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ON THE QUANTITATIVE FATOU PROPERTY

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Abstract. The result of this article together with [1] and [4] gives a full quantitative description of a Fatou type property for functions from Hardy classes in the upper half plane.

We define the Hardy class $H^p(\mathbb{R}^2_+)$ in the classical sense as the set of functions $F(z)$ holomorphic in $\mathbb{R}^2_+$ such that

$$\|F\|^p_{H^p(\mathbb{R}^2_+)} \equiv \sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^p\,dx < \infty.$$

It is well known [5, p. 127] that every $F \in H^p(\mathbb{R}^2_+)$ has a.e. boundary value $\lim_{y\to 0+} F(x+iy) = F(x)$ which is an L^p function with $\|F\|_p = \|F\|_{H^p(\mathbb{R}^2_+)}$. Let us ask the following question:

Suppose that the function $F(x)$ has a certain smoothness property in $L^p(\mathbb{R})$-norm. What is a good/natural rate of a.e. convergence of $F(x+iy)$ towards $F(x)$?

For this we introduce the L^p-modulus of continuity of $F \in L^p(\mathbb{R})$, $0 < p < \infty$, by

$$\omega(F,t)_p = \sup_{|h|<t} \|\Delta_h F\|_{L^p(\mathbb{R})}, \qquad \Delta_h F(x) = F(x+h) - F(x).$$

By the modulus of continuity of an analytic function we will mean the modulus of continuity of the boundary value.

Further, we consider continuous increasing subadditive functions $\omega(t)$ on $(0,\infty)$ with $\lim_{t\to 0+} \omega(t) = 0$; we define smoothness classes H^ω_p by

$$H^\omega_p = \{F \in H^p(\mathbb{R}^2_+) : \omega(F,t)_p \le C\omega(t)\}.$$

Let $\omega(t)$ be a modulus of continuity such that

$$\omega(t)/t \uparrow \infty, \qquad t \to 0+. \tag{1}$$

We define the *Oskolkov sequence* δ_k (see [2]) by

$$\delta_0 = 1,\ \delta_{k+1} = \min\left\{\delta : \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}; \frac{\delta\omega(\delta_k)}{\delta_k\omega(\delta)}\right) = \frac{1}{2}\right\}, \qquad k = 0,1,\ldots \tag{2}$$

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THEOREM I (A. A. Solyanik [4]). *Let* $0 < p < \infty$ *and* $F \in H_p^\omega$ *where* $\omega(\delta)$ *satisfies* (1), *and let* $w(t)$ *be an increasing positive function such that* $\omega(t)/w(t)$ *is also increasing and*

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)} \right)^p < \infty. \tag{3}$$

Then for every $F \in H_p^\omega$ *we have*

$$F(x+it) - F(x) = o_x(w(t)) \quad a.e., \quad t \to 0+. \tag{4}$$

Now it is natural to ask about the sharpness of the estimate (4). For $p \geq 1$ the answer is contained in [1, Theorem 2]. In Theorem II below we extend the result of [1] to the remaining case $0 < p < 1$.

THEOREM II. *Let* $0 < p < 1$, *suppose the modulus of continuity* $\omega(t)$ *satisfies* (1) *and the series in* (3) *diverges*, *i.e.*

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)} \right)^p = \infty. \tag{5}$$

Then there exists an $F \in H_p^\omega$ *such that for almost all* $x \in \mathbb{R}$,

$$\limsup_{t \to 0+} \frac{|F(x+it) - F(x)|}{w(t)} = \infty. \tag{6}$$

Proof. In the following we denote generic constants that are independent of the function (or the variable or sequence) involved by C with different indices. Also, let

$$\psi_k := \left(\frac{w(\delta_k)}{\omega(\delta_k)} \right)^p.$$

We note that the following two simplifications do not restrict generality (see [1, pp. 248–249]).

(i) It is sufficient to prove the existence of some $F \in H_p^\omega$ with

$$\limsup_{t \to 0+} \frac{|F(x+it) - F(x)|}{w(t)} > 0 \quad \text{a.e.} \quad \text{on } \mathbb{R} \tag{7}$$

instead of (6).

