COLLOQUIUM MATHEMATICUM

VOL. 91 2002 NO. 2

TOWARDS A THEORY OF BASS NUMBERS WITH APPLICATION TO GORENSTEIN ALGEBRAS

BY

SHIRO GOTO (Kawasaki) and KENJI NISHIDA (Matsumoto)

Abstract. The notion of Gorenstein rings in the commutative ring theory is generalized to that of Noetherian algebras which are not necessarily commutative. We faithfully follow in the steps of the commutative case: Gorenstein algebras will be defined using the notion of Cousin complexes developed by R. Y. Sharp [Sh1]. One of the goals of the present paper is the characterization of Gorenstein algebras in terms of Bass numbers. The commutative theory of Bass numbers turns out to carry over with no extra changes. Certain algebras having locally finite global dimension are also characterized. The special case where the algebras are free modules over base rings is explored. Thanks to these observations, it is clarified how the Gorensteinness is inherited under flat base changes. In conclusion, a characterization for local algebras to be Gorenstein is given, accounting for the reason why the theory behaves so well in the commutative case. Examples are explored and open problems are given. See [GN2] and [GN3] for further developments.

1. Introduction. The notion of Gorenstein rings is very well established in commutative Noetherian ring theory. In this paper we generalize it to Noetherian algebras that are not necessarily commutative. There might be divers manners for the generalization but our method will faithfully follow in the steps of the commutative case. It is somewhat surprising that almost all of the commutative theory carries over to Noetherian algebras.

Much of the motivation for our work comes from the investigation of minimal injective resolutions in the commutative case. One may say that the theory of Gorenstein rings started from the paper of Bass [B2], and the most glorious stage of the commutative theory was done with it. People have focused, thereafter, on how to adapt the theory to non-commutative rings and algebras. For example, Gorenstein rings had grown out of quasi-Frobenius rings or algebras (cf. [Y]). However, we would like to have a non-commutative Gorenstein ring theory which would reflect the good results of the commutative theory. We now therefore begin by giving an approach towards a unified theory of Gorenstein algebras.

2000 *Mathematics Subject Classification*: Primary 13E05, 16A18; Secondary 13H10, 16A33.

The authors are supported by Grant-in-Aid for Scientific Researches C(2) in Japan.

Throughout this paper, R is a commutative ring and Λ is an R-algebra. For most of this paper we will furthermore assume that R is a Noetherian ring and Λ is finitely generated as an R-module. With this notation the contents of our paper are as follows.

Table of contents

1. Introduction
2. Preliminaries
   2.0. General remarks on Spec Λ
   2.1. Minimal Λ-injective resolutions and localizations over R
   2.2. Associated prime ideals $\mathrm{Ass}_\Lambda M$
   2.3. Flat and projective dimension modulo non-zerodivisors
   2.4. Injective dimension modulo non-zerodivisors
   2.5. The structure of injective Λ-modules
   2.6. Matlis duality
   2.7. Local duality theorem
   2.8. Cousin complexes
3. A lemma of Bass
4. Gorenstein R-algebras
5. Characterization of Gorenstein R-algebras in terms of Bass numbers
6. The case where Λ is R-free
7. Flat base changes
8. Examples

In Section 2 we will summarize some preliminary results on the rings R and Λ for later use and further investigation. Proofs will be mostly sketched. Some of them have their own significance or might not be very familiar to readers. In such cases we shall indicate detailed proofs. A part of the observations in Section 3 was already announced in [G2]. We shall restate a lemma (3.3) obtained by Bass, which was the heart of his paper [B2] and is eventually the heart of ours too. Before moving to the main task, we will recall in Section 3 some direct consequences of the lemma. Also, the normality of the center $\mathrm{C}(\Lambda)$ of Λ will be investigated in the case where the ring Λ satisfies Serre's conditions (S_2) and (R_1) (Proposition 3.12).

In Section 4 we will give the definition of Gorenstein R-algebras (or more generally, Gorenstein modules), using the notion of Cousin complexes developed by R. Y. Sharp [Sh1]. Let R be a commutative Noetherian ring. Then by definition, the ring R is *Gorenstein* if the local ring $R_\mathfrak{p}$ has finite self-injective dimension for all $\mathfrak{p} \in \mathrm{Spec}\, R$ ([B2]). Sharp [Sh1] showed that R is a Gorenstein ring if and only if the Cousin complex $\mathrm{C}^\bullet_R(R)$ of R provides a minimal R-injective resolution for R. Following [Sh2], we say that a finitely generated non-zero R-module M is *Gorenstein* if the Cousin complex $\mathrm{C}^\bullet_R(M)$ of M provides a minimal R-injective resolution for M. Now, let Λ be an R-algebra which is a finitely generated R-module, and let M be a finitely generated non-zero left Λ-module. Then exactly in the same manner as in the commutative case, we say that M is a *Gorenstein Λ-module* if the Cousin complex $\mathrm{C}^{\bullet}_{R}(M)$ of M provides a minimal Λ-injective resolution for M (Definition 4.6). This condition is equivalent to saying that M is a Cohen–Macaulay R-module with $\mathrm{id}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R M$ (Theorem 4.5), where $\mathrm{id}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}$ denotes the injective dimension of the $\Lambda_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and $\mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ stands for the Krull dimension of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. The R-algebra Λ is said to be a *Gorenstein R-algebra* if Λ is a Gorenstein left module over itself. The notion of Gorenstein algebra is left-right symmetric (Corollary 4.8), and the main result (4.12) of Section 4 asserts that every Gorenstein R-algebra is its own "canonical" module, provided the base ring R is local. This enables us to do a closer study of such algebras together with the use of localizations.

One of the sources of our research dates back to [HN], where Hijikata and the second author explored an interesting class of non-commutative Bass orders and certain Gorenstein orders as well. Other sources come from the books of Auslander–Reiten–Smalø [ARS], Simson [Si], and Yoshino [Yo], where some Gorenstein CM-algebras (especially, their representation types) are studied from the representation-theoretic point of view. The class of Gorenstein algebras in our sense contains all the classes of algebras that are studied in [HN], [AR1], [ARS], [Si], and [Yo].

One of the goals of this paper is the characterization of Gorenstein R-algebras in terms of Bass numbers, which will be given in Section 5 (Theorem 5.2). The invariants work very well also in the non-commutative case and it is somewhat surprising to see that the commutative theory of Bass numbers carries over to our algebras with no extra changes. We will also give, as a consequence of Theorem 5.2, a characterization (5.5) of certain algebras having locally finite global dimension. In Section 6 the special case where Λ is a free R-module will be investigated. Some equivalent conditions for the ring Λ to be a Gorenstein R-algebra will be given (Theorem 6.4). Thanks to these observations of Section 6 we can analyze in Section 7 how the Gorensteinness is inherited under flat base changes (Theorem 7.3). In conclusion we will give a characterization (7.7) for local algebras to be Gorenstein, which may account for the reason why the theory behaves so well in the commutative case. Some examples given in Section 8 illustrate our theory.

See [GN2] and [GN3] for further developments of the theory. (In [GN2], the present paper is cited with the temporary title: "On Gorenstein R-algebras".) In [GN2] the structure of minimal injective resolutions of lattices over isolated Cohen–Macaulay (non-commutative) singularities is determined. In [GN3], under a certain mild condition on Cohen–Macaulay algebras, we establish an equivalence between the finitely generated modules of finite projective dimension and the finitely generated modules of finite injective dimension.

Since our definition of Gorenstein R-algebras Λ involves the condition that Λ is a Cohen–Macaulay R-module, we give a brief introduction to Cohen–Macaulay modules. Firstly, let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated non-zero R-module. We put

$$\operatorname{depth}_R M = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq (0)\}$$

and call it the *depth* of M. This invariant equals the length of maximal M-regular sequences contained in $\mathfrak{m}$, and the inequality $\operatorname{depth}_R M \leq \operatorname{Kdim}_R M$ holds true in general, where $\operatorname{Kdim}_R M$ stands for the Krull dimension of M ([Ma], p. 100, Theorem 28). We say that M is a *Cohen–Macaulay R-module* if $\operatorname{depth}_R M = \operatorname{Kdim}_R M$. A Cohen–Macaulay R-module M is called *maximal* if $\operatorname{depth}_R M = \operatorname{Kdim} R$ (since $\operatorname{Kdim}_R M \leq \operatorname{Kdim} R$). In our paper, however, Cohen–Macaulay R-modules do not necessarily mean maximal ones. In the case where the base ring R is not necessarily local, we say that a finitely generated non-zero R-module M is *Cohen–Macaulay* if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is Cohen–Macaulay for all $\mathfrak{p} \in \operatorname{Supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq (0)\}$; see [BH] for detailed investigations. Now, let Λ be an R-algebra and assume that Λ is finitely generated when viewed as an R-module. Let M be a finitely generated non-zero left Λ-module. Then we say that M is a *Cohen–Macaulay Λ-module* if M is Cohen–Macaulay when viewed as an R-module. Similarly, the algebra Λ is called a *Cohen–Macaulay R-algebra* if it is Cohen–Macaulay when viewed as an R-module. Therefore, if $\operatorname{Kdim} R = 0$, Gorenstein R-algebras Λ in our sense are a very special kind of Gorenstein algebras in the sense of Auslander and Reiten [AR2] (see Section 6), because $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{op}} = 0$ in our case. Also, Cohen–Macaulay algebras in the sense of [AR2] are a special kind of Cohen–Macaulay R-algebras in our sense but Cohen–Macaulay R-algebras in our sense are not necessarily Cohen–Macaulay in the sense of [AR2].

Before entering into details, let us fix our standard notation. In what follows let R be a commutative ring and let Λ denote an R-algebra. Let $f : R \to \Lambda$ be the structure map. We denote by $\operatorname{Spec} \Lambda$, $\operatorname{Min} \Lambda$, and $\operatorname{Max} \Lambda$ the set of prime ideals, of minimal prime ideals, and of maximal ideals in Λ, respectively. Let $\mathrm{J}(\Lambda)$ stand for the Jacobson radical of Λ. For each $P \in \operatorname{Spec} \Lambda$ we denote by $\operatorname{ht}_\Lambda P$ the *height* of P, that is,

$$\operatorname{ht}_\Lambda P = \sup\{0 \leq n \in \mathbb{Z} \mid \text{there exists a chain } P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n = P \text{ of prime ideals in } \Lambda\}.$$

We put $\operatorname{Kdim} \Lambda = \sup_{P \in \operatorname{Spec} \Lambda} \operatorname{ht}_\Lambda P$ and call it the *Krull dimension* of Λ. Unless otherwise specified, all modules mean left modules. For commutative algebra we use the same notation and terminology as in [AM], [BH], and [Ma]. See [AF] for general rings and modules terminology. We refer to [MR] for the theory of non-commutative Noetherian rings and modules.

2. Preliminaries. The purpose of this section is to summarize some basic results on R-algebras Λ. We begin with the following.

2.0. *General remarks on* $\operatorname{Spec}\Lambda$. For each ideal I in Λ let $I\cap R = f^{-1}(I)$.

PROPOSITION 2.0.1 ([MR], Chapter 10). *Suppose that Λ is finitely generated as an R-module and the structure map $f : R \to \Lambda$ is injective. Then*:

(1) (Lying-over) *For every prime ideal $\mathfrak{p}$ in R there is a prime ideal P in Λ with $\mathfrak{p} = P \cap R$.*

(2) (Going-up) *Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R and let $P \in \operatorname{Spec}\Lambda$ with $\mathfrak{p} = P \cap R$. Then there is a prime ideal Q in Λ such that $P \subseteq Q$ and $\mathfrak{q} = Q \cap R$.*

(3) (Incomparability) *Let $P \subseteq Q$ be prime ideals in Λ. Then $P = Q$ if and only if $P \cap R = Q \cap R$.*

(4) *Let $Q \in \operatorname{Spec}\Lambda$. Then $Q \in \operatorname{Max}\Lambda$ if and only if $Q \cap R \in \operatorname{Max} R$.*

(5) *Let R be a Noetherian ring. Then for each $P \in \operatorname{Min}\Lambda$ the prime ideal $\mathfrak{p} = P \cap R$ in R consists of zerodivisors for Λ.*

(6) $\operatorname{Kdim} R = \operatorname{Kdim}\Lambda = \operatorname{Kdim}_R \Lambda$, *where* $\operatorname{Kdim}_R \Lambda$ *denotes the Krull dimension of Λ as an R-module.*

(7) *For each $P \in \operatorname{Spec}\Lambda$ we have* $\operatorname{ht}_\Lambda P = \operatorname{ht}_{\Lambda_\mathfrak{p}} P\Lambda_\mathfrak{p} \leq \operatorname{Kdim} R_\mathfrak{p}$, *where $\mathfrak{p} = P \cap R$. Hence* $\operatorname{ht}_\Lambda P$ *is necessarily finite if R is a Noetherian ring.*

(8) (Going-down) *Suppose that R is a Noetherian integrally closed integral domain and Λ is a torsionfree R-module. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R and let $Q \in \operatorname{Spec}\Lambda$ with $\mathfrak{q} = Q \cap R$. Then there is a prime ideal P in Λ such that $P \subseteq Q$ and $\mathfrak{p} = P \cap R$. Hence* $\operatorname{ht}_\Lambda P = \operatorname{ht}_R(P \cap R)$ *for all $P \in \operatorname{Spec}\Lambda$.*

Proof. See [S] for assertion (8). ∎

2.1. *Minimal Λ-injective resolutions and localizations over R.* Let S be a multiplicative system in R with $S^{-1}\Lambda \neq (0)$. The next result is the starting point of our research. We give a brief proof for completeness.

PROPOSITION 2.1.1 (cf. [B1]). (1) *Suppose Λ is a left Noetherian ring. Then for every essential homomorphism $M \xrightarrow{\alpha} N$ of Λ-modules the induced homomorphism $S^{-1}M \xrightarrow{S^{-1}\alpha} S^{-1}N$ of $S^{-1}\Lambda$-modules remains essential.*

(2) *Suppose Λ is a left Noetherian ring. Then the $S^{-1}\Lambda$-module $S^{-1}I$ is injective for every injective Λ-module I.*

(3) *Every injective $S^{-1}\Lambda$-module J is injective as a Λ-module.*

Proof. (1) Let L be an $S^{-1}\Lambda$-submodule of $S^{-1}N$ and let $0 \neq \frac{y}{1} \in L$ with $y \in N$. We put $\mathfrak{S} = \{[(0) :_\Lambda z] \mid z \in \Lambda y \text{ and } [(0) :_\Lambda z] \cap f(S) = \emptyset\}$. Then $\mathfrak{S} \neq \emptyset$ since $[(0) :_\Lambda y] \in \mathfrak{S}$. We have a maximal element $[(0) :_\Lambda z] \in \mathfrak{S}$ with $z \in \Lambda y$ and $[(0) :_\Lambda z] \cap f(S) \neq \emptyset$. Now choose $a \in \Lambda$ and $x \in M$ so that $\alpha(x) = az \neq 0$. Then $\frac{az}{1} \neq 0$. For if $\frac{az}{1} = 0$, then for some $s \in S$ we have $s(az) = a(sz) = 0$. Therefore the ideal $[(0) :_\Lambda sz]$ of Λ strictly contains $[(0) :_\Lambda z]$ and so by the maximality of $[(0) :_\Lambda z]$ in $\mathfrak{S}$ we have $[(0) :_\Lambda sz] \cap f(S) \neq \emptyset$. Hence $(ts)z =$ for some $t \in S$ so that $[(0) :_\Lambda z] \cap f(S) \neq \emptyset$, which is absurd. Thus $\frac{az}{1} \neq 0$ and the homomorphism $S^{-1}M \stackrel{S^{-1}\alpha}{\longrightarrow} S^{-1}N$ is essential, since $(S^{-1}\alpha)\left(\frac{x}{1}\right) = \frac{az}{1} \in L$.

(2) Let L be a finitely generated $S^{-1}\Lambda$-module. We choose a finitely generated Λ-module M so that $L = S^{-1}M$. Then

$$\mathrm{Ext}^1_{S^{-1}\Lambda}(L, S^{-1}I) = \mathrm{Ext}^1_{S^{-1}\Lambda}(S^{-1}M, S^{-1}I) \cong S^{-1}R \otimes_R \mathrm{Ext}^1_\Lambda(M, I) = (0)$$

and the injectivity of the $S^{-1}\Lambda$-module $S^{-1}I$ follows.

(3) The Λ-module J is injective, because the functor S^{-1} is exact and $\mathrm{Hom}_\Lambda(M, J) = \mathrm{Hom}_{S^{-1}\Lambda}(S^{-1}M, J)$ for every Λ-module M. ∎

For each Λ-module M let $\mathrm{E}_\Lambda(M)$ denote the injective envelope of M.

Remark 2.1.2. Proposition 2.1.1(1) is no longer true if Λ is not a left Noetherian ring. For example, let $(R, \mathfrak{m})$ be a Noetherian complete local integral domain with $\mathrm{Kdim}\, R > 0$. Let $E = \mathrm{E}_R(R/\mathfrak{m})$ and $S = R \setminus \{0\}$. We denote by Λ the trivial extension of E over R (cf. [Y]). Then $\Lambda \cong \mathrm{E}_\Lambda(E)$ but $E_{S^{-1}\Lambda}(S^{-1}E) = (0)$.

Corollary 2.1.3 ([B1], Corollary 1.3). *Let Λ be a left Noetherian ring and let*

$$0 \to M \to I^0 \to I^1 \to \ldots \to I^i \to \ldots$$

be a minimal injective resolution of a Λ-module M. Then the sequence

$$0 \to S^{-1}M \to S^{-1}I^0 \to S^{-1}I^1 \to \ldots \to S^{-1}I^i \to \ldots$$

is a minimal injective resolution of the $S^{-1}\Lambda$-module $S^{-1}M$.

2.2. *Associated prime ideals* $\mathrm{Ass}_\Lambda M$. To begin with we record

Lemma 2.2.1. *Let M be a Λ-module. Then* $\mathrm{Supp}_R M \subseteq \mathrm{Supp}_R \Lambda$ *and* $\mathrm{Kdim}_R M \leq \mathrm{Kdim}_R \Lambda$.

Definition 2.2.2. Let M be a Λ-module and $P \in \mathrm{Spec}\, \Lambda$. Then P is said to be an *associated prime ideal* of M if M contains a non-zero Λ-submodule X such that $P = [(0) :_\Lambda Y]$ for every non-zero Λ-submodule Y of X.

This condition is equivalent to saying that M contains a non-zero element x such that $P = [(0) :_\Lambda \Lambda y]$ for every $0 \neq y \in \Lambda x$.

Let $\mathrm{Ass}_\Lambda M$ denote the set of associated prime ideals of M. We have $\mathrm{Ass}_\Lambda \Lambda/P = \{P\}$ for every $P \in \mathrm{Spec}\, \Lambda$. If Λ is a commutative ring, then $\mathrm{Ass}_\Lambda M = \{P \in \mathrm{Spec}\, \Lambda \mid P = [(0) :_\Lambda x] \text{ for some } x \in M\}$ for every Λ-module M. This characterization is no more true if Λ is not commutative, as the following simple example shows. Let $\Lambda = \mathbb{M}_2(k)$ be the full matrix ring over a field k and let $M = \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right)$. Then $\mathrm{Ass}_\Lambda M = \{(0)\}$, but there is no embedding $L \to M$ of Λ-modules.

PROPOSITION 2.2.3. (1) *Let M be a Λ-module and let $\mathfrak{S} = \{[(0) :_\Lambda X] \mid X$ is a non-zero Λ-submodule of $M\}$. Suppose that P is a maximal element in $\mathfrak{S}$. Then P is an associated prime ideal of M.*

(2) *Assume that Λ is a left Noetherian ring and let M be a Λ-module. Then $M = (0)$ if and only if $\operatorname{Ass}_\Lambda M = \emptyset$.*

(3) *Assume that Λ is a left Noetherian ring and let M be a Λ-module. Let $t \in R$. Then t acts on M as a non-zerodivisor if and only if $f(t) \notin P$ for any $P \in \operatorname{Ass}_\Lambda M$.*

(4) $\operatorname{Ass}_\Lambda L \subseteq \operatorname{Ass}_\Lambda M \subseteq \operatorname{Ass}_\Lambda L \cup \operatorname{Ass}_\Lambda N$ *for every exact sequence* $0 \to L \to M \to N \to 0$ *of Λ-modules.*

(5) $\operatorname{Ass}_\Lambda[\bigoplus_{\alpha\in\Omega} M_\alpha] = \bigcup_{\alpha\in\Omega} \operatorname{Ass}_\Lambda M_\alpha$ *for every family $\{M_\alpha\}_{\alpha\in\Omega}$ of Λ-modules.*

(6) $\operatorname{Ass}_\Lambda \mathrm{E}_\Lambda(M) = \operatorname{Ass}_\Lambda M$ *for every Λ-module M.*

(7) *Let M be a Λ-module and let $\Phi \subseteq \operatorname{Ass}_\Lambda M$. Then $\operatorname{Ass}_\Lambda N = \operatorname{Ass}_\Lambda M \setminus \Phi$ and $\operatorname{Ass}_\Lambda M/N = \Phi$ for some Λ-submodule N of M.*

Proof. See [Bo], Chapter 4. The proof given in the case where Λ is commutative still works. ∎

LEMMA 2.2.4. *Let M be a non-zero Λ-module. Then the following conditions are equivalent.*

(1) $\mathrm{E}_\Lambda(M)$ *is indecomposable.*

(2) *M is uniform.*

When this is the case, we have $\# \operatorname{Ass}_\Lambda M \le 1$.

Proof. The equivalence (1)⇔(2) is well known. To check the last assertion, let $P_1, P_2 \in \operatorname{Ass}_\Lambda M$ and choose non-zero Λ-submodules X_i of M so that $P_i = [(0) :_\Lambda Y_i]$ for every non-zero submodule Y_i of X_i ($i = 1, 2$). Let Z_i be any non-zero submodule of X_i. Then as M is uniform, we have $Z = Z_1 \cap Z_2 \ne (0)$, whence $P_1 = [(0) :_\Lambda Z] = P_2$. Thus $\# \operatorname{Ass}_\Lambda M \le 1$. ∎

LEMMA 2.2.5. *Let Λ be a left Noetherian ring and I an indecomposable injective Λ-module. Let S be a multiplicative system in R with $S^{-1}\Lambda \ne (0)$. Then*:

(1) $S^{-1}I = (0)$ *if $sx = 0$ for some $s \in S$ and $0 \ne x \in I$.*

(2) $\# \operatorname{Ass}_R I \le 1$.

Proof. (1) Let $L = \Lambda x$. Then $I = \mathrm{E}_\Lambda(L)$. Hence by 2.1.3, $S^{-1}I \cong \mathrm{E}_{S^{-1}\Lambda}(S^{-1}L) = (0)$.

(2) Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass}_R I$. Assume $\mathfrak{p} \nsubseteq \mathfrak{q}$ and choose $t \in \mathfrak{p}$ so that $t \notin \mathfrak{q}$. Then $tx = 0$ for some $0 \ne x \in I$ since $\mathfrak{p} \in \operatorname{Ass}_R I$. Let $S = R \setminus \mathfrak{q}$. Then $I_\mathfrak{q} = S^{-1}I = (0)$ by (1) since $tx = 0$. This is absurd. Thus $\mathfrak{p} \subseteq \mathfrak{q}$ and $\# \operatorname{Ass}_R I \le 1$. ∎

PROPOSITION 2.2.6. *Suppose Λ is a left Noetherian ring and let I be an indecomposable injective Λ-module. Then*:

(1) $\#\operatorname{Ass}_\Lambda I = \#\operatorname{Ass}_R I = 1$.
(2) *Let* $\operatorname{Ass}_\Lambda I = \{P\}$. *Then* $\operatorname{Ass}_R I = \{P \cap R\}$.

Proof. By 2.2.3(2) and 2.2.4, $\#\operatorname{Ass}_\Lambda I = 1$. Let $\operatorname{Ass}_\Lambda I = \{P\}$ and choose a non-zero Λ-submodule X of I so that $P = [(0) :_\Lambda Y]$ for every non-zero Λ-submodule Y of X. Let $\mathfrak{p} = P \cap R$. Then since $\mathfrak{p} = [(0) :_\Lambda \Lambda x] \cap R = [(0) :_R x]$ for every $0 \neq x \in X$, we see $\mathfrak{p} \in \operatorname{Ass}_R I$. Thus $\operatorname{Ass}_R I = \{\mathfrak{p}\}$ by 2.2.5(2). ∎

COROLLARY 2.2.7. *Let Λ be a left Noetherian ring. Then*:

(1) $\#\operatorname{Ass}_\Lambda M < \infty$ *for every finitely generated Λ-module M.*
(2) $\operatorname{Ass}_R M = \{P \cap R \mid P \in \operatorname{Ass}_\Lambda M\}$ *for every Λ-module M.*

Proof. (1) Let $\mathrm{E}_\Lambda(M) = \bigoplus_{1 \le i \le n} I_i$ be a decomposition into a direct sum of indecomposable submodules ([M], Theorem 2.5). Then $\operatorname{Ass}_\Lambda M = \operatorname{Ass}_\Lambda \mathrm{E}_\Lambda(M) = \bigcup_{1 \le i \le n} \operatorname{Ass}_\Lambda I_i$ (2.2.3(5)&(6)). Hence $\operatorname{Ass}_\Lambda M$ is finite by 2.2.4.

(2) Let $\mathfrak{p} \in \operatorname{Ass}_R M$. Then $\mathfrak{p} \in \operatorname{Ass}_R I$ for some indecomposable injective Λ-submodule I of $\mathrm{E}_\Lambda(M)$ ([M]). Let $\operatorname{Ass}_\Lambda I = \{P\}$. Then by 2.2.6, $\mathfrak{p} = P \cap R$ whence $\operatorname{Ass}_R M \subseteq \{P \cap R \mid P \in \operatorname{Ass}_\Lambda M\}$. The reverse inclusion is clear (cf. proof of 2.2.6). ∎

COROLLARY 2.2.8. *Let Λ be a left Noetherian ring and M a Λ-module. Let N be a Λ-submodule of M and assume that M/N is a finitely generated Λ-module. Let $\mathfrak{F} = \operatorname{Ass}_\Lambda M/N$. Then $\mathfrak{F}$ is a finite set and there exists a family $\{N(P)\}_{P \in \mathfrak{F}}$ of Λ-submodules of M satisfying the following conditions.*

(1) $\operatorname{Ass}_\Lambda M/N(P) = \{P\}$ *for each* $P \in \mathfrak{F}$.
(2) $N = \bigcap_{P \in \mathfrak{F}} N(P)$ *in* M.
(3) $N \neq \bigcap_{P \in \mathfrak{G}} N(P)$ *for any subset $\mathfrak{G}$ of $\mathfrak{F}$ such that* $\mathfrak{G} \neq \mathfrak{F}$.

Proof. Let $\mathrm{E}_\Lambda(M/N) = \bigoplus_{1 \le i \le n} I_i$ be a decomposition into a direct sum of indecomposable Λ-submodules. Then $\operatorname{Ass}_\Lambda I_i = \{P_i\}$ for each $1 \le i \le n$. Hence $\mathfrak{F} = \{P_i \mid 1 \le i \le n\}$. Let $\xi_i : M \xrightarrow{\varepsilon} M/N \to \mathrm{E}_\Lambda(M/N) \xrightarrow{p_i} I_i$ (here ε and p_i denote the canonical epimorphism and the ith projection, respectively) and $N_i = \operatorname{Ker} \xi_i$. Then since M/N_i is a non-zero submodule of I_i ([M], Proposition 2.7), by 2.2.3(2) we get $\operatorname{Ass}_\Lambda(M/N_i) = \operatorname{Ass}_\Lambda I_i = \{P_i\}$. Clearly $N = \bigcap_{1 \le i \le n} N_i$. Let $N(P) = \bigcap_{1 \le i \le n \text{ with } P_i = P} N_i$ for each $P \in \mathfrak{F}$. Then $N = \bigcap_{P \in \mathfrak{F}} N(P)$ and $\operatorname{Ass}_\Lambda M/N(P) = \{P\}$, since $M/N(P)$ is a non-zero Λ-submodule of $\bigoplus_{1 \le i \le n \text{ with } P_i = P} M/N_i$ (2.2.3(2)&(5)). Let $\mathfrak{G} \subseteq \mathfrak{F}$ be such that $\mathfrak{G} \neq \mathfrak{F}$ and assume that $N = \bigcap_{P \in \mathfrak{G}} N(P)$. Then thanks to condition (1), the embedding $M/N \to \bigoplus_{P \in \mathfrak{G}} M/N(P)$ forces $\mathfrak{F} = \operatorname{Ass}_\Lambda M/N \subseteq \mathfrak{G}$ (2.2.3(4)&(5)), which is impossible. Hence condition (3) is satisfied. ∎

2.3. *Flat and projective dimension modulo non-zerodivisors.* Let $t \in R$ and assume t is Λ-regular. Hence t acts on every flat Λ-module as a non-zerodivisor. Let $\overline{M} = M/tM$ for each Λ-module M. The next result may offer the key to a better understanding of the relation between the flat dimension $\mathrm{fd}_\Lambda M$ and $\mathrm{fd}_{\overline{\Lambda}}\overline{M}$.

PROPOSITION 2.3.1. *Let M be a Λ-module and assume that t is a non-zerodivisor for M. Then the following conditions are equivalent.*

(1) *M is Λ-flat.*
(2) *$\overline{M}$ is $\overline{\Lambda}$-flat and M_t is Λ_t-flat.*

Proof. It is enough to show (2)⇒(1). Let X be a Λ^{op}-module. Then from the exact sequence $0 \to M \xrightarrow{t} M \to \overline{M} \to 0$ we get the exact sequence

$$\mathrm{Tor}_2^\Lambda(X, \overline{M}) \to \mathrm{Tor}_1^\Lambda(X, M) \xrightarrow{t} \mathrm{Tor}_1^\Lambda(X, M).$$

We will show $\mathrm{Tor}_2^\Lambda(X, \overline{M}) = (0)$. Let $0 \to Y \to F \to X \to 0$ be a presentation of X with F Λ-projective. Let $\ldots \to F_2 \to F_1 \to F_0 \to Y \to 0$ be a projective resolution of Y. Then since t is a non-zerodivisor for all Y and F_i's, reducing modulo $t\Lambda$, we get the $\overline{\Lambda}$-projective resolution

$$\ldots \to \overline{F}_2 \to \overline{F}_1 \to \overline{F}_0 \to \overline{Y} \to 0$$

of $\overline{Y}$. As $\overline{M}$ is $\overline{\Lambda}$-flat and $\overline{F}_i \otimes_{\overline{\Lambda}} \overline{M} \cong F_i \otimes_\Lambda \overline{M}$, we see that the sequence

$$\ldots \to F_2 \otimes_\Lambda \overline{M} \to F_1 \otimes_\Lambda \overline{M} \to F_0 \otimes_\Lambda \overline{M} \to Y \otimes_\Lambda \overline{M} \to 0$$

is exact. Hence $\mathrm{Tor}_1^\Lambda(Y, \overline{M}) = (0)$, so $\mathrm{Tor}_2^\Lambda(X, \overline{M}) = (0)$ since $\mathrm{Tor}_2^\Lambda(X, \overline{M}) \cong \mathrm{Tor}_1^\Lambda(Y, \overline{M})$. Therefore t acts on $\mathrm{Tor}_1^\Lambda(X, M)$ as a non-zerodivisor and so the canonical map

$$\mathrm{Tor}_1^\Lambda(X, M) \to [\mathrm{Tor}_1^\Lambda(X, \overline{M})]_t = \mathrm{Tor}_1^{\Lambda_t}(X_t, M_t)$$

is injective. Thus $\mathrm{Tor}_1^\Lambda(X, M) = (0)$ as M_t is Λ_t-flat, whence M is Λ-flat. ■

COROLLARY 2.3.2. *Let Λ be a left Noetherian ring and M a finitely generated Λ-module. Assume that t acts on M as a non-zerodivisor. Then M is Λ-projective if and only if $\overline{M}$ is $\overline{\Lambda}$-projective and M_t is L_t-projective.*

COROLLARY 2.3.3. (1) *Let N be a $\overline{\Lambda}$-module. Then* $\mathrm{fd}_\Lambda N = \mathrm{fd}_{\overline{\Lambda}} N + 1$ *if* $\mathrm{fd}_{\overline{\Lambda}} N < \infty$.

(2) ([K], Ch. 4-1, Theorem C) *Suppose Λ is a left Noetherian ring and let N be a finitely generated $\overline{\Lambda}$-module. Then* $\mathrm{pd}_\Lambda N = \mathrm{pd}_{\overline{\Lambda}} N + 1$ *if* $\mathrm{pd}_{\overline{\Lambda}} N < \infty$.

Proof. It suffices to prove (1). Let $n = \mathrm{fd}_{\overline{\Lambda}} N$ and let $0 \to K \to F \to N \to 0$ be a short exact sequence of Λ-modules with F free. Then by the snake lemma we get the exact sequence

$$0 \to N \to \overline{K} \xrightarrow{\alpha} \overline{F} \to N \to 0 \tag{a}$$

of $\overline{\Lambda}$-modules, since t is a non-zerodivisor for F. We have $K_t \cong F_t$ as L_t-modules since $tN = (0)$. Let $N_1 = \operatorname{Im}\alpha$ and split the sequence (a) into

(b) $$0 \to N_1 \to \overline{F} \to N \to 0 \quad \text{and}$$

(c) $$0 \to N \to \overline{K} \to N_1 \to 0.$$

Then $\mathrm{fd}_{\overline{\Lambda}}\overline{K} < \infty$ and $\mathrm{fd}_{\Lambda}K = \mathrm{fd}_{\overline{\Lambda}}\overline{K}$ as K_t is L_t-free (cf. 2.3.1). If $n = 0$, then by (b), N_1 is $\overline{\Lambda}$-flat so that by (c), $\overline{K}$ is $\overline{\Lambda}$-flat. Hence $\mathrm{fd}_{\Lambda}N = 1$ since K is Λ-flat. Similarly if $n > 0$, then by (b), $\mathrm{fd}_{\overline{\Lambda}}N_1 = n-1$ and so $\mathrm{fd}_{\overline{\Lambda}}\overline{K} = n$ by (c). Hence $\mathrm{fd}_{\Lambda}N = n+1$ because $\mathrm{fd}_{\Lambda}K = \mathrm{fd}_{\overline{\Lambda}}\overline{K} = n$. ∎

The following result is due to [R], Proposition 5.6. We give a brief proof in our context.

