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ON SOME REPRESENTATIONS OF ALMOST EVERYWHERE CONTINUOUS FUNCTIONS ON $\mathbb{R}^m$

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Abstract. It is proved that the following conditions are equivalent: (a) f is an almost everywhere continuous function on $\mathbb{R}^m$; (b) $f = g + h$, where g, h are strongly quasicontinuous on $\mathbb{R}^m$; (c) $f = c + gh$, where $c \in \mathbb{R}$ and g, h are strongly quasicontinuous on $\mathbb{R}^m$.

Let λ^* (resp. λ) denote the outer Lebesgue measure (resp. the Lebesgue measure) on $\mathbb{R}^m$. For each $n \in \mathbb{N}$ (the positive integers) and for each sequence $(k_1, \ldots, k_m)$ of integers let

$$P^n_{k_1,\ldots,k_m} = \left[\frac{k_1 - 1}{2^n}, \frac{k_1}{2^n}\right) \times \cdots \times \left[\frac{k_m - 1}{2^n}, \frac{k_m}{2^n}\right).$$

Moreover, let

$$\mathcal{P}_n = \{P^n_{k_1,\ldots,k_m};\ k_1, \ldots, k_m \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{P} = \bigcup_n \mathcal{P}_n.$$

Observe that:

(1) if $(k_1, \ldots, k_m) \neq (l_1, \ldots, l_m)$ then $P^n_{k_1,\ldots,k_m} \cap P^n_{l_1,\ldots,l_m} = \emptyset$;
(2) $\mathbb{R}^m = \bigcup_{k_1,\ldots,k_m \in \mathbb{Z}} P^n_{k_1,\ldots,k_m}$;
(3) if $n_1 > n_2$ then for each sequence $(k_1, \ldots, k_m)$ of integers there is a unique sequence $(l_1, \ldots, l_m)$ of integers such that $P^{n_1}_{k_1,\ldots,k_m} \subset P^{n_2}_{l_1,\ldots,l_m}$;
(4) for each $\mathbf{x} \in \mathbb{R}^m$ and each $n \in \mathbb{N}$ there is a unique integer sequence $(k_1(\mathbf{x}), \ldots, k_m(\mathbf{x}))$ such that $\mathbf{x} \in P^n_{k_1(\mathbf{x}),\ldots,k_m(\mathbf{x})} = P^n(\mathbf{x})$.

For $A \subset \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^m$ denote by

$$d_u(A, \mathbf{x}) = \limsup_{n\to\infty} \frac{\lambda^*(A \cap P^n(\mathbf{x}))}{\lambda(P^n(\mathbf{x}))}, \qquad d_l(A, \mathbf{x}) = \liminf_{n\to\infty} \frac{\lambda^*(A \cap P^n(\mathbf{x}))}{\lambda(P^n(\mathbf{x}))}$$

the upper and lower density of $A \subset \mathbb{R}$ at $\mathbf{x}$ (cf. [2]).

A point $\mathbf{x} \in \mathbb{R}^m$ is called a *density point* of a set $A \subset \mathbb{R}^m$ if there exists a λ-measurable set $B \subset A$ such that $d_l(B, \mathbf{x}) = 1$. The family

$$\mathcal{T}_d = \{A \subset \mathbb{R}^m;\ A \text{ is } \lambda\text{-measurable and } d_l(A, \mathbf{x}) = 1 \text{ for } \mathbf{x} \in A\}$$

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is a topology called the *density topology* (with respect to $\mathcal{P}$) (see [4, 5, 13, 14] for the definitions of the density topologies with respect to other differentiation bases). The topology $\mathcal{T}_d$ introduced above with respect to a fixed sequence of binary nets of half-open cubes is stronger than the ordinary density topology ([13]) with respect to arbitrary cubes containing a given point.

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be *strongly quasi-continuous* (for short, s.q.c.) at a point $\mathbf{x}$ if for every set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ and for every $\varepsilon > 0$ there is an open set U such that $U \cap A \neq \emptyset$ and $|f(\mathbf{t}) - f(\mathbf{x})| < \varepsilon$ for all $\mathbf{t} \in A \cap U$ (cf. [7]).

REMARK 1. In the case $m = 1$ the notion of strong quasicontinuity for functions $f : \mathbb{R} \to \mathbb{R}$ introduced in [7] by Grande with respect to the bilateral density is more general than that above. For example, the function

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0, \end{cases}$$

is s.q.c. at 0 in the sense of Grande, but it is not s.q.c. at 0 in the above sense.

Evidently, if $f : \mathbb{R} \to \mathbb{R}$ is s.q.c. at x in the above sense then it is s.q.c. at x in the sense of Grande.

Observe that if for $\mathbf{x} \in \mathbb{R}^m$ there is an open set $U \subset \mathbb{R}^m$ such that $d_u(U, \mathbf{x}) > 0$ and the restriction $f|_{U \cup \{\mathbf{x}\}}$ is continuous at $\mathbf{x}$ then f is s.q.c. at $\mathbf{x}$.

Moreover, by an elementary proof, we obtain:

REMARK 2. If functions $f_n : \mathbb{R}^m \to \mathbb{R}$, $n = 1, 2, \ldots$, are s.q.c. at a point $\mathbf{x}$ and (f_n) uniformly converges to a function f then f is also s.q.c. at $\mathbf{x}$.

It is known [8] that every s.q.c. function $f : \mathbb{R}^m \to \mathbb{R}$ is almost everywhere continuous. So, the sum and product of two s.q.c. functions are almost everywhere continuous. We will prove the following:

THEOREM 1. *If a function $f : \mathbb{R}^m \to \mathbb{R}$ is almost everywhere continuous then there are two s.q.c. functions $g, h : \mathbb{R}^m \to \mathbb{R}$ such that $f = g + h$.*

Proof. Let cl denote closure and

$$B = \{y \in \mathbb{R};\ \lambda(\mathrm{cl}(f^{-1}(y))) > 0\}.$$

Since f is almost everywhere continuous, the set B is countable. Without loss of generality we can assume that $0 \notin B$, because otherwise we can fix a real $a \notin B$ and consider the function $f - a$.

