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ESTIMATES WITH GLOBAL RANGE FOR OSCILLATORY INTEGRALS WITH CONCAVE PHASE

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Abstract. We consider the maximal function $\|(S^a f)[x]\|_{L^\infty[-1,1]}$ where $(S^a f)(t)^\wedge(\xi) = e^{it|\xi|^a}\widehat{f}(\xi)$ and $0<a<1$. We prove the global estimate

$$\|S^a f\|_{L^2(\mathbb{R},L^\infty[-1,1])} \le C\|f\|_{H^s(\mathbb{R})}, \qquad s > a/4,$$

with C independent of f. This is known to be almost sharp with respect to the Sobolev regularity s.

1. Introduction. In several papers during the last couple of years interest has been focused on summability processes for oscillatory Fourier integrals. The kernels of these summability processes have the feature of being non-summable. An example is given by the integral representing solutions to the time-dependent Schrödinger equation

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\xi+t|\xi|^a)}\widehat{f}(\xi)\,d\xi, \qquad a=2, \qquad t\to 0.$$

Almost everywhere convergence results are established by first deriving a norm inequality for the associated maximal function. Typically, the maximal function is controlled by a Hilbert–Sobolev norm $\|f\|_{H^s(\mathbb{R}^n)}$. The optimal regularity s is equal to $1/4$ for all $a>1$ when $n=1$ (cf. e.g. Carleson [2], Dahlberg, Kenig [5] and Sjölin [8]). Recent results and references in the case $n>1$, where the problem is open, may be found in Moyua, Vargas, Vega [6].

Although it is enough to derive a local estimate for the maximal function, global estimates are of independent interest since they relate global regularity properties of the oscillatory integral to the regularity of the initial datum. The purpose of the present note is to derive such an estimate in the case $0<a<1$ (see Theorem 2.5 with proof in Section 4). This estimate is almost sharp, as shown by a previous counterexample found in [13].

For the specific problem of global maximal estimates in the case $a>0$ we have the following general theorem (cf. e.g. Cowling [3], Cowling, Mauceri [4],

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Rubio de Francia [7], Sogge, Stein [10], Stein [12, §XI.4.1, p. 511], and [14, Theorem 14.1, p. 215]), which is known to be sharp with respect to s in the case $a = 1$ (cf. e.g. [14, Theorem 14.2, p. 216]):

THEOREM. *Assume that the functions w_1 and w_2 belong to $L^2(\mathbb{R})$ and that the function m satisfies the following assumption: there is a number C independent of (t, ξ) such that*

$$|m(t,\xi)| \le Cw_1(t), \qquad |[\partial_t m](t,\xi)| \le C(w_1(t) + w_2(t)|\xi|^a), \qquad a > 0.$$

If $s > a/2$, then there is a number C independent of f such that

$$\Big(\int_{\mathbb{R}^n} \sup_{|t|\le 1} \Big|\int_{\mathbb{R}^n} e^{ix\xi} m(t,\xi)\widehat{f}(\xi)\,d\xi\Big|^2 dx\Big)^{1/2} \le C\|f\|_{H^s(\mathbb{R}^n)}.$$

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2. Notation and statement of the Theorem

2.1. Numbers denoted by C (sometimes with subscripts) may be different at each occurrence even within the same chain of inequalities.

Unless otherwise explicitly stated, all functions f and g are supposed to belong to $\mathcal{S}(\mathbb{R})$.

2.2. *Oscillatory integrals.* For real numbers x, ξ and t and $a > 0$ we let $\Phi_a(\xi, x, t) = x\xi + t|\xi|^a$. We define

$$(S^a f)[x](t) = \int_{\mathbb{R}} e^{i\Phi_a(\xi,x,t)}\widehat{f}(\xi)\,d\xi.$$

Here $\widehat{f}$ is the Fourier transform of f,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx.$$

2.3. *Auxiliary functions and smooth decomposition of Littlewood and Paley.* Let B denote the open unit ball in $\mathbb{R}$. We will use auxiliary functions χ and ψ such that $\chi \in C_0^\infty(\mathbb{R})$ is even,

$$\chi(\mathbb{R} \setminus 2B) = \{0\}, \qquad \chi(\mathbb{R}) \subseteq [0,1], \qquad \chi(B) = \{1\}$$

and $\psi = 1 - \chi$. From χ and ψ we obtain families of functions as follows: for $m > 1$ set $\chi_m(\xi) = \chi(\xi/m)$ and $\psi_m(\xi) = \psi(m\xi)$.