Corollary 2.3.4. *Assume R is a Noetherian ring and Λ is finitely generated as an R-module. Let $t \in \mathrm{J}(R)$. Then* $\mathrm{gl.dim}\,\Lambda = \mathrm{gl.dim}\,\overline{\Lambda} + 1$ *if* $\mathrm{gl.dim}\,\overline{\Lambda} < \infty$.

Proof. Let $n = \mathrm{gl.dim}\,\overline{\Lambda}$. Recall that $\mathrm{gl.dim}\,\Lambda = \sup_M \mathrm{pd}_\Lambda M$ where M runs through finitely generated Λ-modules ([Au1]). Then $\mathrm{gl.dim}\,\Lambda \geq n+1$, since $\mathrm{pd}_\Lambda N = \mathrm{pd}_{\overline{\Lambda}} N + 1$ for every finitely generated $\overline{\Lambda}$-module N. Let M be a finitely generated Λ-module. We will show $\mathrm{pd}_\Lambda M \leq n+1$. Let ${}'M = \{m \in M \mid t^i m = 0 \text{ for some } i \geq 0\}$ and ${}''M = M/{}'M$. Then $\mathrm{pd}_\Lambda M \leq \max\{\mathrm{pd}_\Lambda {}'M, \mathrm{pd}_\Lambda {}''M\}$. The exact sequence $0 \to {}'M \to M \to {}''M \to 0$ divides the problem into the cases (1) $t^k M = (0)$ for some $k > 0$ or (2) every t^k ($k \geq 0$) is a non-zerodivisor for M. For case (1) we choose a filtration

$$M = M_0 \supset M_1 \supset \ldots \supset M_q = (0)$$

so that $t \cdot (M_i/M_{i+1}) = (0)$. Then as $\mathrm{pd}_\Lambda M_i/M_{i+1} \leq n+1$ for all $0 \leq i \leq q-1$, descending induction on i yields $\mathrm{pd}_\Lambda M \leq n+1$. Consider case (2). Let N be a finitely generated Λ-module. Let $i > n$ and look at the exact sequence

$$\mathrm{Ext}^i_\Lambda(M,N) \xrightarrow{t} \mathrm{Ext}^i_\Lambda(M,N) \to \mathrm{Ext}^{i+1}_\Lambda(\overline{M},N)$$

induced from the exact sequence $0 \to M \xrightarrow{t} M \to \overline{M} \to 0$. Then $\mathrm{Ext}^{i+1}_\Lambda(\overline{M},N) = (0)$ as $\mathrm{pd}_\Lambda \overline{M} \leq n+1$. Hence $\mathrm{Ext}^i_\Lambda(M,N) = t \cdot \mathrm{Ext}^i_\Lambda(M,N)$ so that $\mathrm{Ext}^i_\Lambda(M,N) = (0)$ by Nakayama's lemma. Thus $\mathrm{pd}_\Lambda M \leq n$. ∎

We need the following result to compute $\mathrm{fd}_\Lambda \mathrm{E}^i_\Lambda(\Lambda)$.

Theorem 2.3.5. *Let I be a Λ-module and assume that t acts on I as an epimorphism. Let $J = [(0) :_I t]$. Then* $\mathrm{fd}_\Lambda I = \max\{\mathrm{fd}_{\overline{\Lambda}} J + 1, \mathrm{fd}_{\Lambda_t} I_t\}$.

Proof. Let $0 \to I_1 \to F_0 \to I \to 0$ be a presentation of I with F_0 projective. Then the sequence

(a) $$0 \to [I_1]_t \to [F_0]_t \to I_t \to 0$$

of Λ_t-modules is exact. Look at the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & J \arrow[d] & \\
0 \arrow[r] & I_1 \arrow[r] \arrow[d, "t"'] & F_0 \arrow[r] \arrow[d, "t"'] & I \arrow[r] \arrow[d, "t"'] & 0 \\
0 \arrow[r] & I_1 \arrow[r] & F_0 \arrow[r] & I \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

with exact rows and columns. Then by the snake lemma we get a split exact sequence $0 \to J \to \bar{I}_1 \to \overline{F}_0 \to 0$ of $\overline{\Lambda}$-modules. Let

(b) $$\bar{I}_1 = J \oplus \overline{F}_0,$$

$n = \mathrm{fd}_\Lambda I$, and $m = \max\{\mathrm{fd}_{\overline{\Lambda}} J + 1, \mathrm{fd}_{\Lambda_t} I_t\}$. Firstly we will show $n \geq m$. If $n = 0$, then I is flat and t acts on I as an isomorphism. Hence $m = 0$. Assume $n \geq 1$. Then $\mathrm{fd}_{\Lambda_t} I_t \leq n$. We have $\mathrm{fd}_{\overline{\Lambda}} \bar{I}_1 \leq n-1$ as $\mathrm{fd}_\Lambda I_1 = n-1$ and t is a non-zerodivisor for I_1. Hence by (b) we get $\mathrm{fd}_{\overline{\Lambda}} J \leq n-1$ and so $n \geq m$. Let us check $n \leq m$. If $J = (0)$, then $I = I_t$ and $\mathrm{fd}_\Lambda I = \mathrm{fd}_{\Lambda_t} I_t$ as every flat Λ_t-module is Λ-flat, whence $n \leq m$. Let $J \neq (0)$. Then $m \geq 1$ and $\mathrm{fd}_{\overline{\Lambda}} J \leq m-1$. Hence decomposition (b) shows $\mathrm{fd}_{\overline{\Lambda}} \bar{I}_1 \leq m-1$. Let $0 \to L \to F_{m-1} \to \ldots \to F_1 \to I_1 \to 0$ be an exact sequence of Λ-modules with F_i's projective. Then both the induced sequences

(c) $$0 \to L_t \to [F_{m-1}]_t \to \ldots \to [F_1]_t \to [I_1]_t \to 0 \quad \text{and}$$

(d) $$0 \to \bar{L} \to \overline{F}_{m-1} \to \ldots \to \overline{F}_1 \to \bar{I}_1 \to 0$$

are exact. Hence, as $\mathrm{fd}_{\Lambda_t} I_t \leq m$, from sequences (a) and (c) it follows that L_t is Λ_t-flat. On the other hand decomposition (b) shows $\mathrm{fd}_{\overline{\Lambda}} \bar{I}_1 \leq m-1$ as $\mathrm{fd}_{\overline{\Lambda}} J \leq m-1$. Therefore from sequence (d) it follows that $\bar{L}$ is $\overline{\Lambda}$-flat. Hence by 2.3.1, L is Λ-flat so that $\mathrm{fd}_\Lambda I_1 \leq m-1$ and we have $\mathrm{fd}_\Lambda I \leq m$. ∎

2.4. *Injective dimension modulo non-zerodivisors.* Let S be a multiplicative system in R with $S^{-1}\Lambda \neq (0)$. For each Λ-module M let

$$'M = \{m \in M \mid sm = 0 \text{ for some } s \in S\} \text{ and } ''M = M/'M.$$

Then every $s \in S$ is a non-zerodivisor for $''M$. The functors $'[*]$ and $''[*]$ are compatible with direct sums.

LEMMA 2.4.1. *Suppose Λ is a left Noetherian ring and let I be an injective Λ-module. Then*:

(1) *Both* $'I$ *and* $''I$ *are injective* Λ*-modules.*

(2) $I \cong {'I} \oplus {''I}$.

(3) *The canonical map* $''I \to S^{-1}{''I}$ *is bijective and* $S^{-1}I \cong S^{-1}{''I}$.

Proof. (1)&(2) It suffices to show that $'I$ is injective. We may assume I is indecomposable ([M], Theorem 2.5). Then either $'I = I$ or $'I = (0)$ (2.2.5).

(3) Let J be an indecomposable direct summand of $''I$. Then every $s \in S$ acts on J as a monomorphism and so as an isomorphism too. Hence every $s \in S$ acts on $''I$ as an isomorphism so that $''I \cong S^{-1}{''I}$. The second assertion follows from the fact that $S^{-1}{'I} = (0)$. ∎

PROPOSITION 2.4.2. *Let* M *be a* Λ*-module and let* $\mathcal{F}_1 = \{P \mid P \in \operatorname{Ass}_\Lambda M$ *and* $P \cap f(S) \neq \emptyset\}$ *and* $\mathcal{F}_2 = \{P \mid P \in \operatorname{Ass}_\Lambda M$ *and* $P \cap f(S) = \emptyset\}$. *Then*:

(1) $S^{-1}P \in \operatorname{Ass}_{S^{-1}\Lambda} S^{-1}M$ *if* $P \in \mathcal{F}_2$.

(2) $\operatorname{Ass}_{S^{-1}\Lambda} S^{-1}M = \{S^{-1}P \mid P \in \mathcal{F}_2\}$ *if* Λ *is a left Noetherian ring.*

(3) *Suppose* Λ *is a left Noetherian ring. Then* M *contains a unique* Λ*-submodule* L *with* $\operatorname{Ass}_\Lambda L = \mathcal{F}_1$ *and* $\operatorname{Ass}_\Lambda M/L = \mathcal{F}_2$.

Proof. (1) Let $P \in \mathcal{F}_2$ and $Q = S^{-1}P$. Let us show $Q \in \operatorname{Ass}_{S^{-1}\Lambda} S^{-1}M$. Choose a non-zero cyclic submodule $X = \Lambda x$ of M so that $P = [(0) :_\Lambda \Lambda y]$ for every $0 \neq y \in X$. Then every $s \in S$ is a non-zerodivisor for X and the canonical map $X \to S^{-1}X$ is injective. We will check

$$Q = [(0) :_{S^{-1}\Lambda} S^{-1}(\Lambda y)] \quad \text{for } 0 \neq \tfrac{y}{s} \in S^{-1}X \ (y \in X, s \in S).$$

The inclusion $\subseteq$ is clear. To see the opposite inclusion let $\frac{a}{t} \in [(0) :_{S^{-1}\Lambda} S^{-1}(\Lambda y)]$ with $a \in \Lambda$ and $t \in S$. Then $\frac{a}{1} \in [(0) :_{S^{-1}\Lambda} S^{-1}(\Lambda y)]$ and so $a \cdot \Lambda y = (0)$ as the canonical map $\Lambda y \to S^{-1}(\Lambda y)$ is injective. Hence $a \in P = [(0) :_\Lambda \Lambda y]$ so that $\frac{a}{t} \in Q = S^{-1}P$. Therefore $Q = [(0) :_{S^{-1}\Lambda} S^{-1}(\Lambda y)]$ and $Q \in \operatorname{Ass}_{S^{-1}\Lambda} S^{-1}M$.

(2) Let $I = \mathrm{E}_\Lambda(M)$ and look at the decomposition $I = {'I} \oplus {''I}$ given by 2.4.1(2). We put $N = M \cap {''I}$. Then every $s \in S$ is a non-zerodivisor for N and the map $N \to S^{-1}N$ is injective. Let $Q \in \operatorname{Ass}_{S^{-1}\Lambda} S^{-1}M$. Then $Q \in \operatorname{Ass}_{S^{-1}\Lambda} S^{-1}N$ as $S^{-1}N = S^{-1}M$ (2.4.1(3)). Choose a non-zero cyclic $S^{-1}\Lambda$-submodule $\widetilde{X}$ of $S^{-1}N$, say $\widetilde{X} = S^{-1} \cdot \frac{n}{1} = S^{-1}(\Lambda n)$ with $n \in N$, so that $Q = [(0) :_{S^{-1}\Lambda} \widetilde{Y}]$ for every non-zero $S^{-1}\Lambda$-submodule $\widetilde{Y}$ of $\widetilde{X}$. We put $P = Q \cap \Lambda$. We will show $P = [(0) :_\Lambda Z]$ for every non-zero Λ-submodule Z of Λn. Let $a \in P$. Then $\frac{a}{1} \cdot S^{-1}Z = (0)$ as $\frac{a}{1} \in Q$ and $Q = [(0) :_{S^{-1}\Lambda} S^{-1}Z]$. Hence $aZ = (0)$ as the map $Z \to S^{-1}Z$ is injective. Thus $P \subseteq [(0) :_\Lambda Z]$. Conversely, let $a \in [(0) :_\Lambda Z]$. Then $\frac{a}{1} \in Q$ since $\frac{a}{1} \cdot S^{-1}Z = (0)$ and $Q = [(0) :_{S^{-1}\Lambda} S^{-1}Z]$. Therefore $a \in Q \cap \Lambda = P$. Hence $P = [(0) :_\Lambda Z]$ and so $P \in \operatorname{Ass}_\Lambda M$.

(3) Let $L = M \cap {}'I$ this time. Then $S^{-1}L = (0)$ whence $\mathrm{Ass}_\Lambda L \subseteq \mathcal{F}_1$ by (1). We have $\mathrm{Ass}_\Lambda M/L \subseteq \mathcal{F}_2$ by 2.2.3(3), as M/L is a submodule of ${}''I$ and every $s \in S$ acts on ${}''I$ as a non-zerodivisor. Thus $\mathrm{Ass}_\Lambda L = \mathcal{F}_1$ and $\mathrm{Ass}_\Lambda M/L = \mathcal{F}_2$. Let L' be a Λ-submodule of M with $\mathrm{Ass}_\Lambda L' = \mathcal{F}_1$ and $\mathrm{Ass}_\Lambda M/L' = \mathcal{F}_2$. Then by (2), $\mathrm{Ass}_{S^{-1}\Lambda} S^{-1}L' = \emptyset$ whence $S^{-1}L' = (0)$ so that $L' \subseteq M \cap {}'I = L$. We must show $L' = L$. If $L' \neq L$, then $\mathrm{Ass}_\Lambda L/L' \neq \emptyset$ and so $S^{-1}(L/L') \neq (0)$ by (2) because $\mathrm{Ass}_\Lambda L/L' \subseteq \mathrm{Ass}_\Lambda M/L' = \mathcal{F}_2$. This is impossible since $S^{-1}L = (0)$. ∎

We now come to the main result of this subsection.

THEOREM 2.4.3. *Suppose Λ is a left Noetherian ring and let*

$$0 \to M \to \mathrm{E}^0_\Lambda(M) \to \mathrm{E}^1_\Lambda(M) \to \ldots \to \mathrm{E}^i_\Lambda(M) \to \ldots$$

be a minimal injective resolution of a Λ-module M. Then:

(1) $\mathrm{E}^i_{S^{-1}\Lambda}(S^{-1}M) \cong {}''\mathrm{E}^i_\Lambda(M)$ *for all $i \in \mathbb{Z}$. (Here we put $\mathrm{E}^i_\Lambda(M) = (0)$ if $i < 0$ by convention.)*

(2) *Suppose every $s \in S$ acts on M as a non-zerodivisor. Then the Λ-module $N = (S^{-1}M)/M$ has a minimal injective resolution of the form*

$$0 \to N \to {}'\mathrm{E}^1_\Lambda(M) \to {}'\mathrm{E}^2_\Lambda(M) \to \ldots \to {}'\mathrm{E}^{i+1}_\Lambda(M) \to \ldots$$

and $\mathrm{E}^i_\Lambda(M) \cong \mathrm{E}^i_{S^{-1}\Lambda}(N) \oplus \mathrm{E}^i_{S^{-1}\Lambda}(S^{-1}M)$ for all $i \in \mathbb{Z}$.

Proof. (1) This follows from 2.1.3 and 2.4.1(3).

(2) Note that ${}'\mathrm{E}^0_\Lambda(M) = (0)$ since every $s \in S$ is a non-zerodivisor for $\mathrm{E}^0_\Lambda(M)$. We identify ${}''\mathrm{E}^i_\Lambda(M) = \mathrm{E}^i_{S^{-1}\Lambda}(S^{-1}M)$ and look at the following commutative diagram:

$$\begin{tikzcd}
 & & 0 & 0 & & 0 & 0 & \\
0 \arrow[r] & S^{-1}M \arrow[r] & {}''\mathrm{E}^0_\Lambda(M) \arrow[r] \arrow[u] & {}''\mathrm{E}^1_\Lambda(M) \arrow[r] \arrow[u] & \cdots \arrow[r, "\alpha_{i-1}"] & {}''\mathrm{E}^i_\Lambda(M) \arrow[r, "\alpha_i"] \arrow[u] & {}''\mathrm{E}^{i+1}_\Lambda(M) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & M \arrow[r] \arrow[u] & \mathrm{E}^0_\Lambda(M) \arrow[r] \arrow[u] & \mathrm{E}^1_\Lambda(M) \arrow[r] \arrow[u] & \cdots \arrow[r, "\beta_{i-1}"] & \mathrm{E}^i_\Lambda(M) \arrow[r, "\beta_i"] \arrow[u, "\varepsilon_i"'] & \mathrm{E}^{i+1}_\Lambda(M) \arrow[r] \arrow[u, "\varepsilon_{i+1}"'] & \cdots \\
 & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & {}'\mathrm{E}^1_\Lambda(M) \arrow[r] \arrow[u] & \cdots \arrow[r, "\gamma_{i-1}"] & {}'\mathrm{E}^i_\Lambda(M) \arrow[r, "\gamma_i"] \arrow[u] & {}'\mathrm{E}^{i+1}_\Lambda(M) \arrow[r] \arrow[u] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with columns and first two rows exact. Then by the long exact sequence of cohomology modules we see that the sequence

$$(*) \qquad 0 \to N \to {}'\mathrm{E}^1_\Lambda(M) \to \ldots \to {}'\mathrm{E}^i_\Lambda(M) \to \ldots$$

is exact, so that it gives rise to an injective resolution of N. The minimality of $(*)$ follows from the fact that the functor ${}'[*]$ is left exact and preserves essential monomorphisms. We have $\mathrm{E}^i_\Lambda(M) \cong \mathrm{E}^{i-1}_\Lambda(N) \oplus \mathrm{E}^i_{S^{-1}\Lambda}(S^{-1}M)$ by 2.4.1(2). ∎

COROLLARY 2.4.4. *Suppose Λ is a left Noetherian ring. If M is a Λ-module such that every $s \in S$ acts on M as a non-zerodivisor, then*

$$\operatorname{id}_\Lambda M = \max\{\operatorname{id}_\Lambda(S^{-1}M)/M + 1, \operatorname{id}_{S^{-1}\Lambda} S^{-1}M\}.$$

COROLLARY 2.4.5. *Let M be a Λ-module. Let $t \in R$ and assume that t is Λ-regular and acts on M as a non-zerodivisor. Let $\overline{\Lambda} = \Lambda/t\Lambda$ and $\overline{M} = M/tM$. Let $J_i = [(0) :_{\mathrm{E}^i_\Lambda(M)} t]$ for $i \in \mathbb{Z}$. Then:*

(1) *The $\overline{\Lambda}$-module $\overline{M}$ has a minimal injective resolution of the form*

$$0 \to \overline{M} \to J^1 \to \dots \xrightarrow{\beta_{i-1}} J^i \xrightarrow{\beta_i} J^{i+1} \to \dots$$

(2) ([B1], Theorem 2.2) *If Λ is a left Noetherian ring, then*

$$\operatorname{id}_\Lambda M = \max\{\operatorname{id}_{\overline{\Lambda}} \overline{M} + 1, \operatorname{id}_{\Lambda_t} M_t\}.$$

(3) *If Λ is a left Noetherian ring, then*

$$\operatorname{fd}_\Lambda \mathrm{E}^i_\Lambda(M) = \max\{\operatorname{fd}_{\overline{\Lambda}} \mathrm{E}^{i-1}_{\overline{\Lambda}}(\overline{M}) + 1, \operatorname{fd}_{\Lambda_t} \mathrm{E}^i_{\Lambda_t}(M_t)\}$$

for all $i \in \mathbb{Z}$.

Proof. (1) Let $I^i = \mathrm{E}^i_\Lambda(M)$ $(i \in \mathbb{Z})$. Then we get a short exact sequence $0 \to J^i \to I^i \xrightarrow{t} I^i \to 0$ for each $i \in \mathbb{Z}$. Look at the commutative diagram

$$\begin{tikzcd}
& & 0 & 0 & & 0 & 0 & \\
0 \arrow[r] & M \arrow[r] & I^0 \arrow[r] \arrow[u] & I^1 \arrow[r] \arrow[u] & \cdots \arrow[r, "\alpha_{i-1}"] & I^i \arrow[r, "\alpha_i"] \arrow[u] & I^{i+1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & M \arrow[r] \arrow[u, "t"'] & I^0 \arrow[r] \arrow[u, "t"'] & I^1 \arrow[r] \arrow[u, "t"'] & \cdots \arrow[r, "\alpha_{i-1}"] & I^i \arrow[r, "\alpha_i"] \arrow[u, "t"'] & I^{i+1} \arrow[r] \arrow[u, "t"'] & \cdots \\
& & 0 \arrow[r] \arrow[u] & J^1 \arrow[r] \arrow[u] & \cdots \arrow[r, "\beta_{i-1}"] & J^i \arrow[r, "\beta_i"] \arrow[u] & J^{i+1} \arrow[r] \arrow[u] & \cdots \\
& & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with exact columns. Each J^i is an injective $\overline{\Lambda}$-module and as in the proof of 2.4.3 we see the $\overline{\Lambda}$-module $\overline{M}$ has a minimal injective resolution of the form

$$0 \to \overline{M} \to J^1 \to \dots \xrightarrow{\beta_{i-1}} J^i \xrightarrow{\beta_i} J^{i+1} \to \dots$$

(2)&(3) Note that $'I^i = (0)$ if and only if $J^i = (0)$. Then by 2.4.3(2), $\operatorname{id}_\Lambda(M_t/M) = \operatorname{id}_{\overline{\Lambda}} \overline{M}$, whence by 2.4.4 we get $\operatorname{id}_\Lambda M = \max\{\operatorname{id}_{\overline{\Lambda}} \overline{M} + 1, \operatorname{id}_{\Lambda_t} M_t\}$. Assertion (3) is a direct consequence of 2.3.5. ∎

2.5. *The structure of injective Λ-modules.* In this subsection we assume that R is a Noetherian ring and the R-algebra Λ is finitely generated when viewed as an R-module. Let $P \in \operatorname{Spec} \Lambda$ and $\mathfrak{p} = P \cap R$. Then $\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}}$ is a simple Artinian ring. Hence $\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}} \cong \mathrm{M}_m(D)$ for some integer $m > 0$ and a division ring D. The integer m is uniquely determined by P. Write $m = \mathrm{m}(P)$. We have $\mathrm{m}(P) = 1$ if Λ is commutative or more generally if $\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}}$ is a division ring. Let $\mathrm{S}(P)$ denote the simple $\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}}$-module. Let M be a Λ-module. Then by 2.4.2(2), $P \in \operatorname{Ass}_\Lambda M$ if and only if $P\Lambda_{\mathfrak{p}} \in \operatorname{Ass}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}$. The latter condition is equivalent to saying that $\mathrm{S}(P)$ is contained in $M_{\mathfrak{p}}$ as a $\Lambda_{\mathfrak{p}}$-submodule. Since $\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}} \cong \mathrm{S}(P)^m$, we get the equivalence between the first three conditions in the following lemma.

Lemma 2.5.1. *Let M be a Λ-module and $P \in \operatorname{Spec} \Lambda$. Let $\mathfrak{p} = P \cap R$. Then the following conditions are equivalent.*

(1) $P \in \operatorname{Ass}_\Lambda M$.
(2) $\operatorname{Hom}_{\Lambda_{\mathfrak{p}}}(\mathrm{S}(P), M_{\mathfrak{p}}) \neq (0)$.
(3) $\operatorname{Hom}_{\Lambda_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq (0)$.
(4) *There is an embedding $\Lambda/P \to M^n$ of Λ-modules for some integer $n > 0$.*

When this is the case, one may choose $n = \mathrm{m}(P)$.

Proof. It suffices to show (1)⇒(4). Since $\mathrm{S}(P) \subseteq M_{\mathfrak{p}}$ we have an embedding $\beta : \Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}} = [\Lambda/P]_{\mathfrak{p}} \to [M_{\mathfrak{p}}]^m = [M^m]_{\mathfrak{p}}$. Notice that

$$R_{\mathfrak{p}} \otimes_R \operatorname{Hom}_\Lambda(\Lambda/P, M^m) = \operatorname{Hom}_{\Lambda_{\mathfrak{p}}}([\Lambda/P]_{\mathfrak{p}}, [M^m]_{\mathfrak{p}}).$$

We write $\beta = \frac{\alpha}{t}$ with $\alpha \in \operatorname{Hom}_\Lambda(\Lambda/P, M^m)$ and $t \in R \setminus \mathfrak{p}$. Then $\frac{\alpha}{1} : [\Lambda/P]_{\mathfrak{p}} \to [M^m]_{\mathfrak{p}}$ is a monomorphism as so is β, while the canonical map $\Lambda/P \xrightarrow{h} [\Lambda/P]_{\mathfrak{p}}$ is injective. Hence the commutativity of the diagram

$$\begin{tikzcd} \Lambda/P \arrow[r, "\alpha"] \arrow[d, "h"] & M^m \arrow[d, "h"] \\ {[\Lambda/P]_{\mathfrak{p}}} \arrow[r, "\frac{\alpha}{1}"] & {[M^m]_{\mathfrak{p}}} \end{tikzcd}$$

(here $M^m \xrightarrow{h} [M^m]_{\mathfrak{p}}$ denotes the canonical map) implies that $\alpha : \Lambda/P \to M^m$ is also a monomorphism. ∎

By 2.2.6 we see that $\# \operatorname{Ass}_\Lambda I = 1$ for every indecomposable injective Λ-module I. Let us add the following.

Proposition 2.5.2. (1) *Let I be an indecomposable injective Λ-module with $\operatorname{Ass}_\Lambda I = \{P\}$. Then $I^m \cong \mathrm{E}_\Lambda(\Lambda/P)$ for $m = \mathrm{m}(P)$.*

(2) *Let $P \in \operatorname{Spec} \Lambda$ and $m = \mathrm{m}(P)$. Then $I^m \cong \mathrm{E}_\Lambda(\Lambda/P)$ for every indecomposable direct summand I of $\mathrm{E}_\Lambda(\Lambda/P)$.*

(3) *Let I and J be indecomposable injective Λ-modules. Then $I \cong J$ if and only if* $\operatorname{Ass}_\Lambda I = \operatorname{Ass}_\Lambda J$.

Proof. (1) Let $\mathfrak{p} = P \cap R$. Then $I = I_\mathfrak{p}$ and $\mathrm{E}_\Lambda(\Lambda/P) = \mathrm{E}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p})$ since every $s \in R \setminus \mathfrak{p}$ acts on I and $\mathrm{E}_\Lambda(\Lambda/P)$ as an isomorphism, while $I_\mathfrak{p} \cong \mathrm{E}_{\Lambda_\mathfrak{p}}(\mathrm{S}(P))$ and $\mathrm{E}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p}) \cong \mathrm{E}_{\Lambda_\mathfrak{p}}(\mathrm{S}(P))^m$ since $I_\mathfrak{p}$ is an indecomposable injective $\Lambda_\mathfrak{p}$-module with $\mathrm{S}(P) \subseteq I_\mathfrak{p}$ and $\Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p} \cong \mathrm{S}(P)^m$. Hence $I^m = [I_\mathfrak{p}]^m \cong \mathrm{E}_{\Lambda_\mathfrak{p}}(\mathrm{S}(P))^m \cong \mathrm{E}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p}) = \mathrm{E}_\Lambda(\Lambda/P)$.

(2) This follows from (1) since $\operatorname{Ass}_\Lambda I = \{P\}$.

(3) Assume $\operatorname{Ass}_\Lambda I = \operatorname{Ass}_\Lambda J$, say $\operatorname{Ass}_\Lambda I = \operatorname{Ass}_\Lambda J = \{P\}$. Then by (1), $I^m \cong \mathrm{E}_\Lambda(\Lambda/P) \cong J^m$ for $m = \mathrm{m}(P)$. Hence $I \cong J$. ∎

Let $P \in \operatorname{Spec}\Lambda$. Then $\mathrm{E}_\Lambda(\Lambda/P)$ contains a unique (up to isomorphism) indecomposable direct summand, which we denote by $\mathrm{I}(P)$.

Corollary 2.5.3 ([GW], Theorem 8.14). *The correspondence $P \mapsto \mathrm{I}(P)$ yields a bijection between* $\operatorname{Spec}\Lambda$ *and the set of isomorphism classes of indecomposable injective Λ-modules.*

Let M be a Λ-module and $i \in \mathbb{Z}$. Let

$$\mathrm{E}^i_\Lambda(M) = \bigoplus_{\alpha \in \Omega^i(M)} I_\alpha$$

be a decomposition into a direct sum of indecomposable submodules ([M], Theorem 2.5). For each $P \in \operatorname{Spec}\Lambda$ we put

$$\Omega^i(P, M) = \{\alpha \in \Omega^i(M) \mid \operatorname{Ass}_\Lambda I_\alpha = \{P\}\}.$$

Then $\{\Omega^i(P, M)\}_{P \in \operatorname{Spec}\Lambda}$ gives rise to a partition of $\Omega^i(M)$ and

$$\mathrm{E}^i_\Lambda(M) \cong \bigoplus_{P \in \operatorname{Spec}\Lambda} \mathrm{I}(P)^{(\Omega^i(P,M))}.$$

Definition 2.5.4. Let $\mu^i(P, M) = \#\Omega^i(P, M)/\mathrm{m}(P)$ and call it the ith *Bass number* of M with respect to P. Then we have the symbolic direct sum decomposition

$$\mathrm{E}^i_\Lambda(M) = \bigoplus_{P \in \operatorname{Spec}\Lambda} \mathrm{E}_\Lambda(\Lambda/P)^{\mu^i(P,M)}.$$

In general $0 \le \mu^i(P, M) \in \mathbb{Q}$ or $\mu^i(P, M) = \infty$. We explore the invariant $\mu^i(P, M)$ in Section 5.

Proposition 2.5.5. *Let $P \in$* $\operatorname{Spec}\Lambda$ *and $\mathfrak{p} = P \cap R$. Then* $\mathrm{I}(P)$ *is a direct summand of* $\operatorname{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(R/\mathfrak{p}))$.

Proof. Choose $x \in \mathrm{I}(P)$ so that $\mathfrak{p} = [(0) :_R x]$ and let $L = \Lambda x$. Then $\mathrm{I}(P) = \mathrm{E}_\Lambda(L)$ because $\mathrm{I}(P)$ is indecomposable, while $\mathrm{E}_R(L) \cong \mathrm{E}_R(R/\mathfrak{p})^n$ $(n > 0)$ as $\operatorname{Ass}_R L = \{\mathfrak{p}\}$. We have natural embeddings

$$L \subseteq \operatorname{Hom}_R(\Lambda^{\mathrm{op}}, L) \subseteq \operatorname{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(L)) = \operatorname{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(R/\mathfrak{p}))^n$$

of Λ-modules. Therefore $\mathrm{I}(P)$ is a direct summand of the injective Λ-module $\mathrm{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(R/\mathfrak{p}))$ as $\mathrm{I}(P) = \mathrm{E}_\Lambda(L)$. ∎

To end this subsection let us note the following. We always have $\mathrm{Ass}_\Lambda \Lambda \supseteq \mathrm{Min}\, \Lambda$ if Λ is a commutative Noetherian ring ([Ma], p. 50, (7.D), Theorem 9). This is no more true if Λ is not commutative even if Λ is a Cohen–Macaulay R-module. Let k be a field and let $\Lambda = \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$. Then $\mathrm{Ass}_\Lambda \Lambda = \left\{\left[\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}\right]\right\}$ but $\mathrm{Min}\, \Lambda = \left\{\left[\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}\right], \left[\begin{smallmatrix} k & k \\ 0 & 0 \end{smallmatrix}\right]\right\} = \mathrm{Spec}\, \Lambda$. On the other hand we have

PROPOSITION 2.5.6. $\mathrm{Ass}_\Lambda \Lambda \subseteq \mathrm{Min}\, \Lambda$ *if* Λ *is a Cohen–Macaulay* R-*module.*

Proof. Let $Q \in \mathrm{Ass}_\Lambda \Lambda$ and choose $P \in \mathrm{Min}\, \Lambda$ so that $P \subseteq Q$. Then $Q \cap R \in \mathrm{Ass}_R \Lambda$ (2.2.7(2)). Therefore as $Q \cap R$ is minimal in $\mathrm{Supp}_R \Lambda$ ([BH], Theorem 2.1.2(a)) and $P \cap R \in \mathrm{Supp}_R \Lambda$, we get $P \cap R = Q \cap R$. Hence $P = Q$ by 2.0.1(3) and $Q \in \mathrm{Min}\, \Lambda$. ∎

The next result is known. Since we need it later so often, we outline its proof for completeness.

PROPOSITION 2.5.7. *Let* R *be a local ring and assume* Λ *is a Cohen–Macaulay* R-*module. Let* M *be a finitely generated non-zero* Λ-*module and* $k = \mathrm{Kdim}_R \Lambda - \mathrm{Kdim}_R M$. *Let* $\mathfrak{a}$ *be an ideal of* R *such that* $\mathfrak{a} \subseteq [(0) :_R M]$ *and* $\mathrm{Kdim}\, R/\mathfrak{a} = \mathrm{Kdim}_R M$. *Then* $\mathfrak{a}$ *contains a* Λ-*regular sequence of length* k.

