COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 2

ON FIELDS AND IDEALS CONNECTED WITH NOTIONS OF FORCING

BY

W. KUŁAGA (Wrocław)

Abstract. We investigate an algebraic notion of decidability which allows a uniform investigation of a large class of notions of forcing. Among other things, we show how to build σ-fields of sets connected with Laver and Miller notions of forcing and we show that these σ-fields are closed under the Suslin operation.

1. Introduction. We use standard set-theoretic notation. We denote by $\triangle$ symmetric difference, that is, $x \triangle y = (x - y) + (y - x)$.

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set. A subset $X \subseteq P$ is *dense* in $\mathbb{P}$ if $(\forall x \in P)(\exists y \in X)(y \leq x)$. We denote by $D(\mathbb{P})$ the family of all dense subsets of $\mathbb{P}$. For $x \in P$ we put $[x]_\leq = \{y \in P : y \leq x\}$. A subset $X \subseteq P$ is *open* in $\mathbb{P}$ if $(\forall x \in X)([x]_\leq \subseteq X)$.

Let $\mathbb{A} = (A, +, \cdot, 0, 1, \leq)$ be a Boolean algebra. We denote by $\mathbb{A}^+$ the set of all nonzero elements of A. In this case we put $[x]_\leq = \{y \in \mathbb{A}^+ : y \leq x\}$. If B, C are subsets of the Boolean algebra $\mathbb{A}$ then we put

$$B + C = \{b + c : b \in B \,\&\, c \in C\},$$
$$B - C = \{b - c : b \in B \,\&\, c \in C\}.$$

For any nonempty subset X we denote by $\mathbb{P}(X)$ the Boolean algebra of all subsets of X with standard set-theoretic operations.

If $X \subseteq A$ then $\sum_A X$ denotes the supremum (if exists) of X in $\mathbb{A}$. We say that a subalgebra B of an algebra $\mathbb{A}$ *preserves unions* if the following two conditions are satisfied:

(1) $(\forall R \subseteq B)(\forall x \in B)(x = \sum_B R \to x = \sum_A R)$,
(2) $(\forall R \subseteq B)(\forall x \in A)(x = \sum_A R \to (x \in B \wedge x = \sum_B R))$.

The next definitions play a fundamental role in our paper:

Definition 1.1. Let $\mathbb{A}$ be a Boolean algebra and $x, y \in A$. We say that x *decides* y in $\mathbb{A}$ if $x \leq y$ or $x \cdot y = 0$. This relation will be denoted by $x \parallel y$.

2000 *Mathematics Subject Classification*: Primary 28A05, 03G05; Secondary 54E52.

Key words and phrases: Boolean algebras, Suslin operation, Laver forcing, Miller forcing, Marczewski ideal.

I would like to thank J. Cichoń and S. Żeberski for many helpful suggestions.

Definition 1.2. Let $\mathbb{A}$ be a Boolean algebra and let P be a nonempty subset of $\mathbb{A}^+$. Then we put

(1) $\operatorname{dec}(P) = \{x \in A : (\forall u \in P)(\exists v \in P)(v \le u \wedge (v \cdot x = 0 \vee v \le x))\}$,
(2) $s(P) = \{x \in A : (\forall u \in P)(\exists v \in P)(v \le u \wedge v \cdot x = 0)\}$.

The idea of the above definitions is taken from Burstin and Marczewski. A system $(\mathbb{P}(X), \Sigma, I)$, where X is a nonempty set, Σ is a field of subsets of X, I is an ideal included in Σ, for which there exists a nonempty set $P \subseteq P(X)$ such that $P \subseteq \operatorname{dec}(P)$, $\Sigma = \operatorname{dec}(P)$, $s(P) = I$ is called *inner Marczewski–Burstin representable*, or briefly *inner MB-representable* (see [1], [3]).

In this paper we consider general algebraic properties of systems of the form $(\mathbb{A}, \operatorname{dec}(P), s(P))$. We will prove that if P and $\mathbb{A}$ satisfy some general conditions then the above system has the following properties:

- $\operatorname{dec}(P)/s(P)$ preserves unions in $A/s(P)$,
- $\operatorname{dec}(P)/s(P)$ is a complete Boolean algebra,
- if $\mathbb{A} = \mathbb{P}(X)$ then $\operatorname{dec}(P)$ is closed under the Suslin operation.

We give applications of this result to Laver and Miller forcing .

2. Algebraic structure of decidable elements. We start with some observations about the algebraic properties of the above definitions. Some of these facts have been known for the algebra $\mathbb{P}(X)$ (see [2]). Notice that $x \in \operatorname{dec}(P)$ if and only if $\{u \in P : u \parallel x\} \in D(P)$. Moreover, $x \in s(P)$ if and only if $\{u \in P : u \cdot x = 0\} \in D(P)$. The next lemma follows directly from the definitions:

Lemma 2.1. *Let x, y, z be elements of a Boolean algebra $\mathbb{A}$. Then*

(1) $(x \parallel y \wedge z \le x) \to z \parallel y$,
(2) $x \parallel y \to z \cdot x \parallel z \cdot y$,
(3) $(x \parallel y \wedge x \cdot z = 0) \to x \parallel (y + z)$,
(4) $x \parallel y \to x \parallel (-y)$.

