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LOCAL-GLOBAL PRINCIPLE FOR WITT EQUIVALENCE OF FUNCTION FIELDS OVER GLOBAL FIELDS

BY

PRZEMYSŁAW KOPROWSKI (Katowice)

Abstract. We examine the conditions for two algebraic function fields over global fields to be Witt equivalent. We develop a criterion solving the problem which is analogous to the local-global principle for Witt equivalence of global fields obtained by R. Perlis, K. Szymiczek, P. E. Conner and R. Litherland [12]. Subsequently, we derive some immediate consequences of this result. In particular we show that Witt equivalence of algebraic function fields (that have rational places) over global fields implies Witt equivalence of their fields of constants. We also discuss the converse of this implication.

Two fields K and L are said to be *Witt equivalent* when their Witt rings are isomorphic. The classification of fields with respect to Witt equivalence is a natural problem in the algebraic theory of quadratic forms. For fields with a finite number of square classes the classification is complete when the number of square classes does not exceed 32 (see [2]) while for fields with infinite groups of square classes a definitive solution of the problem has been achieved for the global fields of number theory (see [12]) and the algebraic function fields over real closed fields (see [9]).

In this paper we investigate the case of algebraic function fields with global fields of constants. Our approach to classification of Witt rings of function fields is parallel to that of [12] and [9]. In the first section we introduce the notion of quaternion-symbol equivalence and present the main result of this note (Theorem 1.3). This theorem establishes a local-global principle for Witt equivalence of function fields analogous to the local-global principle for global fields presented in [12]. Namely, we show that two function fields are Witt equivalent if and only if they are quaternion-symbol equivalent iff they are uniformly locally Witt equivalent. The proof occupies the entire second section. Finally, in the last section we derive some consequences of our result. In particular we show that Witt equivalence of function fields implies Witt equivalence of their fields of constants (see 3.2), and we show two partial converses. The first of them says that Witt equivalence of global fields implies local Witt equivalence of the rational function fields over them

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(see 3.6), while the second one gives additional sufficient conditions for Witt equivalence of the fields of constants to imply Witt equivalence of algebraic function fields (see 3.7 and 3.9). Those conditions are also of local nature.

Throughout this note we use the letters K, L to denote algebraic (including possibly rational) function fields with fields of constants k and l respectively. Next, we use the symbol $\Omega(K)$ to denote the set of all places of the field K trivial on its field of constants. The completion of the field K with respect to a place $\mathfrak{p} \in \Omega(K)$ will be denoted by $K_\mathfrak{p}$ and its residue field by $K(\mathfrak{p})$. Finally $f(\mathfrak{p})$ is the image of an element $f \in \mathcal{O}_\mathfrak{p}$ in the residue field.

By abuse of notation, we use the same symbol f for both an element of a field and the square class containing this element; analogously, $\langle f_1, \ldots, f_n\rangle$ will denote both a quadratic form and its class in the Witt ring; and finally, $\left(\frac{f,g}{K}\right)$ means (depending on the context) either a quaternion algebra or its class in the Brauer group.

In addition, we use standard notation and terminology for quadratic forms, valuations and function fields as described in [3, 10]. In particular, following [10], by a *local field* we understand a field complete with respect to a discrete valuation (i.e. we do not require its residue field to be finite).

A smooth introduction to the theory of quadratic forms and Witt equivalence can be found in [18].

1. Local-global principle. We need to introduce some local conditions for Witt equivalence which are similar to the conditions defining the *Hilbert-symbol equivalence* between global fields k and l. Recall that the latter is a pair of maps (t, T) in which $t : k^*/k^{*2} \to l^*/l^{*2}$ is an isomorphism of square-class groups, and $T : \Omega(k) \to \Omega(l)$ is a bijection between the sets of all places of the two fields preserving Hilbert symbols in the sense that

$$(a, b)_\mathfrak{p} = 1 \;\Leftrightarrow\; (ta, tb)_{T\mathfrak{p}} = 1$$

for all square classes $a, b \in k^*/k^{*2}$ and all places $\mathfrak{p}$ of k.

When K and L are algebraic function fields the Hilbert symbols have to be replaced with quaternion algebras over the completions of K and L and so we are led to the following definition.

