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DESCRIBING TORIC VARIETIES AND THEIR EQUIVARIANT COHOMOLOGY

BY

MATTHIAS FRANZ (London, Ont.)

Abstract. Topologically, compact toric varieties can be constructed as identification spaces: they are quotients of the product of a compact torus and the order complex of the fan. We give a detailed proof of this fact, extend it to the non-compact case and draw several, mostly cohomological conclusions.

In particular, we show that the equivariant integral cohomology of a toric variety can be described in terms of piecewise polynomials on the fan if the ordinary integral cohomology is concentrated in even degrees. This generalizes a result of Bahri–Franz–Ray to the non-compact case. We also investigate torsion phenomena in integral cohomology.

1. Introduction. Let $T \cong (S^1)^n$ be a torus with Lie algebra $\mathfrak{t} \cong \mathbb{R}^n$, and P a full-dimensional polytope in the dual $\mathfrak{t}^*$ of $\mathfrak{t}$, integral with respect to the weight lattice. The toric variety X_P associated with P is projective, and T as well as its complexification $\mathbb{T} \cong (\mathbb{C}^\times)^n$ act on it.

In the article [J], Jurkiewicz showed that one can recover P as the image of the moment map $X_P \to \mathfrak{t}^*$, and that this map is the quotient of X_P by the action of the compact torus T. (See also Atiyah [A].) Since X_P/T is canonically embedded into X_P as its non-negative part, Jurkiewicz's result immediately yields a T-equivariant homeomorphism

$$X_P \approx (T \times P)/{\sim}, \tag{1.1}$$

where two points $(t_1, x_1), (t_2, x_2) \in T \times P$ are identified if $x_1 = x_2$, say with supporting face f of P, and if $t_1 t_2^{-1}$ lies in the subtorus of T whose Lie algebra is the annihilator of $\mathrm{lin}(f - x_1)$. The aim of this note is to prove a similar description for arbitrary toric varieties.

Let Σ be a not necessarily complete fan in $\mathfrak{t}$, rational with respect to the lattice of 1-parameter subgroups of T, and let $\mathcal{F}(\Sigma)$ be the order complex of Σ. (If Σ is the normal fan of the polytope P, then $\mathcal{F}(\Sigma)$ can be thought of as the barycentric subdivision of P.) Define the T-space

$$Y_\Sigma = (T \times |\mathcal{F}(\Sigma)|)/{\sim}, \tag{1.2}$$

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with the following identification: For $x \in |\mathcal{F}(\Sigma)|$, say with supporting simplex $\alpha = (\sigma_0, \dots, \sigma_p)$, one has $(t_1, x) \sim (t_2, x)$ if $t_1 t_2^{-1}$ lies in the subtorus T_{σ_0} of T whose Lie algebra is the linear span of σ_0.

This construction has appeared in Davis–Januszkiewicz's work on quasitoric manifolds [DJ], but without linking it with algebraic geometry. Fischli and Yavin [Fi], [FY], [Ya] attributed construction (1.2) to MacPherson and used it as the *definition* of a toric variety. Because since then several authors [Jo], [WZŽ], [C], [Pa] have applied (1.2) to toric varieties in the usual sense, we feel that it might be beneficial to supply a justification for this.

THEOREM 1.1. *If Σ is complete, then Y_Σ is T-equivariantly homeomorphic to X_Σ. In general, Y_Σ is a T-equivariant strong deformation retract of X_Σ.*

A basic tool to study transformation groups are equivariant CW complexes. Since Y_Σ is a finite T-CW complex by construction (with T-cells $(T/T_{\sigma_0}) \times |\alpha|$ in the notation used above), we can immediately conclude:

COROLLARY 1.2. *If Σ is complete, then the toric variety X_Σ is a (necessarily finite) T-CW complex. In general, X_Σ has the equivariant homotopy type of a finite T-CW complex.*

These results apply not only to complex, but also to real toric varieties and to their non-negative parts. In Section 2 we introduce the notation necessary to formulate these generalizations; the proof of the topological description then appears in Section 3. Some remarks about cubical subdivisions are made in Section 4.

The second part of this paper studies the ordinary and equivariant singular cohomology of toric varieties. We use Corollary 1.2 to generalize a result of Bahri–Franz–Ray [BFR_1, Prop. 2.2] from the projective to the general case, in particular to non-compact toric varieties. This was announced in [BFR_1, Remark 2].

THEOREM 1.3. *Let X_Σ be a toric variety. If $H^*(X_\Sigma;\mathbb{Z})$ vanishes in odd degrees, then $H_T^*(X_\Sigma;\mathbb{Z})$ is isomorphic, as algebra over the polynomial ring $H^*(BT;\mathbb{Z})$, to $PP(\Sigma;\mathbb{Z})$, the ring of integral piecewise polynomials on Σ.*

The proof appears in Section 5, together with the precise definition of piecewise polynomials. Combining Theorem 1.3 with a result of Payne [P], we get:

COROLLARY 1.4. *Let X_Σ be a toric variety. If $H^*(X_\Sigma;\mathbb{Z})$ vanishes in odd degrees, then $H_T^*(X_\Sigma;\mathbb{Z})$ is isomorphic to $A_T^*(X_\Sigma)$, the equivariant Chow cohomology ring of X_Σ.*

We finally investigate torsion phenomena in the integral cohomology. The celebrated Jurkiewicz–Danilov theorem implies that no torsion appears if X_Σ is smooth and compact. In Section 6 this is extended as follows:

PROPOSITION 1.5. *Assume that X_Σ is smooth or compact. If $H^*(X_\Sigma;\mathbb{Z})$ is concentrated in even degrees, then it is torsion-free.*

2. Toric varieties defined over monoids. In this section we briefly recall how to define toric varieties over submonoids of $\mathbb{C}$, and then state analogues of Theorem 1.1 and Corollary 1.2. Standard references for toric varieties are [O], [F] and [E]; see in particular [O, Sec. 1.3] and [F, Sec. 4.1] for real toric varieties and non-negative parts.

Let N be a free $\mathbb{Z}$-module of rank n with dual $M = N^\vee$. Extensions to real scalars are written in the form $N_\mathbb{R} = N \otimes \mathbb{R}$. (Unless stated otherwise, tensor products are taken over $\mathbb{Z}$.)