The following theorem summarizes basic algebraic properties of the sets $\operatorname{dec}(P)$ and $s(P)$ and also follows directly from the definitions:

Theorem 2.2. *Let $P \subseteq A^+$ be a nonempty subset of a Boolean algebra $\mathbb{A}$. Then*

(1) $\operatorname{dec}(P)$ *is a subalgebra of* $\mathbb{A}$,
(2) $s(P)$ *is an ideal in* $\mathbb{A}$, $s(P) \subseteq \operatorname{dec}(P)$ *and* $P \cap s(P) = \emptyset$,
(3) $(\forall x \in \operatorname{dec}(P) \setminus s(P))(\exists u \in P)(u \le x)$.

In the next theorem we show how the sets $\operatorname{dec}(P)$ and $s(P)$ are related to the basic set P.

For subsets $P, Q \subseteq \mathbb{A}^+$ we will write $P \prec Q$ if $(\forall u \in Q)(\exists v \in P)(v \le u)$.

THEOREM 2.3. *Suppose that $P, Q \subseteq \mathbb{A}^+$ are two nonempty subsets of the Boolean algebra $\mathbb{A}$. Then*:

(1) $(\forall x \in \operatorname{dec}(P) \setminus s(P))(\operatorname{dec}(P \cap [x]_\leq) = (\operatorname{dec}(P) \cap [x]_\leq) + [-x]_\leq)$,
(2) $(\forall x \in \operatorname{dec}(P) \setminus s(P))(s(P \cap [x]_\leq) = (s(P) \cap [x]_\leq) + [-x]_\leq)$,
(3) *if* $P \prec Q$ *and* $Q \prec P$ *then* $\operatorname{dec}(P) = \operatorname{dec}(Q)$ *and* $s(P) = s(Q)$,
(4) *if* $P, Q \subseteq \operatorname{dec}(P) = \operatorname{dec}(Q)$ *and* $s(P) = s(Q)$ *then* $P \prec Q$ *and* $Q \prec P$.

Proof. (1) Suppose that $x \in \operatorname{dec}(P)$. First, we prove the inclusion $\operatorname{dec}(P \cap [x]_\leq) \subseteq (\operatorname{dec}(P) \cap [x]_\leq) + [-x]_\leq$. Suppose that $y \in \operatorname{dec}(P \cap [x]_\leq)$. Since $y = (y \cdot x) + (y \cdot -x)$ it is enough to show that $y \cdot x \in \operatorname{dec}(P)$. So let $u \in P$. There exists $u_1 \in P$ such that $u_1 \leq u$ and $u_1 \parallel x$. If $u_1 \cdot x = 0$ then $u_1 \cdot x \cdot y = 0$. Otherwise $u_1 \leq x$. Then $u_1 \in P \cap [x]_\leq$. Therefore there exists $u_2 \in P$ such that $u_2 \leq u_1$ and $u_2 \parallel y$. Hence, $y \in (\operatorname{dec}(P) \cap [x]_\leq) + [-x]_\leq$.

To prove the opposite inclusion $(\operatorname{dec}(P) \cap [x]_\leq) + [-x]_\leq \subseteq \operatorname{dec}(P \cap [x]_\leq)$, suppose that $y = y_1 + y_2$, where $y_1 \in \operatorname{dec}(P) \cap [x]_\leq$ and $y_2 \in [-x]_\leq$. Let $u \in P \cap [x]_\leq$. Then there exists $v \leq u$ such that $v \in P$ and $v \parallel y_1$. Since $v \cdot y_2 = 0$ we see that $v \parallel y_1 + y_2$.

(2) The proof is similar to the previous one.

(3) Let $x \in \operatorname{dec}(P)$. The assumption $P \prec Q$ implies that for any $v \in Q$ there exists $u \in P \cap [v]_\leq$. There exists $u_1 \in P \cap [u]_\leq$ such that $u_1 \parallel x$. The assumption $Q \prec P$ implies that there exists $v_1 \in Q \cap [u_1]_\leq$. Then $v_1 \in Q \cap [v]_\leq$ and $v_1 \parallel x$. We have proved that $x \in \operatorname{dec}(Q)$.

In a similar way we prove the opposite inclusion.

Both inclusions imply that $\operatorname{dec}(P) = \operatorname{dec}(Q)$. The proof that $s(P) = s(Q)$ is similar.

(4) If $P \subseteq \operatorname{dec}(P)$ then $P \subseteq \operatorname{dec}(P) \setminus s(P)$. Thus $\operatorname{dec}(P) \setminus s(P) = \operatorname{dec}(Q) \setminus s(Q)$ and we have $P \subseteq \operatorname{dec}(Q) \setminus s(Q)$. Theorem 2.2 implies that $Q \prec P$.