DEFINITION 1.1. We say that the function fields K and L are *quaternion-symbol equivalent* when there exists a pair (t, T) of maps in which

- $t : K^*/K^{*2} \to L^*/L^{*2}$ is an isomorphism of square-class groups,
- $T : \Omega(K) \to \Omega(L)$ is a bijection,

preserving the splitting of local quaternion symbols in the sense that

$$\left(\frac{f, g}{K_\mathfrak{p}}\right) = 1 \;\Leftrightarrow\; \left(\frac{tf, tg}{L_{T\mathfrak{p}}}\right) = 1$$

for all square classes $f, g \in K^*/K^{*2}$ and all places $\mathfrak{p}$ of K.

Following [15], we say that two fields are *locally Witt equivalent* if their places can be paired so that the corresponding completions are Witt equivalent. We also say that two fields K, L are *uniformly locally Witt equivalent* if they are locally Witt equivalent and all those local isomorphisms of Witt rings arise from one fixed isomorphism of the Witt rings of the fields K and L:

DEFINITION 1.2. Fields K, L are said to be *uniformly locally Witt equivalent* if there exist

- a strong Witt isomorphism $i : WK \to WL$,
- a bijection $T : \Omega(K) \to \Omega(L)$,
- a set $\{i_{\mathfrak{p}} : WK_{\mathfrak{p}} \to WL_{T\mathfrak{p}} : \mathfrak{p} \in \Omega(K)\}$ of ring isomorphisms

so that for every $\mathfrak{p} \in \Omega(K)$ the following diagram commutes:

$$\begin{tikzcd} WK \arrow[r, "i"] \arrow[d, "\theta_{\mathfrak{p}}"'] & WL \arrow[d, "\theta_{T\mathfrak{p}}"] \\ WK_{\mathfrak{p}} \arrow[r, "i_{\mathfrak{p}}"] & WL_{T\mathfrak{p}} \end{tikzcd}$$

where the θ's are the canonical epimorphisms.

The main result of this paper is the following local-global principle for Witt equivalence of function fields:

THEOREM 1.3. *Let k, l be two fields satisfying*:

(A1) $\operatorname{char} k \neq 2 \neq \operatorname{char} l$.

(A2) *For any finite extensions $k' \supseteq k$, $l' \supseteq l$,*

$$\operatorname{card}(k'^*/k'^{*2}) \geq 4 \quad \textit{and} \quad \operatorname{card}(l'^*/l'^{*2}) \geq 4.$$

Let $K := k(X)$ and $L := l(X)$ denote the rational function fields over k and l respectively. The following conditions are equivalent:

(1.3.1) *K and L are Witt equivalent.*

(1.3.2) *K and L are uniformly locally Witt equivalent.*

(1.3.3) *K and L are quaternion-symbol equivalent.*

Moreover, *the equivalence of* (1.3.1) *and* (1.3.2) *also holds when K and L are algebraic function fields with fields of constants k and l respectively.*

Before we proceed to the proof let us present some basic examples of fields satisfying (A1)–(A2):

- global fields of characteristic $\neq 2$,
- $\mathfrak{p}$-adic local fields,
- rational function fields (of an arbitrary number of variables) over fields satisfying (A1)–(A2),
- algebraic function fields over fields satisfying (A1)–(A2).

The above theorem is analogous to the result of [12], where it is proved that two global fields (in particular, algebraic function fields over finite fields) are Witt equivalent if and only if they are locally Witt equivalent if and only if they are Hilbert-symbol equivalent.

2. Proof of Theorem 1.3

Step 1. The implication (1.3.2)⇒(1.3.1) is trivial. We show (1.3.1)⇒ (1.3.2). The set-up of this proof is somehow analogous to [14, Proposition 4.1]. Unfortunately we cannot in general claim $O_L(H, S)$ to be unexceptional, thus our proof branches off.