Let $L(B)$ be the linear span of the set B over the rationals. Since $L(B)$ is countable, there is a positive number $c \in \mathbb{R} \setminus L(B)$. Fix $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

If
$$\frac{(k-1)c}{2^n} \le f(x) < \frac{kc}{2^n}$$
then we define
$$f_n(x) = \frac{(k-1)c}{2^n}.$$
Observe that every function f_n, $n \in \mathbb{N}$, is almost everywhere continuous and the set $D(f_n)$ of its discontinuity points is closed and of λ-measure zero. Moreover, $D(f_n) \subset D(f_{n+1})$ for $n \in \mathbb{N}$.

STEP 1. Observe that $D(f_1)$ is closed and of λ-measure zero. For each $\mathbf{x} \in D(f_1)$ there is a unique cube $P^1(\mathbf{x}) \in \mathcal{P}_1$ such that $\mathbf{x} \in P^1(\mathbf{x})$. Observe that $\operatorname{diam}(P^1(\mathbf{x})) < m/2$. For the cube $P^1(\mathbf{x})$ there is a finite family of cubes
$$Q_{1,1,\mathbf{x}}, \ldots, Q_{i(1,1,\mathbf{x}),1,\mathbf{x}} \in \mathcal{P}$$
whose closures are pairwise disjoint and contained in $\operatorname{int}(P^1(\mathbf{x})) \setminus D(f_1)$ (int denotes interior) and such that
$$\frac{\lambda(\bigcup_{i=1}^{i(1,1,\mathbf{x})} Q_{i,1,\mathbf{x}})}{\lambda(P^1(\mathbf{x}))} > \frac{1}{2}.$$
Moreover, we assume that if $P^1(\mathbf{x}) = P^1(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in D(f_1)$, then $i(1,1,\mathbf{x}) = i(1,1,\mathbf{y})$ and $Q_{i,1,\mathbf{x}} = Q_{i,1,\mathbf{y}}$ for $i \le i(1,1,\mathbf{x})$. Let
$$S_1^1 = \bigcup_{\mathbf{x} \in D(f_1)} \bigcup_{i \le i(1,1,\mathbf{x})} Q_{i,1,\mathbf{x}}.$$
Observe that
$$\operatorname{cl}(S_1^1) \setminus D(f_1) = \bigcup_{\mathbf{x} \in D(f_1)} \bigcup_{i \le i(1,1,\mathbf{x})} \operatorname{cl}(Q_{i,1,\mathbf{x}}),$$
and the family $\{Q_{i,1,\mathbf{x}};\ i \le i(1,1,\mathbf{x}) \text{ and } \mathbf{x} \in D(f_1)\}$ is *$\mathcal{P}$-locally finite*, i.e. for each $\mathbf{y} \in \mathbb{R}^m$ there is an $l \in \mathbb{N}$ such that the family of triples $(i, 1, \mathbf{x})$ with $\mathbf{x} \in D(f_1)$ and $Q_{i,1,\mathbf{x}} \cap P^l(\mathbf{y}) \ne \emptyset$ is finite.

Now, for each $\mathbf{x} \in D(f_1)$ we find the first positive integer $n(1,2,\mathbf{x})$ such that $\operatorname{diam}(P^{n(1,2,\mathbf{x})}(\mathbf{x})) < 1/2$ and
$$\mathbf{x} \in P^{n(1,2,\mathbf{x})}(\mathbf{x}) \subset P^1(\mathbf{x}) \setminus \operatorname{cl}(S_1^1).$$
There is then a finite family of cubes
$$Q_{1,n(1,2,\mathbf{x}),\mathbf{x}}, \ldots, Q_{i(1,n(1,2,\mathbf{x}),\mathbf{x}),n(1,2,\mathbf{x}),\mathbf{x}} \in \mathcal{P}$$
whose closures are pairwise disjoint and contained in $\operatorname{int}(P^{n(1,2,\mathbf{x})}(\mathbf{x})) \setminus D(f_1)$ and such that
$$\frac{\lambda(\bigcup_{i=1}^{i(1,n(1,2,\mathbf{x}),\mathbf{x})} Q_{i,n(1,2,\mathbf{x}),\mathbf{x}})}{\lambda(P^{n(1,2,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^2}.$$

Moreover, assume that if $n(1,2,\mathbf{x}) = n(1,2,\mathbf{y})$ for some $\mathbf{x},\mathbf{y} \in D(f_1)$, then $i(1,n(1,2,\mathbf{x}),\mathbf{x}) = i(1,n(1,2,\mathbf{y}),\mathbf{y})$ and $Q_{i,n(1,2,\mathbf{x}),\mathbf{x}} = Q_{i,n(1,2,\mathbf{y}),\mathbf{y}}$ for $i \le i(1,n(1,2,\mathbf{x}),\mathbf{x})$. Let

$$S_2^1 = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,n(1,2,\mathbf{x}),\mathbf{x})} Q_{i,n(1,2,\mathbf{x}),\mathbf{x}}.$$

Observe that

$$\mathrm{cl}(S_2^1) \setminus D(f_1) = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,n(1,2,\mathbf{x}),\mathbf{x})} \mathrm{cl}(Q_{i,n(1,2,\mathbf{x}),\mathbf{x}})$$

and the family $\{Q_{i,n(1,2,\mathbf{x}),\mathbf{x}};\ i \le i(1,n(1,2,\mathbf{x}),\mathbf{x})$ and $\mathbf{x} \in D(f_1)\}$ is $\mathcal{P}$-locally finite.