In a similar way we prove that $P \prec Q$. ■

EXAMPLE 2.1. Let $\mathbb{X}$ be a topological space on a set X. We denote by $\operatorname{Open}(\mathbb{X})$ the family of all nonempty open sets in $\mathbb{X}$ and by $N(\mathbb{X})$ the family of all nowhere dense subsets of $\mathbb{X}$. If we treat $\operatorname{Open}(\mathbb{X})$ as a subset of the power set Boolean algebra $\mathbb{P}(\mathbb{X})$ then we have $\operatorname{dec}(\operatorname{Open}^+(\mathbb{X})) = \operatorname{Open}(\mathbb{X}) \triangle N(\mathbb{X})$ and $s(\operatorname{Open}^+(\mathbb{X})) = N(\mathbb{X})$.

DEFINITION 2.1. A subset $P \subseteq \mathbb{A}^+$ of an algebra $\mathbb{A}$ is *separable* in $\mathbb{A}$ if P is nonempty and $P \subseteq \operatorname{dec}(P)$.

EXAMPLE 2.2. Let $B \neq \{0, 1\}$ be a subalgebra of a Boolean algebra $\mathbb{A}$. Let $x \in A$ be such that $x \cdot y > 0$ and $(-x) \cdot y > 0$ for any $y \in B^+$. It is easy to see that B^+ is separable in $\mathbb{A}$ and that $B^+ \cup \{x, -x\}$ is not separable in $\mathbb{A}$.

Lemma 2.4. *Let* $\mathbb{A}$ *be a Boolean algebra. For any nonempty subset* P *of* A *and for any* $x \in A$ *we have*:

(1) $(\forall u \in P)(u \cdot x \in s(P) \to (\exists v \in P \cap [u]_{\leq})(v \cdot x = 0))$,
(2) $(\forall u \in P)(u - x \in s(P) \to (\exists v \in P \cap [u]_{\leq})(v \leq x))$.

Proof. (1) Let $u \in P$ and $u \cdot x \in s(P)$. There exists $v \in P \cap [u]_{\leq}$ such that $v \cdot (u \cdot x) = 0$. The inequality $v \leq u$ implies $v \cdot x = 0$.

The proof of (2) is similar. ∎

Lemma 2.5. *Let* B *be a subalgebra of a Boolean algebra* $\mathbb{A}$ *and let* J *be an ideal in* $\mathbb{A}$ *included in* B. *Then*

$$\{x \in B : x \triangle u \in J\} = \{x \in A : x \triangle u \in J\}.$$

So, B/J *is a subalgebra of* A/J.

If a Boolean algebra $\mathbb{A}$ and a set $P \subseteq \mathbb{A}^+$ are fixed then we denote by $[x]$ the set $\{y \in A : x \triangle y \in s(P)\}$.

Theorem 2.6. *For every Boolean algebra* $\mathbb{A}$ *and any separable subset* P *in* $\mathbb{A}$ *the subalgebra* $\operatorname{dec}(P)/s(P)$ *preserves unions in* $A/s(P)$.

Proof. (1) Suppose that $R \subseteq \operatorname{dec}(P)$, $x \in \operatorname{dec}(P)$ and $[x] = \sum_{\operatorname{dec}(P)/s(P)}[R]$. Let

$$D = \{v \in P : (\exists r \in R)(v \leq r \text{ or } v \cdot x = 0)\}.$$

We claim that D is dense in P. Suppose that $u \in P$. Since $x \in \operatorname{dec}(P)$ there exists $v \in P \cap [u]_{\leq}$ such that $v \cdot x = 0$ or $v \leq x$. If $v \cdot x = 0$ then $D \cap [u]_{\leq} \neq \emptyset$. If $v \leq x$ then we have two possibilities.

If $v \cdot r \in s(P)$ for every $r \in R$ then $r - (x - v) \in s(P)$. Hence $[r] \leq [x - v]$. Since $x - v \in \operatorname{dec}(P)$ and $[x - v] < [x]$ we have $[R] \leq [x - v]$, which contradicts the assumptions on x.

So, there exists $r \in R$ such that $v \cdot r \in \operatorname{dec}(P) \setminus s(P)$. By Theorem 2.2 there exists $v_1 \in P$ such that $v_1 \leq v \cdot r$. Therefore $v_1 \leq r$ and $v_1 \leq u$, which finishes the proof of the density of D.

Suppose that $y \in A$ and $[R] \leq [y]$. Let $v \in D$. We consider two cases.

If $v \cdot x = 0$ then $v \cdot (x - y) = 0$.

Otherwise $v \leq r$ for some $r \in R$. Then $[v] \leq [y]$ and hence $v - y \in s(P)$. By Lemma 2.4 there exists $v_1 \in P \cap [v]_{\leq}$ such that $v_1 \leq y$ and hence $v_1 \cdot (x - y) = 0$. Therefore the set $\{v \in P : v \cdot (x - y) = 0\}$ is dense in P. Then $x - y \in s(P)$ and $[x] \leq [y]$. We have proved that x is the least upper bound of $[R]$ in $\mathbb{A}/s(P)$.

(2) Suppose that $\sum_{\mathbb{A}/s(P)}[R] = [x]$. Let $u \in P$. We consider two cases.

If $[u] \cdot [x] = [0]$ then $u \cdot x \in s(P)$ and by Lemma 2.4 there exists $v \in P \cap [u]_{\leq}$ such that $v \cdot x = 0$.