Proof. By 2.2.1, $k \geq 0$. Let $n = \mathrm{Kdim}_R \Lambda$. Then $\mathrm{Kdim}\, R/\mathfrak{p} = n$ for all $\mathfrak{p} \in \mathrm{Ass}_R \Lambda$ ([BH], 2.1.2(a)). Therefore if $k > 0$, then $\mathfrak{a} \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_R \Lambda$ as $\mathrm{Kdim}_R M = \mathrm{Kdim}\, R/\mathfrak{a}$. Since $\mathrm{Ass}_R \Lambda$ is a finite set (2.2.7(1)), by [AM], 1.11, one may choose $t \in \mathfrak{a}$ so that $t \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R \Lambda} \mathfrak{p}$. Then t is Λ-regular (2.2.3(3)). Let $\overline{\Lambda} = \Lambda/t\Lambda$. Since M is a $\overline{\Lambda}$-module and $\mathrm{Kdim}_R \overline{\Lambda} = \mathrm{Kdim}_R \Lambda - 1$ ([BH], 2.1.2(c)), a successive use of this procedure will guarantee the existence of Λ-regular sequences of length k inside of $\mathfrak{a}$. ∎

2.6. *Matlis duality.* In this subsection we assume that $(R, \mathfrak{m})$ is a Noetherian complete local ring and Λ is a finitely generated R-module. Hence $\mathfrak{m}\Lambda \subseteq P$ for all $P \in \mathrm{Max}\, \Lambda$ and $\mathrm{Max}\, \Lambda$ is a finite set. The ring Λ/J is semisimple and Artinian, where $J = \bigcap_{P \in \mathrm{Max}\, \Lambda} P$ denotes the Jacobson radical of Λ. Let $\mathrm{E} = \mathrm{E}_R(R/\mathfrak{m})$ and let $[*]^\vee = \mathrm{Hom}_R(*, \mathrm{E})$ denote the Matlis duality. We begin with the following.

PROPOSITION 2.6.1. *The functor* $[*]^\vee = \mathrm{Hom}_R(*, \mathrm{E})$ *gives rise to a duality between Noetherian* Λ-*modules and Artinian* Λ^{op}-*modules.*

Proof. Thanks to [M] (Sect. 4), we only have to check that every Artinian Λ-module M is Artinian as an R-module. Let $L = [(0) :_M J]$ be the socle of M. Then L is essential in M and $\mathrm{E}_\Lambda(L) = \mathrm{E}_\Lambda(M)$. Since L

is finitely generated, we have a decomposition $\mathrm{E}_\Lambda(L) = \bigoplus_{1\le\alpha\le n} I_\alpha$ into a finite direct sum of indecomposable Λ-submodules. Recall that $\mathrm{Ass}_\Lambda L = \mathrm{Ass}_\Lambda \mathrm{E}_\Lambda(L) = \bigcup_{1\le\alpha\le n} \mathrm{Ass}_\Lambda I_\alpha$. Then $\bigcup_{1\le\alpha\le n} \mathrm{Ass}_\Lambda I_\alpha \subseteq \mathrm{Max}\,\Lambda$ as $JL = (0)$ and so $\mathrm{Ass}_R I_\alpha = \{\mathfrak{m}\}$ for all $1 \le \alpha \le n$ (2.0.1(4) and 2.2.7(2)). Hence by 2.5.5 each I_α is a direct summand of $\mathrm{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E})$. Therefore by [M] (Sect. 4), I_α is an Artinian R-module. Thus $\mathrm{E}_\Lambda(L) = \mathrm{E}_\Lambda(M)$ is Artinian as an R-module and hence so is M. ∎

The following corollary is a direct consequence of 2.6.1.

Corollary 2.6.2. (1) *Let M be a finitely generated Λ-module. Then M is Λ-projective if and only if $M^\vee$ is Λ^{op}-injective.*

(2) *Let $\gamma : M \to N$ be a homomorphism of finitely generated Λ^{op}-modules and assume that γ is an essential epimorphism. Then the induced map $\gamma^\vee : N^\vee \to M^\vee$ is an essential monomorphism of Λ-modules.*

Corollary 2.6.3. (1) *For each simple Λ^{op}-module T the Λ-module $T^\vee$ is simple and $[(0) :_{\Lambda^{\mathrm{op}}} T^\vee] = [(0) :_\Lambda T]$.*

(2) $([\Lambda/J]^{\mathrm{op}})^\vee \cong \Lambda/J$.

(3) $(\Lambda^{\mathrm{op}})^\vee \cong \mathrm{E}_\Lambda(\Lambda/J)$.

Proof. (1) The fact that $T^\vee$ is simple is now clear. To see $[(0) :_{\Lambda^{\mathrm{op}}} T^\vee] = [(0) :_\Lambda T]$, it suffices to show $[(0) :_{\Lambda^{\mathrm{op}}} T^\vee] \supseteq [(0) :_\Lambda T]$ as $T = T^{\vee\vee}$. Note $(a\varphi)(x) = \varphi(xa) = 0$ for all $a \in [(0) :_\Lambda T]$, $\varphi \in T^\vee$, and $x \in T$. Hence $[(0) :_\Lambda T] \cdot T^\vee = (0)$.

(2) For each $P \in \mathrm{Max}\,\Lambda$ let $\mathrm{T}(P)$ (resp. $\mathrm{S}(P)$) denote the simple $(\Lambda/P)^{\mathrm{op}}$-module (resp. the simple Λ/P-module). Then

$$\Lambda/J \cong \bigoplus_{P\in\mathrm{Max}\,\Lambda} \mathrm{S}(P)^{\mathrm{m}(P)} \quad \text{and} \quad [\Lambda/J]^{\mathrm{op}} \cong \bigoplus_{P\in\mathrm{Max}\,\Lambda} \mathrm{T}(P)^{\mathrm{m}(P)}.$$

Taking the Matlis dual of both sides in the second isomorphism, we have $([\Lambda/J]^{\mathrm{op}})^\vee \cong \bigoplus_{P\in\mathrm{Max}\,\Lambda}[\mathrm{T}(P)^\vee]^{\mathrm{m}(P)}$. Hence $([\Lambda/J]^{\mathrm{op}})^\vee \cong \Lambda/J$ as $\mathrm{T}(P)^\vee \cong \mathrm{S}(P)$ for all $P \in \mathrm{Max}\,\Lambda$ by (1).

(3) The epimorphism $\varepsilon : \Lambda^{\mathrm{op}} \to [\Lambda/J]^{\mathrm{op}}$ yields the essential monomorphism $\varepsilon^\vee : \Lambda/J = ([\Lambda/J]^{\mathrm{op}})^\vee \to (\Lambda^{\mathrm{op}})^\vee$ (cf. 2.6.2). Hence $(\Lambda^{\mathrm{op}})^\vee = \mathrm{E}_\Lambda(\Lambda/J)$. ∎

Corollary 2.6.4. *Suppose* $\mathrm{Kdim}\, R = 0$. *Then*:

(1) $\Lambda \cong (\Lambda^{\mathrm{op}})^\vee$ *if and only if* $\Lambda^{\mathrm{op}} \cong \Lambda^\vee$.

(2) *Λ is an injective Λ-module if and only if Λ^{op} is an injective Λ^{op}-module.*

(3) *Suppose Λ is self-injective and let $[*]^* = \mathrm{Hom}_\Lambda(*, \Lambda)$ denote the Λ-dual. Then the canonical map $\theta_M : M \to M^{**}$ is an isomorphism for all finitely generated Λ-modules M.*

Proof. (1) Note that $\Lambda^{\mathrm{op}} \cong (\Lambda^{\mathrm{op}})^{\vee\vee}$ and we have $\Lambda^{\mathrm{op}} \cong (\Lambda^{\mathrm{op}})^{\vee\vee} = [(\Lambda^{\mathrm{op}})^\vee]^\vee \cong \Lambda^\vee$ if $\Lambda \cong (\Lambda^{\mathrm{op}})^\vee$.

(2) ([ARS], Proposition 3.1) The functor $[*]^\vee = \mathrm{Hom}_R(*, \mathrm{E})$ establishes a duality between Noetherian Λ-modules and Noetherian Λ^{op}-modules as $\mathrm{Kdim}\, R = 0$. Let $\{P_i\}_{1\le i\le n}$ be the non-isomorphic finitely generated indecomposable projective Λ-modules (hence $n = \#\,\mathrm{Max}\,\Lambda$). Then if Λ is an injective Λ-module, each $P_i^\vee$ is Λ^{op}-projective by 2.6.2, since P_i is Λ-injective. Therefore $\{P_i^\vee\}_{1\le i\le n}$ are the non-isomorphic finitely generated indecomposable projective Λ^{op}-modules, because the number of the isomorphism classes of finitely generated indecomposable projective Λ^{op}-modules is $n = \#\,\mathrm{Max}\,\Lambda$. Hence Λ^{op} is Λ^{op}-injective, since it is isomorphic to a direct sum of copies of $P_i^\vee$'s and each $P_i^\vee$ is an injective Λ^{op}-module.

(3) The assertion is obviously true if M is free. Let $0 \to N \to F \to M \to 0$ be an exact sequence of finitely generated Λ-modules with F free. Then as Λ is self-injective, the sequence $0 \to M^* \to F^* \to N^* \to 0$ is exact and so the sequence $0 \to N^{**} \to F^{**} \to M^{**} \to 0$ is still exact since Λ^{op} is self-injective. Look at the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\theta_N"] & F \arrow[r] \arrow[d, "\theta_F"] & M \arrow[r] \arrow[d, "\theta_M"] & 0 \\ 0 \arrow[r] & N^{**} \arrow[r] & F^{**} \arrow[r] & M^{**} \arrow[r] & 0 \end{tikzcd}$$

Then since θ_F is an isomorphism, θ_M is an epimorphism for any finitely generated Λ-module M, while θ_N is always a monomorphism. Therefore θ_N is also an isomorphism, so that θ_M must be an isomorphism too. ∎

2.7. *Local duality theorem.* In this subsection we assume that R is a Noetherian ring and Λ is a left Noetherian ring. We denote by Λ-Mod (resp. R-Mod) the category of Λ-modules (resp. R-modules). Let $\mathfrak{a}$ be an ideal in R. For each $M \in R$-Mod and $i \in \mathbb{Z}$ we define

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \lim_{n\to\infty} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M)$$

and we call it the ith *local cohomology module* of M with respect to $\mathfrak{a}$ ([Gr]). The correspondence $M \mapsto \mathrm{H}^i_{\mathfrak{a}}(M)$ defines a functor and $\{\mathrm{H}^i_{\mathfrak{a}}(*)\}_{i\in\mathbb{Z}}$ are derived functors of

$$\mathrm{H}^0_{\mathfrak{a}}(*) = \lim_{n\to\infty} \mathrm{Hom}_R(R/\mathfrak{a}^n, *) : R\text{-Mod} \to R\text{-Mod}.$$

If $M \in \Lambda$-Mod, then naturally $\mathrm{H}^i_{\mathfrak{a}}(M) \in \Lambda$-Mod and we have defined additive functors $\mathrm{H}^i_{\mathfrak{a}}(*) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$.

Proposition 2.7.1. *Let I be an injective Λ-module. Then $\mathrm{H}^0_{\mathfrak{a}}(I)$ is an injective Λ-module and $\mathrm{H}^i_{\mathfrak{a}}(I) = (0)$ for all $i > 0$.*

Proof. The functors $\mathrm{H}^i_{\mathfrak{a}}(*)$ are compatible with direct sums. So we may assume I is indecomposable. Let $\mathrm{Ass}_\Lambda I = \{P\}$ and put $\mathfrak{p} = P \cap R$. Then each $x \in I$ is killed by some power of $\mathfrak{p}$ as $\mathrm{Ass}_R I = \{\mathfrak{p}\}$, whence $\mathfrak{p} \subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Supp}_R I$. Firstly we consider the case where $\mathfrak{a} \subseteq \mathfrak{p}$. Let $0 \to I \to E^0 \to$

$E^1 \to \ldots \to E^i \to \ldots$ be a minimal injective resolution of the R-module I and let

$$E^i = \bigoplus_{\mathfrak{q} \in \operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{q})^{\mu^i(\mathfrak{q}, I)}.$$

Then $\mu^i(\mathfrak{q}, I) = 0$ for every $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{a} \not\subseteq \mathfrak{q}$. Therefore

$$E^i = \bigoplus_{\mathfrak{q} \in \operatorname{Spec} R \text{ with } \mathfrak{a} \subseteq \mathfrak{q}} \mathrm{E}_R(R/\mathfrak{q})^{\mu^i(\mathfrak{q}, I)}.$$

If $\mathfrak{q} \supseteq \mathfrak{a}$, then each $x \in \mathrm{E}_R(R/\mathfrak{q})$ is killed by some power of $\mathfrak{a}$, so that $\mathrm{H}^0_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{q})) = \mathrm{E}_R(R/\mathfrak{q})$. Hence

$$\mathrm{H}^0_{\mathfrak{a}}(E^i) = \bigoplus_{\mathfrak{q} \in \operatorname{Spec} R \text{ with } \mathfrak{a} \subseteq \mathfrak{q}} \mathrm{H}^0_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{q}))^{\mu^i(\mathfrak{q}, I)} = E^i.$$

Thus $H^0_{\mathfrak{a}}(I) = I$ and $\mathrm{H}^i_{\mathfrak{a}}(I) = (0)$ for all $i > 0$, as the functors $\{\mathrm{H}^i_{\mathfrak{a}}(*)\}_{i \in \mathbb{Z}}$ are defined to be derived functors of $\mathrm{H}^i_{\mathfrak{a}}(*) = \lim_{n \to \infty} \operatorname{Hom}_R(R/\mathfrak{a}^n, *) : R\text{-Mod} \to R\text{-Mod}$. Suppose $\mathfrak{a} \not\subseteq \mathfrak{p}$ and choose $s \notin \mathfrak{a} \backslash \mathfrak{p}$. Then as $\operatorname{Ass}_R I = \{\mathfrak{p}\}$ and I is indecomposable, the element s acts on I as an isomorphism. Hence it also acts on $\mathrm{H}^i_{\mathfrak{a}}(I)$ as an isomorphism. Thus $\mathrm{H}^i_{\mathfrak{a}}(I) = (0)$ for all $i \in \mathbb{Z}$ since each $x \in \mathrm{H}^i_{\mathfrak{a}}(I) = \lim_{n \to \infty} \operatorname{Ext}(R/\mathfrak{a}^n, I)$ is killed by some power of s. ∎

Corollary 2.7.2. *The functors* $\mathrm{H}^i_{\mathfrak{a}}(*) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$ $(i \in \mathbb{Z})$ *are derived functors of*

$$\mathrm{H}^0_{\mathfrak{a}}(*) = \lim_{n \to \infty} \operatorname{Hom}_R(R/\mathfrak{a}^n, *) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}.$$

Hence for each $M \in \Lambda$-Mod, *the* i*th local cohomology module* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *of* M *may be computed as the* i*th cohomology module of the complex of* Λ*-modules*

$$\mathrm{H}^0_{\mathfrak{a}}(I^\bullet): \quad \ldots \to 0 \to \mathrm{H}^0_{\mathfrak{a}}(I^0) \to \mathrm{H}^0_{\mathfrak{a}}(I^1) \to \ldots \to \mathrm{H}^0_{\mathfrak{a}}(I^i) \to \ldots,$$

where $I^\bullet : 0 \to M \to I^0 \to I^1 \to \ldots \to I^i \to \ldots$ *denotes a* Λ*-injective resolution of* M*.*

Proof. The first assertion follows directly from 2.7.1 (use the uniqueness of derived functors). The second assertion is clear. ∎

For the moment suppose that $(R, \mathfrak{m})$ is a local ring with $\operatorname{Kdim} R = d$. When R is $\mathfrak{m}$-adically complete, we put $\mathrm{K}_R = [\mathrm{H}^d_{\mathfrak{a}}(R)]^\vee$ and call it the *canonical module* of R. When R is not necessarily $\mathfrak{m}$-adically complete, an R-module K is said to be the canonical module of R if $\mathrm{K}_{R^\#} \cong R^\# \otimes_R K$, where $R^\#$ denotes the $\mathfrak{m}$-adic completion of R. The canonical module is uniquely determined (up to isomorphism) for R (if it exists) and will be denoted by K_R. If R is a homomorphic image of a Gorenstein local ring S, the ring R has the canonical module K_R and $\mathrm{K}_R \cong \operatorname{Ext}^g_R(R, S)$ $(g = \operatorname{Kdim} S - \operatorname{Kdim} R)$. In the case where R is a Cohen–Macaulay local ring, R has the canonical module K_R if and only if R is a homomorphic image of a Gorenstein local ring ([Re]). See [HK] and [BH] for the basic properties and the general theory of canonical modules.

We close this subsection with the following.

LOCAL DUALITY THEOREM 2.7.3 ([Gr]). *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay complete local ring with* $\operatorname{Kdim} R = d$ *and assume that* Λ *is finitely generated as an* R*-module. Let* K_R *denote the canonical module of* R*. Then for each finitely generated* Λ*-module* M *and* $i \in \mathbb{Z}$ *there is a natural isomorphism*

$$\operatorname{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}_R(R/\mathfrak{m})) \cong \operatorname{Ext}^{d-i}_R(M, \mathrm{K}_R)$$

of Λ^{op}*-modules.*

Proof. For each finitely generated R-module M and $i \in \mathbb{Z}$ we have a natural isomorphism

$$\theta^i_M : \operatorname{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}_R(R/\mathfrak{m})) \to \operatorname{Ext}^{d-i}_R(M, \mathrm{K}_R)$$

of R-modules ([Gr]; see [HK] for a purely ring-theoretic proof). Therefore for a given finitely generated Λ-module M we have the isomorphism θ^i_M of R-modules as well. The isomorphism θ^i_M is not only an isomorphism of R-modules but also an isomorphism of Λ^{op}-modules, because the naturality of $\{\theta^i_M\}_{i\in\mathbb{Z}}$ implies that these maps $\{\theta^i_M\}_{i\in\mathbb{Z}}$ are compatible with the action of the ring Λ^{op}. ∎

2.8. *Cousin complexes.* Cousin complexes for coherent sheaves were originally constructed by Grothendieck [Gr] in algebraic geometry. Subsequently Sharp [Sh1] gave a purely ring-theoretic method of construction. Sharp's method still works for modules over our algebras Λ. Let us give a brief survey of his construction.

For this purpose we need the following.

LEMMA 2.8.1. *Assume that* R *is a Noetherian ring and let* M *be a* Λ*-module. Let* U, U' *be subsets of* $\operatorname{Supp}_R \Lambda$ *and assume that* $U \supseteq \operatorname{Supp}_R M$ *and every* $\mathfrak{p} \in U \setminus U'$ *is minimal in* U*. Then*:

(1) *The map*

$$\xi : M \ni m \mapsto \left\{\frac{m}{1}\right\}_{\mathfrak{p}\in U\setminus U'} \in \bigoplus_{\mathfrak{p}\in U\setminus U'} M_{\mathfrak{p}}$$

is well defined and is an essential homomorphism of Λ*-modules.*

(2) $\operatorname{Supp}_R[\operatorname{Coker}\xi] \cup \operatorname{Supp}_R[\operatorname{Ker}\xi] \subseteq U'$.

Proof. See [Sh1]. ∎

Suppose that R is a Noetherian ring and let M be a Λ-module. For each $i \in \mathbb{Z}$ we put $U^i_R(M) = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \operatorname{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq i\}$. Now let $M_i = (0)$

for $i \leq -2$ and $M^{-1} = M$. Let $i \geq 0$ be an integer and assume that a complex

$$\ldots \xrightarrow{\partial_{i-3}} M^{i-2} \xrightarrow{\partial_{i-2}} M^{i-1}$$

of Λ-modules such that $\mathrm{Supp}_R[\mathrm{Coker}\,\partial_{j-2}] \subseteq U_R^j(M)$ for $j \leq i$ has already been constructed (this assumption is satisfied for $i = 0$). Let

$$M^i = \bigoplus_{\mathfrak{p} \in U_R^i(M) \setminus U_R^{i+1}(M)} [\mathrm{Coker}\,\partial_{i-2}]_{\mathfrak{p}}$$

and let $\partial_{i-1} : M^{i-1} \xrightarrow{\varepsilon} \mathrm{Coker}\,\partial_{i-2} \xrightarrow{\xi} M^i$ where ε is the canonical epimorphism and ξ denotes the homomorphism given by 2.8.1. Then $\partial_{i-1}\partial_{i-2} = 0$ and $\mathrm{Supp}_R[\mathrm{Coker}\,\partial_{i-1}] \subseteq U_R^{i+1}(M)$ by 2.8.1. Hence inductively we get a complex of Λ-modules of the form

$$\ldots \to 0 \to M = M^{-1} \xrightarrow{\partial_{-1}} M^0 \xrightarrow{\partial_0} M^1 \to \ldots \to M^i \xrightarrow{\partial_i} M^{i+1} \to \ldots,$$

which we denote by $\mathrm{C}_R^{\bullet}(M)$ and call the *Cousin complex* for M. The basic properties of Cousin complexes $\mathrm{C}_R^{\bullet}(M)$ and their applications are thoroughly discussed by Sharp [Sh1]–[Sh5]. Let us list some of them which we need later in this paper.

PROPOSITION 2.8.2. (1) $M^i = \bigoplus_{\mathfrak{p} \in \mathrm{Supp}_R M \textit{ with } \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = i} [\mathrm{Coker}\,\partial_{i-2}]_{\mathfrak{p}}$ *for all* $i \geq 0$.

(2) $\mathrm{Supp}_R[\mathrm{Coker}\,\partial_{i-2}] \subseteq U_R^i(M)$ *for all* $i \geq 0$.

(3) *The homomorphism* $\partial_{i-1} : M^{i-1} \to M^i$ *is essential for all* $i \geq 0$.

(4) *Suppose that* $(R, \mathfrak{m})$ *is a local ring and let* M *be a* Λ*-module. Then* $\mathrm{H}_{\mathfrak{m}}^p(M^i) = (0)$ *for all* $0 \leq i < \mathrm{Kdim}_R M$ *and* $p \in \mathbb{Z}$.

(5) *Suppose that* Λ *is finitely generated as an* R*-module and let* M *be a non-zero finitely generated* Λ*-module. Then* M *is a Cohen–Macaulay* R*-module if and only if* $\mathrm{C}_R^{\bullet}(M)$ *is exact. When this is the case*,

$$M^i \cong \bigoplus_{\mathfrak{p} \in \mathrm{Supp}_R M \textit{ with } \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = i} \mathrm{H}_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$$

for all $i \geq 0$.

Proof. See [Sh1], [Sh4], and [Sh5]. ∎

3. A lemma of Bass. In the rest of this paper we assume that R is a Noetherian ring and Λ is finitely generated as an R-module. Let $J = \mathrm{J}(\Lambda)$ denote the Jacobson radical of Λ. The main purpose of this section is to recall Lemma 3.3 below. This lemma was originally given by Bass in his famous paper [B2] on the ubiquity of Gorenstein rings. It is still referred to in commutative algebra but seems less familiar to non-commutative algebraists. We shall give a brief proof and discuss some consequences of it.

Given a finitely generated Λ-module M and $P \in \operatorname{Spec} \Lambda$, we denote by $\mu^i(P, M)$ the ith Bass number of M with respect to P (see Definition 2.5.4).

To begin with we record

LEMMA 3.1. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Let M be a non-zero finitely generated Λ-module and $i \in \mathbb{Z}$. Then the following conditions are equivalent.*

(1) $\operatorname{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)$.
(2) $\mu^i(P, M) > 0$ *for some* $P \in \operatorname{Max} \Lambda$.
(3) $\operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(M) \cap \operatorname{Max} \Lambda \neq \emptyset$.
(4) $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)$.

Proof. Let $0 \to M \to I^0 \xrightarrow{\alpha_0} I^1 \xrightarrow{\alpha_1} \ldots \to I^i \xrightarrow{\alpha_i} I^{i+1} \to \ldots$ be a minimal injective resolution of M. Then $\operatorname{Hom}_\Lambda(\Lambda/J, \alpha_i) = 0$ and $\operatorname{Ext}^i_\Lambda(\Lambda/J, M) = \operatorname{Hom}_\Lambda(\Lambda/J, I^i)$ for all $i > 0$, as Λ/J is a semisimple Artinian ring. Hence $\operatorname{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)$ if and only if I^i contains a non-zero socle if and only if $\operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(M) \cap \operatorname{Max} \Lambda \neq \emptyset$. That is to say, $\mu^i(P, M) > 0$ for some $P \in \operatorname{Max} \Lambda$, which is equivalent to saying that $\mathfrak{m} \in \operatorname{Ass}_R I^i$ (cf. 2.2.7(2)). ■

In the case where $(R, \mathfrak{m})$ is a local ring, for each non-zero finitely generated Λ-module M we put

$$\operatorname{depth}_R M = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq (0)\}$$

and call it the *depth* of M. This invariant equals the length of maximal M-regular sequences contained in the maximal ideal $\mathfrak{m}$ in R ([Ma], p. 100, Theorem 28). See [BH] for detailed investigations. Here let us add the following characterization.

COROLLARY 3.2. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$ and let M be a non-zero finitely generated Λ-module. Then*

$$\begin{aligned}\operatorname{depth}_R M &= \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)\}\\ &= \inf\{i \in \mathbb{Z} \mid \mu^i(P, M) > 0 \textit{ for some } P \in \operatorname{Max} \Lambda\}\\ &= \inf\{i \in \mathbb{Z} \mid \mathfrak{m} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)\}\\ &= \inf\{i \in \mathbb{Z} \mid \mathrm{H}^i_\mathfrak{m}(M) \neq (0)\}.\end{aligned}$$

Proof. Let $m = \operatorname{depth}_R M$ and $n = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)\}$. We will show $m = n$ by induction on m. Suppose $m = 0$. Then $n = 0$ by 3.1, as $\mathfrak{m} \in \operatorname{Ass}_R M \subseteq \operatorname{Ass}_R \mathrm{E}^0_\Lambda(M)$. Let $m > 0$ and assume our equality holds true for $m - 1$. Let $t \in \mathfrak{m}$ be M-regular. Since $t \cdot \operatorname{Ext}^i_\Lambda(\Lambda/J, M) = (0)$, from the exact sequence $0 \to M \xrightarrow{t} M \to M/tM \to 0$ we get the exact sequence

$$(*) \qquad 0 \to \operatorname{Ext}^i_\Lambda(\Lambda/J, M) \to \operatorname{Ext}^i_\Lambda(\Lambda/J, M/tM) \to \operatorname{Ext}^{i+1}_\Lambda(\Lambda/J, M) \to 0$$

of R-modules for each $i \in \mathbb{Z}$. By the hypothesis on m we have $\operatorname{Ext}^{m-1}_\Lambda(\Lambda/J, M/tM) \neq (0)$ and $\operatorname{Ext}^i_\Lambda(\Lambda/J, M/tM) = (0)$ for $i < m - 1$ as $\operatorname{depth}_R M/tM$

$= m-1$. Hence by $(*)$, $\mathrm{Ext}^i_\Lambda(\Lambda/J, M) = (0)$ for all $i < m$, from which again by $(*)$ it follows that $\mathrm{Ext}^m_\Lambda(\Lambda/J, M) = \mathrm{Ext}^{m-1}_\Lambda(\Lambda/J, M/tM) \neq (0)$. Thus $m = n$. See [HK] (Satz 4.10) for the last equality. ■

The next result is due to Bass [B2] and will play a key role in this paper. Let us give an outline of its proof.

LEMMA 3.3 ([B2], (3.1) Lemma). *Let M be a finitely generated Λ-module. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ and assume that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\operatorname{Kdim} R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q} = 1$. Then $\mathfrak{q} \in \operatorname{Ass}_R \mathrm{E}^{i+1}_\Lambda(M)$ if $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)$.*

Proof. Passing to the localization $\Lambda_\mathfrak{q}$, we may assume $(R, \mathfrak{q})$ is a local ring and $\operatorname{Kdim} R/\mathfrak{p} = 1$. Let $\mathfrak{A} = \mathrm{J}(\Lambda_\mathfrak{p}) \cap \Lambda$ and $t \in \mathfrak{q} \setminus \mathfrak{p}$. Then $\operatorname{Kdim} \Lambda/(\mathfrak{A} + t\Lambda) = 0$. Look at the exact sequence

$$\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{A}, M) \xrightarrow{t} \mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{A}, M) \to \mathrm{Ext}^{i+1}_\Lambda(\Lambda/(\mathfrak{A} + t\Lambda), M)$$

of R-modules induced from $0 \to \Lambda/\mathfrak{A} \xrightarrow{t} \Lambda/\mathfrak{A} \to \Lambda/(\mathfrak{A} + t\Lambda) \to 0$. Then since

$$[\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{A}, M)]_\mathfrak{p} = \mathrm{Ext}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/\mathfrak{A}\Lambda_\mathfrak{p}, M_\mathfrak{p}) = \mathrm{Ext}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/\mathrm{J}(\Lambda_\mathfrak{p}), M_\mathfrak{p}) \neq (0),$$

by Nakayama's lemma we see that $\mathrm{Ext}^{i+1}_\Lambda(\Lambda/(\mathfrak{A} + t\Lambda), M) \neq (0)$. Therefore $\mathrm{Ext}^{i+1}_\Lambda(S, M) \neq (0)$ for some composition factor S of $\Lambda/(\mathfrak{A} + t\Lambda)$ whence $\mathfrak{q} \in \operatorname{Ass}_R \mathrm{E}^{i+1}_\Lambda(M)$ by 3.1. ■

REMARK 3.4. Lemma 3.3 is no longer true if we replace $\operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)$ with $\operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(M)$. See Example 8.8.

We summarize some direct consequences of Lemma 3.3.

COROLLARY 3.5. *Let M be a finitely generated non-zero Λ-module.*

(1) *Let $(R, \mathfrak{m})$ be a local ring and let $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)$. Then $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{E}^{i+\operatorname{Kdim} R/\mathfrak{p}}_\Lambda(M)$.*

(2) *Suppose $\operatorname{id}_\Lambda M = n < \infty$. Then $\operatorname{Ass}_R \mathrm{E}^n_\Lambda(M) \subseteq \operatorname{Max} R$ and $\operatorname{Ass}_\Lambda \mathrm{E}^n_\Lambda(M) \subseteq \operatorname{Max} \Lambda$. The Λ-module $\mathrm{E}^n_\Lambda(M)$ contains an essential socle.*

(3) (Auslander) *Let $(R, \mathfrak{m})$ be a local ring. Then*

$$\operatorname{id}_\Lambda M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)\}.$$

Hence $\operatorname{id}_{\Lambda^\#} M^\# = \operatorname{id}_\Lambda M$ where $R^\#$ is the $\mathfrak{m}$-adic completion of R, $\Lambda^\# = R^\# \otimes_R \Lambda$, and $M^\# = R^\# \otimes_R M$.

(4) $\operatorname{Kdim}_R M \leq \operatorname{id}_\Lambda M$.

Proof. (1) See 3.3.

(2) By 3.3 we have $\operatorname{Ass}_R \mathrm{E}^i_\Lambda(M) \subseteq \operatorname{Max} R$, whence $\operatorname{Ass}_\Lambda \mathrm{E}^n_\Lambda(M) \subseteq \operatorname{Max} \Lambda$ by 2.0.1(4) and 2.2.7(2). See 2.5.4 for the last assertion.

(3) Let $m = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(\Lambda/J, M) \neq (0)\} < \infty$. If $\mathrm{E}^i_\Lambda(M) \neq (0)$ with $i > m$, by 3.3 we have $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{E}^j_\Lambda(M)$ for some $j \geq i$. Hence by 3.1

we get $\mathrm{Ext}^j_\Lambda(\Lambda/J, M) \neq (0)$, which is absurd. The second assertion follows from the isomorphism

$$\mathrm{Ext}^i_{\hat\Lambda}(\hat\Lambda/\hat J, \hat M) = R^\# \otimes_R \mathrm{Ext}^i_\Lambda(\Lambda/J, M) \cong \mathrm{Ext}^i_{\Lambda^\#}(\Lambda^\#/\mathrm{J}(\Lambda^\#), M^\#).$$

(4) Let $\mathfrak{p} = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_k = \mathfrak{q}$ be a saturated chain of prime ideals in $\mathrm{Supp}_R\, M$ with $\mathfrak{p} \in \mathrm{Min}_R\, M$ and $\mathfrak{q} \in \mathrm{Max}\, R$. Then by 3.3 we have $\mathfrak{q} \in \mathrm{Ass}_R\, \mathrm{E}^k_\Lambda(M)$, as $\mathfrak{p} \in \mathrm{Ass}_R\, M$. Hence $k \leq \mathrm{id}_\Lambda\, M$, so that $\mathrm{Kdim}_R\, M \leq \mathrm{id}_\Lambda\, M$. ■

COROLLARY 3.6. *Let M be a finitely generated Λ-module. Let $t \in \mathrm{J}(R)$. Assume that t is Λ-regular and acts on M as a non-zerodivisor. Let $\overline{\Lambda} = \Lambda/t\Lambda$ and $\overline{M} = M/tM$. Then* $\mathrm{id}_\Lambda\, M = \mathrm{id}_{\overline{\Lambda}}\, \overline{M} + 1$.

Proof. By 2.4.5(2) it is enough to show that $\mathrm{id}_{\overline{\Lambda}}\, \overline{M} + 1 \geq \mathrm{id}_\Lambda\, M$. We may assume $k = \mathrm{id}_{\overline{\Lambda}}\, \overline{M} < \infty$. Then by 2.4.5(1), t is a non-zerodivisor for $\mathrm{E}^i_\Lambda(M)$ for all $i \geq k+2$. Let $i \geq k+2$ and assume $\mathrm{Ass}_R\, \mathrm{E}^i_\Lambda(M) \neq \emptyset$. Then by 3.3 we have $\mathrm{Ass}_R\, \mathrm{E}^i_\Lambda(M) \cap \mathrm{Max}\, R \neq \emptyset$ for some $j \geq i$. Hence t is a zerodivisor for $\mathrm{E}^i_\Lambda(M) = (0)$ as $t \in \mathrm{J}(R)$. This argument forces $\mathrm{Ass}_R\, \mathrm{E}^i_\Lambda(M) = \emptyset$ for all $i \geq k+2$ so that $\mathrm{E}^i_\Lambda(M) = (0)$. Hence $\mathrm{id}_\Lambda\, M \leq k+1$. ■

The following result generalizes the main theorem of Iwanaga and Sato [IS].

THEOREM 3.7. *Suppose $(R, \mathfrak{m})$ is a local ring and let $t = \mathrm{depth}_R\, \Lambda$. Let M be a finitely generated non-zero Λ-module and assume that $\mathrm{id}_\Lambda\, M = n < \infty$. Then*:

(1) $t \leq n$.

