\|_1}{2}\|T(u-v)\|_1}{\|x^+ - x^-\|_1} = \frac{\|Tu - Tv\|_1}{2}.$$

Together with (2), this implies (4).

Now we are ready to prove the main result of this section, which is a non-commutative version of Bartoszek's result [B].

THEOREM 3.4. *Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a stochastic operator. The following conditions are equivalent*:

(i) *there exist $\varrho > 0$ and $n_0 \in \mathbb{N}$ such that $\alpha(T^{n_0}) \geq \varrho$;*
(ii) *there exists $y \in L^1(M_{\mathrm{sa}}, \tau)$, $y \geq 0$, such that*

$$\lim_{n\to\infty} \|T^n - T_y\| = 0.$$

Proof. (i)⇒(ii). Let $\varrho > 0$ and $n_0 \in \mathbb{N}$ be such that $\alpha(T^{n_0}) \geq \varrho$. Then $\overline{\alpha}(T^{n_0}) \leq 1 - \varrho$. Put $\gamma = 1 - \varrho$. For any $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $\gamma^k < \varepsilon/2$ and set $K = n_0 k$. Since T is a stochastic operator we have $\tau(T^n x - T^m x) = 0$ for every $x \in L^1(M_{\mathrm{sa}}, \tau)$, $x \geq 0$, and $n, m \in \mathbb{N} \cup \{0\}$. Hence using (3) we infer that

$$\begin{aligned}\|T^n x - T^m x\|_1 &= \|T^{n_0}(T^{n-n_0}x - T^{m-n_0}x)\|_1 \le \gamma\|T^{n-n_0}x - T^{m-n_0}x\|_1\\ &\le \gamma^2\|T^{n-2n_0}x - T^{m-2n_0}x\|_1 \le \cdots \le \gamma^k\|T^{n-K}x - T^{m-K}x\|_1\\ &\le \gamma^k(\|T^{n-K}x\|_1 + \|T^{m-K}x\|_1) \le 2\gamma^k\|x\|_1 < \varepsilon\end{aligned}$$

for every $x \in L^1(M_{\mathrm{sa}},\tau)$ with $x \ge 0$, $\|x\|_1 \le 1$, and $n, m \ge K$.

Now in general, keeping in mind (1), for every $x \in L^1(M,\tau)$ with $\|x\|_1 \le 1$ we have $x = \sum_{k=1}^4 i^k x_k$ for some $x_k \ge 0$ with $\|x_k\|_1 \le 1$, therefore the last relation implies that

$$\|T^n x - T^m x\|_1 \le 4\varepsilon.$$

Consequently, $(T^n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the uniform norm. Therefore for $x \in L^1(M,\tau)$ with $x \ge 0$ and $\|x\|_1 = 1$ the sequence $(T^n x)_{n\in\mathbb{N}}$ converges in the norm of $L^1(M,\tau)$ to some $y \in L^1(M,\tau)$. Since $\|Tx\|_1 = \|x\|_1 = 1$ and T is positive, it follows that $y \ge 0$, $\|y\|_1 = 1$ and $Ty = y$. Using this we obtain

$$\|T^n z - y\|_1 = \|T^n z - T^n y\|_1 \le \|T^{n-1}z - T^{n-1}y\|_1 = \|T^{n-1}z - y\|_1$$

for every $z \in L^1(M_{\mathrm{sa}},\tau)$ with $z \ge 0$ and $\|z\|_1 \le 1$. Hence the sequence $(\|T^n z - y\|_1)_{n\in\mathbb{N}}$ is decreasing. As

$$\|T^{mn_0}z - y\|_1 \le 2\gamma^m \quad \text{for every } m \in \mathbb{N}$$

we infer that $(T^n z)_{n\in\mathbb{N}}$ converges to y in the norm topology of $L^1(M_{\mathrm{sa}},\tau)$.