Throughout this paper N will denote a dyadic integer. We will use an even function $\eta \in C_0^\infty(\mathbb{R})$ such that

$$\eta(\mathbb{R}_+ \setminus [1/2, 2]) = \{0\}, \qquad \eta(\mathbb{R}) \subseteq [0,1].$$

We choose η so as to obtain a smooth decomposition of Littlewood and Paley, i.e.

$$\sum_{N>1}\eta(N\xi)+\sum_{N}\eta(\xi/N)=1,\qquad \xi\neq 0.$$

It is well known that this can be achieved. See e.g. Bergh, Löfström [1, Lemma 6.1.7, pp. 135–136].

2.4. *Function spaces.* Let X be a Banach space. For a measurable function $u:\mathbb{R}\to X$ we define

$$\|u\|_{L^p(\mathbb{R},X)}=\Big(\int_{\mathbb{R}}\|u(x)\|_X^p\,dx\Big)^{1/p}.$$

Whenever the integral is convergent we say that u belongs to $L^p(\mathbb{R},X)$. When this notation is used X will be equal to $L^\infty(I)$ for an interval $I\subset\mathbb{R}$.

For a real number s we introduce inhomogeneous fractional Sobolev spaces

$$H^s(\mathbb{R})=\Big\{f\in\mathcal{S}'(\mathbb{R}):\|f\|^2_{H^s(\mathbb{R})}=\int_{\mathbb{R}}\gamma_{2s}(\xi)|\widehat{f}(\xi)|^2\,d\xi<\infty\Big\}.$$

Here

$$\gamma_s(\xi)=\chi(\xi)+\sum_N N^s\eta(\xi/N).$$

Our definition of $H^s(\mathbb{R})$ is consistent with the definition found e.g. in Stein [11, Chapter V, §3.3, pp. 134–135] (there denoted by $\mathscr{L}^2_s(\mathbb{R})$) since there are numbers C_1 and C_2 independent of ξ such that

$$C_1(1+\xi^2)^{s/2}\le\gamma_{2s}(\xi)^{1/2}\le C_2(1+\xi^2)^{s/2}.$$

Note also that there are numbers C_1 and C_2 independent of ξ such that

$$C_1\gamma_{-s}(\xi)\le\gamma_s(\xi)^{-1}\le C_2\gamma_{-s}(\xi).\tag{2.1}$$

2.5. THEOREM. *Assume that $s>a/4$. Then there is a number C independent of f such that*

$$\|S^af\|_{L^2(\mathbb{R},L^\infty(\overline{B}))}\le C\|f\|_{H^s(\mathbb{R})}.\tag{2.2}$$

That the theorem is almost sharp with respect to the number of derivatives s on the right hand side of (2.2) follows from [13, Theorem 1.2(b), p. 486].

3. Some preparation

3.1. LEMMA. *Let φ be a smooth function with $\varphi''(\xi)\neq 0$ for $\xi\in\operatorname{supp}\eta$. Then there exists a number C independent of x and t such that*

$$\Big|\int_{\mathbb{R}}e^{i(x\xi+t\varphi(\xi))}\eta(\xi)\,d\xi\Big|\le C|t|^{-1/2},\qquad |t|\ge 1.$$

Remarks on the proof. The lemma is a consequence of Stein [12, Theorem 1, Chapter VII, p. 348] with $n = 2$.

3.2. LEMMA. *Let* $0 < a < 1$. *Then* $S^a\check{\chi} \in L^1(\mathbb{R}, L^\infty(\overline{2B}))$.

Proof. Let

$$m_{t,1}(\xi) = \Big(\sum_{k=0}^{M} \frac{(it|\xi|^a)^k}{k!}\Big)\chi(\xi). \tag{3.1}$$

Here we have chosen M so that $aM < 1 < a(M+1)$. Let $m_{t,2}(\xi) = \exp(it|\xi|^a)\chi(\xi) - m_{t,1}(\xi)$. From Sjölin [9, p. 110] (estimates for $I_{\mu,\beta,\gamma}$ with notation from the cited paper) it is clear that $\sup_{t\in\overline{2B}} |\check{m}_{t,2}| \in L^1(\mathbb{R})$. Hence it suffices to show that $\sup_{t\in\overline{2B}} |\check{m}_{t,1}| \in L^1(\mathbb{R})$.