For $j > 2$, we proceed analogously and for each $\mathbf{x} \in D(f_1)$ we find the first positive integer $n(1,j,\mathbf{x})$ such that $\mathrm{diam}(P^{n(1,j,\mathbf{x})}(\mathbf{x})) < 1/2^{j-1}$ and

$$\mathbf{x} \in P^{n(1,j,\mathbf{x})}(\mathbf{x}) \subset P^{n(1,j-1,\mathbf{x})}(\mathbf{x}) \setminus \mathrm{cl}(S_{j-1}^1).$$

There is then a finite family of cubes

$$Q_{1,n(1,j,\mathbf{x}),\mathbf{x}}, \dots, Q_{i(1,n(1,j,\mathbf{x}),\mathbf{x}),n(1,j,\mathbf{x}),\mathbf{x}} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{n(1,j,\mathbf{x})}(\mathbf{x})) \setminus D(f_1)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,n(1,j,\mathbf{x}),\mathbf{x})} Q_{i,n(1,j,\mathbf{x}),\mathbf{x}})}{\lambda(P^{n(1,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Moreover, assume that if $n(1,j,\mathbf{x}) = n(1,j,\mathbf{y})$ for some $\mathbf{x},\mathbf{y} \in D(f_1)$, then $i(1,n(1,j,\mathbf{x}),\mathbf{x}) = i(1,n(1,j,\mathbf{y}),\mathbf{y})$ and $Q_{i,n(1,j,\mathbf{x}),\mathbf{x}} = Q_{i,n(1,j,\mathbf{y}),\mathbf{y}}$ for $i \le i(1,n(1,j,\mathbf{x}),\mathbf{x})$. Let

$$S_j^1 = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,n(1,j,\mathbf{x}),\mathbf{x})} Q_{i,n(1,j,\mathbf{x}),\mathbf{x}}.$$

Then

$$\mathrm{cl}(S_j^1) \setminus D(f_1) = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,n(1,j,\mathbf{x}),\mathbf{x})} \mathrm{cl}(Q_{i,n(1,j,\mathbf{x}),\mathbf{x}})$$

and the family $\{Q_{i,n(1,j,\mathbf{x}),\mathbf{x}};\ i \le i(1,n(1,j,\mathbf{x}),\mathbf{x})$ and $\mathbf{x} \in D(f_1)\}$ is $\mathcal{P}$-locally finite.

Let N_l, $l \in \mathbb{Z}$, be pairwise disjoint infinite sets of positive integers such that

$$\mathbb{N} = \bigcup_{l\in\mathbb{Z}} N_l.$$

Observe that for each integer l and for each $\mathbf{x} \in D(f_1)$,

$$d_u\Big(\bigcup_{j\in N_l} \mathrm{int}(S_j^1), \mathbf{x}\Big) = 1.$$

Let

$$g_1(\mathbf{x}) = \begin{cases} kc/2 & \text{if } \mathbf{x} \in S_j^1,\ j \in N_{2k}, \\ f_1(\mathbf{x}) & \text{elsewhere on } \mathbb{R}^m, \end{cases}$$

and let

$$h_1(\mathbf{x}) = f_1(\mathbf{x}) - g_1(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^m.$$

Observe that $f_1 = g_1 + h_1$. Moreover, for each $\mathbf{x} \in \mathbb{R}^m$,

$$(*) \qquad d_u(\operatorname{int}(g_1^{-1}(g_1(\mathbf{x}))), \mathbf{x}) = 1, \qquad d_u(\operatorname{int}(h_1^{-1}(h_1(\mathbf{x}))), \mathbf{x}) = 1.$$

Indeed, if $\mathbf{x} \in D(f_1)$ and $f_1(\mathbf{x}) = (k-1)c/2$ then for each $j \in N_{2k-1}$ there is a cube $P^{n(1,j,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$. But

$$\frac{\lambda(S_j^1 \cap P^{n(1,j,\mathbf{x})}(\mathbf{x}))}{\lambda(P^{n(1,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}$$

and $g_1(\mathbf{x}) = f_1(\mathbf{x})$, so $d_u(\operatorname{int}(g^{-1}((k-1)c/2)), \mathbf{x}) = 1$.

If $\mathbf{x} \in S_j^1$ for some $j \in \mathbb{N}$, then from the construction of g_1 it follows that $d_u(\operatorname{int}((g_1)^{-1}(g_1(\mathbf{x}))), \mathbf{x}) = 1$.

If $\mathbf{x} \in \mathbb{R}^m \setminus (D(f_1) \cup \bigcup_j S_j^1)$ then $d_u((g_1)^{-1}(g_1(\mathbf{x}))), \mathbf{x}) = 1$, since f_1 is continuous at $\mathbf{x}$.

If $\mathbf{x} \in D(f_1)$ then $h_1(\mathbf{x}) = 0$. Since $\bigcup_{k\in\mathbb{Z}} \bigcup_{j \in N_{2k-1}} S_j^1 \subset (h_1)^{-1}(0)$, we have $d_u(h_1^{-1}(h_1(\mathbf{x}))), \mathbf{x}) = 1$.

If $\mathbf{x} \notin D(f_1)$ then $d_u(\operatorname{int}(h_1^{-1}(h_1(\mathbf{x}))), \mathbf{x}) = 1$, since h_1 is the difference of the function f_1 continuous at $\mathbf{x}$ and the function g_1 such that $d_u(\operatorname{int}(g_1^{-1}(g_1(\mathbf{x}))), \mathbf{x}) = 1$ for each $\mathbf{x} \in \mathbb{R}^m$.

From $(*)$ it follows that g_1 and h_1 are s.q.c.