If $z \in L^1(M_{\mathrm{sa}},\tau)$, $z \ge 0$, $\|z\|_1 \ne 0$ then taking into account that

$$T^n z = \|z\|_1 T\left(\frac{z}{\|z\|_1}\right) = \tau(z)T\left(\frac{z}{\|z\|_1}\right)$$

we see that $T^n z \to \tau(z)y$ as $n \to \infty$, since $T(z/\|z\|_1)$ norm converges to y.

If $z \in L^1(M_{\mathrm{sa}},\tau)$, then $z = z^+ - z^-$, therefore

$$T^n z^+ \to \tau(z^+)y \quad \text{and} \quad T^n z^- \to \tau(z^-)y \quad \text{as } n \to \infty.$$

So $T^n z$ converges to $T_y z$ for every $z \in L^1(M_{\mathrm{sa}},\tau)$.

In general, if $z \in L^1(M,\tau)$, then $z = z_1 + iz_2$, where $z_1, z_2 \in L^1(M_{\mathrm{sa}},\tau)$, hence

$$T^n z = T^n z_1 + iT^n z_2 \to \tau(z_1)y + i\tau(z_2)y = \tau(z)y \quad \text{as } n \to \infty.$$

Thus $T^n z$ converges to $T_y z$ for every $z \in L^1(M,\tau)$. Since $(T^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the uniform operator topology it follows that

$$\lim_{n\to\infty}\|T^n - T_y\| = 0.$$

(ii)⇒(i). Let $y \in L^1(M_{\mathrm{sa}},\tau)$ be as in (ii). Fix $\eta \in (0, 1/4)$. Then (ii) implies that there is an $n_0 \in \mathbb{N}$ such that $\|T^n - T_y\| < \eta$ for every $n \ge n_0$. Since $Ty = y$ we get

$$\|T^{n_0}u - T^{n_0}v\|_1 \le \|T^{n_0}u - y\|_1 + \|T^{n_0}v - y\|_1 < 2\eta \tag{5}$$

for every $u, v \in L^1(M_{\mathrm{sa}},\tau)$ with $u, v \ge 0$ and $\|u\|_1 = \|v\|_1 = 1$.

Hence, using Lemma 3.3 (see (4)) we obtain $\overline{\alpha}(T^{n_0}) \le 2\eta$, which yields $\alpha(T^{n_0}) \ge 1 - 2\eta$. The proof is complete.

4. Completely mixing and smoothing contractions. In this section we define completely mixing and smoothing L^1-contractions of noncommutative $L^1(M,\tau)$-spaces. These notions will be used in the next section.

Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear contraction. Define

$$\overline{\varrho}(T) = \sup\Big\{ \lim_{n\to\infty} \frac{\|T^n(u-v)\|_1}{\|u-v\|_1} : u,v \in L^1(M_{\mathrm{sa}},\tau), \quad u,v \ge 0, \|u\|_1 = \|v\|_1 \Big\} \tag{6}$$

and $\varrho(T) = \lim_{n\to\infty} \|T^n\| - \overline{\varrho}(T)$.

The quantity $\varrho(T)$ is called the *asymptotic Dobrushin coefficient of ergodicity* of T. If $\overline{\varrho}(T) = 0$ then T is called *completely mixing*. Note that certain properties of completely mixing quantum dynamical systems have been studied in [AP].

Using the same argument as in the proof of Theorem 3.4 one can prove

THEOREM 4.1. *Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear contraction. Then*

$$\lim_{n\to\infty} \|T^n x\|_1 \le \overline{\varrho}(T)\|x\|_1 + \varrho(T)|\tau(x)| \tag{7}$$

for every $x \in L^1(M_{\mathrm{sa}},\tau)$.