Let k be a multiplicative submonoid of $\mathbb{C}$ containing 0 and 1. We write $\mathbb{T}(k) = \operatorname{Hom}(M, k)$ for the group of monoid homomorphisms $M \to k$ (or, in this case equivalently, of group homomorphisms $M \to k^\times$) and set $T(k) = \mathbb{T}(k \cap S^1)$. Then $\mathbb{T}(\mathbb{C})$ and $T(\mathbb{C})$ are the algebraic torus $\mathbb{T}$ and the compact torus T introduced earlier, and $T(\mathbb{R}) \cong (\mathbb{Z}_2)^n$ is the compact form of $\mathbb{T}(\mathbb{R}) \cong (\mathbb{R}^*)^n$. Taking $k = \mathbb{R}_+ = [0,\infty)$, we get $\mathbb{T}(\mathbb{R}_+) \cong (0,\infty)^n$ and $T(\mathbb{R}_+) = 1$.

For a rational cone $\sigma \subset N_\mathbb{R}$ with dual $\sigma^\vee \subset M_\mathbb{R}$, the affine toric variety $X_\sigma(k)$ is defined as the set of monoid homomorphisms

$$X_\sigma(k) = \operatorname{Hom}(\sigma^\vee \cap M, k). \tag{2.1}$$

Since $\sigma^\vee \cap M$ is finitely generated, we may embed $\operatorname{Hom}(\sigma^\vee \cap M, k)$ into some affine space k^L, which induces a topology on $X_\sigma(k)$. (We always use the metric topology on $\mathbb{C}$, hence on $X_\sigma(k)$.) The "tori" $\mathbb{T}(k)$ and $T(k)$ act on $X_\sigma(k)$ by pointwise multiplication of functions.

As in the introduction, Σ denotes a rational fan in $N_\mathbb{R}$. The toric variety $X_\Sigma(k)$ is obtained by gluing the affine pieces $X_\sigma(k)$ together as prescribed by the fan; it is a Hausdorff space. We write $x_\sigma \in X_\sigma(k)$ for the distinguished point of $X_\sigma(k)$,

$$x_\sigma(m) = \begin{cases} 1 & \text{if } m \in \sigma^\perp, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

and $\mathcal{O}_\sigma(k)$ for its orbit under $\mathbb{T}(k)$. These orbits partition $X_\Sigma(k)$.

Note that $X_\Sigma(\mathbb{C})$ is the usual complex toric variety, $X_\Sigma(\mathbb{R})$ its real part and $X_\Sigma(\mathbb{R}_+)$ its non-negative part (the "associated manifold with corners"). As done in the introduction, we often write $X_\Sigma = X_\Sigma(\mathbb{C})$ and $\mathcal{O}_\sigma = \mathcal{O}_\sigma(\mathbb{C})$. We also use the notation $T_\sigma(k) = \{\, g \in T(k) : \sigma^\perp \cap M \subset \ker g \,\}$ for the subgroup of $T(k)$ determined by $\sigma \in \Sigma$.

Recall that the p-simplices of the order complex $\mathcal{F}(\varSigma)$ are the strictly ascending sequences $\sigma_0 < \cdots < \sigma_p$ of length $p+1$ in the partially ordered set $\varSigma$. (This is the same as the nerve of $\varSigma$, considered as a category with order relations as morphisms.) In particular, vertices of $\mathcal{F}(\varSigma)$ correspond to cones in $\varSigma$. One may think of $\mathcal{F}(\varSigma)$ as the cone over the barycentric subdivision of the "polyhedral complex" obtained by intersecting the unit sphere $S^{n-1} \subset N_{\mathbb{R}}$ with $\varSigma$.

In analogy with (1.2), we define the $T(k)$-space

$$Y_\varSigma(k) = (T(k) \times |\mathcal{F}(\varSigma)|)/\sim, \tag{2.3}$$

where the identification is done as follows: for $x \in |\mathcal{F}(\varSigma)|$, say with supporting simplex $\alpha = (\sigma_0, \dots, \sigma_p)$, one has $(t_1, x) \sim (t_2, x)$ iff $t_1 t_2^{-1} \in T_{\sigma_0}(k)$.

We have the following generalizations of results stated in the introduction, where k denotes either $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{R}_+$. The proof of Theorem 2.1 appears in the following section.

THEOREM 2.1. *If $\varSigma$ is complete, then $Y_\varSigma(k)$ is $T(k)$-equivariantly homeomorphic to $X_\varSigma(k)$. In general, $Y_\varSigma(k)$ is a $T(k)$-equivariant strong deformation retract of $X_\varSigma(k)$.*

Equivariant CW complexes are defined in [AP, Sec. 1.1], for instance.

COROLLARY 2.2. *If $\varSigma$ is complete, then $X_\varSigma(k)$ is a finite $T(k)$-CW complex. In general, $X_\varSigma(k)$ has the equivariant homotopy type of a finite $T(k)$-CW complex.*

Let $D_\varSigma(k)$ be the diagram of spaces over $\varSigma$ that assigns $T(k)/T_\sigma(k)$ to $\sigma \in \varSigma$, and the projection $T(k)/T_\sigma(k) \to T(k)/T_\tau(k)$ to $\sigma \le \tau$. Comparing (2.3) with the standard construction of a homotopy colimit of a diagram of spaces (cf. [WZŽ, Sec. 2]), we arrive at the following observation, which was made in [WZŽ, Prop. 5.3] for compact complex toric varieties:

COROLLARY 2.3. *The space $X_\varSigma(k)$ is the homotopy colimit of the diagram $D_\varSigma(k)$.*

3. A topological description of toric varieties. We write $\varSigma_i \subset \varSigma$ for the subset of i-dimensional cones and $\varSigma_{\max}$ for the set of maximal cones (with respect to inclusion). For a cone $\sigma \in \varSigma$, let N_σ be the intersection of N with the linear hull of σ, and $\pi_\sigma \colon N \to N(\sigma) = N/N_\sigma$ be the quotient map as well as its analogue over $\mathbb{R}$.

3.1. Case of complete $\varSigma$ and $k = \mathbb{R}_+$. Since $T(k) = 1$ is trivial in this case, it suffices to exhibit a triangulation of the non-negative part $X_\varSigma(\mathbb{R}_+)$ isomorphic with $\mathcal{F}(\varSigma)$. For each simplex $\alpha = (\sigma_0, \dots, \sigma_p) \in \mathcal{F}(\varSigma)$, we will construct a p-simplex $B(\alpha)$ in $X_{\sigma_p}(\mathbb{R}_+)$ whose interior ([1]) lies in the orbit $\mathcal{O}_{\sigma_0}(\mathbb{R}_+)$ corresponding to the initial vertex σ_0 of α.