(2) *The Λ-modules $\mathrm{E}^t_\Lambda(\Lambda)$ and $\mathrm{E}^n_\Lambda(M)$ have no common non-zero direct summand if $t \neq n$.*

Proof. Let $\underline{x} = x_1, \ldots, x_t$ be a maximal Λ-regular sequence and let

$$\mathrm{K.} = \mathrm{K.}(\underline{x}; R): \quad 0 \to K_t \xrightarrow{\sigma} K_{t-1} \to \ldots \to K_1 \to K_0$$

be the Koszul complex of R generated by the sequence $\underline{x}$. We identify $K_t = R$ and $K_{t-1} = R^t$. Hence the homomorphism $\sigma : K_t \to K_{t-1}$ is given by $\sigma(1) = (-x_1, x_2, \ldots, (-1)^t x_t)$. Let $\mathfrak{a} = (x_1, x_2, \ldots, x_t)R$. Then the complex

$$\Lambda \otimes_R \mathrm{K.}: \quad 0 \to \Lambda \otimes_R K_t \xrightarrow{\Lambda \otimes_R \sigma} \Lambda \otimes_R K_{t-1} \to \ldots \to \Lambda \otimes_R K_1 \to \Lambda \otimes_R K_0$$

gives rise to a minimal free resolution of the Λ-module $\Lambda/\mathfrak{a}\Lambda$. Apply $\mathrm{Hom}_\Lambda(*, M)$ to it. Then identifying $M^t = \mathrm{Hom}_\Lambda(\Lambda \otimes_R K_{t-1}, M)$ and $M = \mathrm{Hom}_\Lambda(\Lambda \otimes_R K_t, M)$, we get the exact sequence

$$M^t \xrightarrow{\tau} M \to \mathrm{Ext}^t_\Lambda(\Lambda/\mathfrak{a}\Lambda, M) \to 0,$$

where the homomorphism τ is given by

$$\tau\begin{pmatrix} m_1 \\ \vdots \\ m_t \end{pmatrix} = \sum_{1\le i\le t} (-1)^i x_i m_i.$$

Hence

$$M/\mathfrak{a}M = \mathrm{Ext}^t_\Lambda(\Lambda/\mathfrak{a}\Lambda, M) \neq (0)$$

and we have $t \le n$. Assume that $\mathrm{E}^t_\Lambda(\Lambda)$ and $\mathrm{E}^n_\Lambda(M)$ have a common indecomposable direct summand, say I. Then by 3.5(2), $I = \mathrm{E}_\Lambda(S)$ for some simple Λ-module S. Since $\mathrm{E}^0_{\Lambda/\mathfrak{a}\Lambda}(\Lambda/\mathfrak{a}\Lambda) = [(0) :_{\mathrm{E}^t_\Lambda(\Lambda)} \mathfrak{a}]$ by 2.4.5(1) and $S \subseteq \mathrm{E}^t_\Lambda(\Lambda)$, we get $S \subseteq \Lambda/\mathfrak{a}\Lambda$. Look at the exact sequence

$$\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{a}\Lambda, M) \to \mathrm{Ext}^n_\Lambda(S, M) \to \mathrm{Ext}^{n+1}_\Lambda(C, M) = (0)$$

induced from the exact sequence $0 \to S \to \Lambda/\mathfrak{a}\Lambda \to C \to 0$. Then we have $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{a}\Lambda, M) \neq (0)$, as $\mathrm{Ext}^n_\Lambda(S, M) = \mathrm{Hom}_\Lambda(S, \mathrm{E}^n_\Lambda(M)) \neq (0)$ by our choice of S. Hence $n \le t$. ∎

Corollary 3.8 ([IS], Theorem). *Suppose that $0 < \mathrm{id}_\Lambda \Lambda = n < \infty$. Then $\mathrm{E}^0_\Lambda(\Lambda)$ and $\mathrm{E}^n_\Lambda(\Lambda)$ have no common non-zero direct summand.*

Proof. If $\mathrm{E}^0_\Lambda(\Lambda)$ and $\mathrm{E}^n_\Lambda(\Lambda)$ have a common indecomposable direct summand $\mathrm{E}_\Lambda(S)$ with S a simple Λ-module, then $\mathrm{depth}_R \Lambda = 0$ by 3.2. This is impossible. ∎

We say that Λ is a *local ring* if $\Lambda/\mathrm{J}(\Lambda)$ is a simple Artinian ring. Hence Λ contains a unique maximal ideal and there is a unique (up to isomorphism) simple Λ-module.

Corollary 3.9 ([R], Corollary 2.15). *Suppose that both R and Λ are local rings. Then $\mathrm{id}_\Lambda M = \mathrm{depth}_R \Lambda$ for every finitely generated non-zero Λ-module M of $\mathrm{id}_\Lambda M < \infty$.*

Proof. Let $t = \mathrm{depth}_R \Lambda$. Then by 3.2, $\mathrm{E}^t_\Lambda(\Lambda)$ contains at least one simple Λ-submodule, while by 3.5(2), $\mathrm{E}^n_\Lambda(M)$ contains an essential socle. Thus $\mathrm{id}_\Lambda M = t$ by 3.7. ∎

Corollary 3.10 ([V], Theorem 3.1). *Suppose that both R and Λ are local rings. Then Λ is a Cohen–Macaulay R-module and $\mathrm{id}_\Lambda \Lambda = \mathrm{Kdim}\, \Lambda$ if $\mathrm{id}_\Lambda \Lambda < \infty$.*

Proof. We have $\mathrm{id}_\Lambda \Lambda = \mathrm{depth}_R \Lambda$ by 3.9, while $\mathrm{id}_\Lambda \Lambda \ge \mathrm{Kdim}_R \Lambda$ by 3.5(4). Hence Λ is a Cohen–Macaulay R-module and $\mathrm{id}_\Lambda \Lambda = \mathrm{Kdim}_R \Lambda = \mathrm{Kdim}\, \Lambda$. ∎

Suppose R is a regular local ring with $\mathrm{Kdim}\, R = d$ and Λ is local. Let the structure map $f : R \to \Lambda$ be injective and assume that $\mathrm{id}_\Lambda \Lambda < \infty$. Then by

3.10, Λ is a free R-module and $\mathrm{id}_\Lambda \Lambda = \mathrm{Kdim}\,\Lambda = \mathrm{Kdim}\,R$. In Section 6 we shall show that a minimal injective resolution of Λ is given by the complex

$$0 \to \Lambda = \Lambda \otimes_R R \to \Lambda \otimes_R E^0 \to \Lambda \otimes_R E^1 \to \ldots \to L \otimes_R E^d \to 0,$$

where $0 \to R \to E^0 \to E^1 \to \ldots \to E^d \to 0$ denotes a minimal injective resolution of R (cf. 6.5). In Section 7 we will give a characterization of local R-algebras Λ satisfying the condition in 3.10 (cf. 7.7).

QUESTION 3.11. Assume Λ is a local ring. Is it true that Λ is a Cohen–Macaulay R-module if there is a non-zero finitely generated Λ-module M with $\mathrm{id}_\Lambda M < \infty$? This question is an analogue of a famous problem of Bass [B2] that Roberts [Ro3] settled affirmatively.

Following [V], let us make a few remarks on the normality in the center $\mathrm{C}(\Lambda)$ of Λ.

PROPOSITION 3.12. *Suppose Λ satisfies the following two conditions*:

(1) $\mathrm{depth}_{R_\mathfrak{p}} \Lambda_\mathfrak{p} \geq \min\{\mathrm{Kdim}\,\Lambda_\mathfrak{p}, 2\}$ *for every* $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$.
(2) $\mathrm{gl.dim}\,\Lambda_\mathfrak{p} \leq 1$ *for every* $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ *with* $\mathrm{Kdim}\,\Lambda_\mathfrak{p} \leq 1$.

Then $\mathrm{C}(\Lambda)$ *is a normal ring.*

Proof. Let $C = \mathrm{C}(\Lambda)$. Choose $\mathfrak{q} \in \mathrm{Spec}\,C$ and put $\mathfrak{p} = \mathfrak{q} \cap R$. Then $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ and $\mathrm{Kdim}\,C_\mathfrak{q} = \mathrm{Kdim}\,\Lambda_\mathfrak{q} \leq \mathrm{Kdim}\,\Lambda_\mathfrak{p}$ since $\Lambda_\mathfrak{q} = [\Lambda_\mathfrak{p}]_{\mathfrak{q}C_\mathfrak{q}}$. If $\mathrm{Kdim}\,\Lambda_\mathfrak{p} \geq 2$, by condition (1) the ideal $\mathfrak{p}R_\mathfrak{p}$ contains a $\Lambda_\mathfrak{p}$-regular sequence x, y of length two. Hence $\mathrm{depth}_{C_\mathfrak{q}} \Lambda_\mathfrak{q} \geq 2$ as $(x,y)C_\mathfrak{p} \subseteq \mathfrak{q}C_\mathfrak{p}$, so that $\mathrm{Kdim}\,\Lambda_\mathfrak{q} \geq 2$. Therefore if $\mathrm{Kdim}\,\Lambda_\mathfrak{q} \leq 1$, then $\mathrm{Kdim}\,\Lambda_\mathfrak{p} \leq 1$ and $\mathrm{gl.dim}\,\Lambda_\mathfrak{p} \leq 1$ by condition (2), whence $\mathrm{gl.dim}\,\Lambda_\mathfrak{q} \leq 1$ as $\Lambda_\mathfrak{q} = [\Lambda_\mathfrak{p}]_{\mathfrak{q}C_\mathfrak{p}}$. Suppose $\mathrm{Kdim}\,\Lambda_\mathfrak{p} = 1$. Then $\mathfrak{p}R_\mathfrak{p}$ contains at least one $\Lambda_\mathfrak{p}$-regular element, say x. Therefore $\mathrm{depth}_{C_\mathfrak{q}} \Lambda_\mathfrak{q} \geq 1$ since $xC_\mathfrak{p} \subseteq \mathfrak{q}C_\mathfrak{p}$. Thus the C-algebra Λ satisfies conditions (1) and (2). Look at the exact sequence

$$0 \to C \to \Lambda \xrightarrow{\delta} \mathrm{End}_C \Lambda$$

of C-modules, where the map $\delta : \Lambda \to \mathrm{End}_C \Lambda$ is defined by $\delta(a)(x) = ax - xa$. Then the localized sequence

$$(*) \qquad 0 \to C_\mathfrak{q} \to \Lambda_\mathfrak{q} \xrightarrow{\delta} \mathrm{End}_{C_\mathfrak{q}} \Lambda_\mathfrak{q}$$

is still exact. Hence $C_\mathfrak{q} = \mathrm{C}(\Lambda_\mathfrak{q})$. If $\mathrm{Kdim}\,C_\mathfrak{q} \geq 2$, then by $(*)$ we have $\mathrm{depth}_{C_\mathfrak{q}} C_\mathfrak{q} \geq 2$ because $\mathrm{depth}_{C_\mathfrak{q}}(\mathrm{End}_{C_\mathfrak{q}} \Lambda_\mathfrak{q}) \geq 1$ and $\mathrm{depth}_{C_\mathfrak{q}} \Lambda_\mathfrak{q} \geq 2$ (cf. [BH], 1.2.9). If $\mathrm{Kdim}\,C_\mathfrak{q} = 1$, then $\mathrm{depth}\,C_\mathfrak{q} \geq 1$ since $\mathrm{depth}_{C_\mathfrak{q}} \Lambda_\mathfrak{q} \geq 1$. Thus $\mathrm{depth}\,C_\mathfrak{q} \geq \min\{\mathrm{Kdim}\,C_\mathfrak{q}, 2\}$ for all $\mathfrak{q} \in \mathrm{Spec}\,C$. Let $\mathrm{Kdim}\,C_\mathfrak{q} = 1$. Then as $\mathrm{gl.dim}\,\Lambda_\mathfrak{q} \leq 1$, $C_\mathfrak{q} = \mathrm{C}(\Lambda_\mathfrak{q})$ is an integral domain and $\Lambda_\mathfrak{q}$ is $C_\mathfrak{q}$-torsionfree ([V], Lemma 2.1 and Proposition 2.3). Hence by [BC], Theorem 7.1, $C_\mathfrak{q}$ is a DVR. Thus C is a normal ring ([Ma], Theorem 39). ∎

COROLLARY 3.13. *Suppose that Λ is a Cohen–Macaulay R-module. Then* $\mathrm{C}(\Lambda)$ *is a normal ring if* $\mathrm{gl.dim}\,\Lambda_{\mathfrak{p}} \leq 1$ *for every* $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ *with* $\mathrm{Kdim}\,\Lambda_{\mathfrak{p}} \leq 1$.

4. Gorenstein R-algebras. Unless otherwise specified, in this section we assume that R is a local ring with maximal ideal $\mathfrak{m}$ and $\mathrm{Kdim}\,R = d$. Let $n = \mathrm{Kdim}\,\Lambda = \mathrm{Kdim}_R \Lambda$. The purpose is to give the definition and basic properties of Gorenstein R-algebras.

We begin with the following

LEMMA 4.1. *Suppose M is a Cohen–Macaulay R-module with* $\mathrm{id}_\Lambda M = \mathrm{Kdim}_R M = s$. *Then*:

(1) $\mathrm{E}^s_\Lambda(M) \cong \mathrm{H}^s_{\mathfrak{m}}(M)$.

(2) $M_{\mathfrak{p}}$ *is a Cohen–Macaulay $R_{\mathfrak{p}}$-module and* $\mathrm{id}_{\Lambda_{\mathfrak{p}}} M = \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R M$.

(3) *Let* $\mathfrak{p} \in \mathrm{Spec}\,R$. *Then* $\mathfrak{p} \in \mathrm{Ass}_R \mathrm{E}^i_\Lambda(M)$ *if and only if* $\mathfrak{p} \in \mathrm{Supp}_R M$ *and* $\mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = i$.

Proof. (1) Let $0 \to M \to I^0 \to I^1 \to \ldots \to I^s \to 0$ be a minimal injective resolution of M. Then by 3.2 and 3.5(3), $\mathfrak{m} \in \mathrm{Ass}_R I^i$ if and only if $i = s$, while $\mathrm{H}^0_{\mathfrak{m}}(I^s) = I^s$ by 3.5(2). Hence by 2.7.2, $\mathrm{E}^s_\Lambda(M) \cong \mathrm{H}^s_{\mathfrak{m}}(M)$.

(2)&(3) Let $\mathfrak{p} \in \mathrm{Supp}_R M$. It is well known that $M_{\mathfrak{p}}$ is a Cohen–Macaulay $R_{\mathfrak{p}}$-module ([Se], p. 89, Chapter IV, Théorème 6). Let $k = \mathrm{id}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}$ and $m = \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Then $s \geq k$, while $k \geq m$ by 3.5(4). We have $\mathfrak{p} \in \mathrm{Ass}_R I^k$ since $\mathrm{Ass}_{R_{\mathfrak{p}}}[I^k]_{\mathfrak{p}} = \{\mathfrak{p}R_{\mathfrak{p}}\}$ by 3.5(2). Therefore $\mathfrak{m} \in \mathrm{Ass}_R I^{k+\mathrm{Kdim}\,R/\mathfrak{p}}$ so that $k + \mathrm{Kdim}\,R/\mathfrak{p} \leq s$. Hence $k \leq m$ as $s = \mathrm{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \mathrm{Kdim}\,R/\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Supp}_R M$([Se], p. 89, Chapter IV, Théorème 6). Assertion (3) now follows from the proof of (1). ∎

LEMMA 4.2. *Let M be a finitely generated non-zero Λ-module with* $\mathrm{Kdim}_R M = s$ *and assume that M is a Cohen–Macaulay R-module. Then* $\mathrm{H}^s_{\mathfrak{m}}(M)$ *has a minimal injective resolution of the form*

$$0 \to \mathrm{H}^s_{\mathfrak{m}}(M) \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^s_\Lambda(M)) \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^{s+1}_\Lambda(M)) \to \ldots \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^i_\Lambda(M)) \to \ldots$$

Proof. Let $0 \to M \to I^0 \to I^1 \to \ldots \to I^s \to \ldots$ be a minimal injective resolution of M. Then $\mathrm{H}^0_{\mathfrak{m}}(I^i) = (0)$ as $\mathfrak{m} \notin \mathrm{Ass}_R I^i$ for $i < s$ by 3.2, while $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for $i > s$ ([HK], Satz 4.12). Each $\mathrm{H}^0_{\mathfrak{m}}(I^i)$ is an injective Λ-module (cf. 2.7.1). Hence from (2.7.2) we get the injective resolution

$$0 \to \mathrm{H}^s_{\mathfrak{m}}(M) \to \mathrm{H}^0_{\mathfrak{m}}(I^s) \to \mathrm{H}^0_{\mathfrak{m}}(I^{s+1}) \to \ldots \to \mathrm{H}^0_{\mathfrak{m}}(I^i) \to \ldots$$

of $\mathrm{H}^s_{\mathfrak{m}}(M)$, whose minimality follows from the fact that the functor $\mathrm{H}^0_{\mathfrak{m}}(*)$ is left exact and preserves essential monomorphisms. ∎

PROPOSITION 4.3. *Let M be a finitely generated non-zero Λ-module with* $\mathrm{Kdim}_R M = s$. *Then the following conditions are equivalent.*

(1) *M is a Cohen–Macaulay R-module and* $\mathrm{id}_\Lambda M = s$.

(2) *M is a Cohen–Macaulay R-module and* $\mathrm{H}^s_{\mathfrak{m}}(M)$ *is Λ-injective.*

(3) $\mathrm{Ext}^i_\Lambda(\Lambda/J, M) = (0)$ *for* $i \neq s$.

(4) $\mathfrak{m} \notin \mathrm{Ass}_R \mathrm{E}^i_\Lambda(M)$ *for* $i \neq s$.

Proof. (1)⇒(4). See 3.2.

(3)⇔(4). See 3.1.

(4)⇒(2). M is a Cohen–Macaulay R-module since $\mathrm{depth}_R M \geq s$ by 3.2. As $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^i_\Lambda(M)) = (0)$ for $i \neq s$, $\mathrm{H}^s_{\mathfrak{m}}(M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^s_\Lambda(M))$ and $\mathrm{H}^s_{\mathfrak{m}}(M)$ is Λ-injective (cf. 2.7.1 and 2.7.2).

(2)⇒(1). By 4.2, $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^i_\Lambda(M)) = (0)$ for all $i > s$. Assume $\mathrm{E}^i_\Lambda(M) \neq (0)$ for some $i > s$ and choose $\mathfrak{p} \in \mathrm{Ass}_R \mathrm{E}^i_\Lambda(M)$. Let $j = i + \mathrm{Kdim}\, R/p \geq i$. Then, by 3.5(1), $\mathfrak{m} \in \mathrm{Ass}_R \mathrm{E}^j_\Lambda(M)$ and $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{E}^j_\Lambda(M)) \neq (0)$, which is absurd. Thus $\mathrm{E}^i_\Lambda(M) = (0)$ for all $i > s$ whence $\mathrm{id}_\Lambda M = s$, by 3.5(4). ■

COROLLARY 4.4. *Let M be a finitely generated non-zero Λ-module and assume that M is a Cohen–Macaulay R-module with* $\mathrm{Kdim}_R M = s$.

(1) *Suppose that R is a d-dimensional Cohen–Macaulay local ring with canonical module* K_R. *Then* $\mathrm{id}_\Lambda M = s$ *if and only if* $\mathrm{Ext}^{d-s}_R(M, \mathrm{K}_R)$ *is a projective Λ^{op}-module.*

(2) *Suppose that Λ is a Cohen–Macaulay R-module. Then* $\mathrm{Kdim}\, \Lambda = s$ *if* $\mathrm{id}_\Lambda M = s$.

Proof. (1) Let $R^\#$ denote the $\mathfrak{m}$-adic completion of R. Then $\mathrm{K}_{R^\#} \cong R^\# \otimes_R \mathrm{K}_R$ ([HK], Definition 5.6). Hence passing to $R^\#$, by 3.5(3) we may assume R is complete. By 4.3, $\mathrm{id}_\Lambda M = s$ if and only if $\mathrm{H}^s_{\mathfrak{m}}(M)$ is Λ-injective, while by 2.6.2(1) the latter condition is equivalent to saying that $[\mathrm{H}^s_{\mathfrak{m}}(M)]^\vee$ is Λ^{op}-projective. Hence the assertion follows from 2.7.3.

(2) By 3.5(3) we may assume R is complete. Thanks to Cohen's structure theorem [C], we may furthermore assume that R is a Gorenstein local ring. Let $n = \mathrm{Kdim}\, \Lambda$ and $d = \mathrm{Kdim}\, R$. Then by (1), $\mathrm{Ext}^{d-s}_R(M, R)$ is a direct summand of $[\Lambda^{\mathrm{op}}]^k$ with $k > 0$. Hence $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Ext}^{d-s}_R(M, R))$ is a direct summand of $\mathrm{H}^i_{\mathfrak{m}}(\Lambda^{\mathrm{op}})^k$ so that $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Ext}^{d-s}_R(M, R)) = (0)$ for all $i \neq n$ ([HK], 4.10 and 4.12). Thus $\mathrm{Kdim}_R \mathrm{Ext}^{d-s}_R(M, R) = n$ and hence $s = n$ ([BH], 3.3.10). ■

We now come to the definition of Gorenstein R-algebras. In Theorem 4.5 and Definition 4.6 below the ring R is not assumed to be a local ring.

THEOREM 4.5. *Let R be an arbitrary commutative Noetherian ring, Λ a module-finite R-algebra, and M a finitely generated non-zero Λ-module. Then the following two conditions are equivalent.*

(1) *The Cousin complex* $\mathrm{C}^\bullet_R(M)$ *provides a minimal injective resolution for M.*

(2) *M is a Cohen–Macaulay R-module such that* $\mathrm{id}_{\Lambda_\mathfrak{p}} M_\mathfrak{p} = \mathrm{Kdim}_{R_\mathfrak{p}} M_\mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R M$.

Proof. By 2.8.2(3)&(5) it suffices to show that, assuming M is Cohen–Macaulay, M^i is injective for all $i \geq 0$ if and only if $\mathrm{id}_{\Lambda_\mathfrak{p}} M_\mathfrak{p} = \mathrm{Kdim}_{R_\mathfrak{p}} M_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Supp}_R M$. Let $i \geq 0$ be an integer. Then because

$$M^i \cong \bigoplus_{\mathfrak{p} \in \mathrm{Supp}_R M \text{ with } \mathrm{Kdim}_{R_\mathfrak{p}} M_\mathfrak{p} = i} \mathrm{H}^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$$

(cf. 2.8.2(5)), we see by 2.1.1(2)&(3) that M^i is Λ-injective if and only if $\mathrm{H}^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ is $\Lambda_\mathfrak{p}$-injective for all $\mathfrak{p} \in \mathrm{Supp}_R M$ with $\mathrm{Kdim}_{R_\mathfrak{p}} M_\mathfrak{p} = i$. By 4.3 the latter condition is equivalent to saying that $\mathrm{id}_{\Lambda_\mathfrak{p}} M = i$ for all $\mathfrak{p} \in \mathrm{Supp}_R M$ with $\mathrm{Kdim}_{R_\mathfrak{p}} M_\mathfrak{p} = i$. ∎

DEFINITION 4.6. Let R be an arbitrary commutative Noetherian ring, Λ a module-finite R-algebra, and M a finitely generated non-zero Λ-module. Then M is said to be a *Gorenstein Λ-module* if the Cousin complex $\mathrm{C}^\bullet_R(M)$ of M provides a minimal Λ-injective resolution for M. We say that Λ is a *left Gorenstein R-algebra* if Λ is a Gorenstein module over itself.

By Corollary 4.8 proved below, the definition of Gorenstein algebra is left-right symmetric.

Let M be a finitely generated non-zero Λ-module. By 4.5, M is a Gorenstein Λ-module if and only if $M_\mathfrak{p}$ is a Gorenstein $\Lambda_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Supp}_R M$. Hence the condition of M being a Gorenstein Λ-module is local. And when R is local, our alternative definition 4.5(2) of a Gorenstein R-algebra is the same as that of Vasconcelos [V], who used the term *moderated Gorenstein algebra*. In a more general situation Brown and Hajarnavis [BHa2] already investigated this kind of rings, which they call *injectively homogeneous*.

We return to the former assumption that $(R, \mathfrak{m})$ is a local ring. We denote by $R^\#$ the $\mathfrak{m}$-adic completion of R. Let $\Lambda^\# = R^\# \otimes_R \Lambda$ and $M^\# = R^\# \otimes_R M$ for each Λ-module M.

LEMMA 4.7. *Let R be a local ring and M a finitely generated non-zero Λ-module. Then*:

(1) *M is a Gorenstein Λ-module if and only if M is a Cohen–Macaulay R-module and* $\mathrm{id}_\Lambda M = \mathrm{Kdim}_R M$.

(2) *Let $t \in \mathfrak{m}$ be regular for both Λ and M. Let $\overline{\Lambda} = \Lambda/t\Lambda$ and $\overline{M} = M/tM$. Then M is a Gorenstein Λ-module if and only if $\overline{M}$ is a Gorenstein $\overline{\Lambda}$-module.*

(3) *$M^\#$ is a Gorenstein $\Lambda^\#$-module if and only if M is a Gorenstein Λ-module.*

Proof. See 3.5(3), 3.6, and 4.1. ∎

COROLLARY 4.8. *Let R be a local ring. The following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*
(2) *Λ^{op} is a Gorenstein R-algebra.*

When this is the case, $\mathrm{fd}_\Lambda \mathrm{E}^i_\Lambda(\Lambda) = i$ *for all* $0 \le i \le \mathrm{Kdim}\,\Lambda$.

Proof. (1)⇔(2). Reducing modulo an ideal of R generated by a maximal Λ-regular sequence, we may assume that $\mathrm{Kdim}\,\Lambda = 0$ (cf. 4.7(2)) and, passing to the ring $R/[(0):_R \Lambda]$, we may furthermore assume that $\mathrm{Kdim}\,R = 0$. Then Λ is self-injective if and only if so is Λ^{op} (cf. 2.6.4(2)), whence the equivalence follows. We check the last equality by induction on $n = \mathrm{Kdim}\,\Lambda$. We have nothing to prove for $n = 0$. Suppose that $n > 0$ and our equality holds true for $n-1$. Let $t \in \mathfrak{m}$ be Λ-regular and put $\overline{\Lambda} = \Lambda/t\Lambda$. Let $0 \le i \le n$. Then

$$\mathrm{fd}_{\Lambda_t} \mathrm{E}^i_{\Lambda_t}(\Lambda_t) = \sup_{\mathfrak{p}} \mathrm{fd}_{\Lambda_\mathfrak{p}} \mathrm{E}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p})$$

where $\mathfrak{p}$ runs through the prime ideals $\mathfrak{p}$ in $\mathrm{Supp}_R\,\Lambda$ with $t \notin \mathfrak{p}$. Because $\mathfrak{p} \ne \mathfrak{m}$ for any $\mathfrak{p} \in \mathrm{Supp}_R\,\Lambda$ with $t \notin \mathfrak{p}$, we infer by the hypothesis on n that $\mathrm{fd}_{\Lambda_t} \mathrm{E}^i_{\Lambda_t}(\Lambda_t) = \sup_\mathfrak{p} \mathrm{fd}_{\Lambda_\mathfrak{p}} \mathrm{E}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}) \le i$ and $\mathrm{fd}_{\overline{\Lambda}} \mathrm{E}^{i-1}_{\overline{\Lambda}}(\overline{\Lambda}) = i-1$ as well. Hence by 2.4.5(3), $\mathrm{fd}_\Lambda \mathrm{E}^i_\Lambda(\Lambda) = i$ for $0 \le i \le n$. ∎

LEMMA 4.9. *Let Λ be a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$. *Then* $\mathrm{Ext}^n_\Lambda(S, \Lambda) \ne (0)$ *for any simple Λ-module S.*

Proof. Let $t \in \mathfrak{m}$ be Λ-regular and let $\overline{R} = R/tR$, $\overline{\Lambda} = \Lambda/t\Lambda$. Then $\mathrm{Ext}^n_\Lambda(S, \Lambda) \cong \mathrm{Ext}^{n-1}_{\overline{\Lambda}}(S, \overline{\Lambda})$ as $tS = (0)$. Hence passing to the ring $\overline{\Lambda}$, by 4.7(2) we may assume $n = 0$. Also passing to the ring $R/[(0):_R \Lambda]$, we may furthermore assume $\mathrm{Kdim}\,R = 0$. Then $S \cong \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(S, \Lambda), \Lambda)$ by 2.6.4(3) so that $\mathrm{Hom}_\Lambda(S, \Lambda) \ne (0)$. ∎

COROLLARY 4.10 (Iwanaga). *The following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*
(2) $\mathrm{id}_\Lambda \Lambda = k < \infty$ *and* $\mathrm{E}^k_\Lambda(\Lambda) \supseteq S$ *for all simple Λ-modules S.*

Proof. (1)⇒(2). See 4.9.

(2)⇒(1). By 3.2 and 3.7(2) we have $\mathrm{depth}_R\,\Lambda = k$, whence by 3.5(4), Λ is a Gorenstein R-algebra. ∎

PROPOSITION 4.11. *Let Λ be a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$ *and let M be a finitely generated non-zero Λ-module. Then*:

(1) ([N], Proposition 1.6) $\mathrm{depth}_R\,M + \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \ne (0)\} = n$.
(2) $\mathrm{Kdim}_R\,M + \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \ne (0)\} = n$.

(3) *Let* $\mathrm{j}(M) = n - \mathrm{Kdim}_R M$. *Then the* R*-module* M *is Cohen–Macaulay if and only if* $\mathrm{Ext}^i_\Lambda(M,\Lambda) = (0)$ *for all* $i \neq \mathrm{j}(M)$. *When this is the case,* $\mathrm{Ext}^{\mathrm{j}(M)}_\Lambda(M,\Lambda) \neq (0)$.

(4) $\mathrm{depth}_R M + \mathrm{pd}_\Lambda M = n$ *if* $\mathrm{pd}_\Lambda M < \infty$.

(5) $\mathrm{id}_\Lambda M = n$ *if* $\mathrm{id}_\Lambda M < \infty$.

(6) *The following conditions are equivalent.*

(a) M *is a projective* Λ*-module.*

(b) M *is a Gorenstein* Λ*-module.*

(c) M *is a Cohen–Macaulay* R*-module with* $\mathrm{Kdim}_R M = n$ *and* $\mathrm{id}_\Lambda M < \infty$.

(7) $\mathrm{id}_\Lambda M < \infty$ *if and only if* $\mathrm{pd}_\Lambda M < \infty$.

Proof. (1) Induction on $m = \mathrm{depth}_R M$. If $m = 0$, then M contains a simple Λ-submodule S. From the exact sequence $0 \to S \to M \to C \to 0$ we have

$$\mathrm{Ext}^n_\Lambda(M,\Lambda) \to \mathrm{Ext}^n_\Lambda(S,\Lambda) \to \mathrm{Ext}^{n+1}_\Lambda(C,\Lambda) = (0),$$

whence $\mathrm{Ext}^n_\Lambda(M,\Lambda) \neq (0)$ as $\mathrm{Ext}^n_\Lambda(S,\Lambda) \neq (0)$ by 4.9. Assume that $m > 0$ and our assertion holds true for $m-1$. Let $t \in \mathfrak{m}$ be M-regular and put $\overline{M} = M/tM$. Then $\mathrm{depth}_R \overline{M} = m-1$. Let $i \in \mathbb{Z}$ and look at the exact sequence

$$(*) \qquad \mathrm{Ext}^i_\Lambda(M,\Lambda) \xrightarrow{t} \mathrm{Ext}^i_\Lambda(M,\Lambda) \to \mathrm{Ext}^{i+1}_\Lambda(\overline{M},\Lambda) \to \mathrm{Ext}^{i+1}_\Lambda(M,\Lambda)$$

given by the exact sequence $0 \to M \xrightarrow{t} M \to \overline{M} \to 0$. Then $\mathrm{Ext}^i_\Lambda(M,\Lambda) = (0)$ for all $i > n-m$ by Nakayama's lemma, since $\mathrm{Ext}^{i+1}_\Lambda(\overline{M},\Lambda) = (0)$ by hypothesis. On the other hand, as $\mathrm{Ext}^{n-m+1}_\Lambda(\overline{M},\Lambda) \neq (0)$ but $\mathrm{Ext}^{n-m+1}_\Lambda(M,\Lambda) = (0)$, sequence $(*)$ for $i = n-m$ shows $\mathrm{Ext}^{n-m}_\Lambda(M,\Lambda) \neq (0)$.

(2) Induction on n. Firstly suppose $\mathrm{Kdim}_R M = n$ and choose $\mathfrak{p} \in \mathrm{Supp}_R M$ so that $\mathrm{Kdim}\, R/\mathfrak{p} = n$. Then $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ and $\mathrm{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p} = 0$. Hence $\Lambda_\mathfrak{p}$ is self-injective and so by 2.6.4(3),

$$M_\mathfrak{p} \cong \mathrm{Hom}_{\Lambda_\mathfrak{p}}(\mathrm{Hom}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p},\Lambda_\mathfrak{p}),\Lambda_\mathfrak{p}).$$

We have $\mathrm{Hom}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p},\Lambda_\mathfrak{p}) \neq (0)$ as $M_\mathfrak{p} \neq (0)$. Hence $\mathrm{Hom}_\Lambda(M,\Lambda) \neq (0)$. Suppose now that $n > 0$ and our assertion holds true for $n-1$. We may assume that $\mathrm{Kdim}_R M < n$. Choose $t \in [(0) :_R M]$ so that t is Λ-regular (cf. 2.5.7). Let $\overline{\Lambda} = \Lambda/t\Lambda$. Then $\mathrm{Ext}^i_\Lambda(M,\Lambda) \cong \mathrm{Ext}_{\overline{\Lambda}}(M,\overline{\Lambda})$. As $\overline{\Lambda}$ is a Gorenstein R-algebra with $\mathrm{Kdim}\,\overline{\Lambda} = n-1$, $\mathrm{Kdim}_R M + \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_{\overline{\Lambda}}(M,\overline{\Lambda}) \neq (0)\} = n-1$ whence $\mathrm{Kdim}_R M + \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(M,\Lambda) \neq (0)\} = n$.

(3) See (1) and (2).

(4) Recall that $\mathrm{pd}_\Lambda M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(M,\Lambda) \neq (0)\}$ if $\mathrm{pd}_\Lambda M < \infty$.

(5) Let $k = \mathrm{id}_\Lambda M$. Then $n = \mathrm{depth}_R \Lambda \leq k$ by 3.7(1), while $n = k$ by 3.7(2) and 4.9.