Step n $(n \geq 2)$. For a set $A \subset \mathbb{R}^m$ and $\eta > 0$ let

$$\mathcal{O}(A, \eta) = \bigcup_{\mathbf{x}\in A} B(\mathbf{x}, \eta), \quad \text{where} \quad B(\mathbf{x}, \eta) = \{\mathbf{t} \in \mathbb{R}^m;\ |\mathbf{t} - \mathbf{x}| < \eta\}.$$

Assume that there are two functions $g_{n-1}, h_{n-1} : \mathbb{R}^m \to \mathbb{R}$ such that:

- $f_{n-1} = g_{n-1} + h_{n-1}$;
- $g_{n-1}(\mathbb{R}^m) \cup h_{n-1}(\mathbb{R}^m) \subset \{kc/2^{n-1};\ k \in \mathbb{Z}\}$;
- for each $\mathbf{x} \in \mathbb{R}^m$,
  $$d_u(\operatorname{int}(g_{n-1}^{-1}(g_{n-1}(\mathbf{x}))), \mathbf{x}) = 1, \qquad d_u(\operatorname{int}(h_{n-1}^{-1}(h_{n-1}(\mathbf{x}))), \mathbf{x}) = 1.$$

For $k \in \mathbb{Z}$ let

$$G_{n,k} = g_{n-1}^{-1}(kc/2^n) \cap D(f_n), \qquad H_{n,k} = h_{n-1}^{-1}(kc/2^n) \cap D(f_n).$$

(1) If $G_{n,-2^n} \neq \emptyset$ then as in the first step, for each $\mathbf{x} \in G_{n,-2^n}$ we find the first positive integer $r(n, 1, -2^n, \mathbf{x})$ such that $\mathbf{x} \in P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x}) \subset \mathcal{O}(G_{n,-2^n}, 1/2^n)$ and

$$\frac{\lambda(P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x}) \cap g_{n-1}^{-1}(-c))}{\lambda(P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}.$$

Moreover, if $\mathbf{x} \in D(f_n) \setminus D(f_{n-1})$ then $P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x}) \cap D(f_{n-1}) = \emptyset$.

There is a finite family of cubes

$$Q_{1,r(n,1,-2^n,\mathbf{x}),\mathbf{x}},\ Q_{2,r(n,1,-2^n,\mathbf{x}),\mathbf{x}},\dots,\ Q_{i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x}),r(n,1,-2^n,\mathbf{x}),\mathbf{x}} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in

$$(\mathrm{int}(P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x})) \cap g_{n-1}^{-1}(-c)) \setminus D(f_n)$$

and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x})} Q_{i,r(n,1,-2^n,\mathbf{x}),\mathbf{x}})}{\lambda(P^{r(n,1,-2^n,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}.$$

Moreover, we assume that if $\mathbf{y} \in P^{r(n,1,-2^n,\mathbf{x})}$ for some $\mathbf{x},\mathbf{y} \in G_{n,-2^n}$ then $i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x}) = i(1,r(n,1,-2^n,\mathbf{y}),\mathbf{y})$ and $Q_{i,r(n,1,-2^n,\mathbf{x}),\mathbf{x}} = Q_{i,r(n,1,-2^n,\mathbf{y}),\mathbf{y}}$ for $i \le i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x})$. Let

$$S_1^{n,-2^n} = \bigcup_{\mathbf{x}\in G_{n,-2^n}} \ \bigcup_{i\le i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x})} Q_{i,r(n,1,-2^n,\mathbf{x}),\mathbf{x}}.$$

Observe that

$$\mathrm{cl}(S_1^{n,-2^n}) \setminus G_{n,-2^n} = \bigcup_{\mathbf{x}\in G_{n,-2^n}} \ \bigcup_{i\le i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x})} \mathrm{cl}(Q_{i,r(n,1,-2^n,\mathbf{x}),\mathbf{x}})$$

and the family $\{Q_{i,r(n,1,-2^n,\mathbf{x}),\mathbf{x}};\ i \le i(1,r(n,1,-2^n,\mathbf{x}),\mathbf{x}) \text{ and } \mathbf{x} \in G_{n,-2^n}\}$ is $\mathcal{P}$-locally finite.

If $G_{n,-2^n} = \emptyset$ then we put $S_1^{n,-2^n} = \emptyset$.

Next, fix $k \in (-2^n,\ 2^n]$. If $G_{n,k} \ne \emptyset$ then for each $\mathbf{x} \in G_{n,k}$ we find the first positive integer $r(n,1,k,\mathbf{x})$ such that

$$\mathbf{x} \in P^{r(n,1,k,\mathbf{x})}(\mathbf{x}) \subset \mathcal{O}(G_{n,k}, 1/2^n) \setminus \bigcup_{-2^n\le i<k} \mathrm{cl}(S_1^{n,i}),$$

$$\frac{\lambda(P^{r(n,1,k,\mathbf{x})}(\mathbf{x}) \cap g_{n-1}^{-1}(kc/2^n))}{\lambda(P^{r(n,1,k,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}$$

and moreover, if $\mathbf{x} \in D(f_n) \setminus D(f_{n-1})$ then $P^{r(n,1,k,\mathbf{x})}(\mathbf{x}) \cap D(f_{n-1}) = \emptyset$.

There is a finite family of cubes

$$Q_{1,r(n,1,k,\mathbf{x}),\mathbf{x}},\ Q_{2,r(n,1,k,\mathbf{x}),\mathbf{x}},\dots,Q_{i(1,r(n,1,k,\mathbf{x}),\mathbf{x}),r(n,1,k,\mathbf{x}),\mathbf{x}} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in

$$(\mathrm{int}(P^{r(n,1,k,\mathbf{x})}(\mathbf{x})) \cap g_{n-1}^{-1}(kc/2^n)) \setminus D(f_n)$$

and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,r(n,1,k,\mathbf{x}),\mathbf{x})} Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}})}{\lambda(P^{r(n,1,k,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}.$$