Now, assume $[u] \cdot [x] \neq [0]$. Then for some $r \in R$ we have $[u] \cdot [r] \neq [0]$ and $[u] \cdot [r] \leq [x]$. Therefore $u \cdot r - x \in s(P)$. Since $u \cdot r \in \operatorname{dec}(P) \setminus s(P)$, from

Theorem 2.2 there exists $v \in P \cap [u \cdot r]_\leq$. Therefore $v - x \in s(P)$ and then by Lemma 2.4 there exists $w \in P \cap [u]_\leq$ such that $w \leq x$. This proves that the family $\{u \in P : u \parallel x\}$ is a dense subset in P and hence $x \in \mathrm{dec}(P)$. ∎

THEOREM 2.7. *Let* $(\mathbb{A}, B, I)$ *be a system such that* $\mathbb{A}$ *is a Boolean algebra*, B *is a subalgebra of* $\mathbb{A}$ *and* $I \subseteq B$ *is an ideal in* $\mathbb{A}$. *Then the following conditions are equivalent*:

(1) *There exists a subset* P *which is separable in* $\mathbb{A}$ *such that* $\mathrm{dec}(P) = B$ *and* $s(P) = I$.
(2) *The algebra* B/I *preserves unions in* $\mathbb{A}/I$.

Proof. (1)⇒(2). This follows immediately from Theorem 2.6.

(2)⇒(1). For a while we will use the following notation: $[x]_I = \{y \in A : x \triangle y \in I\}$. Let $x \in \mathrm{dec}(B \setminus I)$. By density of $D = \{u \in B \setminus I : u \parallel x\}$ in $B \setminus I$ the set $[D]_I = \{[u]_I : u \in D\}$ is dense in B/I. Let $E \subseteq D$ be such that $[E]_I$ is a maximal partition in B/I. Put $E_1 = \{u \in E : u \leq x\}$ and $E_2 = \{u \in E : u \cdot x = 0\}$. Notice that $[x]_I$ is an upper bound of $[E_1]_I$. Let $y \in A$ be such that $[y]_I$ is an upper bound of $[E_1]_I$. If $u \in E_1$ then $u \cdot (x - y) \in I$, because $u - y \in I$. If $u \in E_2$ then $u \cdot (x - y) = 0$, because $u \cdot x = 0$. Since $[E]_I$ is a maximal partition in B/I, we have $\sum_{B/I} [E]_I = [1]_I$. So $\sum_{A/I} [E]_I = [1]_I$. We have $[x - y]_I = 0$. So, $[x]_I \leq [y]_I$. We have shown that $[x]_I$ is the least upper bound of $[E_1]_I$ in A/I. Because B/I preserves unions, we have $\sum_{B/I} [E_1]_I = [x]_I$. So $x \in B$. We have shown that $\mathrm{dec}(B \setminus I) \subseteq B$.

Because $B \setminus I$ is separable in A we have $B \setminus I \subseteq \mathrm{dec}(B \setminus I)$. So $B = \mathrm{dec}(B \setminus I)$.

In a similar way we show that $s(B \setminus I) = I$. (Notice that $E_1 = \emptyset$.) ∎

3. Disjoint refinement property. In this section we discuss some properties which imply that the Boolean algebra $\mathrm{dec}(P)/s(P)$ is complete.

Let $[E]_\leq = \bigcup \{[x]_\leq : x \in E\}$.

DEFINITION 3.1. A partition E in a Boolean algebra $\mathbb{A}$ is called P-*maximal* for a subset $P \subseteq \mathbb{A}^+$ if $P \cap [E]_\leq$ is a dense open subset in P.

DEFINITION 3.2. We say that a subset $P \subseteq A^+$ has the *disjoint refinement property* if for every open dense subset D in P there exists a P-maximal partition included in D.

LEMMA 3.1. *Let* $\mathbb{A}$ *be a complete Boolean algebra. For* $P \subseteq \mathbb{A}^+$ *and a* P*-maximal partition* E *we have*

(1) $(\forall R \subseteq E)(\sum R \in \mathrm{dec}(P))$,
(2) $(\forall x \in A)(\forall u \in E)(x \cdot u \in s(P) \to x \in s(P))$.

Proof. (1) This follows directly from the inclusion $P \cap [E]_{\leq} \subseteq \{v : v \parallel \sum R\}$.

(2) Suppose that $x \in A$ and for any $u \in E$ we have $u \cdot x \in s(P)$. Let $v \in P$. Then there exist $u \in E$ and $v_1 \in (P \cap [v]_{\leq}) \cap [u]_{\leq}$. Since $u \cdot x \in s(P)$ we have $v_1 \cdot x \in s(P)$. Therefore from Lemma 2.4 there exists $w \in P \cap [v_1]_{\leq}$ such that $w \cdot x = 0$ and moreover $w \leq v$. ∎

THEOREM 3.2. *Let $\mathbb{A}$ be a complete Boolean algebra and let P be a separable subset of A with the disjoint refinement property. Then* $\mathrm{dec}(P)/s(P)$ *is a complete Boolean algebra and it preserves unions in $\mathbb{A}/s(P)$.*

Proof. Let R be any subset of $\mathrm{dec}(P)$. We define

$$D_1 = \{v \in P : (\exists r \in R)(v - r \in s(P))\}$$

and

$$D_2 = \{v \in P : (\forall r \in R)(v \cdot r \in s(P))\}.$$

By separability of P the set $D_1 \cup D_2$ is dense and open in P. By the disjoint refinement property of P there exists a maximal disjoint subset E included in $D_1 \cup D_2$. Let $E_1 = E \cap D_1$ and $E_2 = E \cap D_2$. From Lemma 3.1 we deduce that $\sum E_1 \in \mathrm{dec}(P)$.