Assume that K, L are algebraic function fields with fields of constants k, l respectively. Let $\mathcal{T} := K^* \cap K_{\mathfrak{p}}^{*2}$ and $\mathcal{S} := t(\mathcal{T})$. Combining [14, Lemma (2.1)(3)] and [14, Remark (2.2)] we see that $(1+\mathfrak{p})K^{*2} = \mathcal{T}$. We have

$$K^*/\mathcal{T} = K^*/(K^* \cap K_{\mathfrak{p}}^{*2}) \cong (K^* \cdot K_{\mathfrak{p}}^{*2})/K_{\mathfrak{p}}^{*2} \cong K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2};$$

here $\operatorname{card} K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2} = 2 \cdot \operatorname{card} K(\mathfrak{p})^*/K(\mathfrak{p})^{*2}$ and by (A2) this is at least 8. Therefore

$$[K^* : \mathcal{T}] = [L^* : \mathcal{S}] \geq 8.$$

Observe that the proof of [14, Lemma (3.2)(1)] remains valid also in our case. Hence $B_K(\mathcal{T}) = \pm\mathcal{T}$. Now, [14, Lemma (3.1)(3)] provides us with the equality $B_L(\mathcal{S}) = t(B_K(\mathcal{T})) = \pm\mathcal{S}$, and so

$$[L^* : B_L(\mathcal{S})] \geq 4.$$

Therefore, for $H := B_L(\mathcal{S})$ there exists (by [1, Theorem (2.16)]) a group $\widehat{H} \subset L^*$ such that: $[\widehat{H} : B_L(\mathcal{S})] \leq 2$, the ring $O_L(\widehat{H}, \mathcal{S})$ is an $\mathcal{S}$-compatible valuation ring and $U_{O_L(\widehat{H},\mathcal{S})} \cdot \mathcal{S} \subseteq \widehat{H}$. Since $[L^* : B_L(\mathcal{S})] \geq 4$, we have $[L^* : \widehat{H}] \geq 2$, and so $\widehat{H} \neq L^*$. Now, $U_{O_L(\widehat{H},\mathcal{S})} \subset \widehat{H} \neq L^*$, thus $O_L(\widehat{H}, \mathcal{S})$ is a proper valuation ring. Let $\mathfrak{q}$ denote the place of L associated with $O_L(\widehat{H}, \mathcal{S})$.

The place $\mathfrak{q}$ is $\mathcal{S}$-compatible, and so $1 + \mathfrak{q} \subseteq \mathcal{S}$. Hence $(1 + \mathfrak{q}) \cdot L^{*2} \subseteq \mathcal{S}$. Consider the mappings

$$\begin{array}{ccc} WK & \xrightarrow{i} & WL \\ {\scriptstyle\theta_{\mathfrak{p}}}\downarrow & & \downarrow{\scriptstyle\theta_{\mathfrak{q}}} \\ WK_{\mathfrak{p}} & & WL_{\mathfrak{q}} \end{array}$$

where the θ's are the canonical homomorphisms. Now, [14, Lemma (2.1)] implies that $\ker\theta_{\mathfrak{p}}$ is generated by $\{\langle 1, -a\rangle : a \in K^* \cap K_{\mathfrak{p}}^{*2}\} = \{\langle 1, -a\rangle : a \in \mathcal{T}\}$. Thus $i(\ker\theta_{\mathfrak{p}})$ is generated by $\{\langle t1, t(-a)\rangle : a \in \mathcal{T}\} = \{\langle 1, -b\rangle : b \in \mathcal{S}\}$. But $\ker\theta_{\mathfrak{q}}$ is generated by $\{\langle 1, -c\rangle : c \in (1+\mathfrak{q}) \cdot L^{*2}\}$. Finally,

$$\ker\theta_{\mathfrak{q}} \subseteq i(\ker\theta_{\mathfrak{p}}).$$

The same argument applied to i^{-1} and $\mathfrak{q}$ provides us with a place $\mathfrak{p}' \in \Omega(K)$ such that $\ker\theta_{\mathfrak{p}'} \subseteq i^{-1}(\ker\theta_{\mathfrak{q}})$. Consequently,

$$\ker\theta_{\mathfrak{p}'} \subseteq i^{-1}(\ker\theta_{\mathfrak{q}}) \subseteq \ker\theta_{\mathfrak{p}}.$$

Now [14, Lemma (2.3)(2)] implies $\mathfrak{p} = \mathfrak{p}'$, so all the above inclusions are in fact equalities, thus

$$i(\ker\theta_{\mathfrak{p}}) = \ker\theta_{\mathfrak{q}}.$$

Therefore we conclude that there exists an isomorphism $i_{\mathfrak{p}} : WK_{\mathfrak{p}} \to WL_{\mathfrak{q}}$ completing the above diagram. This means that K and L are uniformly locally Witt equivalent.