(ii) We may assume that

$$\psi_2 = 1, \quad \psi_k \geq k+1. \tag{8}$$

Suppose that the numbers $\{\delta_k\}$ are defined by (2), and q is a fixed positive integer which will be specified later. Define

$$r_k = \max\{m \in \mathbb{Z} : qm\delta_k \leq 1/\psi_k\}, \quad k = 1, 2, \ldots \tag{9}$$

It is easy to verify (see e.g. [2]) that $2\delta_{k+1} \le \delta_k$ and thus

$$\sum_{k=1}^{\infty} \delta_k < \infty$$

while

$$\sum_{k=1}^{\infty} r_k \delta_k = \infty. \tag{10}$$

Since it is easy to choose by induction an increasing sequence n_j such that

$$\sum_{n_{j-1} \le k < n_j,\, r_k \ge j} r_k \delta_k \ge 1,$$

there exists a subsequence of r_k tending to infinity which still has the property (10). We will assume that $r_k \to \infty$ itself, which does not restrict generality, as will be seen below.

For $k \ge 2$ define intervals $I_k = (\alpha_k; \beta_k] \equiv (a_k - \delta_k \psi_k^{1/p}; b_k + \delta_k \psi_k^{1/p}]$, where $b_k - a_k = q r_k \delta_k$, in the following way: Set $\alpha_2 = 0$ and $\alpha_{k+1} = \beta_k$ if $\beta_k < 1$ and $\alpha_{k+1} = 0$ otherwise. Let $s_m \uparrow \infty$ be such that $\alpha_{s_m} = 0$ and consider

$$E_k = \bigcup_{\nu=1}^{r_k - 1} [a_k + (\nu q - 1)\delta_k; a_k + (\nu q + 1)\delta_k].$$

Then $|E_k| = 2(r_k - 1)\delta_k$. Set

$$\mathcal{K} = \{k \in \mathbb{Z}_+ : \psi_k \le k^2\}. \tag{11}$$

It follows from (10) that

$$\sum_{k \in \mathcal{K}} r_k \delta_k = \infty, \tag{12}$$

hence

$$\sum_{k \in \mathcal{K}} |E_k| = \infty.$$

Let

$$\mathcal{L} = \bigcup_{m=1}^{\infty} \mathcal{L}_m, \quad \mathcal{L}_m = \{k \in \mathcal{K} : s_{2m} \le k < s_{2m+1}\}, \quad E_m^* = \bigcup_{k \in \mathcal{L}_m} E_k. \tag{13}$$

Then obviously either

$$\sum_{k \in \mathcal{L}} |E_k| = \infty \tag{14}$$

or $\sum_{k \notin \mathcal{L}} |E_k| = \infty$. Without loss of generality assume (14) and rewrite it as

$$\sum_{m=1}^{\infty} |E_m^*| = \sum_{k \in \mathcal{L}} |E_k| = \infty.$$

By the Borel–Cantelli-type lemma (see e.g. [5, p. 442]), there exist numbers ξ_m such that

$$\limsup_m E^*_{\xi_m} \cup E \equiv \left(\bigcap_{j=1}^{\infty} \bigcup_{m=j}^{\infty} E^*_{\xi_m} \right) \cup E = \mathbb{R}, \tag{15}$$

where $E^*_{\xi_m} = E^*_m - \xi_m$ are translates of E^*_m and E is some set of measure zero. Denote by τ_m the translation $\tau_m(\cdot) \equiv (\cdot - \xi_m)$ and define

$$I^\tau_k = \tau_m(I_k), \qquad s_{2m} \le k < s_{2m+1}.$$

Since now the distribution of I^τ_k is fixed we may denote it again by the same letters, so $I^\tau_k = (\alpha_k; \beta_k]$. For $x \in \mathbb{R}$ we introduce $\mathcal{K}_x = \{k \in \mathcal{L} : I^\tau_k \ni x\}$. It is easy to verify (see [1, p. 250, (28)]) the following important property of $\mathcal{K}_x$: there exists a $k_0 \ge 1$ such that for any $x \in \mathbb{R}$,

$$l, k \in \mathcal{K}_x,\ l > k \ge k_0 \quad \text{implies} \quad l \ge 2k. \tag{16}$$

Let us define a sequence $\{z_{j,k}\}_{j=1}^{r_k}$ of complex numbers by

$$z_{j,k} = a_k + jq\delta_k - i\delta_k, \quad \text{so} \quad \Re z_{j,k} = a_k + jq\delta_k, \tag{17}$$

and let v be the smallest positive integer such that $2vp > 1$. For every $k \in \mathcal{K}$ set

$$F_k(z) = w(\delta_k) \sum_{j=1}^{r_k} \left(\frac{\delta_k}{z_{j,k} - z} \right)^{2v}, \qquad z \in \mathbb{C},\ \Im z > -\delta_k.$$