Notice that the integral

$$\int_{-1}^{1} \sup_{t\in\overline{2B}} |\check{m}_{t,1}(x)|\, dx \tag{3.2}$$

is convergent. Hence, to show that $\sup_{t\in\overline{2B}} |\check{m}_{t,1}| \in L^1(\mathbb{R})$, in view of (3.1) it is enough to show that for $0 < a < 1$ there is a positive number C independent of x such that

$$\Big|\int_{\mathbb{R}} e^{ix\xi}|\xi|^a\chi(\xi)\, d\xi\Big| \le C|x|^{-1-a}, \qquad |x| \ge 1. \tag{3.3}$$

Now we invoke our family of auxiliary functions ψ_m to get

$$\begin{aligned}
\int_{\mathbb{R}} e^{ix\xi}|\xi|^a\chi(\xi)\, d\xi &= \lim_{m\to\infty}\int_{\mathbb{R}} e^{ix\xi}|\xi|^a\psi_m(\xi)\chi(\xi)\, d\xi \\
&= -\frac{1}{ix}\Big(\int_{\mathbb{R}} e^{ix\xi} a\,\mathrm{sgn}(\xi)|\xi|^{a-1}\chi(\xi)\, d\xi \\
&\quad + \int_{\mathbb{R}} e^{ix\xi}|\xi|^a\chi'(\xi)\, d\xi + \lim_{m\to\infty}\int_{\mathbb{R}} e^{ix\xi}|\xi|^a{\psi_m}'(\xi)\chi(\xi)\, d\xi\Big) \\
&= -\frac{1}{ix}(I_1(x) + I_2(x) + \lim_{m\to\infty} I_{3,m}(x))
\end{aligned}$$

where we have used dominated convergence and integration by parts. To show (3.3) it is sufficient to show that there is a number C independent of x such that

$$|I_1(x)| \le C|x|^{-a}, \qquad |x| \ge 1, \tag{3.4}$$

that I_2 decays rapidly at infinity and that $\lim_{m\to\infty} I_{3,m} = 0$.

3.2.1. *Estimate for* I_1. $I_1(x)$ is (apart from multiplication by a nonzero number C independent of x) the convolution at x of the inverse Fourier transform of the homogeneous function $h : \xi \mapsto \mathrm{sgn}(\xi)|\xi|^{a-1}$ with the function $\check{\chi} \in \mathcal{S}(\mathbb{R})$. Since h is odd and homogeneous of degree $a-1$ its inverse

Fourier transform $\check{h}$ is odd and homogeneous of degree $-a$. It follows that the convolution $\check{h} * \check{\chi} = I_1/C$ is bounded and continuous and that it satisfies the estimate (3.4).

3.2.2. *Estimate for I_2.* I_2 is (apart from a transposition) the Fourier transform of the $\mathcal{C}^\infty$-function $\xi \mapsto |\xi|^a \chi'(\xi)$ of compact support. Hence I_2 decays rapidly at infinity.

3.2.3. *Estimate for $I_{3,m}$.* Due to symmetry we have

$$I_{3,m}(x) = 2i \int_0^\infty \sin(x\xi)|\xi|^a \psi_m'(\xi)\chi(\xi)\, d\xi. \tag{3.5}$$

and using the support of ψ_m' we get the estimate

$$|I_{3,m}(x)| \leq 2\left|\int_{1/m}^{2/m} |\xi|^a \psi_m'(\xi)\, d\xi\right| \leq Cm^{-a} \tag{3.6}$$

where C is a number independent of m and x.