Choose a point $v_\sigma \in N$ in the interior of σ, for example the sum of the minimal integral generators of the extremal rays of σ. Let

$$\lambda_\sigma\colon (0,\infty) \to \mathbb{T}(\mathbb{R}_+), \quad t \mapsto (m \mapsto t^{\langle m, v_\sigma\rangle}), \tag{3.1}$$

be the corresponding 1-parameter subgroup. Because Σ is complete, the variety $X_\Sigma(\mathbb{R}_+)$ is compact, so that the limit $\lambda_\sigma(0)x := \lim_{t\to 0}\lambda_\sigma(t)x$ exists for all $x \in X_\Sigma(\mathbb{R}_+)$. For the limits we are interested in, this will become evident during the proof of the following lemma.

Lemma 3.1. *Let $\alpha = (\sigma_0, \dots, \sigma_p) \in \mathcal{F}(\Sigma)$ be a p-simplex, $p \geq 1$. Then the $\mathbb{T}(\mathbb{R}_+)$-action on $X_\Sigma(\mathbb{R}_+)$ induces a continuous map*

$$\varphi_\alpha\colon [0,1]^p \to X_{\sigma_p}(\mathbb{R}_+), \quad t = (t_1, \dots, t_p) \mapsto \lambda_{\sigma_p}(t_p)\cdots\lambda_{\sigma_1}(t_1)x_{\sigma_0}.$$

Moreover, $\varphi_\alpha(t) = x_{\sigma_p}$ if $t_p = 0$, and $\varphi_\alpha(t) \neq \varphi_\alpha(t')$ if $t_p \neq t'_p$.

Proof. By the definition of the action of $\mathbb{T}(k) = \operatorname{Hom}(M,k)$ on $X_\sigma(k) = \operatorname{Hom}(\sigma^\vee \cap M, k)$, we have

$$(\lambda(t)\,x)(m) = \lambda(t)(m)\cdot x(m) = t^{\langle m,v\rangle}\cdot x(m).$$

for $x \in X_\sigma(k)$, $m \in \sigma^\vee \cap M$, and the 1-parameter subgroup λ with differential $v \in N$. In our case, all exponents in

$$\varphi_\alpha(t)(m) = \prod_i t_i^{\langle m, v_{\sigma_i}\rangle}\cdot x_{\sigma_0}(m)$$

are non-negative since all v_{σ_i} lie in σ_p. Hence the map φ_α is well-defined and continuous.

If $m \in \sigma_p^\perp$, then $m \in \sigma_i^\perp$ for all i, hence $\varphi_\alpha(0)(m) = 1$. If $m \notin \sigma_p^\perp$, then $\lambda_{\sigma_p}(0)(m) = x_{\sigma_p}(m) = 0$, hence $\varphi_\alpha(0)(m) = 0$. Therefore, $\varphi_\alpha(0) = x_{\sigma_p}$.

Since σ_{p-1} is a face of σ_p, there is an element $m \in \sigma_p^\vee \cap M$ vanishing on σ_{p-1}, hence on all σ_i, $i < p$, but not on σ_p. Then

$$\varphi_\alpha(t)(m) = t_p^{\langle m, v_{\sigma_p}\rangle},$$

which shows that $\varphi_\alpha(t)$ determines t_p. ∎

Let $\alpha = (\sigma_0, \dots, \sigma_p) \in \mathcal{F}(\Sigma)$ be a p-simplex. Applying Lemma 3.1 repeatedly proves that the image $B(\alpha)$ of φ_α is a p-simplex with vertices $x_{\sigma_0}, \dots, x_{\sigma_p}$. Its interior is contained in $\mathcal{O}_{\sigma_0}(\mathbb{R}_+)$, and its proper faces are the simplices corresponding to proper subsequences of α. It remains to verify the following claim:

Lemma 3.2. *The interiors of the simplices $B(\alpha)$, $\alpha \in \mathcal{F}(\Sigma)$, form a partition of $X_\Sigma(\mathbb{R}_+)$.*

([1]) By "interior" we always mean "relative interior".

Proof. It suffices to show that interiors of the simplices with initial vertex $\sigma_0 = \sigma$ partition $\mathcal{O}_\sigma(\mathbb{R}_+)$. To this end we define a complete fan Σ_σ in $N(\sigma)_\mathbb{R}$ which is the "barycentric subdivision" of the star of σ. Its cones $\tau_\sigma(\alpha)$ are labelled by simplices $\alpha = (\sigma_0, \dots, \sigma_p) \in \mathcal{F}(\Sigma)$ with initial vertex $\sigma_0 = \sigma$, and are spanned by the rays through the vectors $\pi_\sigma(v_{\sigma_1})$, …, $\pi_\sigma(v_{\sigma_p})$. (Observe that $\pi_\sigma(v_{\sigma_i})$ is an interior point of $\pi_\sigma(\sigma_i)$.)

The exponential map $\exp_\sigma \colon N(\sigma)_\mathbb{R} \to \mathcal{O}_\sigma(\mathbb{R}_+)$ is a real analytic isomorphism, in particular bijective. It is clear from Lemma 3.1 that $\exp_\sigma$ maps the interior of the cone $-\tau_\sigma(\alpha)$ onto the interior of $B(\alpha)$. Since the interiors of the cones in Σ_σ partition $N(\sigma)_\mathbb{R}$, this proves the claim. ∎

3.2. Case of complete Σ and arbitrary k. The inclusion $\mathbb{R}_+ \hookrightarrow k$ induces an inclusion $X_\Sigma(\mathbb{R}_+) \hookrightarrow X_\Sigma(k)$, similarly, the norm $k \to \mathbb{R}_+$, $z \mapsto |z|$, induces a retraction $X_\Sigma(k) \to X_\Sigma(\mathbb{R}_+)$. Both maps are compatible with the orbit structures. This implies that the restriction

$$T(k) \times X_\Sigma(\mathbb{R}_+) \to X_\Sigma(k) \tag{3.2}$$

of the $\mathbb{T}(k)$-action on $X_\Sigma(k)$ is surjective and descends to a $T(k)$-equivariant bijection $Y_\Sigma(k) \to X_\Sigma(k)$. This map must be a homeomorphism since $Y_\Sigma(k)$ is compact and $X_\Sigma(k)$ Hausdorff.