Step 2. From now on $K = k(X)$ and $L = l(X)$ are rational function fields. We shall prove (1.3.3)⇒(1.3.1). To this end we need the following lemma, which generalizes the theorem on nondegeneracy of Hilbert symbols (cf. [10, Theorem 2.16, p. 158]) to our case.

LEMMA 2.1. *Let k be a field satisfying* (A1)–(A2) *and let $K = k(X)$ denote the rational function field over k. For an arbitrary element $c \in K \setminus \{0\}$, if all the algebras $\left(\frac{z,c}{K_{\mathfrak{p}}}\right)$ split for $z \in K$ and $\mathfrak{p} \in \Omega(K)$, then c is a square in K.*

Proof. Choose a square-free polynomial $c_0 \in k[X]$ such that the classes of c and c_0 in K^*/K^{*2} are equal. Assume first that $\deg c_0 > 0$. Fix $\mathfrak{p} \in \Omega(K)$ such that $\mathfrak{p} \mid c_0$. Since k is not quadratically closed by (A2), [9, Lemma 2.5] implies that there exists $z \in K$ such that $\left(\frac{z,c_0}{K_{\mathfrak{p}}}\right)\left(= \left(\frac{z,c}{K_{\mathfrak{p}}}\right)\right)$ does not split—a contradiction.

Hence, $\deg c_0 = 0$. Now, for any place $\mathfrak{p}$ of degree 1, c_0 is a square in $K(\mathfrak{p}) = k$ by [9, Lemma 2.4]. Thus $c \in K^{*2}$. ■

We are now ready to prove the implication (1.3.3)⇒(1.3.1). Let the pair (T, t) denote the quaternion-symbol equivalence of the fields K and L.

We claim that $1 \in D_K\langle f, g\rangle$ if and only if $1 \in D_L\langle tf, tg\rangle$ for any $f, g \in K^*/K^{*2}$. Indeed, assume that for some f, g we have $1 \in D_K\langle f, g\rangle$. Then the algebra $\left(\frac{f,g}{K}\right)$ splits, so all the algebras $\left(\frac{f,g}{K_{\mathfrak{p}}}\right)$ split, where $\mathfrak{p}$ runs over all places of K. By the definition of quaternion-symbol equivalence, all the algebras $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ split. This means that the forms $\langle tf, tg\rangle$ and $\langle 1, tftg\rangle$ are equivalent over all $L_{T\mathfrak{p}}$. But T is a bijection of the sets of places of K and L, hence $T\mathfrak{p}$ runs over all places of L. Therefore, by [8, (iv), p. 469], $\langle tf, tg\rangle \simeq \langle 1, tftg\rangle$ over L. In other words $1 \in D_L\langle tf, tg\rangle$, which proves our claim.

Next we show that $t(-1) = -1$. Let $c := t(-1)$. For any $x \in K$ and $\mathfrak{p} \in \Omega(K)$ the algebra $\left(\frac{x,-x}{K_{\mathfrak{p}}}\right)$ splits. Thus in the Brauer group $\mathrm{Br}(L_{T\mathfrak{p}})$ we have

$$1 = \left(\frac{tx, ctx}{L_{T\mathfrak{p}}}\right) = \left(\frac{tx, -tx}{L_{T\mathfrak{p}}}\right) \cdot \left(\frac{tx, -c}{L_{T\mathfrak{p}}}\right) = \left(\frac{tx, -c}{L_T\mathfrak{p}}\right).$$

So $-c$ splits all the quaternion algebras $\left(\frac{z,c}{\mathfrak{q}}\right)$ for all $z \in L^*/L^{*2}$ and $\mathfrak{q} \in \Omega(L)$. Hence, by Lemma 2.1 it is a square. Consequently, t is a Harrison isomorphism, and so the fields in question are Witt equivalent by Harrison's criterion (see [12]).