(6) (a)$\Rightarrow$(c). M is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R M = n$, since M is a direct summand of Λ^k ($k > 0$). We have $\operatorname{id}_\Lambda M < \infty$ as $\operatorname{id}_\Lambda \Lambda < \infty$.

(c)$\Rightarrow$(b). By (5), $\operatorname{id}_\Lambda M = n$. Hence M is Gorenstein.

(b)$\Rightarrow$(a). By (5), $\operatorname{Kdim}_R M = \operatorname{id}_\Lambda M = n$. We will show that M is Λ-projective. Suppose $n = 0$. Passing to the ring $R/[(0) :_R \Lambda]$, we may assume $\operatorname{Kdim} R = 0$. Let $[*]^\vee = \operatorname{Hom}_R(*, \mathrm{E}_R(R/m))$ be the Matlis dual. Then $M^\vee$ is a direct summand of $[\Lambda^{\mathrm{op}}]^k$ with $k > 0$ (cf. 2.6.2(1)). Hence $M = M^{\vee\vee}$ is a direct summand of $([\Lambda^{\mathrm{op}}]^\vee)^k$ and so M is Λ-projective because $[\Lambda^{\mathrm{op}}]^\vee$ is Λ-projective (cf. 2.6.2(1) and 4.8). Assume that $n > 0$ and our assertion holds true for $n-1$. Let $t \in \mathfrak{m}$ be regular for both Λ and M. Let $\overline{\Lambda} = \Lambda/t\Lambda$ and $\overline{M} = M/tM$. Then by hypothesis $\overline{M}$ is $\overline{\Lambda}$-projective. To see that M is Λ-projective, it suffices to show that M_t is Λ_t-projective (cf. 2.3.2). Recall that $\operatorname{pd}_{\Lambda_t} M_t = \sup_{\mathfrak{p}} \operatorname{pd}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through prime ideals $\mathfrak{p} \in \operatorname{Supp}_R M$ with $t \notin \mathfrak{p}$. Let $\mathfrak{p} \in \operatorname{Supp}_R M$ with $t \notin \mathfrak{p}$. Then $\operatorname{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = n - \operatorname{Kdim} R/\mathfrak{p}$ ([Se], p. 89, Chapter IV, Théorème 6). Hence $M_{\mathfrak{p}}$ is a Gorenstein $\Lambda_{\mathfrak{p}}$-module with $\operatorname{Kdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{Kdim} \Lambda_{\mathfrak{p}} < n$. Therefore by the hypothesis on n, $M_{\mathfrak{p}}$ is $\Lambda_{\mathfrak{p}}$-projective so that $\operatorname{pd}_{\Lambda_t} M_t = \sup_{\mathfrak{p}} \operatorname{pd}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$. Thus M_t is Λ_t-projective.

(7) It is enough to show the "only if" part. Let $k = n - \operatorname{depth}_R M$ and choose an exact sequence

$$0 \to L \to F_{k-1} \to F_{k-2} \to \ldots \to F_0 \to M \to 0$$

of Λ-modules so that each F_i is finitely generated and projective. Then L is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R L = n$ ([BH], Proposition 1.2.9). We have $\operatorname{id}_\Lambda L < \infty$ since $\operatorname{id}_\Lambda M < \infty$. Hence by (5), L is a Gorenstein Λ-module so that by (6), L is Λ-projective. Hence $\operatorname{pd}_\Lambda M < \infty$. ∎

For each $i \in \mathbb{Z}$ let $\mathrm{C}_i(\Lambda)$ denote the full subcategory of Λ-Mod consisting of all the finitely generated Λ-modules M such that either $M = (0)$ or M is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R M = i$.

We now come to the main result of this section.

Theorem 4.12. *Assume that Λ is a Cohen–Macaulay R-module with* $\operatorname{Kdim} \Lambda = n$. *Then the following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*
(2) $\operatorname{Ext}^p_\Lambda(M, \Lambda) = (0)$ *for all* $M \in \mathrm{C}_n(\Lambda)$ *and* $p \neq 0$.
(3) $\operatorname{Ext}^p_\Lambda(M, \Lambda) = (0)$ *for all* $0 \le i \le n$, $M \in \mathrm{C}_i(\Lambda)$, *and* $p \neq n - i$.

When this is the case, $\operatorname{Ext}^{n-i}_\Lambda(M, \Lambda) \in \mathrm{C}_i(\Lambda^{\mathrm{op}})$ *for all* $M \in \mathrm{C}_i(\Lambda)$ *and we have a natural isomorphism*

$$M \cong \operatorname{Ext}^{n-i}_\Lambda(\operatorname{Ext}^{n-i}_\Lambda(M, \Lambda), \Lambda).$$

Hence the correspondence $M \mapsto \mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda)$ *yields an equivalence between the categories* $\mathrm{C}_i(\Lambda)$ *and* $\mathrm{C}_i(\Lambda^{\mathrm{op}})$ $(0 \le i \le n)$.

Proof. By 4.11(3) we only have to show that (2) implies (1), and the equivalence of the categories. Look at the exact sequence

$$0 \to M \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \to \Lambda/J \to 0$$

of Λ-modules with each F_i finitely generated and projective. Then $M \in \mathrm{C}_n(\Lambda)$ ([BH], Proposition 1.2.9) and so $\mathrm{Ext}_{\Lambda}^{p}(M, \Lambda) = (0)$ for $p \neq 0$. Hence $\mathrm{Ext}_{\Lambda}^{i}(\Lambda/J, \Lambda) = (0)$ for $i > n$, so that $\mathrm{id}_{\Lambda} \Lambda = n$ (3.5(3)&(4)) and Λ is a Gorenstein R-algebra. Let us show that for all $0 \le i \le n$ and $M \in \mathrm{C}_i(\Lambda)$ there is a natural isomorphism

$$M \cong \mathrm{Ext}_{\Lambda}^{n-i}(\mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda), \Lambda).$$

We begin with the following.

Claim 1. $\mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) \in \mathrm{C}_i(\Lambda^{\mathrm{op}})$ *for all* $0 \le i \le n$ *and* $M \in \mathrm{C}_i(\Lambda)$.

Proof. Firstly note that $\mathrm{Kdim}_R \mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) \le i$ since $[(0) :_R M] \cdot \mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) = (0)$. Therefore we have nothing to prove for $i = 0$. Assume that $i > 0$ and our assertion holds true for $i - 1$. Let $t \in \mathfrak{m}$ be a non-zerodivisor for M. We put $\overline{M} = M/tM$ and apply the functors $\mathrm{Ext}_{\Lambda}^{p}(*, \Lambda)$ $(p \in \mathbb{Z})$ to the exact sequence $0 \to M \xrightarrow{t} M \to \overline{M} \to 0$. Then since $\overline{M} \in \mathrm{C}_{i-1}(\Lambda)$, by condition (3) we get the short exact sequence

$$0 \to \mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) \to \mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) \to \mathrm{Ext}_{\Lambda}^{n-i+1}(\overline{M}, \Lambda) \to 0.$$

Hence $\mathrm{Ext}_{\Lambda}^{n-i}(M, \Lambda) \in \mathrm{C}_i(\Lambda^{\mathrm{op}})$ as $\mathrm{Ext}_{\Lambda}^{n-i+1}(\overline{M}, \Lambda) \in \mathrm{C}_{i-1}(\Lambda^{\mathrm{op}})$ by the assumption on i. ■

We now proceed by descending induction on i. Thanks to 4.8 and condition (2), the proof of the case $i = n$ is the same as that of 2.6.4(3). We assume that $i < n$ and our assertion holds true for $i + 1$.

Claim 2. (1) *Let* M *be a finitely generated non-zero* Λ*-module with* $\mathrm{Kdim}_R M = j$. *Then for each* $j \le k \le n$ *there exists an exact sequence* $0 \to K \to L \to M \to 0$ *of* Λ*-modules with* $L \in \mathrm{C}_k(\Lambda)$. *We have* $K \in \mathrm{C}_{i+1}(\Lambda)$ *if* $M \in \mathrm{C}_i(\Lambda)$ *and* $L \in \mathrm{C}_{i+1}(\Lambda)$.

(2) *Let* $M_1, M_2 \in \mathrm{C}_i(\Lambda)$ *and let* $\varphi : M_1 \to M_2$ *be a homomorphism of* Λ*-modules. Let* $0 \to K_q \to L_q \to M_q \to 0$ $(q = 1, 2)$ *be exact sequences of* Λ*-modules with* $K_q, L_q \in \mathrm{C}_{i+1}(\Lambda)$. *Then one may choose exact sequences* $0 \to \widetilde{K}_q \to \widetilde{L}_q \to M_q \to 0$ $(q = 1, 2)$ *of* Λ*-modules with* $\widetilde{K}_q, \widetilde{L}_q \in \mathrm{C}_{i+1}(\Lambda)$ *and homomorphisms* $\alpha_q : \widetilde{K}_q \to K_q$, $\beta_q : \widetilde{L}_q \to L_q$ $(q = 1, 2)$, *and* $\varrho : \widetilde{K}_1 \to \widetilde{K}_2$,

$\eta : \widetilde{L}_1 \to \widetilde{L}_2$ so that the diagrams

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}_q \arrow[r] \arrow[d, "\alpha_q"] & \widetilde{L}_q \arrow[r] \arrow[d, "\beta_q"] & M_q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_q \arrow[r] & L_q \arrow[r] & M_q \arrow[r] & 0 \end{tikzcd} \qquad (q = 1, 2)$$

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}_1 \arrow[r] \arrow[d, "\varrho"] & \widetilde{L}_1 \arrow[r] \arrow[d, "\eta"] & M_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \widetilde{K}_2 \arrow[r] & \widetilde{L}_2 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

are commutative.

Proof. (1) Let $0 \to N \to F \to M \to 0$ be a presentation of the Λ-module M with F finitely generated and projective. Choose an ideal $\mathfrak{a}$ of R so that $\mathfrak{a}$ is generated by a Λ-regular sequence contained in $[(0) :_R M]$ of length $n - k$ (cf. 2.5.7). Let $L = F/\mathfrak{a}F$ and let $L \to M = M/\mathfrak{a}M$ be the homomorphism induced from $F \to M$. Then the exact sequence $0 \to K \to L \to M \to 0$ satisfies the required conditions (cf. [BH], Proposition 1.2.9).

(2) Firstly note that $\operatorname{Kdim} R/([(0) :_R L_1] \cap [(0) :_R L_2]) = i + 1$ because $\operatorname{Kdim} R/[(0) :_R L_q] = \operatorname{Kdim} L_q = i + 1$ $(q = 1, 2)$. This time we choose the ideal $\mathfrak{a}$ of R so that $\mathfrak{a}$ is generated by a Λ-regular sequence contained in $[(0) :_R L_1] \cap [(0) :_R L_2]$ of length $n - i - 1$. Then $\mathfrak{a}L_q = \mathfrak{a}M_q = (0)$ $(q = 1, 2)$. Let $0 \to N_q \to F_q \to M_q \to 0$ be a presentation of the Λ-module M_q with F_q finitely generated and projective $(q = 1, 2)$. Choose homomorphisms $\psi_q : F_q \to L_q$ $(q = 1, 2)$ and $\psi : F_1 \to F_2$ so that the diagrams

$$\begin{tikzcd} F_q \arrow[r] \arrow[d, "\psi_q"'] & M_q \arrow[d, equal] \\ L_q \arrow[r] & M_q \end{tikzcd} \quad (q = 1, 2) \qquad \begin{tikzcd} F_1 \arrow[r] \arrow[d, "\psi"'] & M_1 \arrow[d] \\ F_2 \arrow[r] & M_2 \end{tikzcd}$$

are commutative. Let $\widetilde{L}_q = F_q/\mathfrak{a}F_q$ and $\beta_q : \widetilde{L}_q \to L_q$, $\eta : \widetilde{L}_1 \to \widetilde{L}_2$ be the induced homomorphisms. Then letting $\widetilde{K}_q = \operatorname{Ker}(\widetilde{L}_q \to M_q)$, we get the required commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}_q \arrow[r] \arrow[d, "\alpha_q"] & \widetilde{L}_q \arrow[r] \arrow[d, "\beta_q"] & M_q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_q \arrow[r] & L_q \arrow[r] & M_q \arrow[r] & 0 \end{tikzcd} \qquad (q = 1, 2)$$

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}_1 \arrow[r] \arrow[d, "\varrho"] & \widetilde{L}_1 \arrow[r] \arrow[d, "\eta"] & M_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \widetilde{K}_2 \arrow[r] & \widetilde{L}_2 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

where the homomorphisms $\alpha_q : \widetilde{K}_q \to K_q$ and $\varrho : \widetilde{K}_1 \to \widetilde{K}_2$ are the restrictions of β_q and η_q respectively. ∎

Now let $M \in \mathrm{C}_i(\Lambda)$ and choose an exact sequence

$$(*) \qquad 0 \to K \to L \to M \to 0$$

of Λ-modules so that $K, L \in \mathrm{C}_{i+1}(\Lambda)$. Then applying the functors $\mathrm{Ext}^p_\Lambda(*, \Lambda)$ $(p \in \mathbb{Z})$, by condition (3) we have the exact sequence

$$0 \to \mathrm{Ext}^{n-i-1}_\Lambda(L, \Lambda) \to \mathrm{Ext}^{n-i-1}_\Lambda(K, \Lambda) \xrightarrow{\Delta} \mathrm{Ext}^{n-i}_\Lambda(M, \Lambda) \to 0$$

of Λ^{op}-modules, where Δ denotes the connecting homomorphism. Since

$$\mathrm{Ext}^{n-i-1}_\Lambda(L, \Lambda), \mathrm{Ext}^{n-i-1}_\Lambda(K, \Lambda) \in \mathrm{C}_{i+1}(\Lambda^{\mathrm{op}})$$

and

$$\mathrm{Ext}^{n-i}_\Lambda(M, \Lambda) \in \mathrm{C}_i(\Lambda^{\mathrm{op}})$$

by Claim 1, by condition (3) applied to the Gorenstein R-algebra Λ^{op} we get the exact sequence

$$(**) \qquad 0 \to \mathrm{Ext}^{n-i-1}_\Lambda(\mathrm{Ext}^{n-i-1}_\Lambda(K, \Lambda), \Lambda) \to \mathrm{Ext}^{n-i-1}_\Lambda(\mathrm{Ext}^{n-i-1}_\Lambda(L, \Lambda), \Lambda) \to \mathrm{Ext}^{n-i}_\Lambda(\mathrm{Ext}^{n-i}_\Lambda(M, \Lambda), \Lambda) \to 0$$

of Λ-modules. By the hypothesis on i we may identify

$$K = \mathrm{Ext}^{n-i-1}_\Lambda(\mathrm{Ext}^{n-i-1}_\Lambda(K, \Lambda), \Lambda)$$

and

$$L = \mathrm{Ext}^{n-i-1}_\Lambda(\mathrm{Ext}^{n-i-1}_\Lambda(L, \Lambda), \Lambda).$$

Then comparing sequence $(**)$ with the original exact sequence $(*)$, we have the required isomorphism

$$\theta_M : M \to \mathrm{Ext}^{n-i}_\Lambda(\mathrm{Ext}^{n-i}_\Lambda(M, \Lambda), \Lambda)$$

of Λ-modules.

Claim 3. *The isomorphism θ_M is natural in M and does not depend on the choice of exact sequences $(*)$ above.*

Proof. Let $j = n - i$ and let

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] \arrow[d, "\varrho"'] & L_1 \arrow[r] \arrow[d, "\eta"'] & M_1 \arrow[r] \arrow[d, "\varphi"'] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] & L_2 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows such that $K_q, L_q \in \mathrm{C}_{i+1}(\Lambda)$ and $M_q \in \mathrm{C}_i(\Lambda)$ $(q = 1, 2)$. Apply the functors $\mathrm{Ext}^p_\Lambda(\mathrm{Ext}^p_\Lambda(*, \Lambda), \Lambda)$ $(p \in \mathbb{Z})$ to get commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & K_q \arrow[r] \arrow[d, equal] & L_q \arrow[r] \arrow[d, equal] & M_q \arrow[r] \arrow[d, "\theta_M"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(K_q, \Lambda)) \arrow[r] & \mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(L_q, \Lambda)) \arrow[r] & \mathrm{Ext}^{j}_\Lambda(\mathrm{Ext}^{j}_\Lambda(M_q, \Lambda)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows $(q = 1, 2)$. Then since $L_1 \to M_1$ is an epimorphism, all the faces in the diagram

$$\begin{tikzcd}
& L_1 \arrow[rr] \arrow[dl, equal] \arrow[dd, "\eta"] & & M_1 \arrow[dl, "\theta_{M_1}"'] \arrow[dd, "\varphi"] \\
\mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(L_1, \Lambda)) \arrow[rr, "\Delta"] \arrow[dd, "\mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(\eta, \Lambda))"'] & & \mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(M_1, \Lambda)) \arrow[dd, "\mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(\varphi, \Lambda))"'] & \\
& L_2 \arrow[rr] \arrow[dl, equal] & & M_2 \arrow[dl, "\theta_{M_2}"'] \\
\mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(L_2, \Lambda)) \arrow[rr, "\Delta"'] & & \mathrm{Ext}^{j-1}_\Lambda(\mathrm{Ext}^{j-1}_\Lambda(M_2, \Lambda)) &
\end{tikzcd}$$

is commutative. Thanks to Claim 2(2), letting $M = M_1 = M_2$ and $\varphi = 1_M$, the commutativity of the particular face

$$\begin{tikzcd}
M_1 \arrow[r, "\theta_{M_1}"] \arrow[d, "\varphi"] & \mathrm{Ext}^j_\Lambda(\mathrm{Ext}^j_\Lambda(M_1, \Lambda), \Lambda) \arrow[d, "\mathrm{Ext}^j_\Lambda(\mathrm{Ext}^j_\Lambda(\varphi, \Lambda), \Lambda)"] \\
M_2 \arrow[r, "\theta_{M_2}"] & \mathrm{Ext}^j_\Lambda(\mathrm{Ext}^j_\Lambda(M_2, \Lambda), \Lambda)
\end{tikzcd}$$

shows that the isomorphism θ_M does not depend on the choice of exact sequences $(*)$ above and hence its naturality does not either. ∎

COROLLARY 4.13. *Suppose Λ is a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$ *and let S be a simple Λ-module. Then* $\mathrm{Ext}^n_\Lambda(S, \Lambda)$ *is a simple* Λ^{op}*-module.*

For each finitely generated non-zero Λ-module M we put $\mathrm{j}(M) = \mathrm{Kdim}_R\,\Lambda - \mathrm{Kdim}_R\,M$.

The next result 4.14(1) shows that if Λ is a Gorenstein R-algebra, then every finitely generated Λ-module M satisfies Auslander's condition so that Λ is an Auslander–Gorenstein ring in the sense of [Bj]. The result 4.14(2) answers a question posed by [Bj], p. 144, in our context.

COROLLARY 4.14. *Let Λ be a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$ *and let M be a finitely generated non-zero Λ-module.*

(1) *Let $j \in \mathbb{Z}$. Then* $\mathrm{Ext}^i_\Lambda(X, \Lambda) = (0)$ *for any Λ^{op}-submodule X of* $\mathrm{Ext}^j_\Lambda(M, \Lambda)$ *and for $i < j$.*

(2) *Let* $j = \mathrm{j}(M)$. *Then* $\mathrm{j}(Y) = j$ *for every non-zero* Λ^{op}*-submodule* Y *of* $\mathrm{Ext}^j_\Lambda(M,\Lambda)$.

Proof. (1) Assume $\mathrm{Ext}^i_\Lambda(X,\Lambda) \neq (0)$ and choose $\mathfrak{p} \in \mathrm{Supp}_R X$ so that $\mathrm{Kdim}_R X = \mathrm{Kdim}\, R/\mathfrak{p}$. Then by 4.11(2), $n \leq \mathrm{Kdim}\, R/\mathfrak{p} + i$, while $n = \mathrm{Kdim}\, R/\mathfrak{p} + \mathrm{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p}$ ([Se], p. 89, Chapter IV, Théorème 6). Hence $\mathrm{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p} = \mathrm{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p} \leq i$ so that $\mathrm{Ext}^j_{\Lambda_\mathfrak{p}}(M_\mathfrak{p},\Lambda_\mathfrak{p}) = (0)$ as $i < j$. Thus $X_\mathfrak{p} = (0)$, which is absurd.

(2) Let $i = \mathrm{Kdim}_R M$ and choose an exact sequence $0 \to K \to L \to M \to 0$ of Λ-modules with $L \in \mathrm{C}_i(\Lambda)$ (Claim 2(1), proof of 4.12). As $\mathrm{Kdim}_R K \leq i$ we get $\mathrm{Ext}^{j-1}_\Lambda(K,\Lambda) = (0)$ by 4.11(2) and so the exact sequence yields the embedding

$$0 \to \mathrm{Ext}^j_\Lambda(M,\Lambda) \to \mathrm{Ext}^j_\Lambda(L,\Lambda).$$

Hence $\mathrm{Ass}_R Y \subseteq \mathrm{Ass}_R \mathrm{Ext}^j_\Lambda(L,\Lambda)$. Therefore $\mathrm{Kdim}_R Y = \mathrm{Kdim}_R \mathrm{Ext}^j_\Lambda(L,\Lambda) = i$ by [BH], Theorem 2.1.2(a), since $\mathrm{Ext}^j_\Lambda(L,\Lambda) \in \mathrm{C}_i(\Lambda^{\mathrm{op}})$ by 4.12. Thus $\mathrm{j}(Y) = j$. ■

Let $\mathcal{A}, \mathcal{P}, \mathcal{X}$ and $\mathcal{Y}$ denote the full subcategories of Λ-Mod such that

$$\begin{aligned}
\mathcal{A} &= \{\text{finitely generated } \Lambda\text{-modules } M\},\\
\mathcal{P} &= \{P \in A \mid P \text{ is } \Lambda\text{-projective}\},\\
\mathcal{X} &= \mathrm{C}_n(\Lambda),\\
\mathcal{Y} &= \{Y \in A \mid \mathrm{id}_\Lambda Y < \infty\}.
\end{aligned}$$

The following result shows that $(\mathcal{X},\mathcal{Y})$ is an AB-context [AB] for A in the sense of Hashimoto.

COROLLARY 4.15. *Suppose* Λ *is a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$. *Then*:

(1) $\mathcal{X} \bigcap \mathcal{Y} = \mathcal{P}$.
(2) $\mathrm{Ext}^p_\Lambda(X,P) = (0)$ *for all* $X \in \mathcal{X}$, $P \in \mathcal{P}$ *and* $p > 0$.
(3) (a) $\mathcal{A}$ *is abelian and* $\mathcal{X}$ *and* $\mathcal{Y}$ *are additive categories.*
   (b) $X \in \mathcal{X}$ *if* $Y \in \mathcal{X}$ *and* X *is a direct summand of* Y.
   (c) $X \in \mathcal{Y}$ *if* $Y \in \mathcal{Y}$ *and* X *is a direct summand of* Y.

(4) *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence in* $\mathcal{A}$. *Then*

   (a) $Y \in \mathcal{X}$ *if* $X, Z \in \mathcal{X}$.
   (b) $X \in \mathcal{X}$ *if* $Y, Z \in \mathcal{X}$.
   (c) $Z \in \mathcal{Y}$ *if* $X, Y \in \mathcal{Y}$.
   (d) $Y \in \mathcal{Y}$ *if* $X, Z \in \mathcal{Y}$.

(5) *Each* $M \in \mathcal{A}$ *has a finite* $\mathcal{X}$*-resolution.*
(6) *Let* $X \in \mathcal{X}$. *Then there exists an exact sequence* $0 \to X \to P \to Y \to 0$ *in* $\mathcal{A}$ *with* $P \in \mathcal{P}$ *and* $Y \in \mathcal{X}$.

Proof. (1) See 4.11(6).

(2) See 4.11(3).

(3)&(4) Use 3.2 and [BH], Proposition 1.2.9.

(5) See the proof of 4.11(7).

(6) Let $[*]^* = \mathrm{Hom}_\Lambda(*, \Lambda)$ be the Λ-dual. Firstly take a presentation $0 \to L \to F \to X^* \to 0$ of the Λ^{op}-module X^* with F finitely generated and projective. Then as $L, X^* \in \mathrm{C}_n(\Lambda^{\mathrm{op}})$ by 4.12, identifying $X = X^{**}$, we get the required exact sequence $0 \to X \to F^* \to L^* \to 0$ of Λ-modules with $Y = L^* \in \mathcal{X} = \mathrm{C}_n(\Lambda)$. ∎

For a Λ-module M let

$$\mathrm{Ext}_\Lambda\text{-}\dim M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq (0)\}.$$

In general $\mathrm{Ext}_\Lambda\text{-}\dim M \le \mathrm{pd}_\Lambda M$, and we have equality if $\mathrm{pd}_\Lambda M < \infty$ and M is finitely generated.

The next result generalizes [G1] (Theorem 1).

COROLLARY 4.16. *Let R be an arbitrary commutative Noetherian ring and Λ a module-finite R-algebra. We consider the following three conditions.*

(1) *Λ is a Gorenstein R-algebra.*

(2) $\mathrm{Ext}_\Lambda\text{-}\dim M < \infty$ *for every finitely generated Λ-module M.*

(3) $\mathrm{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p} < \infty$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$.

Then the implications (1)⇒(2)⇒(3) *hold true.*

Proof. (2)⇒(3). Let $\mathfrak{A} = \mathrm{J}(\Lambda_\mathfrak{p}) \cap \Lambda$ and $k = \mathrm{Ext}_\Lambda\text{-}\dim \Lambda/\mathfrak{A}$. Then

$$\mathrm{Ext}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/\mathrm{J}(\Lambda_\mathfrak{p}), \Lambda_\mathfrak{p}) = R_\mathfrak{p} \otimes_R \mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{A}, \Lambda) = (0) \quad \text{for all } i > k.$$

Hence $\mathrm{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p} < \infty$ (cf. 3.5(3)).

(1)⇒(2). Firstly we note the following, which readily follows from the long exact sequence of $\mathrm{Ext}^i_\Lambda(M_j, \Lambda)$'s.

CLAIM 1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated Λ-modules. Then if any two of M_j's have finite* Ext_Λ-*dimension, so does the remaining one.*

Now assume that Λ is a Gorenstein R-algebra but $\mathrm{Ext}_\Lambda\text{-}\dim M$ is infinite for some finitely generated Λ-module M. By Claim 1 we may assume M to be cyclic, say $M = \Lambda/L$ for some left ideal L in Λ. Choose L so that it is maximal among the left ideals L in Λ with $\mathrm{Ext}_\Lambda\text{-}\dim \Lambda/L$ infinite. Then we have

CLAIM 2. *$\sharp\mathrm{Ass}_\Lambda M = 1$.*

Proof. Let $\mathfrak{F} = \mathrm{Ass}_\Lambda M$ and assume $\sharp\mathfrak{F} > 1$. Choose a family $\{\mathrm{L}(P)\}_{P\in\mathfrak{F}}$ of left ideals of Λ satisfying the three conditions stated in 2.2.8. Choose

$\emptyset \neq \mathfrak{G} \subseteq \mathfrak{F}$ so that $\mathfrak{G} \neq \mathfrak{F}$ and put $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{G}$. Then $L \neq \bigcap_{P \in \mathfrak{G}} \mathrm{L}(P)$ and $L \neq \bigcap_{P \in \mathfrak{G}'} \mathrm{L}(P)$. We look at the exact sequence

$$0 \to M \to \Lambda/\bigcap_{P \in \mathfrak{G}} \mathrm{L}(P) \oplus \Lambda/\bigcap_{P \in \mathfrak{G}'} \mathrm{L}(P) \to \Lambda/\Big[\bigcap_{P \in \mathfrak{G}} \mathrm{L}(P) + \bigcap_{P \in \mathfrak{G}'} \mathrm{L}(P)\Big] \to 0.$$

The maximality of L implies that the Ext_Λ-dimensions of $\Lambda/\bigcap_{P \in \mathfrak{G}} \mathrm{L}(P)$, $\Lambda/\bigcap_{P \in \mathfrak{G}'} \mathrm{L}(P)$, and $\Lambda/[\bigcap_{P \in \mathfrak{G}} \mathrm{L}(P) + \bigcap_{P \in \mathfrak{G}'} \mathrm{L}(P)]$ are finite and so by Claim 1, $\mathrm{Ext}_\Lambda\text{-}\dim M$ must be finite, contrary to assumption. Thus $\sharp \mathrm{Ass}_\Lambda M = 1$. ∎

Let $\mathrm{Ass}_\Lambda M = \{P\}$ and $\mathfrak{p} = P \cap R$. Then $\mathrm{Ass}_R M = \{\mathfrak{p}\}$ (2.2.7(2)).

Claim 3. *$\mathfrak{p}^n \Lambda \subseteq L$ for some integer $n > 0$.*

Proof. As $\mathrm{Ass}_{R_\mathfrak{p}} M_\mathfrak{p} = \{\mathfrak{p}R_\mathfrak{p}\}$, $(\mathfrak{p}R_\mathfrak{p})^n \cdot M_\mathfrak{p} = (0)$ for some $n > 0$. Hence $\mathfrak{p}^n \Lambda \subseteq L$ since the canonical map $M \to M_\mathfrak{p}$ is injective. ∎

Let $k = \mathrm{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p}$. Then since Λ is a Cohen–Macaulay R-module, the prime ideal $\mathfrak{p}$ of R contains a Λ-regular sequence $x_1, x_2, \ldots, x_k$ of length k (cf. proof of [Ma], Theorem 30 iii)). Let $n > 0$ be an integer such that $\mathfrak{p}^n \Lambda \subseteq L$. Then the sequence $x_1^n, x_2^n, \ldots, x_k^n$ is still Λ-regular ([Ma], Theorem 26) and $(x_1^n, x_2^n, \ldots, x_k^n)\Lambda \subseteq L$. Let $\overline{\Lambda} = \Lambda/(x_1^n, x_2^n, \ldots, x_k^n)\Lambda$. Then because $\mathrm{Ext}^i_\Lambda(M, \Lambda) \cong \mathrm{Ext}_{\overline{\Lambda}}(M, \overline{\Lambda})$ for all $i \in \mathbb{Z}$, passing to the ring $\overline{\Lambda}$, we may assume that $k = \mathrm{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p} = 0$. Hence $\mathfrak{p} \in \mathrm{Ass}_R \Lambda$.

Claim 4. *Let $t \in R \setminus \mathfrak{p}$. Then there is an integer $k = k(t)$ depending on t such that the map $\mathrm{Ext}^i_\Lambda(M, \Lambda) \xrightarrow{t} \mathrm{Ext}^i_\Lambda(M, \Lambda)$ is bijective for all $i > k$.*

Proof. As $\mathrm{Ass}_R M = \{\mathfrak{p}\}$, t is a non-zerodivisor for M. Hence $L \neq L + t\Lambda$. Look at the exact sequence $0 \to M \xrightarrow{t} M \to \Lambda/[L + t\Lambda] \to 0$. Then the maximality of L shows that $k = \mathrm{Ext}_\Lambda\text{-}\dim \Lambda/[L + t\Lambda] < \infty$. Therefore $\mathrm{Ext}^i_\Lambda(M, L) \xrightarrow{t} \mathrm{Ext}^i_\Lambda(M, \Lambda)$ is an isomorphism if $i > k$. ∎

This claim allows us, in order to produce a contradiction, to freely localize Λ at any $t \in R \setminus \mathfrak{p}$. For example, choose $t \in \bigcap_{\mathfrak{q} \in \mathrm{Ass}_R \Lambda \setminus \{\mathfrak{p}\}} \mathfrak{q}$ so that $t \notin \mathfrak{p}$ and passing to the algebra Λ_t, assume that $\mathrm{Ass}_R \Lambda = \{\mathfrak{p}\}$. Hence $\mathfrak{p}^N \Lambda = (0)$ for some integer $N > 0$. Then since each $\mathfrak{p}^i \Lambda/\mathfrak{p}^{i+1}\Lambda$ is a finitely generated $R/\mathfrak{p}$-module, we may choose an element $t \in R \setminus \mathfrak{p}$ so that $[\mathfrak{p}^i \Lambda/\mathfrak{p}^{i+1}\Lambda]_t$ is R_t-free for every $0 \leq i \leq N - 1$ (cf. [Bo], Ch. 2, Sect. 5, No. 1). Hence we may assume, from the beginning, that $\mathfrak{p}^i \Lambda/\mathfrak{p}^{i+1}\Lambda$ is $R/\mathfrak{p}$-free for every $0 \leq i \leq N - 1$. Let $\mathfrak{q} \in \mathrm{Supp}_R \Lambda$ and look at the canonical exact sequences

$$0 \to \mathfrak{p}^{i+1}\Lambda_\mathfrak{q} \to \mathfrak{p}^i \Lambda_\mathfrak{q} \to \mathfrak{p}^i \Lambda_\mathfrak{q}/\mathfrak{p}^{i+1}\Lambda_\mathfrak{q} \to 0$$

$(0 \leq i \leq N - 1)$ of $\Lambda_\mathfrak{q}$-modules. Then because each $\mathfrak{p}^i \Lambda_\mathfrak{q}/\mathfrak{p}^{i+1}\Lambda_\mathfrak{q}$ is a free $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$-module and $\mathfrak{p}^N \Lambda_\mathfrak{q} = (0)$, descending induction on i shows that

$\operatorname{depth}_{R_\mathfrak{q}} \Lambda_\mathfrak{q} = \operatorname{depth}_{R_\mathfrak{q}} R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ ([BH], 1.2.9). Since $\operatorname{Ass}_R M = \{\mathfrak{p}\}$, the same technique works to reduce the problem to the case where $\operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} = \operatorname{depth}_{R_\mathfrak{q}} R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Supp}_R \Lambda = \operatorname{Supp}_R M$. Now notice that

$$\begin{aligned}\operatorname{Kdim}_{R_\mathfrak{q}} \Lambda_\mathfrak{q} \ge \operatorname{Kdim}_{R_\mathfrak{q}} M_\mathfrak{q} \ge \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} &= \operatorname{depth}_{R_\mathfrak{q}} R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q} \\ &= \operatorname{depth}_{R_\mathfrak{q}} \Lambda_\mathfrak{q} = \operatorname{Kdim}_{R_\mathfrak{q}} \Lambda_\mathfrak{q}.\end{aligned}$$

Then $M_\mathfrak{q}$ is a Cohen–Macaulay $R_\mathfrak{q}$-module with $\operatorname{Kdim}_{R_\mathfrak{q}} M_\mathfrak{q} = \operatorname{Kdim}_{R_\mathfrak{q}} \Lambda_\mathfrak{q}$. Hence by 4.12,

$$R_\mathfrak{q} \otimes_R \operatorname{Ext}^i_\Lambda(M, \Lambda) = \operatorname{Ext}^i_{\Lambda_\mathfrak{q}}(M_\mathfrak{q}, \Lambda_\mathfrak{q}) = (0)$$

for all $i > 0$ and $\mathfrak{q} \in \operatorname{Supp}_R \Lambda$. Therefore $\operatorname{Ext}^i_\Lambda(M, \Lambda) = (0)$ for all $i > 0$, which is the required contradiction. Thus $\operatorname{Ext}_\Lambda\text{-}\dim M < \infty$ for every finitely generated Λ-module M. ■

5. Characterization of Gorenstein R-algebras in terms of Bass numbers. The purpose of this section is to characterize Gorenstein R-algebras in terms of Bass numbers $\mu^i(P, \Lambda)$ (see Definition 2.5.4). To begin with we record

LEMMA 5.1. *Let $P \in \operatorname{Spec} \Lambda$, $i \in \mathbb{Z}$, M a Λ-module, and $\mu^i(P, M)$ the ith Bass number of M with respect to P.*

(1) *Let S be a multiplicative system in R with $P \cap f(S) = \emptyset$. Then $\mu^i(S^{-1}P, S^{-1}M) = \mu^i(P, M)$.*

(2) *Suppose R is a local ring and $P \in \operatorname{Max} \Lambda$. Then*

$$\mu^i(P, M) = \frac{\ell_{\Lambda/P}(\operatorname{Ext}^i_\Lambda(\Lambda/P, M))}{\mathrm{m}(P)} = \frac{\ell_{\Lambda/P}(\operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{E}^i_\Lambda(M)))}{\mathrm{m}(P)}.$$

(*Here $\ell_{\Lambda/P}(*)$ denotes the length of composition series.*)

(3) *$0 \le \mu^i(P, M) \in \mathbb{Q}$ if M is finitely generated.*

(4) *Suppose $(R, \mathfrak{m})$ is a local ring and $P \in \operatorname{Max} \Lambda$. Let $R^\#$ denote the $\mathfrak{m}$-adic completion of R. Then $\mu^i(P^\#, M^\#) = \mu^i(P, M)$.*

(5) *Let $t \in P \cap R$ be a non-zerodivisor for both Λ and M. Let $\overline{P} = P/t\Lambda$ and $\overline{M} = M/tM$. Then $\mu^{i-1}(\overline{P}, \overline{M}) = \mu^i(P, M)$.*

Proof. Let us maintain the same notation as in 2.5. We look at the direct sum decomposition $\mathrm{E}^i_\Lambda(M) = \bigoplus_{Q \in \operatorname{Spec} \Lambda} \mathrm{I}(Q)^{(\Omega^i(Q,M))}$.