Suppose that $r \in R$ and $u \in E$. If $u \in E_1$ then $u \cdot (r - \sum E_1) = 0$. If $u \in E_2$ then $u \cdot (r - \sum E_1) \in s(P)$. From Lemma 3.1 we see that $r - \sum E_1 \in s(P)$. This proves that $[r] \leq [\sum E_1]$ and therefore $[\sum E_1]$ is an upper bound for the family $[R]$ in $\mathrm{dec}(P)/s(P)$. Let $w \in \mathrm{dec}(P)$ be such that $r - w \in s(P)$ for any $r \in R$. Let $u \in E$. If $u \in E_1$ then there exists $r \in R$ such that $u - r \in s(P)$. Since $r - w \in s(P)$ we have $u - w \in s(P)$. This proves that $u \cdot (\sum E_1 - w) \in s(P)$.

If $u \in E_2$ then $u \cdot (\sum E_1 - w) = 0$. Lemma 3.1 implies that $\sum E_1 - w \in s(P)$. We have proved that $[\sum E_1] \leq [w]$, so that $[\sum E_1]$ is the least upper bound of the family $[R]$ in $\mathrm{dec}(P)/s(P)$. ∎

EXAMPLE 3.1. Let $\mathbb{X}$ be a topological space. With the notation from Example 2.1, $(\mathrm{Open}(\mathbb{X}) \triangle N(\mathbb{X}))/N(\mathbb{X})$ is complete and preserves unions in $\mathbb{P}(X)/N(\mathbb{X})$.

COROLLARY 3.3. *Let κ be an infinite cardinal. Let $\mathbb{A}$ be a complete Boolean algebra and let P be a separable subset of $\mathbb{A}$ of size κ. If $s(P)$ is a κ-complete ideal then P has the disjoint refinement property and the subalgebra* $\mathrm{dec}(P)/s(P)$ *is complete and preserves unions in $A/s(P)$.*

Proof. Let $D = \{u_\xi : \xi \in \eta\}$ be a dense open subset in P, $\eta \leq \kappa$. We construct a sequence $(v_\xi : \xi \in \eta)$. Let $v_0 = u_0$. Assume we have defined $(v_\xi : \xi \in \lambda)$ for some $\lambda \in \eta$. If $\{v_\xi : u_\lambda \cdot v_\xi \notin s(P), \xi \in \lambda\} \neq \emptyset$ then let v_λ be any element of the above set. In the other case $\{u_\lambda \cdot v_\xi : \xi \in \lambda\} \subseteq s(P)$. So, because $\lambda < \kappa$ and $s(P)$ is κ-complete, we have $r_\lambda = \sum\{u_\lambda \cdot v_\xi : \xi \in \lambda\} \in$

$s(P)$. So $u_\lambda - r_\lambda \in \mathrm{dec}(P) \setminus s(P)$. We have $P \cap [u_\lambda - r_\lambda]_\le \neq \emptyset$. By density of D we have $D \cap [u_\lambda - r_\lambda]_\le \neq \emptyset$. Let v_λ be any element of the latter set.

Let $E = \{v_\xi : \xi \in \eta\}$. Directly from the construction it follows that E is included in D and is a partition.

We will show that $\bigcup\{[v]_\le : v \in E\}$ is dense in P. Let $x \in P$. By density of D we can choose $u \in D \cap [x]_\le$. Then $u = u_\xi$ for some $\xi \in \eta$. By the construction of the sequence $(v_\xi : \xi \in \eta)$ we know that $v_\xi \cdot u_\xi \notin s(P)$. So $[v_\xi]_\le \cap [x]_\le \neq \emptyset$. Since $v_\xi \in E$ we have $\bigcup\{[v]_\le : v \in E\} \cap [x]_\le \neq \emptyset$.

Now, we can apply Theorem 3.2 to get the desired conclusion. ■

4. Closedness under the Suslin operation. Recall that a family $\mathcal{B} \subseteq \mathbb{P}(X)$ is *closed under the Suslin operation* if for every function $\varphi : \omega^{<\omega} \to \mathcal{B}$ the set

$$A(\varphi) = \bigcup_{x\in\omega^\omega} \bigcap_{s\subset x} \varphi(s)$$

belongs to $\mathcal{B}$.