Moreover, we assume that if $\mathbf{y} \in P^{r(n,1,k,\mathbf{x})}(\mathbf{x})$ for some $\mathbf{x}, \mathbf{y} \in G_{n,k}$ then $i(1, r(n,1,k,\mathbf{x}), \mathbf{x}) = i(1, r(n,1,k,\mathbf{y}), \mathbf{y})$ and $Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}} = Q_{i,r(n,1,k,\mathbf{y}),\mathbf{y}}$ for $i \leq i(1, r(n,1,k,\mathbf{x}), \mathbf{x})$. Let

$$S_1^{n,k} = \bigcup_{\mathbf{x} \in G_{n,k}} \bigcup_{i \leq i(1,r(n,1,k,\mathbf{x}),\mathbf{x})} Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}}$$

and observe that

$$\mathrm{cl}(S_1^{n,k}) \setminus G_{n,k} = \bigcup_{\mathbf{x} \in G_{n,k}} \bigcup_{i \leq i(1,r(n,1,k,\mathbf{x}),\mathbf{x})} \mathrm{cl}(Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}}).$$

Also observe that the family $\{Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}};\ i \leq i(1, r(n,1,k,\mathbf{x}), \mathbf{x})$ and $\mathbf{x} \in G_{n,k}\}$ is $\mathcal{P}$-locally finite.

If $G_{n,k} = \emptyset$ then we put $S_1^{n,k} = \emptyset$. Now, let

$$S_1^n = \bigcup_{-2^n \leq k \leq 2^n} S_1^{n,k}.$$

(1′) Analogously, if $H_{n,k} \neq \emptyset$ for $k \in [-2^n,\ 2^n]$ then as in (1) for the sets $H_{n,k}$ we construct sets $K_1^{n,k}$ and

$$K_1^n = \bigcup_{-2^n \leq k \leq 2^n} K_1^{n,k}$$

which are counterparts of $S_1^{n,k}$ and S_1^n constructed in (1) for the sets $G_{n,k}$, contained in the complement of S_1^n and having analogous properties.

(2) For $j \geq 2$ and $k \in [-2^{n+j-1},\ 2^{n+j-1}]$ with $G_{n,k} \neq \emptyset$ we find families of cubes $Q_{1,r(n,j,k,\mathbf{x}),\mathbf{x}},\ Q_{2,r(n,j,k,\mathbf{x}),\mathbf{x}}, \ldots, Q_{i(1,r(n,j,k,\mathbf{x}),\mathbf{x}),r(n,j,k,\mathbf{x}),\mathbf{x}} \in \mathcal{P}$ whose closures are pairwise disjoint and contained in

$$(\mathrm{int}(P^{r(n,j,k,\mathbf{x})}(\mathbf{x})) \cap g_{n-1}^{-1}(kc/2^n)) \setminus D(f_n)$$

and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,r(n,j,k,\mathbf{x}),\mathbf{x})} Q_{i,r(n,1,k,\mathbf{x}),\mathbf{x}})}{\lambda(P^{r(n,j,k,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Let

$$S_j^{n,k} = \bigcup_{\mathbf{x} \in G_{n,k}} \bigcup_{i \leq i(1,r(n,j,k,\mathbf{x}),\mathbf{x})} Q_{i,r(n,j,k,\mathbf{x}),\mathbf{x}}.$$

Then

$$\mathrm{cl}(S_j^{n,k}) \setminus G_{n,k} = \bigcup_{\mathbf{x} \in G_{n,k}} \bigcup_{i \leq i(1,r(n,j,k,\mathbf{x}),\mathbf{x})} \mathrm{cl}(Q_{i,r(n,j,k,\mathbf{x}),\mathbf{x}}).$$

Also let

$$S_j^n = \bigcup_{-2^{n+j-1} \leq k \leq 2^{n+j-1}} S_j^{n,k}.$$

If $G_{n,k} = \emptyset$ then we put $S_1^{n,k} = \emptyset$.

(2′) Now, if $H_{n,k} \neq \emptyset$ for $k \in [-2^{n+j-1}, 2^{n+j-1}]$ and $j \geq 2$, then for the sets $H_{n,k}$ we construct sets $K_j^{n,k}$ and

$$K_j^n = \bigcup_{-2^{n+j-1} \leq k \leq 2^{n+j-1}} K_j^{n,k}$$

which are counterparts of $S_j^{n,k}$ and S_j^n constructed in (2) for the sets $G_{n,k}$, contained in the complement of S_j^n and having analogous properties.

From the above construction it follows that for every $k \in \mathbb{Z}$ we have

- $d_u(\bigcup_{j \in N_k} \operatorname{int}(S_j^n), \mathbf{x}) = 1$ for each $\mathbf{x} \in G_{n,k}$,
- $d_u(\bigcup_{j \in N_k} \operatorname{int}(K_j^n), \mathbf{x}) = 1$ for each $\mathbf{x} \in H_{n,k}$.

Finally, we define $g_n, h_n : \mathbb{R}^m \to \mathbb{R}$ as follows:

1) $g_n(\mathbf{x}) = g_{n-1}(\mathbf{x}) + c/2^n$ for $\mathbf{x} \in S_j^n,\ j \in N_{2k},\ k \in \mathbb{Z}$;
2) $h_n(\mathbf{x}) = h_{n-1}(\mathbf{x}) - c/2^n$ for $\mathbf{x} \in K_j^n,\ j \in N_{2k-1},\ k \in \mathbb{Z}$;
3) for $\mathbf{x} \in \mathbb{R}^m \setminus \bigcup_{k \in \mathbb{Z}} (\bigcup_{j \in N_{2k}} S_j^n \cup \bigcup_{j \in N_{2k-1}} K_j^n)$ let

$$g_n(\mathbf{x}) = g_{n-1}(\mathbf{x}) + (f_n(\mathbf{x}) - f_{n-1}(\mathbf{x}));$$

4) $h_n(\mathbf{x}) = f_n(\mathbf{x}) - g_n(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

As in the first step, we verify that g_n, h_n are s.q.c. Moreover, observe that:

- by 1) and 3), for $\mathbf{x} \in \mathbb{R}^m \setminus \bigcup_{k \in \mathbb{Z}} \bigcup_{j \in N_{2k-1}} K_j^n$ we have

$$|g_n(\mathbf{x}) - g_{n-1}(\mathbf{x})| \leq c/2^n;$$

- by 4) (and 2)), for $\mathbf{x} \in K_j^n,\ j \in N_{2k-1},\ k \in \mathbb{Z}$ we have

$$\begin{aligned} |g_n(\mathbf{x}) - g_{n-1}(\mathbf{x})| &= |(f_n(\mathbf{x}) - h_n(\mathbf{x})) - g_{n-1}(\mathbf{x})| \\ &= |(f_n(\mathbf{x}) - h_{n-1}(\mathbf{x}) + c/2^n) - g_{n-1}(\mathbf{x})| \\ &= |f_n(\mathbf{x}) - (f_{n-1}(\mathbf{x}) - g_{n-1}(\mathbf{x})) + c/2^n - g_{n-1}(\mathbf{x})| \\ &\leq |f_n(\mathbf{x}) - f_{n-1}(\mathbf{x})| + c/2^n < c/2^{n-1}. \end{aligned}$$

So, $|g_n - g_{n-1}| < c/2^{n-1}$ everywhere on $\mathbb{R}^m$. Similarly we can check that $|h_n - h_{n-1}| < c/2^{n-2}$ everywhere on $\mathbb{R}^m$.

The sequences $(g_n)_n$ and $(h_n)_n$ uniformly converge to some functions g and h respectively, which, by Remark 1, are s.q.c. Moreover,

$$g + h = \lim_{n\to\infty} g_n + \lim_{n\to\infty} h_n = \lim_{n\to\infty} (g_n + h_n) = \lim_{n\to\infty} f_n = f.$$

This finishes the proof.

REMARK 3. If the function f from Theorem 1 is of Baire class α ($\alpha > 0$) then the functions g, h can be found in the same class.

REMARK 4. From the proof of Theorem 1 it follows immediately that if U is an open set in $\mathbb{R}^m$ and if $f : U \to \mathbb{R}$ is an almost everywhere continuous function then there are s.q.c. functions $g, h : U \to \mathbb{R}$ such that $f = g + h$.

REMARK 5. If $U \subset \mathbb{R}^m$ is a nonempty open set and $f : U \to \mathbb{R}$ is a s.q.c. function then for each cube $P \in \mathcal{P}$ the restricted functions $f|_P$ and $f|_{\mathbb{R}^m \setminus P}$ are also s.q.c. at all points of their domains.

Now we will investigate products of s.q.c. functions.

THEOREM 2. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be an almost everywhere continuous function such that $\lambda(\mathrm{cl}(f^{-1}(0)) \setminus \mathrm{int}(f^{-1}(0))) = 0$. Then there are s.q.c. functions g, h such that $f = g \cdot h$.*

Proof. Set $A = \{\mathbf{x};\ f(\mathbf{x}) > 0\}$, $B = \{\mathbf{x};\ f(\mathbf{x}) < 0\}$ and observe that

$$\lambda(\mathbb{R}^m \setminus (\mathrm{int}(A) \cup \mathrm{int}(B) \cup \mathrm{int}(f^{-1}(0))) = 0.$$

If $\mathrm{int}(A) \neq \emptyset$ and if O is a component of $\mathrm{int}(A)$ then $\mathbf{x} \mapsto \ln(f(\mathbf{x}))$ for $\mathbf{x} \in O$ is an almost everywhere continuous function, and by Theorem 1, there are s.q.c. functions $g_O, h_O : O \to \mathbb{R}$ such that $\ln(f(\mathbf{x})) = g_O(\mathbf{x}) + h_O(\mathbf{x})$ for $\mathbf{x} \in O$. Consequently, $f|_O = (e^{\ln(f)})|_O = e^{g|_O} \cdot e^{h|_O}$ is the product of two s.q.c. functions.

Analogously, if $\mathrm{int}(B) \neq \emptyset$ and if O' is a component of $\mathrm{int}(B)$ then $-f|_{O'}$ is the product of two s.q.c. functions, and consequently, so is $f|_{O'}$. Hence, there are s.q.c. functions $g_1, h_1 : (\mathrm{int}(A) \cup \mathrm{int}(B)) \to \mathbb{R}$ such that $f|_{\mathrm{int}(A) \cup \mathrm{int}(B)} = g_1 \cdot h_1$.

Now, let

$$F = \{\mathbf{x} \in \mathrm{cl}(\mathrm{int}(f^{-1}(0))) \setminus \mathrm{int}(f^{-1}(0));\ d_u(\mathrm{int}(f^{-1}(0)), \mathbf{x}) > 0\}.$$

As in the proof of Theorem 1, we can prove that there is a $\mathcal{P}$-locally finite (in $\mathrm{int}(f^{-1}(0))$) family of cubes $Q_i^j \in \mathcal{P}$ ($i, j \in \mathbb{N}$) whose closures $\mathrm{cl}(Q_i^j)$ are pairwise disjoint and contained in $\mathrm{int}(f^{-1}(0))$ and such that:

- if $S^j = \bigcup_{i \in \mathbb{N}} Q_i^j$ then $\mathrm{cl}(S^j) = \bigcup_{i \in \mathbb{N}} \mathrm{cl}(Q_i^j)$;
- the sequence $(\mathrm{cl}(S^j))_j$ converges in the Hausdorff metric to $\mathrm{cl}(F)$;
- for each infinite set $N_0 \subset \mathbb{N}$ and for each $\mathbf{x} \in F$,

$$d_u\Big(\bigcup_{j \in N_0} S^j, \mathbf{x}\Big) > 0.$$

Let $\{N_{k,l}\}$ be a family of pairwise disjoint infinite subsets of $\mathbb{N}$ such that

$$\mathbb{N} = \bigcup_{k,l=1}^{\infty} N_{k,l}$$

and let $(w_n)_n$ be a one-to-one enumeration of all non-zero rationals.