3.3. Case of arbitrary Σ. Any rational fan is a subfan of a complete rational fan $\tilde{\Sigma}$ (cf. [E, Thm. 9.3]); equivalently, any toric variety $X_\Sigma(k)$ is a $\mathbb{T}(k)$-stable open subvariety of a complete toric variety $X_{\tilde{\Sigma}}(k)$. The order complex $\mathcal{F}(\tilde{\Sigma})$ contains $\mathcal{F}(\Sigma)$ as a full subcomplex.

Recall from Section 3.1 that the interior of the simplex $B(\alpha)$ is contained in the orbit corresponding to the initial vertex of α. Therefore, the closed $\mathbb{T}(\mathbb{R}_+)$-subvariety

$$Z = X_{\tilde{\Sigma}}(\mathbb{R}_+) \setminus X_\Sigma(\mathbb{R}_+) \tag{3.3}$$

is the union of all simplices $B(\sigma_0, \dots, \sigma_p)$ such that σ_0, hence all vertices σ_i are not in Σ. Denote this subcomplex of $\mathcal{F}(\tilde{\Sigma})$ by L. Then $\mathcal{F}(\Sigma)$ and L are full subcomplexes of $\mathcal{F}(\tilde{\Sigma})$ on complementary vertex sets. This implies that $|\mathcal{F}(\Sigma)|$ is a strong deformation retract of $|\mathcal{F}(\tilde{\Sigma})| \setminus |L|$ (cf. [M, Lemma 70.1]).

The $T(k)$-equivariant homeomorphism $Y_{\tilde{\Sigma}}(k) \to X_{\tilde{\Sigma}}(k)$ given by (3.2) induces a $T(k)$-homeomorphism between

$$Y = (T(k) \times (|\mathcal{F}(\tilde{\Sigma})| \setminus |L|))/{\sim} \tag{3.4}$$

and $X_{\tilde{\Sigma}}(k) \setminus Z = X_\Sigma(k)$. The canonical strong deformation retraction $|\mathcal{F}(\tilde{\Sigma})| \setminus |L| \to |\mathcal{F}(\Sigma)|$ finally induces a $T(k)$-equivariant strong deformation retraction $Y \to Y_\Sigma(k)$.

4. Cubical subdivisions. Any simple polytope P admits a "cubical subdivision" with one full-dimensional cube per vertex (cf. [BP$_2$, Sec. 4.2]).

If Σ is a complete simplicial fan (as is the case for the normal fan of a simple polytope), then the homeomorphism $Y_\Sigma(\mathbb{R}_+) \approx X_\Sigma(\mathbb{R}_+)$ permits us to define a cubical structure on $X_\Sigma(\mathbb{R}_+)$ by setting

$$I_\tau^\sigma = \bigcup_{\substack{(\sigma_0,\ldots,\sigma_p)\in\mathcal{F}(\Sigma)\\ \tau\le\sigma_0,\ \sigma_p\le\sigma}} B(\sigma_0,\ldots,\sigma_p) \tag{4.1}$$

for $\tau \le \sigma$. (The right-hand side of (4.1) is the standard triangulation of a cube along the main diagonal [BP$_2$, Constr. 4.4], so I_τ^σ is indeed a cube of dimension $\dim\sigma - \dim\tau$.)

There is another, more intrinsic description of this subdivision, which in particular shows that it is canonical and does not depend on the choice of interior points v_σ used to define the simplices $B(\alpha)$ in Section 3. In fact,

$$I_0^\sigma = X_\sigma(I) \subset X_\sigma(\mathbb{R}_+), \tag{4.2}$$

where I denotes the multiplicative monoid $[0,1]$. To see this, observe that the union of the interiors of the simplices $B(\alpha)$ with initial vertex σ_0 and final vertex σ_p is the image of the interior of the cone $-\pi_{\sigma_0}(\sigma_p) \subset N(\sigma_0)_\mathbb{R}$ under the exponential map $\exp_{\sigma_0}\colon N(\sigma_0)_\mathbb{R} \to \mathcal{O}_{\sigma_0}(\mathbb{R}_+)$.

Note also that this proof shows that the canonical inclusion $X_\Sigma(I) \to X_\Sigma(\mathbb{R}_+)$ is in fact surjective. One sees similarly that for the disc $D^2 = \{z \in \mathbb{C} : |z| \le 1\}$ one has $X_\Sigma(D^2) = X_\Sigma(\mathbb{C})$. If Σ is regular, we therefore obtain a canonical decomposition of the smooth compact toric variety $X_\Sigma(\mathbb{C})$ into balls $(D^2)^n$. If Σ is not complete, then it is clear from (4.2) that we still have $X_\Sigma(I) = Y_\Sigma(\mathbb{R}_+)$, and similarly $X_\Sigma(D^2) = Y_\Sigma(\mathbb{C})$.

If Σ is a subfan of the cone spanned by a basis of N, then $X_\Sigma(\mathbb{C})$ is the complement of a complex coordinate subspace arrangement, and $X_\Sigma(D^2)$ is the moment-angle complex associated with the simplicial fan Σ, considered as a simplicial complex (see [BP$_2$, Ch. 6]). Therefore, Theorem 2.1 includes the well-known fact that moment-angle complexes and complements of complex coordinate subspace arrangements are equivariantly homotopy-equivalent [S, Prop. 20] (see also [BP$_1$, Lemma 2.13]).

5. Piecewise polynomials. An (integral) piecewise polynomial on the fan Σ is a function f from the support

$$|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subset N_\mathbb{R} \tag{5.1}$$

of Σ to $\mathbb{Z}$ such that for any $\sigma \in \Sigma$ the restriction $f|_\sigma$ of f to σ coincides with the restriction of some polynomial, integral with respect to the lattice N. The set $PP(\Sigma;\mathbb{Z})$ of all piecewise polynomials on Σ is a ring under pointwise addition and multiplication. Moreover, the canonical identification of $H^*(BT;\mathbb{Z})$ with the integral polynomials on N (see Step 1 below) gives a morphism of rings $H^*(BT;\mathbb{Z}) \to PP(\Sigma;\mathbb{Z})$ by restriction of functions to $|\Sigma|$, hence endows $PP(\Sigma;\mathbb{Z})$ with the structure of an $H^*(BT;\mathbb{Z})$-algebra.