LEMMA 4.1. *Let P be a separable subset of a complete Boolean algebra $\mathbb{A}$. If $s(P)$ is a κ-complete ideal then* $\mathrm{dec}(P)$ *is a κ-complete subalgebra of $\mathbb{A}$.*

Proof. Suppose that $R \subseteq \mathrm{dec}(P)$ and $|R| < \kappa$. Let $v \in P$. If $v \cdot r \in s(P)$ for any $r \in R$ then $v \cdot (\sum R) \in s(P)$ because $s(P)$ is κ-complete. By Lemma 2.4 there exists $w \in P \cdot [v]_\le$ such that $w \cdot (\sum R) = 0$. If $r \in R$ is such that $v \cdot r \notin s(P)$ then $v \cdot r \in \mathrm{dec}(P) \setminus s(P)$ and from Theorem 2.2 there exists $w \in P \cap [v \cdot r]_\le$. This implies that $w \in P \cap [v]_\le$ and $w \le \sum R$. ■

The starting point of the proof of the next theorem is the following classical result of Marczewski:

THEOREM 4.2 (Marczewski). *Let $\mathbf{B}$ be a σ-field of subsets of a set X and let J be an ideal in $\mathbb{P}(X)$ included in $\mathbf{B}$ such that*

$$(\forall Z \subseteq X)(\exists M \in \mathbf{B})(Z \subseteq M \wedge (\forall N \in \mathbf{B})(Z \subseteq N \to M \setminus N \in J)).$$

Then B is closed under the Suslin operation.

LEMMA 4.3. *Assume that $\mathbb{A}$ is a Boolean algebra. Let B be a subalgebra of $\mathbb{A}$ and let I be an ideal of $\mathbb{A}$ included in B. Suppose B/I is complete and*

$$(\forall R \subseteq B)(\textstyle\sum_{B/I}[R] = [x] \to \sum_{A/I}[R] = [x]).$$

Then for every $y \in A$ there exists $x \in B$ such that $y \le x$ and

$$(\forall r)(r \in B \wedge y \le r \to x - r \in I).$$

Proof. Let $y \in A$. Put $R = \{r \in B : y \le r\}$. By completeness of B/I we have $\prod_{B/I}[R] = [z]$ for some $z \in B$. By assumption $\prod_{A/I}[R] = [z]$. Notice that $[y] \le [r]$ for every $r \in R$. So $[y] \le \prod_{A/I}[R] = [z]$. Thus, $[y] \le [z]$. Put

$x = z \vee y - z$. Notice that $y - z \in I$. So $x \in B$ and $y \leq x$. If $r \in B$, $y \leq r$ then $r \in R$. So, $[z] \leq [r]$ and $[z] = [x]$. Thus $x - r \in I$. ∎

Recall that an ideal $\mathcal{I} \subseteq \mathbb{P}(X)$ is *σ-closed* if it is closed under countable unions.

Similarly, an ideal $\mathcal{I} \subseteq \mathbb{P}(X)$ is *ω_1-closed* if for every family $\mathcal{A} \subseteq \mathcal{I}$ such that $|\mathcal{A}| \leq \omega_1$ we have $\bigcup \mathcal{A} \in \mathcal{I}$.

THEOREM 4.4. *Suppose that P is separable and has the disjoint refinement property in $\mathbb{P}(X)$ and that the ideal $s(P)$ is σ-closed. Then the algebra* $\operatorname{dec}(P)$ *is closed under the Suslin operation.*

Proof. By Theorem 3.2 the Boolean algebra $\operatorname{dec}(P)/s(P)$ is complete and preserves unions in $\mathbb{P}(X)/s(P)$.

We show that the assumptions of Theorem 4.2 are satisfied. Because P is separable and $s(P)$ is a σ-closed ideal, Lemma 4.1 shows that $\operatorname{dec}(P)$ is σ-field.

Let $Z \subseteq X$. By Lemma 4.3, putting $B = \operatorname{dec}(P)$, $I = s(P)$, $A = P(X)$, there exists $M \in \operatorname{dec}(P)$ such that $Z \subseteq M$ and for every $N \in \operatorname{dec}(P)$ with $Z \subseteq N$ we have $M - N \in s(P)$.

Hence $\operatorname{dec}(P)$ is closed under the Suslin operation. ∎

COROLLARY 4.5. *Let P be a separable subset of $\mathbb{P}(\kappa)$ for a regular cardinal number κ. If $s(P)$ is κ-complete and $|P| \leq \kappa$ then*

(1) *P has the disjoint refinement property.*
(2) $\operatorname{dec}(P)/s(P)$ *is complete and preserves unions in* $\mathbb{P}(\kappa)/s(P)$.
(3) *If $\kappa \geq \omega_1$ then* $\operatorname{dec}(P)$ *is closed under the Suslin operation.*

Proof. (1) and (2) follow from Corollary 3.3.

(3) If $\kappa \geq \omega_1$ then $\operatorname{dec}(P)$ is a σ-field by Lemma 4.1. So, the assertion follows from Theorem 4.4. ∎

COROLLARY 4.6. *If P is a separable subset in $\mathbb{P}(X)$ such that $s(P)$ is ω_1-closed then* $\operatorname{dec}(P)$ *is closed under the Suslin operation.*

Proof. It is a classical fact that if $A \in \operatorname{Suslin}(B)$ and B is σ-closed then there exists a family $\{A_\xi\}_{\xi<\omega_1} \subseteq B$ such that

$$A = \bigcup_{\xi \in \omega_1} A_\xi.$$

From Lemma 4.1 we deduce that $\operatorname{dec}(P)$ is ω_1-closed. Using the above fact we deduce that $\operatorname{dec}(P)$ is closed under the Suslin operation. ∎

EXAMPLE 4.1. Let (X, S, μ) be a complete measure space such that $\mu(X) < \infty$. Let $J = \{A \in S : \mu(A) = 0\}$ and $S^+ = \{A \in S : \mu(A) > 0\}$. Then S^+ is separable and has the disjoint refinement property in $\mathbb{P}(X)$.