(1) Let $Q \in \operatorname{Spec} \Lambda$ with $Q \cap f(S) = \emptyset$. Then every $s \in S$ acts on $\mathrm{E}_\Lambda(\Lambda/Q)$ as an isomorphism so that by 2.1.3,

$$\mathrm{E}_\Lambda(\Lambda/Q) = S^{-1}\mathrm{E}_\Lambda(\Lambda/Q) = \mathrm{E}_{S^{-1}\Lambda}(S^{-1}\Lambda/S^{-1}Q),$$

whence $\mathrm{m}(Q) = \mathrm{m}(S^{-1}Q)$ and $\mathrm{I}(Q) = S^{-1}\mathrm{I}(Q) = \mathrm{I}(S^{-1}Q)$. On the other hand $S^{-1}\mathrm{I}(Q) = (0)$ if $Q \cap f(S) \neq \emptyset$, since $S^{-1}\mathrm{E}_\Lambda(\Lambda/Q) = (0)$. Consequently,

$$\begin{aligned} S^{-1}\mathrm{E}^i_\Lambda(M) &= \bigoplus_{Q \in \operatorname{Spec}\Lambda} S^{-1}\mathrm{I}(Q)^{(\Omega^i(Q,M))} \\ &= \bigoplus_{Q \in \operatorname{Spec}\Lambda \text{ with } Q \cap f(S) = \emptyset} \mathrm{I}(S^{-1}Q)^{(\Omega^i(Q,M))} \end{aligned}$$

and we have $\mu^i(S^{-1}P, S^{-1}M) = \mu^i(P, M)$ as $\mathrm{m}(S^{-1}P) = \mathrm{m}(P)$.

(2) Let $\mathrm{S}(P)$ denote the simple Λ/P-module. Then from the isomorphisms

$$\begin{aligned} \operatorname{Ext}^i_\Lambda(\Lambda/P, M) &= \operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{E}^i_\Lambda(M)) \\ &= \bigoplus_{Q \in \operatorname{Spec}\Lambda} \operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{I}(Q))^{(\Omega^i(Q,M))} \\ &= \operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{I}(P))^{(\Omega^i(P,M))} \qquad (2.5.1) \\ &= \mathrm{S}(P)^{(\Omega^i(P,M))} \end{aligned}$$

of Λ/P-modules we have

$$\ell_{\Lambda/P}(\operatorname{Ext}^i_\Lambda(\Lambda/P, M)) = \ell_{\Lambda/P}(\operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{E}^i_\Lambda(M))) = \sharp\Omega^i(P, M).$$

Hence the results follow.

(3) Passing to the localization $\Lambda_\mathfrak{p}$ with $\mathfrak{p} = P \cap R$, by (1) we may assume that R is a local ring and $P \in \operatorname{Max}\Lambda$. Hence assertion (3) immediately follows from (2) as $\ell_{\Lambda/P}(\operatorname{Ext}^i_\Lambda(\Lambda/P, M))$ is finite.

(4) This follows from (2) and the isomorphisms

$$\operatorname{Ext}^i_\Lambda(\Lambda/P, M) = R^\# \otimes_R \operatorname{Ext}^i_\Lambda(\Lambda/P, M) = \operatorname{Ext}^i_{\Lambda^\#}(\Lambda^\#/P^\#, M^\#).$$

Note that $\mathrm{m}(P) = \mathrm{m}(P^\#)$ since $\Lambda/P = R^\# \otimes_R \Lambda/P = \Lambda^\#/P^\#$.

(5) Let $\overline{\Lambda} = \Lambda/t\Lambda$. We have

$$\begin{aligned} \mathrm{E}^{i-1}_{\overline{\Lambda}}(\overline{\Lambda}) &= \operatorname{Hom}_\Lambda(\overline{\Lambda}, \mathrm{E}^i_\Lambda(\Lambda)) \qquad (2.4.5(1)) \\ &= \bigoplus_{Q \in \operatorname{Spec}\Lambda} \operatorname{Hom}_\Lambda(\overline{\Lambda}, \mathrm{I}(Q))^{(\Omega^i(Q,M))} \\ &= \bigoplus_{Q \in \operatorname{Spec}\Lambda \text{ with } f(t) \in Q} \operatorname{Hom}_\Lambda(\overline{\Lambda}, \mathrm{I}(Q))^{(\Omega^i(Q,M))} \qquad (2.2.3(3)). \end{aligned}$$

Let $Q \in \operatorname{Spec}\Lambda$ be such that $t \in Q$ and put $\overline{Q} = Q/t\Lambda$. Then because $\operatorname{Hom}_\Lambda(\overline{\Lambda}, \mathrm{E}_\Lambda(\Lambda/Q)) = \mathrm{E}_{\overline{\Lambda}}(\overline{\Lambda}/\overline{Q})$, we have $\operatorname{Hom}_\Lambda(\overline{\Lambda}, \mathrm{I}(Q)) = \mathrm{I}(\overline{Q})$ and $\mathrm{m}(Q) = \mathrm{m}(\overline{Q})$. Hence

$$\mathrm{E}^{i-1}_{\overline{\Lambda}}(\overline{\Lambda}) = \bigoplus_{Q \in \operatorname{Spec}\Lambda \text{ with } f(t) \in Q} \mathrm{I}(\overline{Q})^{(\Omega^i(Q,M))}$$

so that $\mu^{i-1}(\overline{P}, \overline{M}) = \mu^i(P, M)$. ■

We now give the characterization (5.2) of Gorenstein R-algebras Λ in terms of Bass numbers $\mu^i(P,\Lambda)$. Condition (3) in it corresponds to the homogeneity condition in [BHa2].

THEOREM 5.2. *The following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*
(2) (a) $\operatorname{Kdim}\Lambda_{P\cap R} = \operatorname{ht}_\Lambda P$ *for every* $P \in \operatorname{Spec}\Lambda$.
   (b) *Let* $P \in \operatorname{Spec}\Lambda$ *and* $i \in \mathbb{Z}$. *Then* $\mu^i(P,\Lambda) > 0$ *if and only if* $i = \operatorname{ht}_\Lambda P$.
(3) (a) $\operatorname{Kdim}\Lambda_{P\cap R} = \operatorname{ht}_\Lambda P$ *for every* $P \in \operatorname{Max}\Lambda$.
   (b) *Let* $P \in \operatorname{Max}\Lambda$ *and* $i \in \mathbb{Z}$. *Then* $\mu^i(P,\Lambda) > 0$ *if and only if* $i = \operatorname{ht}_\Lambda P$.

To prove the theorem we need the following, which assures the catenarity in Cohen–Macaulay R-algebras.

PROPOSITION 5.3 ([GN1], Corollary (1.3)). *Assume that R is a local ring and Λ is a Cohen–Macaulay R-module. Then Λ is a catenary ring and* $\operatorname{Kdim}\Lambda = \operatorname{Kdim}\Lambda/Q + \operatorname{ht}_\Lambda Q$ *for every* $Q \in \operatorname{Spec}\Lambda$. *The equality* $k = \operatorname{ht}_\Lambda Q - \operatorname{ht}_\Lambda P$ *holds true for every pair $P \subseteq Q$ of prime ideals in Λ and for every saturated chain $P = P_0 \subset P_1 \subset \ldots \subset P_k = Q$ of prime ideals between P and Q.*

Proof of Theorem 5.2. (1)$\Rightarrow$(2). Let $P \in \operatorname{Spec}\Lambda$ and $\mathfrak{p} = P\cap R$. Then by 5.3 we have $\operatorname{Kdim}\Lambda_\mathfrak{p} = \operatorname{ht}_{\Lambda_\mathfrak{p}} P\Lambda_\mathfrak{p} = \operatorname{ht}_\Lambda P$, because $\Lambda_\mathfrak{p}$ is a Cohen–Macaulay $R_\mathfrak{p}$-module and $P\Lambda_\mathfrak{p} \in \operatorname{Max}\Lambda_\mathfrak{p}$. If $\mu^i(P,\Lambda) > 0$, then $P \in \operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(\Lambda)$. Therefore $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(\Lambda)$ by 2.2.7(2) and so $\operatorname{Kdim}\Lambda_\mathfrak{p} = i$ by 4.1(3), whence $\operatorname{ht}_\Lambda P = i$. Conversely, let $i = \operatorname{ht}_\Lambda P$. Then since $P\Lambda_\mathfrak{p} \in \operatorname{Max}\Lambda_\mathfrak{p}$ and $\operatorname{Kdim}\Lambda_\mathfrak{p} = i$, by 4.9 and 2.5.1 we deduce that $P\Lambda_\mathfrak{p} \in \operatorname{Ass}_{\Lambda_\mathfrak{p}} \mathrm{E}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p})$. Hence $P \in \operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(\Lambda)$ by 2.4.2(2) and so $\mu^i(P,\Lambda) > 0$.

(2)$\Rightarrow$(3). This is clear.

(3)$\Rightarrow$(1). Let $\mathfrak{p} \in \operatorname{Max}R \cap \operatorname{Supp}_R\Lambda$ and choose $P \in \operatorname{Max}\Lambda$ so that $\mathfrak{p} = P \cap R$. Let $i = \operatorname{ht}_\Lambda P$. Then by (a), $\operatorname{Kdim}\Lambda_\mathfrak{p} = i$. Now assume $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{E}^k_\Lambda(\Lambda)$ for some $k \neq i$. Then by 2.2.7(2) we may choose $Q \in \operatorname{Ass}_\Lambda \mathrm{E}^k_\Lambda(\Lambda)$ so that $\mathfrak{p} = Q \cap R$. Then $Q \in \operatorname{Max}\Lambda$ since $\mathfrak{p} \in \operatorname{Max}R$ (2.0.1(4)). Therefore by assumption (b), $k = \operatorname{ht}_\Lambda Q$ while by (a), $\operatorname{ht}_\Lambda Q = \operatorname{Kdim}\Lambda_\mathfrak{p} = i$. This is impossible. Thus $P \notin \operatorname{Ass}_R \mathrm{E}^k_\Lambda(\Lambda)$ if $k \neq i$. Hence by 2.4.2(2) and 4.3, $\Lambda_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-algebra. ∎

For the moment let $P \in \operatorname{Max}\Lambda$ and $\mathfrak{p} = P\cap R$. Then $\ell_R(\Lambda/P) < \infty$ as $\mathfrak{p} \in \operatorname{Max}R$. Hence every $s \in R\setminus\mathfrak{p}$ acts on Λ/P as an automorphism, so that the canonical map $\operatorname{Ext}^i_\Lambda(\Lambda/P, M) \to [\operatorname{Ext}^i_\Lambda(\Lambda/P, M)]_\mathfrak{p} = \operatorname{Ext}^i_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p}, M_\mathfrak{p})$ is bijective for every Λ-module M and $i \in \mathbb{Z}$. Therefore $\operatorname{Ext}^i_\Lambda(\Lambda/P, M) \neq (0)$ if and only if $\mu^i(P,M) > 0$ (cf. 5.1(1)) and $\operatorname{pd}_\Lambda \Lambda/P = \operatorname{pd}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}/P\Lambda_\mathfrak{p}$. In particular, if Λ is a Gorenstein R-algebra, by 5.2 we get $\mathrm{Ext}_\Lambda\text{-}\dim \Lambda/P = \mathrm{ht}_\Lambda P = \mathrm{Kdim}\, \Lambda_{\mathfrak{p}} < \infty$. This observation shows the implication (1)⇒(2) in

Corollary 5.4. *The following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*

(2) $\mathrm{Ext}_\Lambda\text{-}\dim \Lambda/P = \mathrm{Ext}_\Lambda\text{-}\dim \Lambda/Q < \infty$ *for all* $P, Q \in \mathrm{Max}\, \Lambda$ *with* $P \cap R = Q \cap R$.

When this is the case, $\mathrm{Ext}_\Lambda\text{-}\dim \Lambda/P = \mathrm{ht}_\Lambda P = \mathrm{Kdim}\, \Lambda_{P\cap R}$ *for every* $P \in \mathrm{Max}\, \Lambda$.

Proof. (2)⇒(1). Let $\mathfrak{p} \in \mathrm{Max}\, R \cap \mathrm{Supp}_R \Lambda$. Let $k = \mathrm{Ext}_\Lambda\text{-}\dim \Lambda/P$ with $P \in \mathrm{Max}\, \Lambda$ such that $\mathfrak{p} = P \cap R$. Then for all $Q \in \mathrm{Max}\, \Lambda$ with $\mathfrak{p} = Q \cap R$ we get

$$\mathrm{Ext}^k_{\Lambda_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}/Q\Lambda_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) = (0)$$

for $i > k$. Therefore $\mathrm{id}_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}} = k$ by 3.5(3) and hence $\Lambda_{\mathfrak{p}}$ is a Gorenstein $R_{\mathfrak{p}}$-algebra by 4.10. ∎

Corollary 5.5. *The following conditions are equivalent.*

(1) *Λ is a Cohen–Macaulay R-module and* $\mathrm{gl.dim}\, \Lambda_{\mathfrak{p}} = \mathrm{Kdim}\, \Lambda_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$.

(2) $\mathrm{pd}_\Lambda \Lambda/P = \mathrm{pd}_\Lambda \Lambda/Q < \infty$ *for all* $P, Q \in \mathrm{Max}\, \Lambda$ *with* $P \cap R = Q \cap R$.

When this is the case, $\mathrm{pd}_\Lambda \Lambda/P = \mathrm{ht}_\Lambda P = \mathrm{Kdim}\, \Lambda_{P\cap R}$ *for all* $P \in \mathrm{Max}\, \Lambda$ *and* $\mathrm{pd}_\Lambda M < \infty$ *for every finitely generated Λ-module M. The center* $\mathrm{C}(\Lambda)$ *of Λ is a normal ring.*

Proof. (1)⇒(2). We have $\mathrm{id}_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}} = \mathrm{gl.dim}\, \Lambda_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ since $\mathrm{gl.dim}\, \Lambda_{\mathfrak{p}} < \infty$, so that Λ is a Gorenstein R-algebra. Let $P \in \mathrm{Max}\, \Lambda$ and put $\mathfrak{p} = P \cap R$. Then $\mathrm{pd}_\Lambda \Lambda/P = \mathrm{Ext}_\Lambda\text{-}\dim \Lambda/P = \mathrm{Kdim}\, \Lambda_{\mathfrak{p}}$ by 5.4 since $\mathrm{pd}_\Lambda \Lambda/P = \mathrm{pd}_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/P\Lambda_{\mathfrak{p}} < \infty$.

(2)⇒(1). By 5.4, Λ is a Gorenstein R-algebra. Let $\mathfrak{p} \in \mathrm{Max}\, R \cap \mathrm{Supp}_R \Lambda$ and let $k = \mathrm{Kdim}\, \Lambda_{\mathfrak{p}}$. Then $\mathrm{pd}_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/Q\Lambda_{\mathfrak{p}} = k$ for all $Q \in \mathrm{Max}\, \Lambda$ with $\mathfrak{p} = Q \cap R$, whence $\mathrm{gl.dim}\, \Lambda_{\mathfrak{p}} = k$. Therefore $\mathrm{gl.dim}\, \Lambda_{\mathfrak{p}} = \mathrm{Kdim}\, \Lambda_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R \Lambda$ (cf. 4.1(2)).

To see the last assertions let M be a finitely generated Λ-module. Then $\mathrm{Ext}_\Lambda\text{-}\dim M < \infty$ by 4.15. Let $k = \mathrm{Ext}_\Lambda\text{-}\dim M$. We want to show $\mathrm{pd}_\Lambda M = k$. Let $i \in \mathbb{Z}$ and assume $\mathrm{Ext}^i_\Lambda(M, N) \neq (0)$ for some Λ-module N. Choose $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{Ext}^i_\Lambda(M, N)$ and let $j = \mathrm{pd}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{Ext}_{\Lambda_{\mathfrak{p}}} - \dim M_{\mathfrak{p}}$. Then $i \le j$ since $\mathrm{Ext}^i_\Lambda(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq (0)$, while $j \le k$ since $\mathrm{Ext}^j_\Lambda(M_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) \neq (0)$. Thus $i \le k$ so that $\mathrm{pd}_\Lambda M = k$. See 3.13 for the normality of $\mathrm{C}(\Lambda)$. ∎

Corollary 5.6. *Suppose that R is an integrally closed integral domain and Λ is R-torsionfree. Then the following conditions are equivalent.*

(1) *Λ is a Gorenstein R-algebra.*
(2) *Let $P \in \operatorname{Max}\Lambda$ and $i \in \mathbb{Z}$. Then $\mu^i(P,\Lambda) > 0$ if and only if $i = \operatorname{ht}_\Lambda P$.*

Proof. See 5.2(3) and use 2.0.1(8). ∎

For commutative Gorenstein R-algebras Λ the Bass numbers $\mu^i(P,\Lambda)$ are always integers and are equal to $\delta_{i,\operatorname{ht}_\Lambda P}$ ([B2]). It is however a total fallacy to suppose that this is still true if Λ is non-commutative (cf. Example 8.6). Brown and Hajarnavis erroneously claimed [BHa2] (Theorem 5.5) that this holded for injectively homogeneous rings with finite self-injective dimension. And this drives us to the question when the equality $\mu^i(P,L) = \delta_{i,\operatorname{ht}_\Lambda P}$ holds true for general Gorenstein R-algebras Λ. Here we give some basic observations (5.7 and 5.9), which we continue in Section 7.

Theorem 5.7. *Suppose that R is a Cohen–Macaulay local ring with canonical module K_R and assume that Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R \Lambda = \operatorname{Kdim} R = n$. Let $L = \operatorname{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{K}_R)$. Then $\mu^i(P,L) = \delta_{i,\operatorname{ht}_\Lambda P}$ for every $P \in \operatorname{Spec}\Lambda$ and $i \in \mathbb{Z}$. In particular L is a Gorenstein Λ-module with $\operatorname{Kdim}_R L = n$.*

Proof. By [BH] (Theorem 3.3.10), L is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R \Lambda = n$. Let $P \in \operatorname{Spec}\Lambda$ and $i \in \mathbb{Z}$. We put $\mathfrak{p} = P \cap R$. Then $\mathfrak{p} \in \operatorname{Supp}_R \Lambda$ and $\operatorname{Kdim}_{R_\mathfrak{p}} \Lambda_\mathfrak{p} = n - \operatorname{Kdim} R/\mathfrak{p} = \operatorname{Kdim} R_\mathfrak{p}$ ([Se], p. 89, Chapter IV, Théorème 6). Since $\mathrm{K}_{(R_\mathfrak{p})} \cong (\mathrm{K}_R)_\mathfrak{p}$ ([BH], 3.3.5(b)), $L_\mathfrak{p} \cong \operatorname{Hom}_R((\Lambda_\mathfrak{p})^{\mathrm{op}}, \mathrm{K}_{(R_\mathfrak{p})})$. Therefore by 5.1(1), passing to the localization $\Lambda_\mathfrak{p}$, we may assume that $\mathfrak{p} = \mathfrak{m}$ and $P \in \operatorname{Max}\Lambda$. Then $\operatorname{ht}_\Lambda P = n$ by 5.3. Suppose $n > 0$ and choose $t \in \mathfrak{m}$ so that t is R-regular. Note that t is also regular for K_R and Λ. Let $\overline{\Lambda} = \Lambda/t\Lambda$, $\overline{P} = P/t\Lambda$, and $\overline{R} = R/tR$. Then $L/tL \cong \operatorname{Hom}_{\overline{R}}((\overline{\Lambda})^{\mathrm{op}}, \mathrm{K}_{\overline{R}})$ ([BH], 3.3.3) since $\mathrm{K}_R/t\mathrm{K}_R \cong \mathrm{K}_{\overline{R}}$ ([BH], 3.3.5(a)), while $\operatorname{ht}_{\overline{\Lambda}} \overline{P} = \operatorname{Kdim} \overline{\Lambda} = n-1$ by 5.3. Therefore thanks to 5.1(5), passing to the ring $\Lambda/(x_1,\ldots,x_n)\Lambda$ for some system $x_1,\ldots,x_n$ of parameters of R, we may assume $n = 0$. Let $J = \mathrm{J}(\Lambda)$ be the Jacobson radical of Λ. Then since $L = [\Lambda^{\mathrm{op}}]^\vee$ (the Matlis dual of Λ^{op}) ([BH], 3.3.4(a)), by 2.6.3(3) we have $L = \mathrm{E}_\Lambda(\Lambda/J)$. Hence $\mu^i(P,L) = 0$ for $i \neq 0$, while $\mu^0(P,L) = 1$ as $\operatorname{Hom}_\Lambda(\Lambda/P, L) = \operatorname{Hom}_\Lambda(\Lambda/P, \mathrm{E}_\Lambda(\Lambda/J)) = \operatorname{Hom}_\Lambda(\Lambda/P, \Lambda/J) = \Lambda/P$. ∎

Corollary 5.8. *Suppose that R is a Cohen–Macaulay local ring with canonical module K_R and assume that Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R \Lambda = \operatorname{Kdim} R = n$. Then the following conditions are equivalent.*

(1) *$\Lambda^{\mathrm{op}} \cong \operatorname{Hom}_R(\Lambda, \mathrm{K}_R)$ as Λ^{op}-modules.*
(2) *$\mu^n(P,\Lambda) = 1$ for every $P \in \operatorname{Max}\Lambda$.*
(3) *$\mu^i(P,\Lambda) = \delta_{i,\operatorname{ht}_\Lambda P}$ for every $P \in \operatorname{Spec}\Lambda$ and $i \in \mathbb{Z}$.*

Proof. (1)⇒(3). See 5.7.

(3)⇒(2). This is clear.

(2)⇒(1). We argue by induction on n. Let $n = 0$. Since $\mu^0(P, L) = 1$ for all $P \in \operatorname{Max} \Lambda$, we get

$$\mathrm{E}^0_\Lambda(\Lambda) = \bigoplus_{P \in \operatorname{Max} \Lambda} \mathrm{E}_\Lambda(\Lambda/P) = \mathrm{E}_\Lambda(\Lambda/\mathrm{J}(\Lambda)) = [\Lambda^{\mathrm{op}}]^\vee \qquad (2.6.3(3)),$$

where $[*]^\vee$ stands for the Matlis dual. Note that $\ell_R(\Lambda) = \ell_R([\Lambda^{\mathrm{op}}]^\vee)$ and $\Lambda = \mathrm{E}^0_\Lambda(\Lambda) = [\Lambda^{\mathrm{op}}]^\vee$ as $\Lambda \subseteq \mathrm{E}^0_\Lambda(\Lambda) = [\Lambda^{\mathrm{op}}]^\vee$. Assume that $n > 0$ and our assertion holds true for $n-1$. Choose $t \in \mathfrak{m}$ so that t is R-regular. Let $\overline{R} = R/tR$ and $\overline{\Lambda} = \Lambda/t\Lambda$. Then by 5.1(5), $\mu^{n-1}(Q, \overline{\Lambda}) = 1$ for every $Q \in \operatorname{Max} \overline{\Lambda}$ and so by the hypothesis on n we have $(\overline{\Lambda})^{\mathrm{op}} \cong \operatorname{Hom}_{\overline{R}}(\overline{\Lambda}, \mathrm{K}_{\overline{R}})$. Let $L = \operatorname{Hom}_R(\Lambda, \mathrm{K}_R)$. Then by Nakayama's lemma L is a cyclic Λ^{op}-module, since $L/tL \cong \operatorname{Hom}_{\overline{R}}(\overline{\Lambda}, \mathrm{K}_{\overline{R}})$ ([BH], 3.3.3 and 3.3.5). Let $\varphi : \Lambda^{\mathrm{op}} \to L$ be an epimorphism of Λ^{op}-modules and put $K = \operatorname{Ker} \varphi$. We want to show $K = (0)$. Assume the contrary and choose $\mathfrak{p} \in \operatorname{Ass}_R K$. Then since $\mathfrak{p} \in \operatorname{Ass}_R \Lambda^{\mathrm{op}}$, we see $\operatorname{Kdim} R/\mathfrak{p} = n$, whence $\operatorname{Kdim} R_\mathfrak{p} = 0$. Because $L_\mathfrak{p} \cong \operatorname{Hom}_R([\Lambda_\mathfrak{p}]^{\mathrm{op}}, \mathrm{K}_{(R_\mathfrak{p})})$ and $\mathrm{K}_{(R_\mathfrak{p})} = \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$, we have $\ell_{R_\mathfrak{p}}([\Lambda_\mathfrak{p}]^{\mathrm{op}}) = \ell_{R_\mathfrak{p}}([\Lambda_\mathfrak{p}]^\vee) = \ell_{R_\mathfrak{p}}(L_\mathfrak{p}) < \infty$. Therefore the induced epimorphism $\varphi_\mathfrak{p} : \Lambda_\mathfrak{p}^{\mathrm{op}} \to L_\mathfrak{p}$ is an isomorphism, which forces $K_\mathfrak{p} = (0)$. This contradicts the fact that $\mathfrak{p} \in \operatorname{Ass}_R K$. Hence $K = (0)$ and $\varphi : \Lambda^{\mathrm{op}} \to L$ is an isomorphism. ∎

QUESTION 5.9. Suppose R is a local ring and let $n = \operatorname{Kdim} \Lambda$. Is it true that Λ is a Gorenstein R-algebra if $\mu^n(P, \Lambda) = 1$ for every $P \in \operatorname{Max} \Lambda$? This is true when Λ is commutative and $\operatorname{Kdim} \Lambda = \operatorname{ht}_\Lambda P$ for all $P \in \operatorname{Max} \Lambda$ (cf. [Ro2]).

6. The case where Λ is R-free. In this section we assume that R is a local ring with maximal ideal $\mathfrak{m}$. Let $\kappa = R/\mathfrak{m}$ and $\Delta = \kappa \otimes_R \Lambda$. The purpose is to prove Theorem 6.4 below.

We begin with the following.

LEMMA 6.1. *Assume Λ is a finitely generated free R-module. Then* $\operatorname{id}_\Lambda \Lambda \otimes_R \mathrm{E}_R(\kappa) = \operatorname{id}_\Delta \Delta$.

Proof. Let $E = \mathrm{E}_R(\kappa)$ and let

$$0 \to \Lambda \otimes_R E \to I^0 \to I^1 \to \ldots \to I^i \to \ldots$$

be a minimal injective resolution of $\Lambda \otimes_R E$. Then because

$$\operatorname{Hom}_\Lambda(\Delta, \Lambda \otimes_R E) \cong \Lambda \otimes_R \operatorname{Hom}_R(\kappa, E) \cong \Lambda \otimes_R \kappa = \Delta$$

and $\operatorname{Ext}^i_R(\kappa, \Lambda \otimes_R E) = (0)$ for all $i > 0$, the complex

$$0 \to \operatorname{Hom}_\Lambda(\Delta, \Lambda \otimes_R E) \to \operatorname{Hom}_\Lambda(\Delta, I^0)$$
$$\to \operatorname{Hom}_\Lambda(\Delta, I^1) \to \ldots \to \operatorname{Hom}_\Lambda(\Delta, I^i) \to \ldots$$

of Δ-modules is exact and gives rise to a minimal injective resolution of Δ. Notice that $\mathrm{Hom}_\Lambda(\Delta, I^i) \neq (0)$ if and only if $\mathfrak{m} \in \mathrm{Ass}_R I^i$. The latter condition is equivalent to saying that $I^i \neq (0)$, because $\mathrm{Supp}_R \Lambda \otimes_R E = \{\mathfrak{m}\}$. Thus $\mathrm{id}_\Lambda \Lambda \otimes_R E = \mathrm{id}_\Delta \Delta$. ∎

PROPOSITION 6.2. *Assume Λ is a finitely generated free R-module. Then*:

(1) *Every injective Λ-module I is R-injective.*
(2) $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_\Delta \Delta + \mathrm{id}_R R$.
(3) *R is a regular local ring if* $\mathrm{gl.dim}\, \Lambda < \infty$.
(4) $\mathrm{gl.dim}\, \Lambda = \mathrm{gl.dim}\, \Delta + \mathrm{Kdim}\, R$ *if* $\mathrm{gl.dim}\, \Delta < \infty$ *and R is regular.*

Proof. (1) We may assume $I = \mathrm{I}(P)$ for some $P \in \mathrm{Spec}\, \Lambda$. Then I is a direct summand of $\mathrm{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(R/\mathfrak{p}))$ with $\mathfrak{p} = P \cap R$ (2.5.5), so that the Λ-module I is R-injective since $\mathrm{Hom}_R(\Lambda^{\mathrm{op}}, \mathrm{E}_R(R/\mathfrak{p})) = \mathrm{E}_R(R/\mathfrak{p})^r$ with $r = \mathrm{rank}\, \Lambda$.

(2) To see that $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_R R + \mathrm{id}_\Delta \Delta$, we may assume R is a Gorenstein ring. In fact, if $\mathrm{id}_R R + \mathrm{id}_\Delta \Delta < \infty$, R is certainly Gorenstein. Let $0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^i \to \ldots$ be a minimal injective resolution of Λ. Then by (1) it is an injective resolution of the R-module Λ as well, whence R is Gorenstein if $\mathrm{id}_\Lambda \Lambda < \infty$. Thus, in order to prove $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_R R + \mathrm{id}_\Delta \Delta$, without loss of generality we may assume R is a Gorenstein ring. Passing to the $R/\mathfrak{q}$-algebra $\Lambda/\mathfrak{q}\Lambda$ for some ideal $\mathfrak{q}$ of R generated by a system of parameters, by 3.6 we may furthermore assume that $d = 0$. But then the equality $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_\Delta \Delta$ follows from 6.1 since $\mathrm{E}_R(\kappa) = R$.

(3) As Λ is R-free and $\Delta = \kappa \otimes_R \Lambda$, the Λ-projective resolution of Δ involves an R-free resolution of κ. Hence R is a regular local ring if $\mathrm{gl.dim}\, \Lambda < \infty$.

(4) See 2.3.4 and use the fact that the maximal ideal $\mathfrak{m}$ of R is generated by a regular sequence of length d. ∎

COROLLARY 6.3. *Assume Λ is a finitely generated free R-module and R is a Gorenstein ring. Then*

$$\mathrm{id}_\Lambda \Lambda \otimes_R \mathrm{E}_R(R/\mathfrak{p}) = \mathrm{id}_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p} \leq \mathrm{id}_\Delta \Delta$$

for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

Proof. The first equality follows from 6.1, because

$$\begin{aligned}\mathrm{id}_\Lambda \Lambda \otimes_R \mathrm{E}_R(R/\mathfrak{p}) &= \mathrm{id}_\Lambda \Lambda_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \\ &= \mathrm{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}).\end{aligned}$$

To prove the inequality we may assume $\mathrm{id}_\Delta \Delta < \infty$. Hence $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_\Delta \Delta + \mathrm{id}_R R < \infty$ (6.2(2)). Therefore letting $k = \mathrm{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}$, we have $\mathfrak{p} \in \mathrm{Ass}_R \mathrm{E}^k_\Lambda(\Lambda)$ since $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}} \mathrm{E}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p})$ by 3.5(2). Thus $\mathfrak{m} \in \mathrm{Ass}_R \mathrm{E}^{k+\mathrm{Kdim}\, R/\mathfrak{p}}_\Lambda(\Lambda)$ by

3.5(1) so that

$$k + \operatorname{Kdim} R/\mathfrak{p} \leq \operatorname{id}_\Lambda \Lambda = \operatorname{Kdim} R + \operatorname{id}_\Delta \Delta.$$

Thus $k \leq \operatorname{Kdim} R_\mathfrak{p} + \operatorname{id}_\Delta \Delta$, while $k = \operatorname{Kdim} R_\mathfrak{p} + \operatorname{id}_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$ by 6.2(2). Hence $\operatorname{id}_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p} \leq \operatorname{id}_\Delta \Delta$. ■

We now come to the main result of this section (Theorem 6.4), in which the equivalence of assertions (4) and (5) was given by Endo [En]. We are grateful to him for pointing out this.