Using this theorem we can prove

THEOREM 4.2. *If T is a stochastic operator then $\overline{\varrho}(T) = 0$ or 1.*

Proof. From (6) one can easily see that $0 \le \overline{\varrho}(T) \le 1$. Now suppose that $\overline{\varrho}(T) < 1$. This means that there is a number $\gamma \ge 0$ such that $\overline{\varrho}(T) \le \gamma < 1$. Let $x \in X$, $x \ne 0$. It follows that

$$\lim_{n\to\infty} \|T^n x\|_1 \le \overline{\varrho}(T)\|x\|_1 \le \gamma\|x\|_1,$$

so there is $n_1 \in \mathbb{N}$ such that $\|T^{n_1}x\|_1 \le \gamma\|x\|_1$. If $T^{n_1}x = 0$ then

$$\lim_{n\to\infty} \|T^n x\|_1 = 0.$$

If $T^{n_1}x \ne 0$ then $\tau(T^{n_1}x) = \tau(x) = 0$ since T is stochastic. Thus by means of (7) we get

$$\lim_{n\to\infty} \|T^{n+n_1}x\|_1 \le \overline{\varrho}(T)\|T^{n_1}x\|_1 \le \gamma\|T^{n_1}x\|_1 \le \gamma^2\|x\|_1.$$

It follows that there exists $n_2 > n_1$ such that $\|T^{n_2}x\|_1 \le \gamma^2\|x\|_1$. Continuing in this way, if $T^n x \ne 0$ for every $n \in \mathbb{N}$ then we can find a strictly increasing sequence (n_k) such that $\|T^{n_k}x\|_1 \le \gamma^k\|x\|_1$ for every $k \in \mathbb{N}$. Since T is a contraction we conclude that $\|T^n x\|_1 \to 0$ as $n \to \infty$, which implies that $\overline{\varrho}(T) = 0$.

Let T be a positive contraction of $L^1(M,\tau)$, and let $x \in L^1(M,\tau)$ be such that $x \geq 0$, $x \neq 0$. We say that T is *smoothing* with respect to x if for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(pT^n x) < \varepsilon$ for every $p \in \nabla$ such that $\tau(p) < \delta$ and for every $n \geq n_0$. A commutative counterpart of this notion was introduced in [ZZ], [KT]. The following result has been proved in [MTA]; for the sake of completeness we include the proof.

THEOREM 4.3. *Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a positive contraction. Assume that there is a positive element $y \in L^1(M,\tau)$ such that T is smoothing with respect to y. Then either $\lim_{n\to\infty} \|T^n y\|_1 = 0$ or there is a non-zero positive $z \in L^1(M,\tau)$ such that $Tz = z$.*

Proof. Since T is a contraction, the limit

$$\lim_{n\to\infty} \|T^n y\|_1 = \alpha$$

exists. Assume that $\alpha \neq 0$. Define $\lambda : M_{\mathrm{sa}} \to \mathbb{R}$ by

$$\lambda(x) = L((\tau(xT^n y)_{n\in\mathbb{N}}))$$

for every $x \in M_{\mathrm{sa}}$, where L is a Banach limit (see [K]). We have

$$\lambda(\mathbb{1}) = L((\tau(T^n x)_{n\in\mathbb{N}})) = \lim_{n\to\infty} \|T^n x\|_1 = \alpha \neq 0,$$

so $\lambda \neq 0$. Moreover, λ is a positive functional, since for $x \in M_{\mathrm{sa}}$, $x \geq 0$, we have

$$\tau(xT^n y) = \tau(x^{1/2} T^n y x^{1/2}) \geq 0$$

for every $n \in \mathbb{N}$.