The idea of the proof of Theorem 1.3 is to identify the piecewise polynomials on the fan Σ with the kernel of some "Mayer–Vietoris differential" for X_Σ and then to relate this kernel to the so-called "Atiyah–Bredon sequence" for X_Σ.

5.1. Step 1. Recall that any toric variety X_Σ is covered by the affine toric subvarieties X_σ where σ runs through $\Sigma_{\max}$, the set of maximal cones. The intersection of any two affine toric subvarieties X_σ and X_τ is the affine toric subvariety $X_{\sigma\cap\tau}$.

Any affine toric variety X_σ can be equivariantly retracted onto its unique closed orbit $\mathcal{O}_\sigma$ and the latter onto the T-orbit T/T_σ of x_σ. This establishes canonical isomorphisms

$$H_T^*(X_\sigma;\mathbb{Z}) = H_T^*(\mathcal{O}_\sigma;\mathbb{Z}) = H_T^*(T_\sigma;\mathbb{Z}) = \mathbb{Z}[\sigma], \tag{5.2}$$

where $\mathbb{Z}[\sigma]$ denotes the polynomials on N_σ (or, equivalently, on $\sigma\cap N$) with integer coefficients. The induced grading on polynomials is twice the usual degree. Moreover, for any pair $\tau \le \sigma$ the map $H_T^*(X_\sigma;\mathbb{Z}) \to H_T^*(X_\tau;\mathbb{Z})$ induced by the inclusion $X_\tau \hookrightarrow X_\sigma$ corresponds under the isomorphism (5.2) to the restriction of polynomials from N_σ to N_τ. In the following, we will not distinguish between polynomials and elements in the various cohomology groups in (5.2).

Fix some ordering of $\Sigma_{\max}$. A piecewise polynomial on Σ can be given uniquely by a collection of polynomials $f_\sigma \in \mathbb{Z}[\sigma]$, $\sigma\in\Sigma_{\max}$, that agree on common intersections. In other words, we can identify $PP(\Sigma;\mathbb{Z})$ with the kernel of the map

$$\begin{gathered}\delta\colon \bigoplus_{\sigma_0\in\Sigma_{\max}} H_T^*(X_{\sigma_0};\mathbb{Z}) \to \bigoplus_{\substack{\sigma_0,\sigma_1\in\Sigma_{\max}\\ \sigma_0<\sigma_1}} H_T^*(X_{\sigma_0\cap\sigma_1};\mathbb{Z}),\\ (\delta f)_{\sigma_0\sigma_1} = f_{\sigma_1}|_{\sigma_0\cap\sigma_1} - f_{\sigma_0}|_{\sigma_0\cap\sigma_0}\end{gathered} \tag{5.3}$$

where we have used the same notation as in [BT, §8]. (In fact, the map δ is the differential between the first two columns of the E_2 term of the Mayer–Vietoris spectral sequence associated to our covering of X_Σ by maximal affine toric subvarieties.)

5.2. Step 2. Our first observation is standard (at least for cohomology with field coefficients).

LEMMA 5.1. *The following conditions are equivalent for a toric variety X_Σ:*

(1) *$H^*(X_\Sigma;\mathbb{Z})$ is concentrated in even degrees.*

(2) *The Serre spectral sequence for the Borel construction of X_Σ degenerates at the E_2 level.*
(3) *The canonical map $H_T^*(X_\Sigma;\mathbb{Z}) \to H^*(X_\Sigma;\mathbb{Z})$ is a surjection.*

Proof. The implication (1)⇒(2) and the equivalence (2)⇔(3) hold for any T-space. For (3)⇒(1) we use that $H_T^*(X_\Sigma;\mathbb{Z})$ injects into $H_T^*(X_\Sigma^T;\mathbb{Z})$ (see [FP] or the proof of Proposition 5.2 below). Since X_Σ^T is discrete, this forces $H_T^*(X_\Sigma;\mathbb{Z})$, hence also $H^*(X_\Sigma;\mathbb{Z})$, to be concentrated in even degrees. ∎

PROPOSITION 5.2. *If X_Σ satisfies the conditions in Lemma* 5.1, *then all maximal cones in Σ are full-dimensional.*

Proof. We abbreviate $X_\Sigma = X$.

Since in the case of toric varieties all isotropy groups are connected, the conditions listed in Lemma 5.1 imply by a result of Franz–Puppe [FP] that the "Atiyah–Bredon sequence"

$$(5.4)\quad 0 \to H_T^*(X;\mathbb{Z}) \xrightarrow{\iota^*} H_T^*(X_0;\mathbb{Z}) \xrightarrow{\delta^0} H_T^{*+1}(X_1,X_0;\mathbb{Z}) \xrightarrow{\delta^1} \cdots$$
$$\cdots \xrightarrow{\delta^{n-1}} H_T^{*+n}(X_n,X_{n-1};\mathbb{Z}) \to 0$$

is exact. (The first part of (5.4) up to $H_T^*(X_1,X_0;\mathbb{Z})$ is also called the "Chang–Skjelbred sequence".) Here X_i denotes the equivariant i-skeleton of X, i.e., the union of all orbits of dimension at most i. In particular, $X_0 = X^T$, the fixed point set. The map ι^* is induced by the inclusion $\iota : X^T \hookrightarrow X$, and δ^i is the differential in the long exact cohomology sequence for the triple (X_{i+1},X_i,X_{i-1}). Note that while Franz–Puppe [FP] work in the setting of finite T-CW complexes, Corollary 1.2 allows us to apply this result to X even in the non-compact case; here we use that the canonical T-homotopy described in Section 3.3 preserves orbit types.

We have

$$(5.5)\qquad H_T^*(X_i,X_{i-1};\mathbb{Z}) = \bigoplus_{\sigma\in\Sigma_{n-i}} H_T^*(\bar{\mathcal{O}}_\sigma,\partial\mathcal{O}_\sigma;\mathbb{Z}),$$

and the differential

$$(5.6)\qquad \delta^i\colon H_T^*(X_i,X_{i-1};\mathbb{Z}) \to H_T^{*+1}(X_{i+1},X_i;\mathbb{Z})$$

is a "block matrix" whose components are the maps

$$(5.7)\qquad H_T^*(\bar{\mathcal{O}}_\tau,\partial\mathcal{O}_\tau;\mathbb{Z}) \to H_T^{*+1}(\bar{\mathcal{O}}_\sigma,\partial\mathcal{O}_\sigma;\mathbb{Z})$$

for those pairs (σ,τ) where $\mathcal{O}_\tau \subset \bar{\mathcal{O}}_\sigma$, i.e., where $\tau\in\Sigma_{n-i}$ is a facet of $\sigma\in\Sigma_{n-i+1}$.