THEOREM 6.4. *Assume Λ is a finitely generated free R-module and R is a Gorenstein ring. Let*

$$0 \to R \to E^0 \to E^1 \to \ldots \to E^d \to 0$$

be a minimal injective resolution of R. Then the following conditions are equivalent.

(1) *Λ is a Gorenstein R-algebra.*
(2) $\operatorname{Hom}_R(\Lambda, R)$ *is a projective Λ^{op}-module.*
(3) $\operatorname{id}_\Lambda \Lambda = d$.
(4) $\operatorname{id}_\Delta \Delta = 0$.
(5) $\operatorname{id}_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
(6) *The Λ-module $\Lambda \otimes_R E$ is injective for every injective R-module E.*
(7) *The minimal injective resolution of Λ is given by*

$$0 \to \Lambda = \Lambda \otimes_R R \to \Lambda_R E^0 \to \Lambda \otimes_R E^1 \to \ldots \to \Lambda \otimes_R E^d \to 0.$$

Hence $\mathrm{E}^i_\Lambda(\Lambda) \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p} = i} \Lambda \otimes_R \mathrm{E}_R(R/\mathfrak{p})$ *for all* $i \in \mathbb{Z}$.

Proof. Recall Λ is a Cohen–Macaulay R-module with $\dim_R \Lambda = d$. See 4.4(1), 6.1–6.3 for the implications (1)⇔(2)⇔(3)⇔(4)⇔(5) and (6)⇒(4). We have (5)⇒(6) and (7)⇒(6), since every injective R-module is a direct sum of copies of $\{\mathrm{E}_R(R/\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Spec} R}$. It suffices to show the implication (1)⇒(7). As Λ is R-free, the sequence

$$0 \to \Lambda = \Lambda \otimes_R R \to \Lambda \otimes_R E^0 \to \Lambda \otimes_R E^1 \to \ldots \to \Lambda \otimes_R E^d \to 0$$

is exact and gives rise to an injective resolution of Λ. Choose a minimal injective resolution

$$0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^d \to 0$$

of Λ and a family $\{\varphi^i : I^i \to \Lambda \otimes_R E^i\}_{0 \leq i \leq d}$ of monomorphisms so that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda = \Lambda \otimes_R R \arrow[r] & \Lambda \otimes_R E^0 \arrow[r] & \Lambda \otimes_R E^1 \arrow[r] & \cdots \arrow[r] & \Lambda \otimes_R E^d \arrow[r] & 0 \\
0 \arrow[r] & \Lambda \arrow[u, equal] \arrow[r] & I^0 \arrow[u, "\varphi^0"] \arrow[r] & I^1 \arrow[u, "\varphi^1"] \arrow[r] & \cdots \arrow[r] & I^d \arrow[u, "\varphi^d"] \arrow[r] & 0
\end{tikzcd}$$

is commutative. We will show by induction on d that each φ^i is an isomorphism. We may assume that $d > 0$ and our assertion holds true for $d-1$. Hence $[\varphi^i]_{\mathfrak{p}}$ is an isomorphism for all $0 \le i \le d$ and $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Let $0 \le i \le d$ be an integer and assume that φ^i is not an epimorphism. Let $C = \operatorname{Coker} \varphi^i$. Then $\operatorname{Ass}_R C = \{\mathfrak{m}\}$ whence $\mathfrak{m} \in \operatorname{Ass}_R E^i$ as $\Lambda \otimes_R E^i \cong I^i \oplus C$. Consequently, we have $i = d$, since

$$\Lambda \otimes_R E^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \text{ with } \operatorname{Kdim} R_{\mathfrak{p}} = i} \Lambda \otimes_R \mathrm{E}_R(R/\mathfrak{p})$$

and Λ is R-free (2.2.3(5)). Thus φ^i is an isomorphism for all $0 \le i < d$ so that φ^d is also an isomorphism. ∎

We say that Λ is a *local ring* if $\Lambda/\mathrm{J}(\Lambda)$ is a simple Artinian ring.

COROLLARY 6.5 (cf. [R], Theorem 2.16). *Let R be regular and assume Λ is a local ring such that the structure map $f : R \to \Lambda$ is injective and $\operatorname{id}_\Lambda \Lambda < \infty$. Then Λ is a Gorenstein R-algebra which is a free R-module with* $\operatorname{Kdim} \Lambda = \operatorname{Kdim} R$. *The minimal injective resolution of Λ is given by the complex*

$$0 \to \Lambda = \Lambda \otimes_R R \to \Lambda \otimes_R E^0 \to \Lambda \otimes_R E^1 \to \ldots \to \Lambda \otimes_R E^d \to 0,$$

where $0 \to R \to E^0 \to E^1 \to \ldots \to E^d \to 0$ *denotes a minimal injective resolution of R.*

7. Flat base changes. In this section we assume that R is a local ring with maximal ideal $\mathfrak{m}$. Let $\kappa = R/\mathfrak{m}$. The purpose is to prove Theorem 7.5 below. It sharpens Corollary 5.8 concerning the question when the equality $\mu^i(P, \Lambda) = \delta_{i, \operatorname{ht}_\Lambda P}$ holds true for Gorenstein R-algebras Λ. To do this we need some technique of reduction to the case where R is complete.

Let $\varphi : (R, \mathfrak{m}, \kappa) \to (S, \mathfrak{n}, K)$ be a local homomorphism of Noetherian local rings. We put $\Gamma = S \otimes_R \Lambda$. Let $m = \operatorname{Kdim} \Gamma, n = \operatorname{Kdim} \Lambda$, and $k = \operatorname{Kdim} S/\mathfrak{m}S$. We consider the problem of when Γ inherits Gorensteinness from Λ. We begin with the following.

LEMMA 7.1. *Let $P \in \operatorname{Max} \Lambda$. Then*:

(1) $\operatorname{id}_{S \otimes_R (\Lambda/P)} S \otimes_R (\Lambda/P) = \operatorname{id}_{S/\mathfrak{m}S} S/\mathfrak{m}S$.

(2) *The following conditions are equivalent.*

  (a) *$S \otimes_R (\Lambda/P)$ is a Gorenstein S-algebra.*

  (b) *$S/\mathfrak{m}S$ is a Gorenstein ring.*

When this is the case, $\mu^k(Q, S \otimes_R (\Lambda/P)) = 1$ for all $Q \in$ $\operatorname{Max}(S \otimes_R (\Lambda/P))$.

Proof. (1) We have $\mathrm{Hom}_\kappa(\Lambda/P,\kappa)\cong[\Lambda/P]^{\mathrm{op}}$ since Λ/P is a simple κ-algebra. Therefore

$$\mathrm{Hom}_K(K\otimes_\kappa(\Lambda/P),K)\cong K\otimes_\kappa\mathrm{Hom}_\kappa(\Lambda/P,\kappa)\cong[K\otimes_\kappa(\Lambda/P)]^{\mathrm{op}}$$

so $\mathrm{id}_{K\otimes_\kappa(\Lambda/P)}\,K\otimes_\kappa(\Lambda/P)=0$. Hence $\mathrm{id}_{S\otimes_R(\Lambda/P)}\,S\otimes_R(\Lambda/P)=\mathrm{id}_{S/\mathfrak{m}S}\,S/\mathfrak{m}S$ by 6.2(2), as $S\otimes_R(\Lambda/P)=(S/\mathfrak{m}S)\otimes_\kappa(\Lambda/P)$ is $S/\mathfrak{m}S$-free.

(2) See 6.4. By 5.8 we get $\mu^k(Q,S\otimes_R(\Lambda/P))=1$ for all $Q\in\mathrm{Max}(S\otimes_R(\Lambda/P))$. ∎

LEMMA 7.2. *Suppose the homomorphism $\varphi:R\to S$ is flat. Let $Q\in\mathrm{Max}\,\Gamma$ and put $P=Q\cap\Lambda$. Then $\mu^0(Q,\Gamma)=\mu^0(Q,S\otimes_R(\Lambda/P))\cdot\mu^0(P,\Lambda)$.*

Proof. Note that $P\in\mathrm{Max}\,\Lambda$ and $\Gamma/P\Gamma\cong S\otimes_R(\Lambda/P)$. Then the equality follows from the isomorphisms

$$\begin{aligned}\mathrm{Hom}_\Gamma(\Gamma/Q,\Gamma)&\cong\mathrm{Hom}_\Gamma(\Gamma/Q,\mathrm{Hom}_\Gamma(\Gamma/P\Gamma,\Gamma))\\&\cong\mathrm{Hom}_\Gamma(\Gamma/Q,S\otimes_R\mathrm{Hom}_\Lambda(\Lambda/P,\Lambda))\\&\cong\mathrm{Hom}_\Gamma(\Gamma/Q,S\otimes_R[(\Lambda/P)^{\mu^0(P,\Lambda)}])\\&\cong\mathrm{Hom}_\Gamma(\Gamma/Q,S\otimes_R(\Lambda/P))^{\mu^0(P,\Lambda)}\\&\cong(\Gamma/Q)^{\mu^0(Q,S\otimes_R(\Lambda/P))\cdot\mu^0(P,\Lambda)}.\ \blacksquare\end{aligned}$$

THEOREM 7.3. *Suppose that the morphism $\varphi:R\to S$ is flat. Then the following conditions are equivalent.*

(1) *Γ is a Gorenstein S-algebra.*

(2) *Λ is a Gorenstein R-algebra and $S/\mathfrak{m}S$ is a Gorenstein ring.*

When this is the case, $\mu^m(Q,\Gamma)=\mu^n(Q\cap\Lambda,\Lambda)$ for all $Q\in\mathrm{Max}\,\Gamma$.

Proof. Recall that $m=n+k$ and $\mathrm{depth}_S\,\Gamma=\mathrm{depth}_R\,\Lambda+\mathrm{depth}_{S/\mathfrak{m}S}\,S/\mathfrak{m}S$ ([Ma], Theorems 19 and 50). Hence Γ is a Cohen–Macaulay S-module if and only if Λ is a Cohen–Macaulay R-module and $S/\mathfrak{m}S$ is a Cohen–Macaulay ring. Therefore we may assume that the S-module Γ is Cohen–Macaulay. Thanks to 4.7(3) and 5.1(4), we may assume that both the local rings R and S are complete. Passing to the ring $R/[(0):_R\Lambda]$, we may furthermore assume that $\mathrm{Kdim}\,R=n$ and $\mathrm{Kdim}\,S=m$.

(2)⇒(1). We have $\mathrm{K}_S=S\otimes_R\mathrm{K}_R$ ([BH], 3.3.14) as $S/\mathfrak{m}S$ is a Gorenstein ring. Since

$$\mathrm{Hom}_S(\Gamma,\mathrm{K}_S)\cong\mathrm{Hom}_S(\Gamma,S\otimes_R\mathrm{K}_R)\cong S\otimes_R\mathrm{Hom}_R(\Lambda,\mathrm{K}_R)$$

as Γ^{op}-modules and $\mathrm{Hom}_R(\Lambda,\mathrm{K}_R)$ is Λ^{op}-projective (4.4(1)), $\mathrm{Hom}_S(\Gamma,\mathrm{K}_S)$ is Γ^{op}-projective. Hence by 4.4(1), Γ is Gorenstein.

(1)⇒(2) and the last assertion. Passing to the ring $S/\mathfrak{b}$ with an ideal $\mathfrak{b}$ generated by a maximal $S/\mathfrak{m}S$-regular sequence contained in $\mathfrak{n}$, we may assume that $k=0$ ([BH], Lemma 1.2.17; see also 4.7(2) and 5.1(5)); hence $m=n$. Passing to the ring $R/\mathfrak{a}$ with an ideal $\mathfrak{a}$ generated by a system of parameters of R, we may assume that $m = n = 0$. Also, passing to the ring $R/[(0) :_R \Lambda]$, we may assume that R and S are contained in Λ and Γ. Now let M be a finitely generated Λ-module. Then

$$S \otimes_R \mathrm{Ext}^1_\Lambda(M, \Lambda) \cong \mathrm{Ext}^1_\Gamma(S \otimes_R M, \Gamma) = (0)$$

since $\mathrm{id}_\Gamma \Gamma = 0$; and as the morphism $\varphi : R \to S$ is faithfully flat, we get $\mathrm{Ext}^1_\Lambda(M, \Lambda) = (0)$. Thus $\mathrm{id}_\Lambda \Lambda = 0$. Hence $\mathrm{Hom}_R(\Lambda, \mathrm{K}_R)$ is Λ^{op}-projective (2.6.2(1)) and so $\mathrm{Hom}_S(\Gamma, S \otimes_R \mathrm{K}_R) = S \otimes_R \mathrm{Hom}_R(\Lambda, \mathrm{K}_R)$ is Γ^{op}-projective. Therefore $\mathrm{Hom}_S(\Gamma, S \otimes_R \mathrm{K}_R)$ is Γ^{op}-injective so that by [E], Theorem 2, $S \otimes_R \mathrm{K}_R$ is an injective S-module. Since $\mathrm{K}_R = \mathrm{E}_R(\kappa)$, we see that

$$\mathrm{Hom}_S(S \otimes_R \mathrm{K}_R, S \otimes_R \mathrm{K}_R) \cong S \otimes_R \mathrm{Hom}_R(\mathrm{K}_R, \mathrm{K}_R) \cong S \otimes_R R = S$$

([M]), whence the S-module $S \otimes_R \mathrm{K}_R$ is indecomposable. Thus $\mathrm{K}_S = S \otimes_R \mathrm{K}_R$ (note $\mathrm{K}_S = \mathrm{E}_S(K)$) and $S/\mathfrak{m}S$ is a Gorenstein ring ([BH], 3.3.14). The last assertion follows from 7.1 and 7.2. ■

Since we cannot find a reference for the following result, we give a brief proof for completeness.

Proposition 7.4. *Suppose the morphism $\varphi : R \to S$ is flat. Then for each $P \in \mathrm{Spec}\,\Lambda$ there is at least one $Q \in \mathrm{Spec}\,\Gamma$ such that $P = Q \cap \Lambda$.*

Proof. We may assume $P = (0)$. Passing to the ring $R/[(0) :_R \Lambda]$, we may furthermore assume the structure map $f : R \to \Lambda$ is injective. Let Q be a minimal prime ideal in Γ and let $\mathfrak{q} = Q \cap R$. Then each $t \in \mathfrak{q}$ is a zerodivisor for Λ since $Q \in \mathrm{Min}\,\Gamma$ (2.0.1(5)). Therefore thanks to the flatness of φ, t is also a zerodivisor for Λ. This forces $\mathfrak{q} = (0)$ since Λ is a prime ring. Thus $Q \cap \Lambda = (0)$ (2.0.1(3)). ■

We now turn to the main subject.

Theorem 7.5. *Let Λ be a Gorenstein R-algebra with* $\mathrm{Kdim}\,\Lambda = n$ *and assume that $\mu^n(P, \Lambda) = 1$ for all $P \in \mathrm{Max}\,\Lambda$. Then $\mu^i(P, \Lambda) = \delta_{i, \mathrm{ht}_\Lambda P}$ for all $i \in \mathbb{Z}$ and $P \in \mathrm{Spec}\,\Lambda$. Hence*

$$\mathrm{E}^i_\Lambda(\Lambda) = \bigoplus_{P \in \mathrm{Spec}\,\Lambda \text{ with } \mathrm{ht}_\Lambda P = i} \mathrm{E}_\Lambda(\Lambda/P)$$

for $i \in \mathbb{Z}$.

Proof. If R is complete, thanks to the structure theorem of Cohen [C], R is a homomorphic image of a regular local ring, say T. Let $g = \mathrm{Kdim}\,T - \mathrm{Kdim}\,\Lambda$. Then $[(0) :_T \Lambda]$ contains a T-regular sequence of length g. Let $\mathfrak{a}$ denote the ideal generated by this sequence. Then passing to $T/\mathfrak{a}$, we may assume R is a Gorenstein local ring with $\mathrm{Kdim}\,R = \mathrm{Kdim}\,\Lambda$, so that the assertion follows from 5.8.

To study the general case we look at the completion $\Lambda^\# = R^\# \otimes_R \Lambda$ of Λ. Note that $\mathrm{Max}\,\Lambda^\# = \{P\Lambda^\# \mid P \in \mathrm{Max}\,\Lambda\}$ since $\Lambda^\#/\mathrm{J}(\Lambda^\#) = \Lambda/\mathrm{J}(\Lambda)$. Then by 5.1(4), $\mu^n(P,\Lambda) = 1$ for all $P \in \operatorname{Max}\Lambda^\#$ and hence by 5.8, $\mu^i(Q,\Lambda^\#) = \delta_{i,\mathrm{ht}_{\Lambda^\#} Q}$ for all $i \in \mathbb{Z}$ and $Q \in \operatorname{Spec}\Lambda^\#$. Let $P \in \operatorname{Spec}\Lambda$ with $\mathrm{ht}_\Lambda P = i$ and choose $Q \in \operatorname{Spec}\Lambda^\#$ so that $P = Q \cap \Lambda$ (7.4). Let $j = \mathrm{ht}_\Lambda Q$, $\mathfrak{q} = Q \cap R^\#$, and $\mathfrak{p} = P \cap R$. Then we have a flat local homomorphism $R_\mathfrak{p} \to R^\#{}_\mathfrak{q}$ since $\mathfrak{p} = \mathfrak{q} \cap R$. As $\Lambda^\#{}_\mathfrak{q}$ is a Gorenstein $R^\#{}_\mathfrak{q}$-algebra and $P\Lambda_\mathfrak{p} = Q\Lambda^\#{}_\mathfrak{q} \cap \Lambda_\mathfrak{p}$, by 5.1(1) and 7.3 we see $\mu^i(P,\Lambda) = \mu^i(P\Lambda_\mathfrak{p},\Lambda_\mathfrak{p}) = \mu^j(Q\Lambda^\#{}_\mathfrak{q},\Lambda^\#{}_\mathfrak{q}) = \mu^j(Q,\Lambda^\#) = 1$, which implies the assertion since $\mu^k(P,\Lambda) = 0$ if $k \neq i$ (cf. 5.2). ■

COROLLARY 7.6. *Let Λ be a Gorenstein R-algebra with* $\operatorname{Kdim}\Lambda = n$ *and suppose that all* $\mathrm{m}(P)$ $(P \in \operatorname{Max}\Lambda)$ *have the same value independent of the choice of $P \in \operatorname{Max}\Lambda$. Let $i \in \mathbb{Z}$. Then $\mu^i(P,\Lambda) = \delta_{i,\mathrm{ht}_\Lambda P}$ for every $P \in \operatorname{Spec}\Lambda$ and* $\mathrm{E}^i_\Lambda(\Lambda) = \bigoplus_{P \in \operatorname{Spec}\Lambda \textit{ with } \mathrm{ht}_\Lambda P = i} \mathrm{E}_\Lambda(\Lambda/P)$.

Proof. By 7.5 it is enough to show $\mu^n(P,\Lambda) = 1$ for all $P \in \operatorname{Max}\Lambda$. As $\mathrm{ht}_\Lambda P = n$ for all $P \in \operatorname{Max}\Lambda$ (5.3), after reduction modulo a system of parameters of R we may assume by 5.1(1) that $\operatorname{Kdim}\Lambda = 0$. Also, after reduction modulo $[(0) :_R \Lambda]$, we may furthermore assume that $\operatorname{Kdim} R = 0$. Hence the maximal ideal $\mathfrak{m}$ of R is nilpotent and any idempotent of $\Lambda/\mathrm{J}(\Lambda)$ can be lifted to one of Λ. Now our proof follows that of [DK], Theorem 9.3.2. Let us finish it for completeness.

Let $P \in \operatorname{Max}\Lambda$ and put $n = \mathrm{m}(P)$. Then $\Lambda/J(\Lambda) = \prod_{P \in \operatorname{Max}\Lambda} \mathrm{M}_n(\mathrm{D}(P))$ with $\mathrm{D}(P)$ division rings. Let $\mathrm{S}(P)$ be a simple Λ/P-module and let $\mathrm{P}(P)$ be the Λ-projective cover of it. Then $\Lambda = [\bigoplus_{P \in \operatorname{Max}\Lambda} \mathrm{P}(P)]^n$ since $\Lambda/\mathrm{J}(\Lambda) = [\bigoplus_{P \in \operatorname{Max}\Lambda} \mathrm{S}(P)]^n$. Let $[*]^\vee$ be the Matlis dual. Then since Λ is self-injective, by 2.6.2, $\{\mathrm{P}(P)^\vee\}_{P \in \operatorname{Max}\Lambda}$ are finitely generated indecomposable projective Λ^{op}-modules. It follows that $\Lambda^{\mathrm{op}} \cong [\bigoplus_{P \in \operatorname{Max}\Lambda} \mathrm{P}(P)^\vee]^n$ because $\Lambda/\mathrm{J}(\Lambda) = \prod_{P \in \operatorname{Max}\Lambda} \mathrm{M}_n(\mathrm{D}(P))$. Thus $\Lambda^{\mathrm{op}} \cong \Lambda^\vee$ and so $\Lambda \cong [\Lambda^{\mathrm{op}}]^\vee$ (2.6.4(1)), whence $\mu^0(P,\Lambda) = 1$ for all $P \in \operatorname{Max}\Lambda$ (5.8). ■

Concluding this section, we apply our observations to local R-algebras Λ, that is, to the case where the ring $\Lambda/\mathrm{J}(\Lambda)$ is a simple ring. The next result may account for the reason why the theory behaves so well in the commutative case.

COROLLARY 7.7. *Suppose that Λ is a local ring with* $\operatorname{Kdim}\Lambda = n$. *Let $\mathfrak{M} = \mathrm{J}(\Lambda)$. Then the following conditions are equivalent.*

(1) $\mathrm{id}_\Lambda \Lambda < \infty$.

(2) *Λ is a Gorenstein R-algebra.*

(3) *Λ is a Cohen–Macaulay R-module and* $\mathrm{H}^n_\mathfrak{m}(\Lambda) \cong \mathrm{E}^n_\Lambda(\Lambda)$.

(4) *Λ is a Cohen–Macaulay R-module and* $\mathrm{H}^n_\mathfrak{m}(\Lambda)$ *is Λ-injective.*

(5) $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M},\Lambda) = (0)$ *if $i \neq n$ and* $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{M},\Lambda) \cong \Lambda/\mathfrak{M}$.

(6) $\mathfrak{m} \notin \operatorname{Ass}_R \mathrm{E}^i_\Lambda(\Lambda)$ *if $i \neq n$.*

(7) $\mu^i(\mathfrak{M}, \Lambda) = \delta_{i,n}$ *for every* $i \in \mathbb{Z}$.

(8) $\mu^i(P, \Lambda) = \delta_{i,\mathrm{ht}_\Lambda P}$ *for every* $i \in \mathbb{Z}$ *and* $P \in \operatorname{Spec} \Lambda$, *that is*, Λ *has the minimal injective resolution* $0 \to \Lambda \to \mathrm{E}^0 \to \mathrm{E}^1 \to \ldots \to \mathrm{E}^i \to \ldots \to \mathrm{E}^n \to 0$, *where*

$$\mathrm{E}^i = \bigoplus_{P \in \operatorname{Spec} \Lambda \textit{ with } \mathrm{ht}_\Lambda P = i} \mathrm{E}_\Lambda(\Lambda/P).$$

When Λ *is a Cohen–Macaulay* R*-module and* R *is an* n*-dimensional Cohen–Macaulay local ring with canonical module* K_R, *then each of the conditions* (1)–(8) *is equivalent to the following*:

(9) $\mathrm{Hom}_R(\Lambda, \mathrm{K}_R)$ *is* Λ^{op}*-projective.*

(10) $\Lambda^{\mathrm{op}} \cong \mathrm{Hom}_R(\Lambda, \mathrm{K}_R)$ *as* Λ^{op}*-modules.*

Proof. See 3.10, 4.3, 4.4(1), 5.8 and 7.6. ∎

8. Examples. In this section we gather several examples to illustrate our theorems. Throughout let R denote a commutative Noetherian ring. To begin with we record

EXAMPLE 8.1. *Let* R *be Gorenstein and let* $\Lambda = \mathrm{M}_n(R)$ $(n > 0)$. *Then* Λ *is a Gorenstein* R*-algebra with* $\mu^i(P, \Lambda) = \delta_{i,\mathrm{ht}_\Lambda P}$ *for every* $i \in \mathbb{Z}$ *and* $P \in \operatorname{Spec} \Lambda$. *There is a bijection* $\mathfrak{p} \mapsto \mathfrak{p}\Lambda = \mathrm{M}_n(\mathfrak{p})$ *between* $\operatorname{Spec} R$ *and* $\operatorname{Spec} \Lambda$ *and*

$$\mathrm{E}^i_\Lambda(\Lambda) = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p} = i} \mathrm{E}_\Lambda(\Lambda/\mathfrak{p}\Lambda)$$

for all $i \in \mathbb{Z}$.

EXAMPLE 8.2. *Let* R *be a Gorenstein normal local ring with* $\operatorname{Kdim} R = 2$ *and let* M *be a* 2*-dimensional Cohen–Macaulay* R*-module. Let* $\Lambda = \mathrm{End}_R M$. *Then* Λ *is a Gorenstein* R*-algebra with* $\operatorname{Kdim} \Lambda = 2$ *and* $\mu^i(P, \Lambda) = \delta_{i,\mathrm{ht}_\Lambda P}$ *for every* $i \in \mathbb{Z}$ *and* $P \in \operatorname{Spec} \Lambda$.

Proof. Since $\mathrm{depth}_R \Lambda \geq 2$, Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim} \Lambda = 2$, while $\Lambda \cong \mathrm{Hom}_R(\Lambda, R)$ as Λ-bimodules ([Au2]). Hence by 4.4(1), Λ is a Gorenstein R-algebra. See 5.8 for the last assertion. ∎

Let R be a Cohen–Macaulay local ring with canonical module K_R. Let Λ be an R-algebra which is finitely generated as an R-module. Assume that Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim}_R \Lambda = \operatorname{Kdim} R = n$. Let $L = \mathrm{Hom}_R(\Lambda, \mathrm{K}_R)$. We denote by $\Gamma = \Lambda \ltimes L$ the trivial extension of the Λ-bimodule L. (Hence $\Gamma = \Lambda \oplus L$ as additive groups, the multiplication in Γ is given by $(a, x)\cdot(b, y) = (ab, ay + xb)$, and the R-algebra structure $g : R \to \Gamma$ of Γ is defined so that $g(r) = (f(r), 0)$ for each $r \in R$, where

$f : R \to \Lambda$ denotes the R-algebra structure of Λ; see [Y].). Then we have the following.

EXAMPLE 8.3. *Γ is a Gorenstein R-algebra with $\mu^i(P,\Gamma) = \delta_{i,\mathrm{ht}_\Gamma P}$ for every $P \in \operatorname{Spec}\Gamma$ and $i \in \mathbb{Z}$.*

Proof. It is clear that Γ is a Cohen–Macaulay R-module with $\operatorname{Kdim}\Gamma = n$. It is routine to check that the canonical isomorphism

$$\begin{aligned}\Gamma &= \Lambda \oplus L\\ &\cong L \oplus \Lambda \quad \text{(twist)}\\ &\cong L \oplus \operatorname{Hom}_R(\operatorname{Hom}_R(\Lambda, \mathrm{K}_R), \mathrm{K}_R)\\ &= \operatorname{Hom}_R(\Lambda, \mathrm{K}_R) \oplus \operatorname{Hom}_R(L, K_R)\\ &\cong \operatorname{Hom}_R(\Gamma, \mathrm{K}_R)\end{aligned}$$

of R-modules is actually a homomorphism of Γ-bimodules. Hence Γ is a Gorenstein R-algebra and $\mu^i(P,\Gamma) = \delta_{i,\mathrm{ht}_\Gamma P}$ for every $P \in \operatorname{Spec}\Gamma$ and $i \in \mathbb{Z}$ (4.4(1) and 5.8). ∎

EXAMPLE 8.4. *Let $(R, \mathfrak{m}, \kappa)$ be a Cohen–Macaulay local ring with* $\operatorname{Kdim} R = d$ *and let M be a Gorenstein R-module with $\mu^d(\mathfrak{m}, M) = r$. Let $\Lambda = \operatorname{End}_R M$. Then*:

(1) *Λ is a local ring and the correspondence $\mathfrak{p} \mapsto \mathfrak{p}\Lambda$ yields a bijection between* $\operatorname{Spec} R$ *and* $\operatorname{Spec}\Lambda$*.*

(2) *M is an indecomposable Gorenstein Λ-module with $\mu^d(\mathfrak{m}\Lambda, M) = 1/r$.*

(3) *For every $i \in \mathbb{Z}$,*

$$\mathrm{E}^i_\Lambda(M) = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p} = i} \mathrm{E}_\Lambda(\Lambda/\mathfrak{p}\Lambda)^{1/r}.$$

(4) *Any Gorenstein Λ-module L is isomorphic to a finite direct sum of copies of M.*

Proof. (1) Let $R^\#$ be the $\mathfrak{m}$-adic completion of R. Then $M^\# \cong [\mathrm{K}_{R^\#}]^r$ ([Sh3]). Hence $\Lambda^\# = \mathrm{M}_r(R^\#)$ as $R^\# = \operatorname{End}_{R^\#}(\mathrm{K}_{R^\#})$ ([BH], 3.3.4). Therefore Λ is a free R-module of rank r^2. The ring Λ is a local ring with unique maximal ideal $\mathfrak{m}\Lambda$ since $\Lambda/\mathfrak{m}\Lambda = \Lambda^\#/\mathfrak{m}\Lambda^\# = \mathrm{M}_r(R/\mathfrak{m})$. Hence the algebra $\Lambda_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}\Lambda_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ because $M_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-module ([Sh2]). We have $\mathfrak{p}\Lambda = \mathfrak{p}\Lambda_\mathfrak{p} \cap \Lambda$ since Λ is R-free. Therefore $\mathfrak{p}\Lambda$ is a prime ideal of Λ and the correspondence $\mathfrak{p} \mapsto \mathfrak{p}\Lambda$ yields a bijection between $\operatorname{Spec} R$ and $\operatorname{Spec}\Lambda$.

(2) Let $\mathfrak{a}$ be an ideal of R generated by a system of parameters. Then $M/\mathfrak{a}M$ is a Gorenstein R/a-module and $\Lambda/\mathfrak{a}\Lambda = \operatorname{End}_{R/\mathfrak{a}} M/\mathfrak{a}M$ ([Sh2] and

[BH], 3.3.3). Hence to show (2), passing to the ring $R/\mathfrak{a}$, by 4.7 and 5.1(5) we may assume $d = 0$. Then $M = [\mathrm{E}_R(\kappa)]^r$ and $\Lambda = \mathrm{M}_r(R)$. Therefore M is an indecomposable injective Λ-module and $\mu^0(\mathfrak{m}\Lambda, M) = 1/r$ since $\mathrm{m}(\mathfrak{m}\Lambda) = r$.

(3) Let $P \in \operatorname{Spec}\Lambda$ and $\mathfrak{p} = P \cap R$. Then $P = \mathfrak{p}\Lambda$ by (1) and so by (2) and 5.1(1),

$$\mu^i(P, M) = \mu^i(P\Lambda_{\mathfrak{p}}, M_{\mathfrak{p}}) = \mu^i(\mathfrak{p}\Lambda_{\mathfrak{p}}, M_{\mathfrak{p}}) = (1/r)\delta_{i,\mathrm{ht}_\Lambda \mathfrak{p}}$$

for every $i \in \mathbb{Z}$. Hence

$$\mathrm{E}^i_\Lambda(M) = \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R \text{ with } \operatorname{Kdim} R_{\mathfrak{p}} = i} \mathrm{E}_\Lambda(\Lambda/\mathfrak{p}\Lambda)^{1/r}.$$

(4) Passing to the completion $R^{\#}$ of R, we may assume that R is complete. We have $\operatorname{Kdim}_R L = d$ by 4.4(2). Moreover, by (2) and 4.4(1), $\operatorname{Hom}_R(M, \mathrm{K}_R)$ and $\operatorname{Hom}_R(L, \mathrm{K}_R)$ are finitely generated projective Λ^{op}-modules. Therefore since the ring Λ^{op} is local (by (1)) and $\operatorname{Hom}_R(M, \mathrm{K}_R)$ is indecomposable as a Λ^{op}-module ((2) and [BH], 3.3.10), $\operatorname{Hom}_R(L, \mathrm{K}_R) \cong \operatorname{Hom}_R(M, \mathrm{K}_R)^k$ for some $k > 0$. Hence

$$\begin{aligned} L &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(L, \mathrm{K}_R), \mathrm{K}_R) \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M^k, \mathrm{K}_R), \mathrm{K}_R) \cong M^k. \ \blacksquare \end{aligned}$$

REMARK 8.5. For each integer $r \geq 1$ there is a Cohen–Macaulay normal local ring $(R, \mathfrak{m})$ with $\operatorname{Kdim} R = 2$ having an indecomposable Gorenstein R-module M with $\mu^2(\mathfrak{m}, M) = r$. See [Ni].

EXAMPLE 8.6. *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring with* $\operatorname{Kdim} R = d > 0$ *and let $0 \neq t \in \mathfrak{m}$. The R-algebra*

$$\Lambda = \begin{bmatrix} R & R & R \\ tR & R & R \\ tR & R & R \end{bmatrix}$$

has the following properties.

(1) *Λ is a Gorenstein R-algebra with* $\operatorname{Kdim}\Lambda = d$.