Step 3. It remains to prove the implication (1.3.2)$\Rightarrow$(1.3.3). Fix any place $\mathfrak{p} \in \Omega(K)$ and let $f, g \in K^*/K^{*2}$ be two square classes such that $\left(\frac{f,g}{K_\mathfrak{p}}\right)$ splits. Then $\theta_\mathfrak{p}\langle\langle f, g\rangle\rangle = 0$, hence also $i_\mathfrak{p} \circ \theta_\mathfrak{p}\langle\langle f, g\rangle\rangle = 0$. By the assumption (1.3.2) we have $\theta_{T\mathfrak{p}} \circ i\langle\langle f, g\rangle\rangle = 0$. Moreover, i is a strong isomorphism of Witt rings, hence $\theta_{T\mathfrak{p}}\langle\langle tf, tg\rangle\rangle = 0$. Consequently, the algebra $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ splits. Using the same reasoning for inverse maps one shows that if $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ splits, then $\left(\frac{f,g}{K_\mathfrak{p}}\right)$ splits as well. Consequently, the pair (T, t) constitutes a quaternion-symbol equivalence.

3. Some consequences. The following lemma generalizes, to the case of arbitrary local fields, a fact which is well known for local fields with finite residue fields. Unfortunately the proof given for $\mathfrak{p}$-adic fields in [10] does not hold in the general case. Hence we feel obliged to state this result explicitly.

LEMMA 3.1. *Let E, F be two local fields with respect to nondyadic places $\mathfrak{p}$ and $\mathfrak{q}$. Then E, F are Witt equivalent if and only if $E(\mathfrak{p})$ and $E(\mathfrak{q})$ are Witt equivalent.*

Proof. By [13, Corollary 2.6, Chapter 6] we have

$$WE \cong W(E(\mathfrak{p}))[T]/(T^2 - \langle 1\rangle) \cong W(F(\mathfrak{q}))[T]/(T^2 - \langle 1\rangle) \cong WF.$$

Hence if the residue fields are Witt equivalent then so are the local fields.

We will show the opposite implication. Assume that E, F are Witt equivalent and let $t: E^*/E^{*2} \to F^*/F^{*2}$ denote an associated Harrison map. We claim that there is $p \in E^*/E^{*2}$ such that $\operatorname{ord}_\mathfrak{p} p \equiv \operatorname{ord}_\mathfrak{p} tp \equiv 1 \pmod 2$. Indeed, if we suppose that there is no such p, then from the surjectivity of t we can find $u \in E^*/E^{*2}$ such that $\operatorname{ord}_\mathfrak{p} u \equiv 0 \pmod 2$ and $\operatorname{ord}_\mathfrak{p} tu \equiv 1 \pmod 2$. Now, take any $p \in E^*/E^{*2}$ with $\operatorname{ord}_\mathfrak{p} p \equiv 1 \pmod 2$. Then $\operatorname{ord}_\mathfrak{q} tp \equiv 0 \pmod 2$, hence $\operatorname{ord}_\mathfrak{p} up \equiv \operatorname{ord}_\mathfrak{q} t(up) \equiv 1 \pmod 2$, which leads to a contradiction. Thus, we have proved our claim.

Fix an element $p \in E^*/E^{*2}$ such that $\operatorname{ord}_\mathfrak{p} p \equiv \operatorname{ord}_\mathfrak{q} tp \equiv 1 \pmod 2$. Let $\mathcal{A}$ be a basis of the $\mathbb{F}_2$-linear space $E(\mathfrak{p})^*/E(\mathfrak{p})^{*2}$. Observe that $-1 \in E(\mathfrak{p})^{*2} \Leftrightarrow -1 \in F(\mathfrak{q})^{*2}$. If any (hence both) of these are not satisfied let us assume that $-1 \in \mathcal{A}$. Now we can find an $\mathbb{F}_2$-linear homomorphism $t': E(\mathfrak{p})^*/E(\mathfrak{p})^{*2} \to F(\mathfrak{q})^*/F(\mathfrak{q})^{*2}$ which takes the following values at the elements of the basis $\mathcal{A}$:

$$t'(u(\mathfrak{p})) := \begin{cases} (tu)(\mathfrak{q}) & \text{if } \operatorname{ord}_\mathfrak{p} tu \equiv 0 \pmod 2, \\ (t(pu))(\mathfrak{q}) & \text{if } \operatorname{ord}_\mathfrak{p} tu \equiv 1 \pmod 2. \end{cases}$$