Since the boundaries $\mathrm{Fr}(A)$ and $\mathrm{Fr}(B)$ are of λ-measure zero, similarly we can prove that there is a locally finite (in $\mathrm{int}(A) \cup \mathrm{int}(B)$) family of cubes

$W_i^j \in \mathcal{P}$ whose closures are pairwise disjoint and contained in $\operatorname{int}(A) \cup \operatorname{int}(B)$ and such that:

- if $V^j = \bigcup_{i\in\mathbb{N}} W_j^j$ then $\operatorname{cl}(V^j) = \bigcup_{i\in\mathbb{N}} \operatorname{cl}(W_i^j)$;
- $(\operatorname{cl}(V^j))_j$ converges to $\operatorname{cl}((\operatorname{Fr}(A) \cup \operatorname{Fr}(B)) \setminus F)$ in the Hausdorff metric;
- for each infinite set $N_0 \subset \mathbb{N}$ and for each $\mathbf{x} \in (\operatorname{Fr}(A) \cup \operatorname{Fr}(B)) \setminus F$,

$$d_u\Big(\bigcup_{j\in N_0} V^j, \mathbf{x}\Big) > 0.$$

Since g_1 is almost everywhere continuous on its domain, in each cube W_i^j we can find a finite family of cubes

$$U_{i,1}^j, \ldots, U_{i,k(i,j)}^j \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in $\operatorname{int}(W_i^j)$ and such that for every cube $U_{i,k}^j$ there is a positive real $r(i,j,k)$ such that:

- $|g_1(\mathbf{x})| > r(i,j,k)$ for $\mathbf{x} \in \operatorname{cl}(U_{i,k}^j)$, $i, j \in \mathbb{N}$ and $k \le k(i,j)$;
- $\operatorname{osc}_{\operatorname{cl}(U_{i,k}^j)} g_1 < r(i,j,k)/jw_l$ for $i, l \in \mathbb{N}$, $j \in N_{l,i}$, $k \le k(i,j)$;
- for each $\mathbf{x} \in (\operatorname{Fr}(A) \cup \operatorname{Fr}(B)) \setminus F$ and for each infinite set $N_0 \subset \mathbb{N}$,

$$d_u\Big(\bigcup_{j\in N_0} \bigcup_{i\in\mathbb{N}} \bigcup_{k\le k(i,j)} U_{i,k}^j, \mathbf{x}\Big) = d_u\Big(\bigcup_{j\in N_0} V^j, \mathbf{x}\Big) > 0.$$

Fix a point $\mathbf{x}_{i,k}^j$ in each cube $\operatorname{int}(U_{i,k}^j)$, $i, j \in \mathbb{N}$, $k \le k(i,j)$. Put

$$g(\mathbf{x}) = \begin{cases} w_k & \text{if } \mathbf{x} \in S^n,\ n \in N_{2k,1},\ k \in \mathbb{N},\\ 0 & \text{if } \mathbf{x} \in S^n,\ n \in N_{2k-1,1},\ k \in \mathbb{N},\\ 0 & \text{elsewhere on } f^{-1}(0),\\ \dfrac{g_1(\mathbf{x})w_j}{g_1(\mathbf{x}_{i,k}^n)} & \text{if } \mathbf{x} \in U_{i,k}^n,\ i \in \mathbb{N},\ n \in N_{j,1},\ k \le k(i,n),\\ g_1(\mathbf{x}) & \text{elsewhere on } \operatorname{int}(A) \cup \operatorname{int}(B),\\ f(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbb{R}^m \setminus (\operatorname{int}(A) \cup \operatorname{int}(B) \cup f^{-1}(0)), \end{cases}$$

and

$$h(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in S^n,\ n \in N_{2k,1},\ k \in \mathbb{N},\\ 1 & \text{if } \mathbf{x} \in S^n,\ n \in N_{2k-1,1},\ k \in \mathbb{N},\\ 0 & \text{elsewhere on } f^{-1}(0),\\ \dfrac{h_1(\mathbf{x})g_1(\mathbf{x}_{i,k}^n)}{w_j} & \text{if } \mathbf{x} \in U_{i,k}^n,\ i \in \mathbb{N},\ n \in N_{j,1},\ k \le k(i,n),\\ h_1(\mathbf{x}) & \text{elsewhere on } \operatorname{int}(A) \cup \operatorname{int}(B),\\ 1 & \text{if } \mathbf{x} \in \mathbb{R}^m \setminus (\operatorname{int}(A) \cup \operatorname{int}(B) \cup f^{-1}(0)). \end{cases}$$

Since $g(U^n_{i,k}) \subset (w_j - 1/j, w_j + 1/j)$ for all $i, j \in \mathbb{N}$, $n \in N_{j,1}$, $k \le k(i,n)$, and for each infinite set $N_0 \subset \mathbb{N}$ we have

$$d_u\Big(\bigcup_{j\in N_0} U^j_{i,k},\ \mathbf{x}\Big) > 0$$

for each

$$\mathbf{x} \in H = \mathbb{R}^m \setminus (\mathrm{int}(A) \cup \mathrm{int}(B) \cup \mathrm{int}(f^{-1}(0)) \cup F),$$

the function g is s.q.c. at every $\mathbf{x} \in H$.

Evidently, it is also s.q.c. elsewhere on $\mathbb{R}^m$. Analogously, h is s.q.c. Obviously, $f = g \cdot h$ and the proof is complete.

THEOREM 3. *If $f : \mathbb{R}^m \to \mathbb{R}$ is an almost everywhere continuous function then there are a constant $c \in \mathbb{R}$ and two s.q.c. functions g, h such that $f = c + g \cdot h$.*

Proof. Let $c \in \mathbb{R}$ be such that

$$\lambda(\mathrm{cl}(f^{-1}(c))) = 0.$$

Then the function $f_1 = f - c$ satisfies the assumptions of Theorem 2, and consequently, there are s.q.c. functions g, h such that $f_1 = g\cdot h$. So, $f = c+g\cdot h$ and the proof is finished.