Now assume that $\tau\in\Sigma$ is maximal and of codimension $k>0$. Then, by maximality, $H_T^*(\bar{\mathcal{O}}_\tau,\partial\mathcal{O}_\tau;\mathbb{Z}) = H_T^*(\mathcal{O}_\tau;\mathbb{Z})$ is a direct summand of the module $H_T^*(X_k, X_{k-1}; \mathbb{Z})$, and no non-zero element of $H_T^*(\mathcal{O}_\tau; \mathbb{Z})$ can be in the image of the differential δ^{k-1}.

Let σ be a facet of τ. (If τ were the zero cone, then X would a complex torus and $H^1(X; \mathbb{Z}) \neq 0$, contrary to our assumptions.) The toric variety $\bar{\mathcal{O}}_\sigma$ is described by the star of σ in Σ (cf. [F, Sec. 3.1]), which we denote by $\hat{\sigma}$. To compute the map (5.7), we replace $\bar{\mathcal{O}}_\sigma$ by the T-equivariantly homotopy equivalent T-CW complex $Y = Y_{\hat{\sigma}}(\mathbb{C}) \subset \bar{\mathcal{O}}_\sigma$ and $\partial\mathcal{O}_\sigma$ by $Z = Y \cap \partial\mathcal{O}_\sigma$. Note that $Y \cap \mathcal{O}_\tau$ is a single orbit T_τ because τ is maximal. Let Y' be the space obtained from Y by replacing this orbit T_τ by T_σ, and similarly for Z'. (This means changing the identification for the the points above the vertex $\tau \in \mathcal{F}(\hat{\sigma})$ in (2.3).) From the projection $Y' \to Y$ we see that the map

$$H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau; \mathbb{Z}) = H_T^*(T_\tau; \mathbb{Z}) \to H_T^{*+1}(Y, Z; \mathbb{Z}) = H_T^{*+1}(\bar{\mathcal{O}}_\sigma, \partial\mathcal{O}_\sigma; \mathbb{Z}) \tag{5.8}$$

factors through

$$H_T^*(T_\tau; \mathbb{Z}) \to H_T^*(T_\sigma; \mathbb{Z}) \to H_T^{*+1}(Y', Z'; \mathbb{Z}) = H_T^{*+1}(Y, Z; \mathbb{Z}). \tag{5.9}$$

The map $H_T^*(T_\tau; \mathbb{Z}) \to H_T^*(T_\sigma; \mathbb{Z})$ is the canonical projection $\mathbb{Z}[\tau] \to \mathbb{Z}[\sigma]$. Pick a non-zero element f_σ in its kernel. Then the product of all these f_σ, as σ runs through the facets of τ, is a non-zero element in the kernel of the differential δ^k. As it does not lie in the image of δ^{k-1}, we get a contradiction to the exactness of the Atiyah–Bredon sequence. ∎

5.3. Step 3. Equations (5.2) and (5.5) (for $i = 0$) together give a canonical isomorphism

$$H_T^*(X_0; \mathbb{Z}) = \bigoplus_{\sigma \in \Sigma_n} H_T^*(X_\sigma; \mathbb{Z}). \tag{5.10}$$

We finally show that under this isomorphism the kernel of the differential

$$\delta^0 \colon H^*(X_0; \mathbb{Z}) \to H_T^{*+1}(X_1, X_0; \mathbb{Z}) = \bigoplus_{\sigma \in \Sigma_{n-1}} H_T^*(\bar{\mathcal{O}}_\sigma, \partial\mathcal{O}_\sigma; \mathbb{Z}) \tag{5.11}$$

coincides with that of the map (5.3).

Since no cone $\tau \in \Sigma_{n-1}$ is maximal, it is contained in either one or two full-dimensional cones. In the first case we have

$$H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau; \mathbb{Z}) = H_T^*(\mathbb{C}, \{0\}; \mathbb{Z}) = 0, \tag{5.12}$$

and in the second case

$$H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau; \mathbb{Z}) = H_T^*(\mathbb{CP}^1, \{0, \infty\}; \mathbb{Z}) \cong \mathbb{Z}[\tau][+1], \tag{5.13}$$

where the last isomorphism is chosen such that if τ is the common facet of σ_0 and σ_1, $\sigma_0 < \sigma_1$, then the differential is of the form

$$\begin{aligned} H_T^*(\mathcal{O}_{\sigma_0}; \mathbb{Z}) \oplus H_T^*(\mathcal{O}_{\sigma_1}; \mathbb{Z}) &\to H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau; \mathbb{Z}), \\ (f_0, f_1) &\mapsto f_1|_{N_\tau} - f_0|_{N_\tau}. \end{aligned} \tag{5.14}$$

Consider the following diagram, where the vertical map on the right sends each summand $H_T^*(X_{\sigma_0\cap\sigma_1};\mathbb{Z}) = \mathbb{Z}[\tau]$ to 0 if $\tau = \sigma_0 \cap \sigma_1$ is a not common facet, and identically onto $H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau;\mathbb{Z}) = \mathbb{Z}[\tau][+1]$ otherwise:

$$\begin{tikzcd}
0 \arrow[r] & H_T^*(X;\mathbb{Z}) \arrow[r, "\iota^*"] & \bigoplus_{\sigma\in\Sigma_n} H_T^*(\mathcal{O}_\sigma;\mathbb{Z}) \arrow[r, "\delta^0"] & \bigoplus_{\tau\in\Sigma_{n-1}} H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau;\mathbb{Z}) \\
& H_T^*(X;\mathbb{Z}) \arrow[r, "\iota^*"] & \bigoplus_{\sigma\in\Sigma_n} H_T^*(X_\sigma;\mathbb{Z}) \arrow[r, "\delta"] \arrow[u, equal] & \bigoplus_{\substack{\sigma_0,\sigma_1\in\Sigma_n\\ \sigma_0<\sigma_1}} H_T^*(X_{\sigma_0\cap\sigma_1};\mathbb{Z}) \arrow[u]
\end{tikzcd} \tag{5.15}$$

The commutativity of the right square follows from formulas (5.3) and (5.14).