The mapping t' is obviously injective. We claim that t' is actually an isomorphism. Indeed, take any $v(\mathfrak{q}) \in F(\mathfrak{q})^*/F(\mathfrak{q})^{*2}$ and let $w := t^{-1}(v) \in E^*/E^{*2}$. We consider two cases. If $\operatorname{ord}_{\mathfrak{p}} w \equiv 0 \pmod 2$, then $t'(w(\mathfrak{p})) = v(\mathfrak{q})$. If $\operatorname{ord}_{\mathfrak{p}} w \equiv 1 \pmod 2$, then $t'((pw)(\mathfrak{p})) = (t(p \cdot pw))(\mathfrak{q}) = (tw)(\mathfrak{q}) = v(\mathfrak{q})$. Hence, we have proved our claim.

We will show that t' is a Harrison map. The condition $t'(-1) = -1$ is obvious. Assume that $1 \in D_{E(\mathfrak{p})}\langle u(\mathfrak{p}), v(\mathfrak{p})\rangle$ for some $u(\mathfrak{p}), v(\mathfrak{p}) \in E(\mathfrak{p})^*/E(\mathfrak{p})^{*2}$. Hence $1 \in D_E\langle u, v\rangle$. We need to consider three cases.

If $\operatorname{ord}_{\mathfrak{q}} tu \equiv \operatorname{ord}_{\mathfrak{q}} tv \equiv 0 \pmod 2$, then $t'(u(\mathfrak{p})) = (tu)(\mathfrak{q})$ and $t'(v(\mathfrak{p})) = (tv)(\mathfrak{q})$. Since t is a Harrison map, we have $1 \in D_F\langle tu, tv\rangle$, and consequently $1 \in D_{F(\mathfrak{q})}\langle t'(u(\mathfrak{p})), t'(v(\mathfrak{p}))\rangle$.

Suppose now that $\operatorname{ord}_{\mathfrak{q}} tu \equiv 1 \pmod 2$ and $\operatorname{ord}_{\mathfrak{q}} tv \equiv 0 \pmod 2$ (the case $\operatorname{ord}_{\mathfrak{q}} tu \equiv 0 \pmod 2$ and $\operatorname{ord}_{\mathfrak{q}} tv \equiv 1 \pmod 2$ is totally analogous). Since $1 \in D_F\langle tu, tv\rangle$, [9, Lemma 2.4] implies that $tv = 1 \in F^*/F^{*2}$. Thus $v = 1 \in E^*/E^{*2}$, hence $t'(v(\mathfrak{p})) = 1$. Consequently, 1 is represented over $F(\mathfrak{q})$ by the form $\langle t'(u(\mathfrak{p})), t'(v(\mathfrak{p}))\rangle$.

Finally assume that $\operatorname{ord}_{\mathfrak{q}} tu \equiv \operatorname{ord}_{\mathfrak{q}} tv \equiv 1 \pmod 2$. Again $1 \in D_F\langle tu, tv\rangle$, hence $tu = -tv$. Finally, $1 \in D_{F(\mathfrak{q})}\langle t'(u(\mathfrak{p})), t'(v(\mathfrak{p}))\rangle = D_{F(\mathfrak{q})}\langle t'(u(\mathfrak{p})), -t'(u(\mathfrak{p}))\rangle$, and thus t' is a Harrison map. By Harrison's criterion, $E(\mathfrak{p}), F(\mathfrak{q})$ are Witt equivalent. ∎

PROPOSITION 3.2. *Let k, l be two global fields of characteristic $\neq 2$ and let K, L denote algebraic function fields with fields of constants k, l. Assume that both K and L have rational places. If K, L are Witt equivalent then so are k, l.*

Proof. By Theorem 1.3 the fields K and L are uniformly locally Witt equivalent. Let $T : \Omega(K) \to \Omega(L)$ denote the corresponding bijection of the sets of places. The previous lemma shows that the global fields $K(\mathfrak{p})$ and $L(T\mathfrak{p})$ are Witt equivalent. Let us consider two cases. First assume that k is a number field. Since $k = K(\mathfrak{p})$ for a rational place $\mathfrak{p}$ and it is Witt equivalent to $L(T\mathfrak{p}) \supseteq l$, it follows that l is also a number field by [15, Theorem (1.5)(i)]. Since the degree over $\mathbb{Q}$ is an invariant of Witt equivalence of number fields (see [15, Corollary (1.6)(i)]), it follows that $[l : \mathbb{Q}]$ divides $[k : \mathbb{Q}]$. By symmetry, $[k : \mathbb{Q}]$ divides $[l : \mathbb{Q}]$. Consequently, $L(T\mathfrak{p}) = l$ and $Wk \cong Wl$.