4. Proof of the Theorem

4.1. *Linearization of the maximal operator.* Fix $s > a/4$. To prove our theorem it is necessary and sufficient to prove that there is a number C independent of f such that

$$\left(\int_{\mathbb{R}} \sup_{t\in\overline{B}}\left|\int_{\mathbb{R}} e^{i\Phi_a(\xi,x,t)}\gamma_{-2s}(\xi)^{1/2} f(\xi)\, d\xi\right|^2 dx\right)^{1/2} \leq C\|f\|_{L^2(\mathbb{R})}. \tag{4.1}$$

This follows from the definition of $H^s(\mathbb{R})$, from (2.1) and from Parseval's formula. As in Sjölin [9], for a measurable function t with $t(\mathbb{R}) \subseteq \overline{B}$ we define

$$[R_t f](x) = \int_{\mathbb{R}} e^{i\Phi_a(\xi,x,t(x))}\gamma_{-2s}(\xi)^{1/2} f(\xi)\, d\xi.$$

To prove (4.1) it is sufficient to prove that R_t is bounded on $L^2(\mathbb{R})$ uniformly with respect to t.

4.2. *Approximation of R_t and reduction to kernel estimate.* To approximate R_t we define, for m and μ both greater than 1,

$$[R_{m\mu,t} f](x) = \int_{\mathbb{R}} \chi_m(x) e^{i\Phi_a(\xi,x,t(x))}\gamma_{-2s}(\xi)^{1/2}\chi_\mu(\xi) f(\xi)\, d\xi.$$

Because of the cutoffs both in range and frequency the boundedness on $L^2(\mathbb{R})$ for $R_{m\mu,t}$ can easily be verified. To reduce to a kernel estimate we study the adjoint given by

$$[R^*_{m\mu,t} g](\xi) = \int_{\mathbb{R}} \chi_m(x) e^{-i\Phi_a(\xi,x,t(x))}\gamma_{-2s}(\xi)^{1/2}\chi_\mu(\xi) g(x)\, dx.$$

We will prove that the operator $R^*_{m\mu,t}$ is bounded on $L^2(\mathbb{R})$ uniformly with respect to m, μ and t. Then $R_{m\mu,t}$ will be bounded on $L^2(\mathbb{R})$ uniformly with respect to m, μ and t. Since

$$[R_t f](x) = \lim_{m\to\infty} [R_{mm,t} f](x),$$

by Fatou's lemma we can conclude that R_t is bounded on $L^2(\mathbb{R})$ and that the bound is independent of t.

A computation involving Fubini's theorem shows that

$$\int_{\mathbb{R}} |[R^*_{m\mu,t} g](\xi)|^2 \, d\xi \le \iint_{\mathbb{R}^2} K_{m\mu}(x-x')|g(x)g(x')| \, dx \, dx', \tag{4.2}$$

where

$$K_{m\mu}(x) = \chi_{2m}(x) \sup_{t\in\overline{2B}} \left| \int_{\mathbb{R}} e^{i\Phi_a(\xi,x,t)} \gamma_{-2s}(\xi)\chi_\mu(\xi)^2 \, d\xi \right|.$$

(When $m=\infty$ we replace χ_{2m} by 1.) We shall prove that there is a number C independent of m and μ such that

$$\|K_{m\mu}\|_{L^1(\mathbb{R})} \le C. \tag{4.3}$$

Once this *kernel estimate* is proved, the desired uniform boundedness is immediate. See §4.4.

4.3. *Proof of the kernel estimate.* To show (4.3) it is enough to show that there is a number C independent of μ such that

$$\|K_{\infty\mu}\|_{L^1(\mathbb{R})} \le C. \tag{4.4}$$

Now we apply the smooth decomposition of Littlewood and Paley, referred to in §§2.3 and 2.4, expressed by the function γ_{-2s}. We define

$$L_0(x) = \|(S^a\check{\chi})[x]\|_{L^\infty(\overline{2B})},$$
$$L_N(x) = N^{-2s} \sup_{t\in\overline{2B}} \left| \int_{\mathbb{R}} e^{i\Phi_a(\xi,x,t)} \eta(\xi/N) \, d\xi \right|.$$

To show (4.4) it is then enough to show that

$$\|L_0\|_{L^1(\mathbb{R})} + \sum_N \|L_N\|_{L^1(\mathbb{R})} \tag{4.5}$$

is convergent.