For arbitrary $x = x_1 + ix_2 \in M$ define

$$\lambda(x) = \lambda(x_1) + i\lambda(x_2).$$

Let T^{**} be the second dual of T, i.e. $T^{**} : M^{**} \to M^{**}$. The functional λ is T^{**}-invariant. Indeed,

$$\begin{aligned}(T^{**}\lambda)(x) &= \langle x, T^{**}\lambda\rangle = \langle T^* x, \lambda\rangle = L((\tau(T^n y T^* x)_{n\in\mathbb{N}})) \\ &= L((\tau(xT^{n+1}y)_{n\in\mathbb{N}})) = L((\tau(xT^n y)_{n\in\mathbb{N}})) = \lambda(z).\end{aligned}$$

Let $\lambda = \lambda_{\mathrm{n}} + \lambda_{\mathrm{s}}$ be the Takesaki decomposition (see [T]) of λ into the normal and singular components. Since T is normal and $T^{**}\lambda = \lambda$, using the idea of [J] it can be proved that $T^{**}\lambda_{\mathrm{n}} = \lambda_{\mathrm{n}}$. Now we will show that λ_{n} is non-zero. Consider $\mu := \lambda|_{\nabla}$. It is clear that μ is an additive measure on ∇. Let us prove that it is σ-additive. To this end, it is enough to show that $\mu(p_k) \to 0$ whenever $p_{k+1} \leq p_k$ and $p_k \searrow 0$, $p_k \in \nabla$.

Let $\varepsilon > 0$. From $p_k \searrow 0$ we infer that $\tau(p_k) \to 0$ as $k \to \infty$. It follows that there exists $k_\varepsilon \in \mathbb{N}$ such that $\tau(p_k) < \varepsilon$ for all $k \geq k_\varepsilon$. Since T is smoothing with respect to y we obtain

$$\tau(p_k T^n y) < \varepsilon, \quad \forall k \geq k_\varepsilon,$$

for every $n \geq n_0$. From the properties of Banach limits we get

$$\lambda(p_k) = L((\tau(p_k T^n y)_{n\in\mathbb{N}}) < \varepsilon \quad \forall k \geq k_\varepsilon,$$

which implies $\mu(p_k) \to 0$ as $k \to \infty$. This means that the restriction of λ_{n} to ∇ coincides with μ. Since

$$\tau(p^\perp T^n y) > \tau(T^n y) - \varepsilon \geq \inf \|T^n y\|_1 - \varepsilon = \alpha - \varepsilon,$$

and we can assume that $\alpha - \varepsilon > 0$ as ε has been arbitrary, it follows that $\mu(p^\perp) > 0$ for all $p \in \nabla$ such that $\tau(p) < \delta$. Therefore $\mu \neq 0$, and consequently, $\lambda_{\mathrm{n}} \neq 0$.

From this we infer that there exists a positive element $z \in L^1(M,\tau)$ such that

$$\lambda_{\mathrm{n}}(x) = \tau(zx), \quad \forall x \in M.$$

The last equality and $T^{**}\lambda_{\mathrm{n}} = \lambda_{\mathrm{n}}$ yield

$$\tau(zx) = \langle x, T^{**}\lambda_{\mathrm{n}}\rangle = \langle T^* x, \lambda_{\mathrm{n}}\rangle = \tau(zT^*x) = \tau(Tzx)$$

for every $x \in M$, which implies that $Tz = z$.

Remark 4.1 Theorem 4.3 is a non-commutative analog of Akcoglu and Sucheston's result [AS]. However, they used weak convergence instead of smoothing. In fact, smoothing is less restrictive, since if a sequence $T^n x$ with $x \geq 0$ weakly converges then it is a weakly pre-compact set, and from [T, Theorem III.5.4] we infer that T is smoothing with respect to x.

Using Theorem 4.3, in [MTA] we have proved a non-commutative analog of the result of [KS] which indicates a relation between mixing and complete mixing.