Now assume that both k and l are global function fields. By [15, Theorem (1.3)] it is enough to verify that both fields have equal levels, which is straightforward. ∎

The above proposition together with the fact that $\mathbb{Q}$ and $\mathbb{Q}(i)$ are the *lonely* global fields (cf. [12]) allows us to derive an analog of [9, Corollary 5.3].

COROLLARY 3.3. *For any global field $k \neq \mathbb{Q}$ (resp. $k \neq \mathbb{Q}(\sqrt{-1})$), the rational function fields $\mathbb{Q}(X)$ and $k(X)$ (resp. $\mathbb{Q}(\sqrt{-1})(X)$ and $k(X)$) are not Witt equivalent.*

The rest of this article will be devoted to developing two partial converses of Proposition 3.2.

LEMMA 3.4. *Let k, l be two Witt equivalent global fields. Then for every proper field extension $k' \supsetneq k$, $k' \neq \mathbb{Q}(\sqrt{-1})$, there exists an infinite family $\{l'_m\}_{m\in\mathbb{N}}$ of fields $l'_m \supset l$ which are Witt equivalent to k'.*

Proof. The proof runs similarly to the proofs in [15, 17], hence we only sketch it here. The main ideas behind this proof are depicted in the diagram (3.5). Let (T_0, t_0) denote a Hilbert-symbol equivalence of k and l (cf. [12]). The field k is a solution over either $\mathbb{Q}$, when the level $s(k) \neq 1$, or $\mathbb{Q}(\sqrt{-1})$ otherwise, of a finite system of prescriptions (see [6, Chapter IV] and [17, §2]) consisting of: all real primes of k, all dyadic ones and possibly (if $s(k) = 2$ and every dyadic completion has level 1) of one more prime, congruent to 3 $(\mathrm{mod}\, 4)$. Then l is a solution over $\mathbb{Q}$, or $\mathbb{Q}(\sqrt{-1})$ respectively, of the system of prescriptions obtained from the previous ones by replacing all completions $k_{\mathfrak{p}}$ with $l_{T_0\mathfrak{p}}$.

$$
\begin{tikzcd}
 & \{k_{\mathfrak{P}}\}_{\mathfrak{P}|\mathfrak{p}} \arrow[dl, dash] \arrow[dd, dash] \arrow[rr, dotted, dash] & & \{l_{\mathfrak{Q}}\}_{\mathfrak{Q}|T_0\mathfrak{p}} \arrow[dl, dash] \arrow[dd, dash] \\
k_{\mathfrak{p}} \arrow[rr, dotted, dash] \arrow[dd, dash] & & l_{T_0\mathfrak{p}} \arrow[dd, dash] & \\
 & k' \arrow[rr, dotted, dash] \arrow[dl, dash] & & l'_m \arrow[dl, dash] \\
k \arrow[rr, dotted, dash] \arrow[dr, dash] & & l \arrow[dl, dash] & \\
 & \mathbb{Q} & &
\end{tikzcd}
\tag{3.5}
$$

Now, for every finite extension of any $k_{\mathfrak{p}}$ we can find a Witt equivalent finite extension of $l_{T_0\mathfrak{p}}$. Thus, since k' is a solution over k of a finite system $\mathcal{T}$ of prescriptions, we can construct any l'_m as a solution of the finite system of prescriptions (cf. [5, Satz 7 and Korollar on page 97]) obtained from $\mathcal{T}$ by replacing extensions of $k_{\mathfrak{p}}$ with the Witt equivalent extensions of $l_{T_0\mathfrak{p}}$. If we take, in addition, a sequence of primes of l congruent to 1 $(\mathrm{mod}\, 4)$ and require that the first $m-1$ of them split completely in l'_m while the mth does not split, then this requirement is describable in terms of prescriptions. All the solutions l'_m ($m \in \mathbb{N}$) are distinct and Witt equivalent to k'. ∎