We note that F_k restricted to the real line is bounded,

$$\|F_k\|_\infty \le C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^{2v}}{(jq\delta_k)^{2v}} \le C_p w(\delta_k) \tag{18}$$

and, therefore,

$$\|F_k\|_p^p \le \|F_k\|_\infty^p \int_{x \in 3I_k} dx + \int_{x \notin 3I_k} |F_k(x)|^p \, dx \le C_p \omega(\delta_k)^p + \dots$$

since $|I_k| < \psi_k^{-1} = o(1)$ (see (8)). Also

$$\begin{aligned}
\int_{x \notin 3I_k} |F_k(x)|^p \, dx &\le w(\delta_k)^p \delta_k^{2vp} \sum_{j=1}^{r_k} \int_{x \notin 3I_k} \frac{dx}{|z_{j,k} - x|^{2vp}} \\
&\le C_p w(\delta_k)^p r_k \delta_k^{2vp} \int_{x \ge |I_k|} \frac{dx}{(\delta_k^2 + x^2)^{vp}} \\
&\le C_p \omega(\delta_k)^p \psi_k r_k \delta_k^{2vp} |I_k|^{1-2vp} \\
&\le C_p \omega(\delta_k)^p r_k^{1-2vp} = o(\omega(\delta_k)),
\end{aligned}$$

hence

$$\|F_k\|_p \le C_p \omega(\delta_k). \tag{19}$$

Further

$$|F_k'(x)| \le C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^{2v}}{(jq\delta_k)^{2v+1}} \le C_p \frac{w(\delta_k)}{q^{2v+1}\delta_k} \sum_{j=1}^{\infty} j^{-(2v+1)} \le C_p \frac{w(\delta_k)}{\delta_k}$$

whence

$$\|F_k'\|_\infty \le C_p \frac{w(\delta_k)}{\delta_k}. \tag{20}$$

If $x \notin I_k$ then

$$\begin{aligned} |F_k(x)| &\le C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^{2v}}{(\delta_k + jq\delta_k + \delta_k \psi_k^{1/p})^{2v}} \\ &\le C_p w(\delta_k) \sum_{j=1}^{\infty} (jq + \psi_k^{1/p})^{-2v} \le C_p w(\delta_k)(\psi_k^{1/p})^{1-2v} \\ &\le C_p \omega(\delta_k)(\psi_k^{1/p})^{2-2v} = O(\omega(\delta_k)) \end{aligned}$$

and

$$\|F_k \chi_{I_k^c}\|_\infty \le C_p \omega(\delta_k). \tag{21}$$

Also

$$\begin{aligned} |F_k'(x)| &\le C_p w(\delta_k) \sum_{j=1}^{r_k} \frac{\delta_k^{2v}}{(\delta_k + jq\delta_k + \delta_k \psi_k^{1/p})^{2v+1}} \\ &\le C_p \frac{w(\delta_k)}{\delta_k} \sum_{j=1}^{\infty} (jq + \psi_k^{1/p})^{-(2v+1)} \\ &\le C_p \frac{\omega(\delta_k)}{\delta_k} (\psi_k^{1/p})^{1-2v} \le C_p \frac{\omega(\delta_k)}{\delta_k} \end{aligned}$$

and

$$\|F_k' \chi_{I_k^c}\|_\infty \le C_p \frac{\omega(\delta_k)}{\delta_k}. \tag{22}$$

Now define $F = \sum_{k \in \mathcal{L}} F_k$. In view of (2) and (8) the estimates (18) and (19) imply that F is bounded analytic in $\mathbb{R}^2_+$ and belongs to $H^p(\mathbb{R}^2_+)$. We show that $F \in H_p^\omega$.