Moreover $\mathrm{dec}(S^+) = S$ and $s(S^+) = J$. From Theorem 4.4 we obtain the classical result of Sierpiński about closedness of S under the Suslin operation.

EXAMPLE 4.2 (Marczewski sets). Let X be a Polish space without isolated points. We denote by $\mathrm{Perf}(X)$ the family of all nonempty compact dense-in-themselves subsets of X. Marczewski (see [7]) introduced the notion of sets with property S and the ideal s^0. in our terminology these objects may be defined as follows: $S = \mathrm{dec}(\mathrm{Perf}(X))$ and $s^0 = s(\mathrm{Perf}(X))$. Marczewski proved that s^0 is a σ-closed ideal.

Suppose that U is an open subset in X and $F \in \mathrm{Perf}(X)$. If $U \cap F \neq \emptyset$ then there exists $H \in \mathrm{Perf}(X) \cap [F]_{\leq}$ such that $H \subseteq U \cap F$. It follows that $\mathrm{Open}(X) \subseteq S$ and in consequence the family $\mathrm{Perf}(X)$ is separable. In $\mathrm{Perf}(X)$ any dense open family D has size $\mathbf{c}$. If $A \subseteq X$ and $F \in \mathrm{Perf}(X)$ and $|A| < \mathbf{c}$ then there exists $H \in \mathrm{Perf}(X) \cap [F]_{\leq}$ such that $H \cap A = \emptyset$. Using this property, in a standard way we may conclude that $\mathrm{Perf}(X)$ has the disjoint refinement property in $\mathbb{P}(\mathbb{X})$. From Theorem 3.2 we conclude that the subalgebra S/s^0 is complete (see [9]) and preserves unions in $\mathbb{P}(X)/s^0$ and, moreover, S is closed under the Suslin operation (see [7]) by Theorem 4.4.

Let p be a closed dense-in-itself subset of ω^ω in the standard topology. For $s \in \omega^{<\omega}$ we put $p(s) = p \cap \{x \in \omega^\omega : s \subseteq x\}$. Let P_{M} denote the family of all nonempty closed dense-in-themselves subsets p of ω^ω such that for any $s \in \omega^{<\omega}$ with $p(s) \neq \emptyset$ there exists $t \supseteq s$ such that $|\{n \in \omega : p(t^\frown n) \neq \emptyset\}| = \aleph_0$. The family P_{M} is called the *Miller forcing*. It is known that the ideal $s(P_{\mathrm{M}})$ is σ-closed (see [8]).

Let P_{L} denote the family of all nonempty closed dense-in-themselves subsets p of ω^ω such that there exists s for which $p(s) \neq \emptyset$ and, for every t, if $p(t) \neq \emptyset$ then $t \subseteq s$ or $|\{n \in \omega : p(t^\frown n) \neq \emptyset\}| = \aleph_0$. The family P_{L} is called the *Laver forcing*. It is known that the ideal $s(P_{\mathrm{L}})$ is σ-closed (see [6]).

COROLLARY 4.7. (CH) *Let $Q = P_{\mathrm{M}}$ or $Q = P_{\mathrm{L}}$. Then*

(1) *Q is separable and has the disjoint refinement property*,
(2) $\mathrm{dec}(Q)/s(Q)$ *is complete and preserves unions in* $\mathbb{P}(\omega^\omega)$,
(3) $\mathrm{dec}(Q)$ *is a σ-closed field*, *contains all Borel subsets of ω^ω and is closed under the Suslin operation.*

Proof. Similarly to Example 4.2 we prove that $\mathrm{Open}(\omega^\omega)$ is included in $\mathrm{dec}(Q)$. From this it is easy to see that $\mathrm{dec}(Q)$ is separable. Since $s(Q)$ is σ-closed, the Borel subsets are contained in $\mathrm{dec}(Q)$. The disjoint refinement property follows from Corollary 3.3. Assertions (2) and (3) follow immediately from Theorem 4.4. ∎

5. A generalization of first category sets. We will generalize the notion of first category sets to the class of complete Boolean algebras.

Definition 5.1.

(1) For a complete Boolean algebra $\mathbb{A}$ and a separable subset P in A we say that $x \in \mathbb{A}$ is of the *first category* for $\operatorname{dec}(P)$ if x is the supremum of a countable family included in $s(P)$.
(2) The family of all subsets of the first category for $\operatorname{dec}(P)$ will be denoted by $I(P)$.