Take now any place $\mathfrak{p}$ of the rational function field $k(X)$. Then there are infinitely many extensions of l that are Witt equivalent to the residue field of $\mathfrak{p}$ and every such extension is the residue field for a place of $l(X)$. Thus if we introduce an equivalence relation on $\Omega(k(X))$ (respectively $\Omega(l(X))$) defined by the condition that two places $\mathfrak{p}, \mathfrak{q}$ are related if their residue fields are Witt equivalent, then every equivalence class of this relation is infinite (and countable). Moreover for every such class $A \subset \Omega(k(X))$ there is a unique class $B \subset \Omega(l(X))$ such that all residue fields of places from A are Witt equivalent to residue fields of places from B. So we can find a bijection $\Omega(k(X)) \to \Omega(l(X))$ that factors through this equivalence relation. Finally, we have proved the following partial converse of 3.2.

COROLLARY 3.6. *Let k, l be two global fields of characteristic $\neq 2$ and let $K = k(X)$ and $L = l(X)$ be the rational function fields over k, l. If k, l are Witt equivalent then K, L are locally Witt equivalent.*

In order to get the (partial) converse of 3.2 incorporating algebraic function fields we must introduce a more geometrical point of view. Recall that a curve X over a field k is called *geometrically integral* if $X \times_k k_{\mathrm{alg}}$ is integral (cf. e.g. [7, Chapter II.3]). Here k_{alg} denotes the algebraic closure of k.

PROPOSITION 3.7. *Let k, l be two Witt equivalent number fields and let (T_0, t_0) denote an associated Hilbert-symbol equivalence. Let further X and Y be two smooth, projective, geometrically integral curves over k and l respectively and assume that both have rational points. Denote by K and L the fields of rational functions over X and Y respectively. Finally, assume that the 2-torsion subgroups ${}_2Ш(J(X))$ and ${}_2Ш(J(Y))$ of the Tate–Shafarevich groups of the Jacobians of X and Y are trivial. If there exists an isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ extending t_0 (i.e. $t|_{k^*/k^{*2}} = t_0$) such that*

$$1 = \left(\frac{f, g}{K_\nu}\right) \Leftrightarrow 1 = \left(\frac{tf, tg}{L_{T_0\nu}}\right), \tag{3.8}$$

for all square classes $f, g \in K^*/K^{*2}$ and all places ν of k (here $K_\nu, L_{T_0\nu}$ denote the function fields of $X \times_k k_\nu, Y \times_l l_{T_0\nu}$ respectively) then the fields K, L are Witt equivalent.

Proof. We have $t(-1) = t_0(-1) = -1$. Moreover, if $1 \in D_K\langle f, g\rangle$, then for every place $\nu \in \Omega(k)$ the algebra $\left(\frac{f,g}{K_\nu}\right)$ splits, and so do all the algebras $\left(\frac{tf,tg}{L_\mu}\right)$ for $\mu \in \Omega(l)$. Hence, [11, Theorem 1.1] implies that $1 \in D_L\langle tf, tg\rangle$. By the Harrison criterion, K and L are Witt equivalent. ∎

COROLLARY 3.9. *Let K, L be two rational function fields over Witt equivalent number fields k, l respectively. If there exists an extension (in the sense of (3.8) of the previous proposition) of a Hilbert-symbol equivalence (T_0, t_0) of k and l, then the fields K, L are Witt equivalent.*

REMARK 1. Notice that if f and g of the above proposition are constants (i.e. $f, g \in k$) then $\left(\frac{f,g}{K_\nu}\right) = 1$ if and only if $\left(\frac{f,g}{k_\nu}\right) = 1$, and so the condition (3.8) reduces to the condition defining Hilbert-symbol equivalence.

Remark 2. Although Corollary 3.9 follows immediately from the preceding Proposition 3.7 due to simplicity of the geometry of a projective line, one can actually prove it directly using a simpler and totally nongeometrical argument. Namely, replace [11, Theorem 1.1] with [4, Proposition 1.1] in the proof.

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Institute of Mathematics
Silesian University
40-007 Katowice, Poland
E-mail: perry@ux2.math.us.edu.pl

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