Choose $\delta_{s+1} < h \le \delta_s$. Then

$$\begin{aligned} \omega(F,h)_p^p &\le \sum_{k \le s,\, k \in \mathcal{L}} \omega(F_k, h)_p^p + 2 \sum_{k > s,\, k \in \mathcal{L}} \|F_k\|_p^p \\ &\le \sum_{k \le s,\, k \in \mathcal{L}} \omega(F_k, h)_p^p + C_p \omega(\delta_{s+1})^p \end{aligned}$$

by (2) and (19). Now

$$\|F_k(x+h)-F_k(x)\|_p^p \leq \int_{x\in 5I_k} |F_k(x+h)-F_k(x)|^p\,dx + \int_{x\notin 5I_k} |F_k(x+h)-F_k(x)|^p\,dx \equiv I_1+I_2.$$

By (20),

$$I_1 \leq h^p \|F_k'\|_\infty^p \int_{x\in 5I_k} dx \leq C_p h^p w(\delta_k)^p \delta_k^{-p}\psi_k^{-1} = C_p h^p \omega(\delta_k)^p \delta_k^{-p}.$$

Further,

$$I_2 \leq C_p h^p w(\delta_k)^p \delta_k^{2vp} \sum_{j=1}^{r_k} \int_{x\notin 5I_k} \frac{dx}{|z_{j,k}-x-\xi_j|^{(2v+1)p}}$$

with some $0\leq \xi_j < h \leq \delta_k$. Since $x+\xi_j \notin 3I_k$ we have

$$\int_{x\notin 5I_k} \frac{dx}{|z_{j,k}-x-\xi_j|^{(2v+1)p}} \leq C_p \int_{|x|\geq |I_k|} \frac{dx}{|x|^{(2v+1)p}} \leq C_p \psi_k^{(2v+1)p-1}.$$

Hence

$$I_2 \leq C_p w(\delta_k)^p \delta_k^{2vp} h^p r_k \psi_k^{(2v+1)p-1} = C_p h^p \frac{\omega(\delta_k)^p}{\delta_k^p} r_k^{1-(2v+1)p} \leq C_p h^p \frac{\omega(\delta_k)^p}{\delta_k^p}$$

and thus $\omega(F_k,h)_p \leq C_p h\omega(\delta_k)\delta_k^{-1}$. Since

$$\sum_{k\leq s,\, k\in\mathcal{L}} \omega(F_k,h)_p^p \leq C_p h^p \sum_{k\leq s} \omega(\delta_k)^p \delta_k^{-p} \leq C_p h^p \omega(\delta_s)^p \delta_s^{-p} \leq C_p \omega(h)^p$$

we obtain

$$\omega(F,h)_p \leq C_p \omega(h),$$

i.e., $F\in H_p^\omega$.

Next we examine the behavior of $F(x+it)-F(x)$. Take $t=\delta_s$ with $s\in\mathcal{L}$. Then

$$|F(x+it)-F(x)| \geq |F_s(x+it)-F_s(x))| - \sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)| - \sum_{k>s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)|.$$

We discuss the contributions of these terms. First we have

$$\sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it)-F_k(x)| = \sum_{k<s,\, k\in\mathcal{K}_x} |F_k(x+it)-F_k(x)| + \sum_{k<s,\, k\in\mathcal{K}_x^{\mathrm{c}}} |F_k(x+it)-F_k(x)| \equiv \varSigma_1+\varSigma_2$$

where again $\mathcal{K}_x^{\mathrm{c}} = \mathcal{L} \setminus \mathcal{K}_x$. Then, by (16) and (11),

$$\begin{aligned}\Sigma_1 &\leq \sum_{k<s,\, k\in\mathcal{K}_x} \delta_s \|F_k'\|_\infty \leq C_{p,q}\delta_s \sum_{k\leq s/2} \omega(\delta_k)\psi_k^{1/p}\delta_k^{-1} \\ &\leq C_{p,q}\delta_s \frac{\omega(\delta_s)}{\delta_s} s^2 \sum_{k\leq s/2} 2^{k-s} \leq C_{p,q} s^2 2^{-s/2}\omega(\delta_s) = o(\omega(\delta_s))\end{aligned}$$

and

$$\Sigma_2 \leq \sum_{k<s,\, k\in\mathcal{L}} \delta_s \|F_k'\chi_{I_k^{\mathrm{c}}}\|_\infty \leq C_{p,q}\delta_s \sum_{k<s} \frac{\omega(\delta_k)}{\delta_k} \leq C_{p,q}\omega(\delta_s).$$

Combining these two estimates we have, for sufficiently large s,

$$\sum_{k<s,\, k\in\mathcal{L}} |F_k(x+it) - F_k(x)| \leq C_{p,q}\omega(\delta_s). \tag{23}$$