Lemma 5.1. *Let $\mathbb{A}$ be a Boolean algebra and let P be a separable subset with the disjoint refinement property in $\mathbb{A}$. Let I be an ideal in $\mathbb{A}$ such that* $\operatorname{dec}(P) \cap I = s(P)$. *Then $P - I$ is separable and has the disjoint refinement property.*

Proof. Notice that $(P-I)\cap I = \emptyset$. We will use the letters r, s for elements of I. Let $u-r$ and $v-s$ be any elements of $P-I$. Since there is $w \in P\cap[v]_{\leq}$ such that $w \parallel u$, we have $w-(r+s) \parallel u-r$. This proves that $P-I$ is separable in $\mathbb{A}$.

If D is any dense open subset in $P-I$ then

$$H = \{u \in P : (\exists r)(r \in I \,\&\, u-r \in D)\}$$

is open dense in P. Let E be a P-maximal disjoint family included in H. For any u let r_u be such that $u-r_u \in D$. Then $\{u-r_u : u \in E\}$ is a $(P-I)$-maximal disjoint family included in D. ∎

The next lemma is a reformulation of the well-known Banach lemma ([4]) in our language.

Lemma 5.2. *Suppose that P is a separable subset in a complete Boolean algebra $\mathbb{A}$ and E is a P-maximal partition included in P. Then*

(1) *If $x \in A$ is such that $x \cdot u \in I(P)$ for any $u \in E$ then $x \in I(P)$.*
(2) *For any subset $M \subseteq E$ we have $\sum M \in \operatorname{dec}(P)$.*

Proof. (1) For any $u \in E$ take a family $\{r_n(u) : n \in \omega)\}$ with least upper bound $x \cdot u$. Set $r_n = \sum\{r_n(u) : u \in E\}$. It follows from Lemma 3.1 that $x \cdot r_n \in s(P)$ for any $n \in \omega$. From the equality $x = \sum\{(x\cdot u) : u \in E\} + x \cdot (-\sum E)\}$ it follows that $x = \sum\{(x \cdot r_n) : n \in \omega\} + x\cdot(-\sum E)$. Since $(-\sum E) \in s(P)$ we have $x \in I(P)$.

(2) This follows immediately from the definition of a P-maximal partition. ∎

Theorem 5.3. *Let $\mathbb{A}$ be a complete Boolean algebra, let P be separable and $P \cap I(P) = \emptyset$ and let P have the disjoint refinement property. Then*

(1) *$P - I(P)$ is separable and has the disjoint refinement property in $\mathbb{A}$.*
(2) $\operatorname{dec}(P - I(P)) = \operatorname{dec}(P) \triangle I(P)$ *and* $s(P - I(P)) = I(P)$.

Proof. (1) follows directly from Lemma 5.1.

(2) Fix $x \in \mathrm{dec}(P - I(P))$ and let

$$D = \{v \in P : (\exists r \in I(P))(v - r \parallel x)\}.$$

Let E be a P-maximal disjoint family in D. Let $E_1 = \{u \in E : u - x \in I(P)\}$. In a standard way we prove that $u \cdot (x \triangle \sum_A E_1) \in I(P)$ for any $u \in E$. Lemma 5.2 yields $x \triangle \sum_A E_1 \in I(P)$ and $\sum_A E_1 \in \mathrm{dec}(P)$. This shows that $\mathrm{dec}(P - I(P)) \subseteq \mathrm{dec}(P) \triangle I(P)$.

In a similar way we prove that $s(P - I(P)) \subseteq I(P)$.

The reverse inclusions can be proved similarly. ∎

We get the following example (see [5]):

EXAMPLE 5.1. Let $\mathbb{X}$ be a topological space and let $I(\mathbb{X})$ denote the ideal of the first category subsets of X. Then $\mathrm{dec}(\mathrm{Open}^+(\mathbb{X}) - I(\mathbb{X})) = \mathrm{Baire}(\mathbb{X})$ and $s(\mathrm{Open}^+(\mathbb{X}) - I(\mathbb{X})) = I(\mathbb{X})$.

REFERENCES

[1] M. Balcerzak, A. Bartoszewicz and K. Ciesielski, *Algebras with inner MB-representation*, Real Anal. Exchange 29 (2003/2004), 265–274.

[2] M. Balcerzak, A. Bartoszewicz, J. Rzepecka and S. Wroński, *Marczewski fields and ideals*, ibid. 26 (2000/2001), 703–715.

[3] S. Baldwin, *The Marczewski hull property and complete Boolean algebras*, ibid. 28 (2002/2003), 415–428.

[4] S. Banach, *Théorème sur les ensembles de première catégorie*, Fund. Math. 16 (1930), 395–398.

[5] J. B. Brown and H. Elalaoui-Talibi, *Marczewski–Burstin characterizations of σ-algebras, ideals, and measurable functions*, Colloq. Math. 82 (1999), 277–286.

[6] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. 137 (1976), 151–169.

[7] E. Marczewski, *Sur un classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. 24 (1937), 17–34.

[8] A. Miller, *Rational perfect set forcing*, in: Contemp. Math. 31, Amer. Math. Soc., 1984, 143–159.

[9] J. Morgan II, *Point Set Theory*, Dekker, New York, 1990.

Estońska 40a/26
54-401 Wrocław, Poland
E-mail: wieslaw.kulaga@wp.pl

Received 8 March 2005;
revised 4 December 2005 (4574)