Since the differential δ^0 is the composition of δ and another map, the kernel of δ^0 contains that of δ. We know that $\ker\delta^0 = H_T^*(X;\mathbb{Z})$. We also know that the map ι^* induced by the inclusion of the fixed point set is injective, and its image is contained in the kernel of δ. Hence $H_T^*(X;\mathbb{Z}) \subset \ker\delta \subset \ker\delta^0 = H_T^*(X;\mathbb{Z})$, so the two kernels coincide. This finishes the proof of Theorem 1.3.

6. Torsion-free cohomology. We now turn our attention to toric varieties whose equivariant cohomology is not only concentrated in even degrees, but also torsion-free. This property can be characterized nicely in terms of equivariant cohomology; moreover, it behaves well when passing to orbit closures.

LEMMA 6.1. *The ordinary cohomology $H^*(X_\Sigma;\mathbb{Z})$ is torsion-free and concentrated in even degrees iff the equivariant cohomology $H_T^*(X_\Sigma;\mathbb{Z})$ is free over $H^*(BT;\mathbb{Z})$.*

Proof. If $H^{\mathrm{odd}}(X_\Sigma;\mathbb{Z})$ vanishes, then the map $\iota^*\colon H_T^*(X_\Sigma;\mathbb{Z}) \to H^*(X_\Sigma;\mathbb{Z})$ is surjective by Lemma 5.1. If moreover $H^*(X_\Sigma;\mathbb{Z})$ is free over $\mathbb{Z}$, then there exists a section to ι^*, and $H_T^*(X_\Sigma;\mathbb{Z}) \cong H^*(X_\Sigma;\mathbb{Z}) \otimes H^*(BT;\mathbb{Z})$ is free over $H^*(BT;\mathbb{Z})$ by the Leray–Hirsch Theorem.

Conversely, if $H_T^*(X_\Sigma;\mathbb{Z})$ is free over $H^*(BT;\mathbb{Z})$, then the sequence (5.4) is exact, and $H_T^*(X_\Sigma;\mathbb{Z})$ injects into $H_T^*(X_\Sigma^T;\mathbb{Z}) = H^*(X_\Sigma^T;\mathbb{Z})\otimes H^*(BT;\mathbb{Z})$. Since X_Σ^T is finite, this shows that $H_T^*(X_\Sigma;\mathbb{Z})$ is concentrated in even degrees. Therefore, $H^*(X_\Sigma;\mathbb{Z}) = H_T^*(X_\Sigma;\mathbb{Z}) \otimes_{H^*(BT;\mathbb{Z})} \mathbb{Z}$ is torsion-free and concentrated in even degrees. ∎

PROPOSITION 6.2. *If $H^*(X_\Sigma;\mathbb{Z})$ is torsion-free and concentrated in even degrees, then the same holds true for any orbit closure $\bar{\mathcal{O}}_\sigma \subset X_\Sigma$.*

Recall that $\bar{\mathcal{O}}_\sigma$ is again a toric variety, defined by the star of σ in Σ.

Proof. Note first that it is enough to prove that $H^*(\bar{\mathcal{O}}_\sigma;\mathbb{F}_p)$ is concentrated in even degrees for all primes p. Moreover, since $\bar{\mathcal{O}}_\sigma$ is a component of $X_\Sigma^{T_\sigma}$, it suffices to consider fixed point sets X_Σ^G, where $G \subset T$ is any subtorus.

We use that for a (sufficiently "nice") T-space X one has

$$\dim H^*(X;\mathbb{F}_p) \geq \dim H^*(X^T;\mathbb{F}_p) \tag{6.1}$$

with equality iff $H_T^*(X;\mathbb{F}_p)$ is free over $H^*(BT;\mathbb{F}_p)$ (cf. [AP, Cor. 3.1.14 & 3.1.15]). (There rational coefficients are used. However, a look at the proof shows that in the case of connected isotropy groups coefficients can be taken in any field.)

Set $X = X_\Sigma$ and $Y = X_\Sigma^G$. In this case we have

$$\dim H^*(X;\mathbb{F}_p) \geq \dim H^*(Y;\mathbb{F}_p) \geq \dim H^*(Y^T;\mathbb{F}_p). \tag{6.2}$$

Since $Y^T = X^T$, all inequalities must be equalities. Therefore $H_T^*(Y^G;\mathbb{F}_p)$ surjects onto $H^*(Y^G;\mathbb{F}_p)$, so the latter is concentrated in even degrees. ∎

Question 6.3. Is the property "$H^{\mathrm{odd}}(X_\Sigma;\mathbb{Z}) = 0$" inherited by orbit closures even in the presence of torsion?

In the course of the proof of Lemma 6.1 we showed that if the odd-dimensional cohomology of X_Σ vanishes, then $H_T^*(X_\Sigma;\mathbb{Z})$ injects into the free $H^*(BT;\mathbb{Z})$-module $H_T^*(X_\Sigma^T;\mathbb{Z})$, so $H_T^*(X_\Sigma;\mathbb{Z})$ cannot have $\mathbb{Z}$-torsion. For a general T-space X, this last property together with the degeneration of the Serre spectral sequence does *not* guarantee that $H^*(X;\mathbb{Z})$ itself is torsion-free. (See [FP, Ex. 5.2] for a counterexample.) But Proposition 1.5 asserts that this conclusion is valid for toric varieties which are smooth or compact.

Proof of Proposition 1.5. We again write $X = X_\Sigma$. Assume first that X is compact. Then all terms $H_T^{*+i}(X_i, X_{i-1};\mathbb{Z})$ in the Atiyah–Bredon sequence (5.4) are free over $\mathbb{Z}$: in fact, equation (5.5) becomes

$$\begin{aligned} H_T^*(X_i, X_{i-1};\mathbb{Z}) &= \bigoplus_{\sigma\in\Sigma_{n-i}} H_T^*(\bar{\mathcal{O}}_\sigma, \partial\mathcal{O}_\sigma;\mathbb{Z}) \\ &= \bigoplus_{\sigma\in\Sigma_{n-i}} H_T^*(\mathcal{O}_\sigma;\mathbb{Z})[+i] \cong \bigoplus_{\sigma\in\Sigma_{n-i}} \mathbb{Z}[\sigma][+i]. \end{aligned} \tag{6.3}$$

(This is the E_1 term of the spectral sequence considered in [Fi].) $H_T^*(X;\mathbb{Z})$, being a submodule of $H_T^*(X_0;\mathbb{Z})$, is free over $\mathbb{Z}$ as well. Hence, the Atiyah–Bredon sequence over a finite field $\mathbb{F}_p$ is obtained by tensoring the integral version (5.4) with $\mathbb{F}_p$, and this does not affect exactness.