(2) *For each* $\mathfrak{p} \in \operatorname{Spec} R$ *let* $S(\mathfrak{p}) = \{P \in \operatorname{Spec}\Lambda \mid P \cap R = \mathfrak{p}\}$. *Then*

$$S(\mathfrak{p}) = \begin{cases} \left\{ \begin{bmatrix} \mathfrak{p} & R & R \\ tR & R & R \\ tR & R & R \end{bmatrix}, \begin{bmatrix} R & R & R \\ tR & \mathfrak{p} & \mathfrak{p} \\ tR & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \right\} & \text{if } t \in \mathfrak{p}, \\ \left\{ \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \right\} & \text{if } t \notin \mathfrak{p}. \end{cases}$$

(3) *For each* $i \in \mathbb{Z}$,

$$\mathrm{E}^i_\Lambda(\Lambda) = \left[\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p}=i \textit{ and } t\in\mathfrak{p}} \mathrm{E}_\Lambda\left(\Lambda/\begin{bmatrix} \mathfrak{p} & R & R \\ tR & R & R \\ tR & R & R\end{bmatrix}\right)^2\right]$$
$$\oplus \left[\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p}=i \textit{ and } t\in\mathfrak{p}} \mathrm{E}_\Lambda\left(\Lambda/\begin{bmatrix} R & R & R \\ tR & \mathfrak{p} & \mathfrak{p} \\ tR & \mathfrak{p} & \mathfrak{p}\end{bmatrix}\right)^{1/2}\right]$$
$$\oplus \left[\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p}=i \textit{ and } t\notin\mathfrak{p}} \mathrm{E}_\Lambda\left(\Lambda/\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p}\end{bmatrix}\right)\right].$$

Proof. (1) Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim}\Lambda = d$ since it is R-free. Let $[*]^* = \operatorname{Hom}_R(*, R)$. We put

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}.$$

Then $\Lambda \cong \Lambda e_1 \oplus [\Lambda e_2]^2$. We have $[\Lambda e_1]^* \cong e_2\Lambda$ and $[\Lambda e_2]^* \cong e_1\Lambda$. Therefore $\Lambda^* \cong [e_1\Lambda]^2 \oplus e_2\Lambda$, whence by 4.4 our R-algebra Λ is Gorenstein.

(2) Let $J = \mathrm{J}(\Lambda)$. Then

$$\mathrm{J}(\Lambda) = \begin{bmatrix} \mathfrak{m} & R & R \\ tR & \mathfrak{m} & \mathfrak{m} \\ tR & \mathfrak{m} & \mathfrak{m}\end{bmatrix}$$

and $\Lambda/J = \kappa \times \mathbb{M}_2(\kappa)$. The maximal ideals of Λ are given by

$$\mathfrak{m}_1 = \begin{bmatrix} \mathfrak{m} & R & R \\ tR & R & R \\ tR & R & R\end{bmatrix} \quad \text{and} \quad \mathfrak{m}_2 = \begin{bmatrix} R & R & R \\ tR & \mathfrak{m} & \mathfrak{m} \\ tR & \mathfrak{m} & \mathfrak{m}\end{bmatrix}.$$

The right Λ-module $e_1\Lambda$ is the projective cover of $\Lambda/\mathfrak{m}_1$ and $[e_2\Lambda]^2$ is that of $\Lambda/\mathfrak{m}_2$. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $t \in \mathfrak{p}$, then

$$\Lambda_\mathfrak{p} = \begin{bmatrix} R_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p} \\ tR_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p} \\ tR_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p}\end{bmatrix}$$

and the above observation shows the maximal ideals of $\Lambda_\mathfrak{p}$ are

$$\mathfrak{P}_1(\mathfrak{p}) = \begin{bmatrix} \mathfrak{p}R_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p} \\ tR_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p} \\ tR_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p}\end{bmatrix} \quad \text{and} \quad \mathfrak{P}_2(\mathfrak{p}) = \begin{bmatrix} R_\mathfrak{p} & R_\mathfrak{p} & R_\mathfrak{p} \\ tR_\mathfrak{p} & \mathfrak{p}R_\mathfrak{p} & \mathfrak{p}R_\mathfrak{p} \\ tR_\mathfrak{p} & \mathfrak{p}R_\mathfrak{p} & \mathfrak{p}R_\mathfrak{p}\end{bmatrix}.$$

Hence

$$S(\mathfrak{p}) = \{\mathfrak{P}_i(\mathfrak{p}) \cap \Lambda \mid i = 1, 2\} = \left\{ \begin{bmatrix} \mathfrak{p} & R & R \\ tR & R & R \\ tR & R & R \end{bmatrix}, \begin{bmatrix} R & R & R \\ tR & \mathfrak{p} & \mathfrak{p} \\ tR & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \right\}.$$

If $t \notin \mathfrak{p}$, then $\Lambda_{\mathfrak{p}} = \mathbb{M}_3(R_{\mathfrak{p}})$, which is a local ring with maximal ideal $\mathfrak{p}\Lambda_{\mathfrak{p}}$. Hence

$$S(\mathfrak{p}) = \left\{ \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \right\}.$$

(3) Let $R^{\#}$ be the $\mathfrak{m}$-adic completion of R and let $[*]^{\vee}$ denote the Matlis dual. Then since $\mathrm{E}^d_{\Lambda^{\#}}(\Lambda^{\#}) \cong \mathrm{H}^d_{\mathfrak{m}^{\#}}(\Lambda^{\#}) \cong [\mathrm{Hom}_{R^{\#}}(\Lambda^{\#}, R^{\#})]^{\vee}$ by 2.7.3 and 4.1, we have $\mathrm{E}^d_{\Lambda^{\#}}(\Lambda^{\#}) \cong ([e_1\Lambda^{\#}]^{\vee})^2 \oplus [e_2\Lambda^{\#}]^{\vee}$. On the other hand $\mathrm{E}_{\Lambda^{\#}}(\Lambda^{\#}/\mathfrak{m}_1^{\#}) \cong [e_1\Lambda^{\#}]^{\vee}$ and $\mathrm{E}_{\Lambda^{\#}}(\Lambda^{\#}/\mathfrak{m}_2^{\#}) \cong ([e_2\Lambda^{\#}]^{\vee})^2$ by 2.6.2, since $e_1\Lambda^{\#}$ is the projective cover of $\Lambda^{\#}/\mathfrak{m}_1^{\#}$ and $[e_2\Lambda^{\#}]^2$ is that of $\Lambda^{\#}/\mathfrak{m}_2^{\#}$. Hence $\mathrm{E}^d_{\Lambda^{\#}}(\Lambda^{\#}) \cong \mathrm{E}_{\Lambda^{\#}}(\Lambda^{\#}/\mathfrak{m}_1^{\#})^2 \oplus \mathrm{E}_{\Lambda^{\#}}(\Lambda^{\#}/\mathfrak{m}_2^{\#})^{1/2}$. Therefore $\mu^d(\mathfrak{m}_1, \Lambda) = 2$ and $\mu^d(\mathfrak{m}_2, \Lambda) = 1/2$ by 5.1(4), so that we have

$$\mathrm{E}^d_{\Lambda}(\Lambda) \cong \mathrm{E}_{\Lambda}(\Lambda/\mathfrak{m}_1)^2 \oplus \mathrm{E}_{\Lambda}(\Lambda/\mathfrak{m}_2)^{1/2}.$$

Now let $P \in \operatorname{Spec} \Lambda$ and put $\mathfrak{p} = P \cap R$. Let $i = \operatorname{Kdim} R_{\mathfrak{p}}$. Then $\operatorname{ht}_{\Lambda} P = i$ (5.3). We have

$$\mu^i(P, \Lambda) = \mu^i(P\Lambda_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) = \begin{cases} 2 & \text{if } t \in \mathfrak{p} \text{ and } P = \begin{bmatrix} \mathfrak{p} & R & R \\ tR & R & R \\ tR & R & R \end{bmatrix}, \\ \dfrac{1}{2} & \text{if } t \in \mathfrak{p} \text{ and } P = \begin{bmatrix} R & R & R \\ tR & \mathfrak{p} & \mathfrak{p} \\ tR & \mathfrak{p} & \mathfrak{p} \end{bmatrix}. \end{cases}$$

If $t \notin \mathfrak{p}$, then

$$P = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ t\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$$

and $\Lambda_{\mathfrak{p}} = \mathbb{M}_3(R_{\mathfrak{p}})$, whence $\mu^i(P, \Lambda) = \mu^i(P\Lambda_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) = 1$ (8.1). Thus $\mathrm{E}^i_{\Lambda}(\Lambda)$ has the required form. ∎

EXAMPLE 8.7. *Let R be a Gorenstein ring and let $n \geq 2$ be an integer. The R-algebra*

$$\Lambda = \{[a_{ij}] \in \mathbb{M}_n(R) \mid a_{ij} = 0 \textit{ if } i < j\}$$

of lower triangular matrices has the following properties.

(1) $\operatorname{Kdim} \Lambda = \operatorname{Kdim} R$ *and* $\operatorname{id}_{\Lambda} \Lambda = \operatorname{id}_R R + 1$. *Hence Λ is not a Gorenstein R-algebra.*

(2) *Let* $S(\mathfrak{p}) = \{P \in \operatorname{Spec} \Lambda \mid P \cap R = \mathfrak{p}\}$ *for* $\mathfrak{p} \in \operatorname{Spec} R$*. Then the set* $S(\mathfrak{p})$ *consists of the* n *elements* $\mathfrak{P}_k(\mathfrak{p}) = \{[a_{ij} \in \Lambda \mid a_{kk} \in \mathfrak{p}\}$ $(1 \le k \le n)$.

(3) *For each* $i \in \mathbb{Z}$,

$$\mathrm{E}^i_\Lambda(\Lambda) = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p} = i} \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_n(\mathfrak{p}))^n \oplus \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \textit{ with } \operatorname{Kdim} R_\mathfrak{p} = i-1} \ \bigoplus_{1 \le k \le n-1} \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_k(\mathfrak{p})).$$

Proof. Since Λ is R-free, Λ is a Cohen–Macaulay R-module with $\operatorname{Kdim} \Lambda = \operatorname{Kdim} R$. We have $S \otimes_R \Lambda = \{[a_{ij}] \in \mathbb{M}_n(S) \mid a_{ij} = 0 \text{ if } i < j\}$ for every commutative R-algebra S.

(1) Note that $\operatorname{id}_\Lambda \Lambda = \sup_{\mathfrak{p} \in \operatorname{Spec} R} \operatorname{id}_{\Lambda_\mathfrak{p}} \Lambda_\mathfrak{p}$. Passing to the localization $\Lambda_\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Spec} R$, we may assume $(R, \mathfrak{m}, \kappa)$ is a local ring. Then the κ-algebra $k \otimes_R \Lambda$ is hereditary and $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_R R + 1$ by 6.2(2).

(2) For the moment suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let $J = \mathrm{J}(\Lambda)$. Then

$$J = \{[a_{ij}] \in \Lambda \mid a_{ii} \in \mathfrak{m} \text{ for all } 1 \le i \le n\},$$

$\Lambda/J = \kappa \times \ldots \times \kappa$ (n times), and $\operatorname{Max} \Lambda = \{\mathfrak{P}_k(\mathfrak{m}) \mid 1 \le k \le n\}$. We have $\mathrm{m}(P) = 1$ for every $P \in \operatorname{Max} \Lambda$. As $\Lambda_\mathfrak{p} = \{[a_{ij}] \in \mathbb{M}_n(R_\mathfrak{p}) \mid a_{ij} = 0 \text{ if } i < j\}$, passing to $\Lambda_\mathfrak{p}$, we get $S(p) = \{\mathfrak{P}_k(\mathfrak{p}R_\mathfrak{p}) \cap \Lambda \mid 1 \le k \le n\} = \{\mathfrak{P}_k(\mathfrak{p}) \mid 1 \le k \le n\}$.

(3) Let $\mathfrak{p} \in \operatorname{Spec} R$ and $P = \mathfrak{P}_k(\mathfrak{p})$ with $1 \le k \le n$. Let $h = \operatorname{Kdim} R_\mathfrak{p}$. Then $h = \operatorname{ht}_\Lambda P$ (cf. 5.3). It suffices to show

$$\mu^i(P, \Lambda) = \begin{cases} n & \text{if } k = n \text{ and } i = h, \\ 0 & \text{if } k = n \text{ and } i \ne h, \\ 1 & \text{if } k \ne n \text{ and } i = h + 1, \\ 0 & \text{if } k \ne n \text{ and } i \ne h + 1. \end{cases}$$

To check it, first we pass to the localization $\Lambda_\mathfrak{p}$ and secondly we reduce Λ by a system of parameters of the base ring. Moreover, by 5.1 we may reduce the problem to the case where $(R, \mathfrak{m})$ is a local ring with $\operatorname{Kdim} R = 0$. Hence $\mathfrak{p} = \mathfrak{m}$ and $\operatorname{id}_\Lambda \Lambda = 1$ by (1). For each $1 \le k \le n$ let $e_k \in \Lambda$ with $[e_k]_{ij} = 1$ if $i = j = k$ and $[e_k]_{ij} = 0$ otherwise. Let $P_k = \Lambda e_k$ and $Q_k = e_k \Lambda$. Let $[*]^\vee = \operatorname{Hom}_R(*, R)$ denote the Matlis dual. Then the canonical exact sequence

$$0 \to \Lambda \to P_1^n \to \bigoplus_{2 \le k \le n} P_1/P_k \to 0$$

of Λ-modules provides a minimal injective resolution for Λ, because $\operatorname{id}_\Lambda \Lambda = 1$ and $P_1 = [Q_n]^\vee$ is an indecomposable injective Λ-module (2.6.2(1)). It is routine to check $P_1/P_k = [Q_{k-1}]^\vee$ for $2 \le k \le n$ so that

$$\mathrm{E}^0_\Lambda(\Lambda) = \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_n(\mathfrak{m}))^n \quad \text{and} \quad \mathrm{E}^1_\Lambda(\Lambda) = \bigoplus_{1\le k\le n-1} \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_k(\mathfrak{m})).$$

Thus the assertions follow (recall that $\mathrm{m}(P) = 1$ for all $P \in \operatorname{Max}\Lambda$). ∎

EXAMPLE 8.8. *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring with* $\operatorname{Kdim} R = d$ *and let $K = \mathrm{Q}(R)$ be the quotient field of R. The R-algebra*

$$L = \begin{bmatrix} R & R \\ \mathfrak{m} & R \end{bmatrix}$$

has the following properties.

(1) $\operatorname{Kdim}\Lambda = d$, $\operatorname{depth}_R \Lambda = \min\{d, 1\}$, *and* $\operatorname{gl.dim}\Lambda = \operatorname{id}_\Lambda \Lambda = \max\{d, 1\}$.
(2) *For each $\mathfrak{p} \in \operatorname{Spec} R$ let $S(\mathfrak{p}) = \{P \in \operatorname{Spec}\Lambda \mid P \cap R = \mathfrak{p}\}$. Then*

$$S(\mathfrak{p}) = \begin{cases} \left\{ \mathfrak{P}_1 = \begin{bmatrix} \mathfrak{m} & R \\ \mathfrak{m} & R \end{bmatrix}, \ \mathfrak{P}_2 = \begin{bmatrix} R & R \\ \mathfrak{m} & \mathfrak{m} \end{bmatrix} \right\} & \text{if } \mathfrak{p} = \mathfrak{m}, \\ \left\{ \begin{matrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{matrix} \right\} & \text{if } \mathfrak{p} \ne \mathfrak{m}. \end{cases}$$

(3) *The minimal injective resolution of Λ has the form*:

$$\begin{aligned} &(d = 0) \quad 0 \to \Lambda \to \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_1)^2 \to \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_2) \to 0, \\ &(d = 1) \quad 0 \to \Lambda \to \mathbb{M}_2(K) \to \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_1) \oplus \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_2) \to 0, \\ &(d \ge 2) \quad 0 \to \Lambda \to \mathbb{M}_2(K) \to \mathrm{E}^1 \to \ldots \to \mathrm{E}^d \to 0, \end{aligned}$$

where $\mathrm{E}^1 = \big[\bigoplus_{\mathfrak{p}\in\operatorname{Spec} R \textit{ with } \operatorname{ht}_R \mathfrak{p}=1} \mathrm{E}_\Lambda\big(\Lambda/\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right]\big)\big] \oplus \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_2)$ *for $i = 1$,* $\mathrm{E}^i = \big[\bigoplus_{\mathfrak{p}\in\operatorname{Spec} R \textit{ with } \operatorname{ht}_R \mathfrak{p}=i} \mathrm{E}_\Lambda\big(\Lambda/\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right]\big)\big] \oplus \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_1)^{\binom{d}{i-1}}$ *for $2 \le i < d$, and* $\mathrm{E}^d = \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_1)^{d+1}$.

Proof. It suffices to show that $\operatorname{gl.dim}\Lambda = \max\{d, 1\}$, since $\operatorname{gl.dim}\Lambda = \operatorname{id}_\Lambda \Lambda$ if $\operatorname{gl.dim}\Lambda < \infty$. Let $J = \mathrm{J}(\Lambda)$. Then $J = \left[\begin{smallmatrix} \mathfrak{m} & R \\ \mathfrak{m} & \mathfrak{m}\end{smallmatrix}\right]$ and $\Lambda/J = \kappa \times \kappa$. Hence $\operatorname{Max}\Lambda = S(\mathfrak{m}) = \{\mathfrak{P}_1, \mathfrak{P}_2\}$ where $\mathfrak{P}_1 = \left[\begin{smallmatrix} \mathfrak{m} & R \\ \mathfrak{m} & R\end{smallmatrix}\right]$, $\mathfrak{P}_2 = \left[\begin{smallmatrix} R & R \\ \mathfrak{m} & \mathfrak{m}\end{smallmatrix}\right]$. We have $\mathrm{m}(P) = 1$ for all $P \in \operatorname{Max}\Lambda$. Let $S_i = \Lambda/\mathfrak{P}_i$ $(i = 1, 2)$ and $e_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right]$, $e_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right]$. Then S_1 and S_2 have the presentations

$$0 \to \mathfrak{m}\cdot\Lambda e_2 \to \Lambda e_1 \to S_1 \to 0 \quad \text{and} \quad 0 \to \Lambda e_1 \to \Lambda e_2 \to S_2 \to 0.$$

Hence $\operatorname{pd}_\Lambda S_2 = 1$. Let $L = \Lambda e_2$. Let $\underline{x} = x_1, \ldots, x_d$ be a minimal system of generators of $\mathfrak{m}$ and let

$$\mathrm{K}_\bullet = \mathrm{K}_\bullet(\underline{x}; R): \quad 0 \to K_d \to K_{d-1} \to \ldots \to K_1 \xrightarrow{\varepsilon} R = K_0$$

be the Koszul complex of R generated by the sequence. Then $\mathrm{K}_\bullet$ is a minimal free resolution of $\kappa = R/\mathfrak{m}$ and the complex

$$L \otimes_R \mathrm{K}_\bullet : 0 \to L \otimes_R K_d \to L \otimes_R K_{d-1} \to \ldots \to L \otimes_R K_1 \xrightarrow{L\otimes_R\varepsilon} L = L \otimes_R K_0$$

gives rise to a minimal projective resolution of the Λ-module $L/\mathfrak{m}L$, since $L \cong R^2$ as R-modules and $\mathfrak{m}\Lambda \subseteq J$. Hence $\operatorname{pd}_\Lambda S_1 = d$ (note $\operatorname{Im}(L \otimes_R K_1 \xrightarrow{L\otimes_R\varepsilon} L = L \otimes_R K_0) = \mathfrak{m}L$). Thus $\operatorname{gl.dim}\Lambda = \max\{\operatorname{pd}_\Lambda S_1, \operatorname{pd}_\Lambda S_2\} = \max\{d, 1\}$.

(2) If $\mathfrak{p} \neq \mathfrak{m}$, then $\Lambda_\mathfrak{p} = \mathbb{M}_2(R_\mathfrak{p})$, whence $\Lambda_\mathfrak{p}$ contains the unique maximal ideal $\mathfrak{p}\Lambda_\mathfrak{p} = \mathbb{M}_2(\mathfrak{p}R_\mathfrak{p})$. Thus $S(\mathfrak{p}) = \left\{\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right]\right\}$, because $\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right] = \mathfrak{p}\Lambda_\mathfrak{p} \cap \Lambda$.

(3) Let $\mathfrak{p} \in \operatorname{Spec} R$ be different from $\mathfrak{m}$ and put $P = \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right]$. Then by 8.1 we have $\mu^i(P, \Lambda) = \delta_{i,\operatorname{ht}_\Lambda P}$ for every $i \in \mathbb{Z}$ because $\mu^i(P, \Lambda) = \mu^i(\mathbb{M}_2(\mathfrak{p}R_\mathfrak{p}), \mathbb{M}_2(R_\mathfrak{p}))$ by 5.1(1). We will show

Claim. *If $i \in \mathbb{Z}$, then $\mu^i(\mathfrak{P}_2, \Lambda) = \delta_{i,1}$. Furthermore:*

(a) $\mu^i(\mathfrak{P}_1, \Lambda) = 2\delta_{i,0}$ *for* $d = 0$, $\mu^i(\mathfrak{P}_1, \Lambda) = \delta_{i,1}$ *for* $d = 1$, *and for* $d \geq 2$ *we have*

(b)
$$\mu^i(\mathfrak{P}_1, \Lambda) = \begin{cases} \binom{d}{i-1} & \text{if } 2 \leq i < d, \\ d+1 & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $[*]^* = \operatorname{Hom}_\Lambda(*, \Lambda)$. Take the Λ-dual of the presentation $0 \to \Lambda e_1 \to \Lambda e_2 \to S_2 \to 0$ of S_2. Then because $S_2^* = (0)$ and $[\Lambda e_i]^* \cong e_i\Lambda$ $(i = 1, 2)$, we get the short exact sequence

$$0 \to e_2\Lambda \xrightarrow{\varepsilon} e_1\Lambda \to \operatorname{Ext}^1_\Lambda(S_2, \Lambda) \to 0$$

where $\varepsilon : e_2\Lambda \to e_1\Lambda$ is the map $\varepsilon\left(\left[\begin{smallmatrix} 0 & 0 \\ a & b\end{smallmatrix}\right]\right) = \left(\left[\begin{smallmatrix} a & b \\ 0 & 0\end{smallmatrix}\right]\right)$. Hence $\ell_{\Lambda/\mathfrak{P}_2}(\operatorname{Ext}^1_\Lambda(S_2, \Lambda)) = \ell_\kappa(\operatorname{Ext}^1_\Lambda(S_2, \Lambda)) = 1$ so that $\mu^i(\mathfrak{P}_2, \Lambda) = \delta_{i,1}$ for all $i \in \mathbb{Z}$.

(a) $(d = 0)$ Note that $S_1 \cong \Lambda e_1$ and $[\Lambda e_1]^* \cong e_1\Lambda = \left[\begin{smallmatrix} R & R \\ 0 & 0\end{smallmatrix}\right]$.

$(d = 1)$ See 7.6. Recall that Λ is a Gorenstein R-algebra with $\mathrm{m}(P) = 1$ for all $P \in \operatorname{Max}\Lambda$ (cf. (1)).

(b) $(d \geq 2)$ We have $\mu^i(\mathfrak{P}_1, \Lambda) = 0$ if $i \leq 1$ or $i > d$ (cf. 3.7). Let $2 \leq i \leq d$. Then by the presentation $0 \to \mathfrak{m}L \to \Lambda e_1 \to S_1 \to 0$ of S_1 with $L = \Lambda e_2$ we naturally have the isomorphism

$$\operatorname{Ext}^i_\Lambda(S_1, \Lambda) \cong \operatorname{Ext}^i_\Lambda(L/\mathfrak{m}L, \Lambda).$$

Now take the Λ-dual of the above projective resolution $L \otimes_R \mathrm{K}_\bullet$ of $L/\mathfrak{m}L$ and note that $\operatorname{Hom}_\Lambda(L \otimes_R \mathrm{K}_\bullet, \Lambda) = \operatorname{Hom}_R(\mathrm{K}_\bullet, L^*)$ as complexes of Λ^{op}-modules. Then

$$\operatorname{Ext}^i_\Lambda(L/\mathfrak{m}L, \Lambda) \cong \operatorname{Ext}^i_R(R/\mathfrak{m}, L^*).$$

Therefore since $L^* \cong e_2\Lambda = \left[\begin{smallmatrix} 0 & 0 \\ \mathfrak{m} & R\end{smallmatrix}\right]$, we get

$$\begin{aligned} \ell_{\Lambda/\mathfrak{P}_1}(\operatorname{Ext}^i_\Lambda(L/\mathfrak{m}L, \Lambda)) &= \ell_\kappa(\operatorname{Ext}^i_\Lambda(L/\mathfrak{m}L, \Lambda)) = \ell_\kappa(\operatorname{Ext}^i_R(R/\mathfrak{m}, L^*)) \\ &= \ell_\kappa(\operatorname{Ext}^i_R(R/\mathfrak{m}, \mathfrak{m})) + \ell_\kappa(\operatorname{Ext}^i_R(R/\mathfrak{m}, R)). \end{aligned}$$

Hence $\mu^i(\mathfrak{P}_1, \Lambda) = \binom{d}{i-1}$ if $2 \leq i < d$ and $\mu^i(\mathfrak{P}_1, \Lambda) = d + 1$ because R is a regular local ring with $\operatorname{Kdim} R = d \geq 2$. ∎

In Example 8.8 let $d = 2$. Then the minimal injective resolution of Λ has the following form:

$$0 \to \Lambda \to \mathbb{M}_2(K) \to \left[\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R,\, \operatorname{ht}_R \mathfrak{p}=1} \mathrm{E}_\Lambda\left(\Lambda/\begin{bmatrix}\mathfrak{p} & \mathfrak{p}\\ \mathfrak{p} & \mathfrak{p}\end{bmatrix}\right)\right] \oplus \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_2)$$
$$\to \mathrm{E}_\Lambda(\Lambda/\mathfrak{P}_1)^3 \to 0.$$

Let $\mathfrak{p}$ be any height 1 prime ideal in R and put $P = \left[\begin{smallmatrix}\mathfrak{p} & \mathfrak{p}\\ \mathfrak{p} & \mathfrak{p}\end{smallmatrix}\right]$. Then $\operatorname{ht}_\Lambda P = 1$, $\operatorname{ht}_\Lambda \mathfrak{P}_2 = 2$, and $P \subseteq \mathfrak{P}_2$. We have $\mu^2(\mathfrak{P}_2, \Lambda) = 0$ while $\mu^1(P, \Lambda) = 1$. This shows that Lemma 3.3 of Bass is no more true if we replace $\operatorname{Ass}_R \mathrm{E}^i_\Lambda(M)$ with $\operatorname{Ass}_\Lambda \mathrm{E}^i_\Lambda(M)$. We have $\operatorname{pd}_\Lambda \Lambda/\mathfrak{P}_2 = 1 < \operatorname{ht}_\Lambda \mathfrak{P}_2 = 2$ and Λ is not a Cohen–Macaulay R-module, since $\operatorname{Kdim} \Lambda = 2$ but $\operatorname{depth}_R \Lambda = 1$. This provides a counterexample to [A] ((1.1) Theorem (ii)) and the claim of Brown and Hajarnavis [BHa1] (p. 199, ↓16, proof of Theorem 2.5) that $\operatorname{rank}(M) \le \operatorname{prdim}_R(R/M)$ as well.

Question 8.9. Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\mu^i(\mathfrak{m}, R) > 0$ if $\operatorname{depth}_R R \le i \le \operatorname{id}_R R$ ([Ro1]). Let Λ be a module-finite R-algebra and $i \in \mathbb{Z}$. Is it true that $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{E}^i_\Lambda(\Lambda)$ if and only if $\operatorname{depth}_R \Lambda \le i \le \operatorname{id}_\Lambda \Lambda$? The answer is affirmative if Λ is Cohen–Macaulay as an R-module.

REFERENCES

[AF] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer, 1991.

[A] M. Artin, *Maximal orders of global dimension and Krull dimension two*, Invent. Math. 84 (1986), 195–222.

[AM] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[Au1] M. Auslander, *On the dimension of modules and algebras (III). Homological dimension*, Nagoya Math. J. 9 (1955), 67–77.

[Au2] —, *Rational singularities and almost split exact sequences*, Trans. Amer. Math. Soc. 293 (1986), 511–531.

[AB] M. Auslander and R. O. Buchweitz, *The homological theory of maximal Cohen–Macaulay approximations*, Mem. Soc. Math. France 38 (1989), 5–37.

[AR1] M. Auslander and I. Reiten, *The CM type of CM algebras*, Adv. Math. 73 (1989), 1–23.

[AR2] —, —, *Applications of contravariantly finite subcategories*, ibid. 86 (1991), 111–152.

[ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[B1] H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. 102 (1962), 18–29.

[B2] —, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–28.

[BC] G. M. Bergman and P. M. Cohn, *The centres of 2-firs and hereditary rings*, Proc. London Math. Soc. (3) 23 (1971), 83–98.

[Bj] J.-E. Björk, *The Auslander condition on Noetherian rings*, in: Sém. d'Algèbre P. Dubreil et M.-P. Malliavin, 1987–88 (M.-P. Malliavin, ed.), Lecture Notes in Math. 1404, Springer, 1989, 137–173.

[Bo] N. Bourbaki, *Elements of Mathematics*, *Commutative Algebra*, Hermann, Paris, 1972.

[BHa1] A. Brown and R. Hajarnavis, *Homologically homogeneous rings*, Trans. Amer. Math. Soc. 281 (1984), 197–208.

[BHa2] —, —, *Injectively homogeneous rings*, J. Pure Appl. Algebra 51 (1988), 65–77.

[BH] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, 1993.

[C] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. 59 (1946), 54–106.

[DK] Yu. A. Drozd and V. V. Kirichenko, *Finite Dimensional Algebras*, Springer, 1994.

[E] D. Eisenbud, *Subrings of Artinian and Noetherian rings*, Math. Ann. 185 (1970), 247–279.

[En] S. Endo, *Completely faithful modules and quasi-Frobenius algebras*, J. Math. Soc. Japan 19 (1967), 437–456.

[GW] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Student Texts 16, Cambridge Univ. Press, 1989.

[G1] S. Goto, *Vanishing of* $\mathrm{Ext}^i_A(M, A)$, J. Math. Kyoto Univ. 22 (1982), 481–484.

[G2] —, *Injective dimension in Noetherian algebras*, in: Proc. 29th Symposium on Ring Theory and Representation Theory (Kashikojima, 1996), Univ. of Tsukuba, 1997, 7–16.

[GN1] S. Goto and K. Nishida, *Catenarity in module-finite algebras*, Proc. Amer. Math. Soc. 127 (1999), 3495–3502.

[GN2] —, —, *Minimal injective resolutions of Cohen–Macaulay isolated singularities*, Arch. Math. (Basel) 73 (1999), 249–255.

[GN3] —, —, *Finite modules of finite injective dimension over a Noetherian algebra*, J. London Math. Soc. 63 (2001), 319–335.

[Gr] A. Grothendieck, *Local Cohomology*, Lecture Notes in Math. 41, Springer, 1967.

[HK] J. Herzog and E. Kunz (eds.), *Der kanonische Modul eines Cohen–Macaulay-Rings*, Lecture Notes in Math. 238, Springer, 1971.

[HN] H. Hijikata and K. Nishida, *Bass orders in non-semisimple algebras*, J. Math. Kyoto Univ. 34 (1994), 797–837.

[IS] Y. Iwanaga and H. Sato, *Minimal injective resolutions of Gorenstein rings*, Comm. Algebra 18 (1990), 3835–3856.

[K] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, 1970.

[M] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. 8 (1958), 511–528.

[Ma] H. Matsumura, *Commutative Algebra*, 2nd ed., Benjamin/Cummings, 1980.

[MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, 1987.

[N] K. Nishida, *A characterization of Gorenstein orders*, Tsukuba J. Math. 12 (1988), 459–468.

[Ni] J. Nishimura, *A few examples of local rings*, preprint, Hokkaido University of Education, 1998.

[R] M. Ramras, *Maximal orders over regular local rings of dimension two*, Trans. Amer. Math. Soc. 142 (1969), 457–479.

[Re] I. Reiten, *The converse to a theorem of Sharp on Gorenstein modules*, Proc. Amer. Math. Soc. 32 (1972), 417–420.

[Ro1] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) 9 (1976), 103–106.

[Ro2] —, *Rings of type 1 are Gorenstein*, Bull. London Math. Soc. 15 (1983), 48–50.

[Ro3] —, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I 304 (1987), 177–180.

[S] W. Schelter, *Non-commutative affine P. I. rings are catenary*, J. Algebra 51 (1978), 12–18.

[Se] J.-P. Serre, *Algèbre Locale*, *Multiplicités*, 3rd ed., Lecture Notes in Math. 11, Springer, 1975.

[Sh1] R. Y. Sharp, *The Cousin complex for a module over a commutative ring*, Math. Z. 112 (1969), 340–356.

[Sh2] —, *Gorenstein modules*, ibid. 115 (1970), 117–139.

[Sh3] —, *On Gorenstein modules over a complete Cohen–Macaulay local ring*, Quart. J. Math. Oxford Ser. (2) 22 (1971), 425–434.

[Sh4] —, *Local cohomology and the Cousin complex for a commutative Noetherian ring*, Math. Z. 153 (1977), 19–22.

[Sh5] —, *A Cousin complex characterization of balanced big Cohen–Macaulay modules*, Quart. J. Math. Oxford Ser. (2) 33 (1982), 471–485.

[Si] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon and Breach, 1992.

[V] W. V. Vasconcelos, *On quasi-local regular algebras*, Symposia Math. 11 (1973), 11–22.

[Y] K. Yamagata, *Frobenius algebras*, in: Handbook of Algebra, M. Hazewinkel (ed.), Vol. 1, North-Holland Elsevier, Amsterdam, 1996, 841–887.

[Yo] Y. Yoshino, *Cohen–Macaulay Modules over Cohen–Macaulay Rings*, London Math. Soc. Lecture Note Ser. 146, Cambridge Univ. Press, 1990.

Department of Mathematics
Meiji University
Kawasaki 214-8571, Japan
E-mail: goto@math.meiji.ac.jp

Department of Mathematical Sciences
Shinshu University
Matsumoto 390-8621, Japan
E-mail: kenisida@math.shinshu-u.ac.jp

Received 12 April 2000;
revised 8 March 2001 (3918)