Remark 4.2. It should be noted that Theorem 4.3 is not valid if the von Neumann algebra is only semi-finite. Indeed, let $B(\ell_2)$ be the algebra of all bounded linear operators on the Hilbert space ℓ_2. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be the standard basis of ℓ_2, i.e.

$$\phi_n = (\underbrace{0,\ldots,0,1}_{n}, 0, \ldots).$$

The matrix units of $B(\ell_2)$ can be defined by

$$e_{ij}(\xi) = (\xi, \phi_i)\phi_j, \quad \xi \in \ell_2,\ i,j \in \mathbb{N}.$$

A trace on $B(\ell_2)$ is defined by

$$\tau(x) = \sum_{k=1}^{\infty} (x\phi_k, \phi_k).$$

We denote by ℓ_∞ the maximal commutative subalgebra generated by $\{e_{ii} : i \in \mathbb{N}\}$. Let $E : B(\ell_2) \to \ell_\infty$ be the canonical conditional expectation (see [T]). Define a map $s : \ell_\infty \to \ell_\infty$ as follows: for $a \in \ell_\infty$, $a = \sum_{k=1}^{\infty} a_k e_{kk}$, put

$$s(a) = \sum_{k=1}^{\infty} a_k e_{k+1,k+1}.$$

Define $T : B(\ell_2) \to B(\ell_2)$ by $T(x) = s(E(x))$ for $x \in B(\ell_2)$. It is clear that T is positive and $\tau(T(x)) \le \tau(x)$ for every $x \in L^1(B(\ell_2), \tau) \cap B(\ell_2)$ with $x \ge 0$. Hence, T is a positive L^1-contraction. But for this T there is no non-zero x such that $Tx = x$. Moreover, for every $y \in L^1(B(\ell_2), \tau)$ we have $\lim_{n\to\infty} \|T^n y\|_1 \ne 0$.

5. Strongly asymptotically stable contractions. In this section we give a criterion for strong asymptotic stability of contractions in terms of complete mixing.

Theorem 5.1. *Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a positive contraction. The following conditions are equivalent*:

(i) *T is completely mixing and smoothing with respect to some $h \in L^1(M, \tau)$, $h \ge 0$;*
(ii) *there exists $y \in L^1(M, \tau)$, $y \ge 0$, such that for every $x \in L^1(M, \tau)$,*

$$\lim_{n\to\infty} \|T^n x - T_y x\|_1 = 0.$$

Proof. (i)⇒(ii). Let $h \in L^1(M, \tau)$, $h \ge 0$, $h \ne 0$, be such that T is smoothing with respect to h. Without loss of generality we may assume that $\|h\|_1 = 1$. By Theorem 4.3 there are only two possibilities:

(a) $\lim_{n\to\infty} \|T^n h\|_1 = 0$;
(b) there exists $y \in L^1(M, \tau)$, $y \ge 0$, $y \ne 0$, such that $Ty = y$.

In case (a), for every $x \in L^1(M, \tau)$ with $x \ge 0$ and $\|x\|_1 = 1$, using complete mixing one gets

$$\lim_{n\to\infty} \|T^n x\|_1 \le \lim_{n\to\infty} \|T^n x - T^n h\|_1 + \lim_{n\to\infty} \|T^n h\|_1 = 0.$$

Let $x \in L^1(M, \tau)$. Then $x = \sum_{k=1}^{4} i^k x_k$, where $x_k \ge 0$. Hence using the last relation one finds that T^n converges strongly to T_0.

In case (b) we may assume that $\|y\|_1 = 1$. Since T is completely mixing,

$$\lim_{n\to\infty} \|T^n x - y\|_1 = 0$$

for every $x \in L^1(M, \tau)$ with $x \ge 0$ and $\|x\|_1 = 1$. Arguments similar to those used towards the end of the proof of Theorem 3.4 show the desired relation holds.

(ii)⇒(i). If $g \in X$ then $T^n g$ norm converges to $\tau(g)y = 0$, and hence T is completely mixing.

If $x \in L^1(M,\tau)$, $x \geq 0$, $\|x\|_1 = 1$, then $T^n x$ norm converges to y. So according to Remark 4.1 we find that T is smoothing with respect to x.

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