Analogously, we decompose

$$\begin{aligned}&\sum_{k>s,\, k\in\mathcal{L}} |F_k(x+it) - F_k(x)| \\ &= \sum_{k>s,\, k\in\mathcal{K}_x} |F_k(x+it) - F_k(x)| + \sum_{k>s,\, k\in\mathcal{K}_x^{\mathrm{c}}} |F_k(x+it) - F_k(x)| \equiv \Sigma^1 + \Sigma^2.\end{aligned}$$

Then, by (16), (11), and (18),

$$\begin{aligned}\Sigma^1 &\leq 2 \sum_{k>s,\, k\in\mathcal{K}_x} \|F_k\|_\infty \leq C_{p,q} \sum_{k\geq 2s} w(\delta_k) \\ &\leq C_{p,q}\omega(\delta_s) \sum_{k\geq 2s} 2^{s-k} k^{2/p} \leq C_{p,q}\omega(\delta_s) s^2 2^{-s} = o(\omega(\delta_s))\end{aligned}$$

and by (21),

$$\Sigma^2 \leq \sum_{k>s,\, k\in\mathcal{K}_x^{\mathrm{c}}} \|F_k\chi_{I_k^{\mathrm{c}}}\|_\infty \leq C_{p,q} \sum_{k>s} \omega(\delta_k) \leq C_{p,q}\omega(\delta_s).$$

Thus, for sufficiently large s,

$$\sum_{k>s,\, k\in\mathcal{L}} |F_k(x+it) - F_k(x)| \leq C_{p,q}\omega(\delta_s) \tag{24}$$

(recall that we have set $t = \delta_s$, $s \in \mathcal{L}$). Therefore, as a consequence of (23) and (24), we have

$$|F(x+it) - F(x)| \geq |F_s(x+it) - F_s(x)| + O(\omega(\delta_s)). \tag{25}$$

For $x \in E_s$ and j with $|\Re z_{j,s} - x| \le \delta_s$ it follows that

$$
\begin{aligned}
&|F_s(x+it) - F_s(x)| \\
&\qquad \ge w(\delta_s)\biggl(\frac{\delta_s^{2v}}{|z_{j,s}-x|^{2v}} - \frac{\delta_s^{2v}}{|z_{j,s}-x-it|^{2v}} \\
&\qquad\qquad - \sum_{n\neq j}\frac{\delta_s^{2v}}{|z_{n,s}-x|^{2v}} - \sum_{n\neq j}\frac{\delta_s^{2v}}{|z_{n,s}-x-it|^{2v}}\biggr) \\
&\qquad \equiv w(\delta_s)(A-B-C-D).
\end{aligned}
$$

But it is easy to see that $A - B \ge 1/4$ and $D \le C$. Finally, by (17),

$$
\begin{aligned}
C &\le \sum_{n\neq j}\frac{\delta_s^{2v}}{|\Re(z_{n,s}-x)|^{2v}} \le \sum_{n\neq j}\frac{\delta_s^{2v}}{(|\Re z_{n,s}-\Re z_{j,s}| - |\Re z_{j,s}-x|)^{2v}} \\
&\le \sum_{n\neq j}\frac{\delta_s^{2v}}{(q|n-j|\delta_s-\delta_s)^{2v}} \le 2\sum_{n=1}^{\infty}\frac{\delta_s^{2v}}{|(qn-1)\delta_s|^{2v}} \le cq^{-2v}.
\end{aligned}
$$

Now choose q such that $cq^{-2v} \le 1/16$. Then

$$
|F_s(x+it) - F_s(x)| \ge w(\delta_s)/8,
$$

which together with (25) implies that for the given x,

$$
|F(x+it) - F(x)| \ge \tfrac{1}{8}w(\delta_s) + O(\omega(\delta_s)) = \tfrac{1}{8}w(t) + o(w(t))
$$

from which (7) follows and Theorem II is proved.

Remark 1. Theorems I and II are the non-periodic versions of the results due to A. A. Solyanik [3, 4], which are extensions of Oskolkov's results [2] concerning Steklov means of periodic functions. The present construction is much simpler than that in [4] due to an application of the Borel–Cantelli-type lemma, which allows us to avoid tantalizing technical difficulties solved by Solyanik in the periodic case.

Remark 2. One of the possible future directions for the above subject could be multidimensional generalizations. However, Solyanik's theorem has a rather complex proof and the first step toward the multidimensional case might be the investigation of the problem for real Hardy classes.

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