It actually holds true for any field k as coefficients that the exactness of the Atiyah–Bredon sequence implies the freeness of $H_T^*(X;k)$ over $H^*(BT;k)$. (This can be seen by inspecting the proofs in [B$_2^+$, Sec. 4.8] or [FP, Sec. 4].) Since $H_T^*(X;k)$ injects into $H_T^*(X_0;k)$ and the latter module is concentrated in even degrees, the same applies to the former, hence also to its quotient $H^*(X;k)$. By considering the Universal Coefficient Theorem for prime fields $k=\mathbb{F}_p$, one sees that this is impossible if $H^{\mathrm{even}}(X;\mathbb{Z})$ has torsion.

Suppose now that X is smooth, and let $\tilde{X}$ be a toric compactification of X (cf. Section 3.3). Set $Z=\tilde{X}\setminus X$. By Lefschetz duality (cf. [M, Thm. 70.2]), $H^*(\tilde{X},Z;\mathbb{Z})=H_{2n-*}(X;\mathbb{Z})$ is also concentrated in even degrees, and the reasoning for the compact case carries over to the pair $(\tilde{X},Z)$ instead of X. Hence, $H^*(\tilde{X},Z;\mathbb{Z})$ is torsion-free, and therefore $H^*(X;\mathbb{Z})$ as well. ∎

Assume that $H^*(X_\Sigma;\mathbb{R})$ is concentrated in even degrees. Then a reasoning analogous to that in Section 5 shows that $H_T^*(X_\Sigma;\mathbb{R})=PP(X_\Sigma;\mathbb{R})$ is free over the polynomial ring $H^*(BT;\mathbb{R})$. A result of Yuzvinsky [Y, Cor. 3.6] implies that the reduced homology of all links in Σ vanishes except in top degrees. (In the case of a simplicial fan, this is Reisner's Cohen–Macaulay criterion [BH, Cor. 5.3.9].) For compact toric varieties, we can give a short topological proof of this fact, even with integer coefficients.

PROPOSITION 6.4. *If X_Σ is compact and $H^*(X_\Sigma;\mathbb{Z})$ concentrated in even degrees, then $\tilde{H}_i(\operatorname{lk}\sigma;\mathbb{Z})=0$ for all $\sigma\in\Sigma$ and all $i<n-\dim\sigma-1$.*

Proof. By Propositions 1.5 and 6.2, it is enough to consider the case $\sigma=0$.

Because $X=X_\Sigma$ is compact, we have

$$H_T^*(X_i,X_{i-1};\mathbb{Z})\cong\bigoplus_{\sigma\in\Sigma_{n-i}}\mathbb{Z}[\sigma][+i], \tag{6.4}$$

where the isomorphism is determined by a choice of orientations of the cones. Hence, the part

$$H_T^0(X_0;\mathbb{Z})\xrightarrow{\delta^0}H_T^1(X_1,X_0;\mathbb{Z})\xrightarrow{\delta^1}\cdots\xrightarrow{\delta^{n-1}}H_T^n(X_n,X_{n-1};\mathbb{Z})\to 0 \tag{6.5}$$

of the Atiyah–Bredon sequence computes the homology with closed support of $|\Sigma|$. This is, up to a degree shift by 1, the homology of link of the zero cone. Since $H^*(X;\mathbb{Z})$ is concentrated in even degrees, this sequence is exact, which means $\tilde{H}_i(\operatorname{lk}0;\mathbb{Z})=0$ for all $i<n-1$. ∎

We conclude with a remark about hereditary fans. A fan Σ is called *hereditary* if all maximal cones are full-dimensional and if for every $\tau\in\Sigma$ one has that

$$\left\{\begin{array}{l}\text{any two maximal cones } \sigma,\ \sigma' \text{ in the star of } \tau \text{ can be joined by}\\ \text{a sequence } \sigma=\sigma_0,\ \ldots,\ \sigma_k=\sigma' \text{ of maximal cones in the star}\\ \text{of } \tau \text{ such that } \sigma_{i-1} \text{ and } \sigma_i \text{ have a common facet, } 1\le i\le k;\end{array}\right. \tag{6.6}$$

see [BR].

Proposition 6.5. *If $H^*(X_\Sigma;\mathbb{Z})$ is concentrated in even degrees, then Σ is hereditary.*

Proof. We know from Proposition 5.2 that all maximal cones in Σ are full-dimensional. Group these cones in Σ into "connected components" in the sense that all cones in a component can be connected by full-dimensional cones sharing a common facet. Then the number of these components is the dimension of the free $\mathbb{Z}$-module of "piecewise constant functions" on Σ, in other words, the dimension of the kernel of the differential δ^0 in the Atiyah–Bredon sequence. But this equals $\dim H^0_T(X_\Sigma;\mathbb{Z}) = 1$ because the sequence is exact. So condition (6.6) holds for $\tau = 0 \in \Sigma$, the zero cone.

To reduce the case of general $\tau \in \Sigma$ to the case $\tau = 0$, we consider the orbit closure $\bar{\mathcal{O}}_\tau \subset X_\Sigma$. By Proposition 6.2, $H^*(\bar{\mathcal{O}}_\tau;\mathbb{Z})$ is torsion-free and concentrated in even degrees. Therefore, condition (6.6) holds for the zero cone in the star of any τ, which means that it holds for all $\tau \in \Sigma$. ∎

We leave it to the reader to check that it would actually be enough to assume $H^{\mathrm{odd}}(X_\Sigma;\mathbb{Q}) = 0$. But even over the rationals one cannot hope for a converse to Proposition 6.5: An example originally due to Eikelberg and further studied by Barthel–Brasselet–Fieseler–Kaup [B_1^+, Ex. 3.5] shows that two combinatorially equivalent complete fans Σ and Σ' in $\mathbb{R}^3$ can lead to toric varieties X_Σ and $X_{\Sigma'}$ with $H^{\mathrm{odd}}(X_\Sigma;\mathbb{Q}) = 0$ and $H^{\mathrm{odd}}(X_{\Sigma'};\mathbb{Q}) = H^3(X_{\Sigma'};\mathbb{Q}) \cong \mathbb{Q}$.

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Matthias Franz
Department of Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada
E-mail: mfranz@uwo.ca

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