REMARK 6. If $f : \mathbb{R}^m \to \mathbb{R}$ is the product of a finite family of s.q.c. functions f_i, where $i \le n$, then f satisfies the following condition:

(H) if $A \subset \mathrm{cl}(f^{-1}(0)) \setminus f^{-1}(0)$ is such that $d_l(f^{-1}(0), \mathbf{x}) = 1$ for each $\mathbf{x} \in A$ then the set A is nowhere dense in $f^{-1}(0)$.

Proof. I repeat the proof of Remark 5 from [6]. Let

$$B = \{\mathbf{x};\ f(\mathbf{x}) = 0 \text{ and } d_l(f^{-1}(0), \mathbf{x}) = 1\}.$$

If $B \ne \emptyset$ and A is not nowhere dense in $f^{-1}(0)$, then there is $\mathbf{x} \in A$ and a positive integer $i \le n$ such that $\mathbf{x}$ is a density point of $f_i^{-1}(0)$. Since $f_i(\mathbf{x}) \ne 0$ and f_i is s.q.c. at $\mathbf{x}$, we obtain a contradiction. If $B = \emptyset$ then $A = \emptyset$ and the proof is complete.

Let $(w_n)_n$ be a sequence of all rationals. From the last remark it follows that the function

$$f(x_1, \ldots, x_m) = \begin{cases} 1/n & \text{for } x_1 = w_n, \\ 0 & \text{elsewhere on } \mathbb{R}^m, \end{cases}$$

is almost everywhere continuous, but it is not the product of any finite family of s.q.c. functions.

Each strongly quasicontinuous function is also quasicontinuous in the sense of Kempisty.

Recall ([9], [10]) that a function $f : \mathbb{R}^m \to \mathbb{R}$ is *quasicontinuous* at a point $\mathbf{x} \in \mathbb{R}^m$ (in the sense of Kempisty) if for each $\varepsilon > 0$ and each open set $U \ni \mathbf{x}$ there is a nonempty open set $V \subset U$ such that $f(V) \subset (f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)$.

This is a purely topological notion while the notion of almost continuity is a measure-theoretic one. The sums and products of finitely many quasicontinuous real functions on $\mathbb{R}^m$ are *cliquish* functions, i.e. are discontinuous only at points of some first category sets ([10]). In Borsík's articles [1], [2] it is proved that each cliquish function $f : \mathbb{R}^m \to \mathbb{R}$ is the sum of two quasicontinuous functions. The results of the present article have similar corollaries:

COROLLARY 1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a cliquish function. Then*

$$f = f_1 + f_2 \quad \textit{and} \quad f = c + f_3 \cdot f_4,$$

where $c \in \mathbb{R}$ is a constant and $f_1,\ f_2,\ f_3,\ f_4$ are quasicontinuous functions.

Proof. By [11] and [12] there is a homeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda(h^{-1}(D(f))) = 0$. The function $\phi(\mathbf{x}) = f(h(\mathbf{x}))$, $\mathbf{x} \in \mathbb{R}^m$, is almost everywhere continuous. So, by Theorems 1 and 3 there are a constant $c \in \mathbb{R}$ and strongly quasicontinuous functions $\phi_1, \phi_2, \phi_3, \phi_4 : \mathbb{R}^m \to \mathbb{R}$ such that

$$\phi = \phi_1 + \phi_2 \quad \text{and} \quad \phi = c + \phi_3 \cdot \phi_4.$$

Observe that for $i = 1, 2, 3, 4$ the functions $f_i = \phi_i \circ h^{-1}$ are quasicontinuous,

$$f = \phi \circ h^{-1} = \phi_1 \circ h^{-1} + \phi_2 \circ h^{-1} = f_1 + f_2$$

and

$$f = f \circ h^{-1} = c + (\phi_3 \circ h^{-1}) \cdot (\phi_4 \circ h^{-1}) = c + f_3 \cdot f_4.$$

This finishes the proof.

COROLLARY 2. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a cliquish function such that the set $\mathrm{cl}(f^{-1}(0)) \setminus \mathrm{int}(f^{-1}(0))$ is nowhere dense in $\mathbb{R}^m$. Then there are two quasicontinuous functions $f_1, f_2 : \mathbb{R}^m \to \mathbb{R}$ with $f = f_1 \cdot f_2$.*

Proof. As in the proof of Corollary 1, by [11] and [12], there is a homeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda(h^{-1}(\mathrm{cl}(f^{-1}(0)) \setminus \mathrm{int}(f^{-1}(0)))) = 0$. Let $\phi(\mathbf{x}) = f(h(\mathbf{x}))$ for $\mathbf{x} \in \mathbb{R}^m$. Then

$$\begin{aligned}\lambda(\mathrm{cl}(\phi^{-1}(0)) \setminus \mathrm{int}(\phi^{-1}(0))) &= \lambda(\mathrm{cl}(f \circ h)^{-1}(0)) \setminus \mathrm{int}(f \circ h)^{-1}(0))) \\ &= \lambda(\mathrm{cl}(h^{-1}(f^{-1}(0))) \setminus \mathrm{int}(h^{-1}(f^{-1}(0)))) \\ &= \lambda(h^{-1}(\mathrm{cl}(f^{-1}(0)) \setminus \mathrm{int}(f^{-1}(0)))) = 0.\end{aligned}$$

So, by Theorem 2 there are strongly quasicontinuous functions $\phi_1, \phi_2 : \mathbb{R}^m \to \mathbb{R}$ such that $\phi = \phi_1 \cdot \phi_2$. Put $f_1 = \phi_1 \circ h$ and $f_2 = \phi_2 \circ h$ and observe that the functions f_1 and f_2 are quasicontinuous and

$$f = \phi \circ h = (\phi_1 \circ h) \cdot (\phi_2 \circ h) = f_1 \cdot f_2.$$

This completes the proof.

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