A change of variables gives

$$L_N(x) = N^{1-2s} \sup_{t\in\overline{2B}} |(S^a\check{\eta})[Nx](N^a t)|.$$

4.3.1. *Estimate for $(S^a\check{\eta})[Nx](N^a t)$ when $t \in 2^{a-2}N^{1-a}|x|\overline{B}$.* It is clear that

$$\sup_{x\in\mathbb{R},\, t\in\overline{2B}} |(S^a\check{\eta})[Nx](N^a t)| \le \|\eta\|_{L^1(\mathbb{R})}. \tag{4.6}$$

Let

$$B_{N,x} = 2^{a-2} N^{1-a} |x| B.$$

Set $\Phi'_a = \partial_1 \Phi_a$. Our assumption on t together with $|\xi| \geq 1/2$ implies by the triangle inequality that

$$|\Phi'_a(\xi, Nx, N^a t)| \geq |Nx|/2. \tag{4.7}$$

In the integral

$$(S^a \check{\eta})[Nx](N^a t) = \int_{\mathbb{R}} e^{i\Phi_a(\xi, Nx, N^a t)} \eta(\xi)\, d\xi$$

we use integration by parts twice (cf. Stein [12, p. 331]). When we carry out the differentiation

$$\left(\frac{1}{i\Phi'_a} \left(\frac{\eta}{i\Phi'_a} \right)' \right)'$$

we find that there is a number C independent of N and x such that

$$\sup_{t \in \overline{B_{N,x}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)| \leq C(1 + |Nx|^2)^{-1} \tag{4.8}$$

taking (4.6), (4.7) and the support of η into account.

4.3.2. *Estimate for* $(S^a \check{\eta})[Nx](N^a t)$ *when* $t \in 2^{a-2} N^{1-a} |x| \overline{B}^{\mathrm{c}}$. From Lemma 3.1 with $\varphi(\xi) = |\xi|^a$, $\xi \in \operatorname{supp} \eta$, and our assumption on t in this case it follows that there is a number C independent of N and x such that

$$\sup_{t \in \overline{B_{N,x}}^{\mathrm{c}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)| \leq C |Nx|^{-1/2}. \tag{4.9}$$

4.3.3. *Estimate for* L_N. Let

$$B_N = 2^{3-a} N^{a-1} B.$$

We have

$$\begin{aligned} &\sup_{t \in \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)| \\ &\qquad \leq \sup_{t \in \overline{B_{N,x}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)| + \sup_{t \in \overline{B_{N,x}}^{\mathrm{c}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)|. \end{aligned}$$

Now we estimate $\|L_N\|_{L^1(\mathbb{R})}$. We get

$$\begin{aligned} &\int_{\mathbb{R}} \sup_{t \in \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)|\, dx \\ &\quad \leq \int_{\mathbb{R}} \sup_{t \in \overline{B_{N,x}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)|\, dx + \int_{B_N} \sup_{t \in \overline{B_{N,x}}^{\mathrm{c}} \cap \overline{2B}} |(S^a \check{\eta})[Nx](N^a t)|\, dx \\ &\quad \leq C \Big(\int_{\mathbb{R}} (1 + |Nx|^2)^{-1}\, dx + \int_{B_N} |Nx|^{-1/2}\, dx \Big) \leq C N^{a/2-1} \end{aligned}$$

where we have used (4.8) and (4.9) in the second last inequality. (The numbers C are independent of N.) In conclusion

$$\|L_N\|_{L^1(\mathbb{R})} \le CN^{1-2s}N^{a/2-1} = CN^{a/2-2s}. \tag{4.10}$$

Together with Lemma 3.2 this shows that the expression in (4.5) is finite.

4.4. *Conclusion.* Having proved our *kernel estimate* we now proceed from (4.2) as follows:

$$\begin{aligned}\int_{\mathbb{R}} |[R^*_{m\mu,t}g](\xi)|^2\,d\xi &\le \|K_{m\mu} * |g|\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\\ &\le \|K_{m\mu}\|_{L^1(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}\end{aligned}$$

where we have used Minkowski's inequality in the last inequality. We now conclude the proof by invoking (4.3).

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