ON OPERATORS FROM $\ell_s$ TO $\ell_p \hat{\otimes} \ell_q$ OR TO $\ell_p \hat{\hat{\otimes}} \ell_q$

BY

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Abstract. We show that every operator from $\ell_s$ to $\ell_p \hat{\otimes} \ell_q$ is compact when $1 \leq p, q < s$ and that every operator from $\ell_s$ to $\ell_p \hat{\hat{\otimes}} \ell_q$ is compact when $1/p + 1/q > 1 + 1/s$.

1. Introduction. We recall Pitt’s theorem: for $1 \leq p < s < \infty$, every operator from $\ell_s$ to $\ell_p$ is compact [7], [8]. This result has been extended to different settings. Among the more recent contributions we mention [1] and [3]. The aim of this paper is to show that every operator from $\ell_s$ to $\ell_p \hat{\otimes} \ell_q$ is compact when

$$1 \leq p, q < s \quad (1.1)$$

and that every operator from $\ell_s$ to $\ell_p \hat{\hat{\otimes}} \ell_q$ is compact when

$$\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{s}, \quad (1.2)$$

A proof of the injective case, using $\tau_\alpha$-convergence, is given in [1]. Here we use a different method and the same technique in both cases. Let $r = s'$ be the conjugate exponent of $s$ (i.e. $1/s + 1/s' = 1$). We show that under condition (1.1) (resp. (1.2)) the space $[\ell_p \hat{\otimes} \ell_q] \hat{\otimes} \ell_r$ (resp. $[\ell_p \hat{\hat{\otimes}} \ell_q] \hat{\otimes} \ell_r$) does not contain a subspace isomorphic to $c_0$. The conclusions will then follow from [11].

2. Notation. We shall make use of standard Banach space facts and terminology as may be found in [6], [7].

The term operator means bounded linear operator. Subspace means closed linear subspace.

Let $E, F$ be Banach spaces. We denote by:

- $\mathcal{L}(E, F)$ the space of operators from $E$ to $F$.
- $\mathcal{N}(E, F)$ the space of nuclear operators from $E$ to $F$, and by $\|u\|_{\text{nuc}}$ the nuclear norm of a nuclear operator $u$.

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• $B(E, F)$ the space of continuous bilinear forms on $E \times F$.
• $E \hat{\otimes} F$ the completion of $E \otimes F$ endowed with the projective norm [4], [5].
• $E \hat{\otimes} F$ the completion of $E \otimes F$ endowed with the injective norm [4], [5].
• $\mathcal{I}(E, F)$ the space of bilinear integral forms on $E \times F$. We have $\mathcal{I}(E, F) = [E \hat{\otimes} F]^\ast$. The norm of an integral form $\varphi$ is denoted by $\|\varphi\|_{\text{int}}$.
• $\ell_m^p$ the $m$-dimensional space $\ell_p(\{1, \ldots, m\})$.

Let $r$ be a real number $\geq 1$; we define

$$sl_r(E) = \left\{ x = (x_n)_{n \geq 1}; \text{ for all } n \geq 1, \ x_n \in E, \quad \text{and for all } x^* \in E^*, \ \sum_{n=1}^{\infty} |x^*(x_n)|^r < \infty \right\}.$$  

We recall that for $x = (x_n)_n \in sl_r(E)$ we have

$$\|x\| = \sup_{\|x^*\| \leq 1} \left( \sum_{n=1}^{\infty} |x^*(x_n)|^r \right)^{1/r} < \infty.$$  

The space $(sl_r(E), \|\cdot\|)$ is a Banach space. For every integer $m$, let $R_m$ be the projection of $sl_r(E)$ defined, for every $x = (x_k)_k$, by $R_m(x) = (x_1, \ldots, x_m, 0, 0, \ldots)$. The subspace

$$F_r(E) = \{ x \in sl_r(E); \ x = \lim_{m \to \infty} R_m(x) \}$$

of $sl_r(E)$ is isometrically isomorphic to $\ell_r \hat{\otimes} E$ (see [9]). We shall use this isometric isomorphism without any reference.

3. Lemmas. Let $1 \leq p, q, r < \infty$. We denote by $(P_m)_m$ the natural projections associated to the unit vector basis of $\ell_p$ and by $(Q_m)_m$ the natural projections associated to the unit vector basis of $\ell_q$. We denote by $\tilde{P}_m, \tilde{Q}_m$ the norm 1 projections of $\ell_r \hat{\otimes} (\ell_p \hat{\otimes} \ell_q)$ or $\ell_r \hat{\otimes} (\ell_p \hat{\otimes} \ell_q)$ which are defined by $\tilde{P}_m = I_{\ell_r} \otimes (P_m \otimes I_{\ell_q})$ and $\tilde{Q}_m = I_{\ell_r} \otimes (I_{\ell_p} \otimes Q_m)$. For every $x = (x_k)_k \in F_r(\ell_p \hat{\otimes} \ell_q)$ or $x = (x_k)_k \in F_r(\ell_p \hat{\otimes} \ell_q)$ we have

$$\tilde{P}_m(x) = ((P_m \otimes I_{\ell_q})(x_1), \ldots, (P_m \otimes I_{\ell_q})(x_k), \ldots),$$

$$\tilde{Q}_m(x) = ((I_{\ell_p} \otimes Q_m)(x_1), \ldots, (I_{\ell_p} \otimes Q_m)(x_k), \ldots).$$

For all $m, n$ we have $\tilde{P}_m \circ R_n = R_n \circ \tilde{P}_m$, $\tilde{Q}_m \circ R_n = R_n \circ \tilde{Q}_m$ and $\tilde{P}_m \circ \tilde{Q}_n = \tilde{Q}_n \circ \tilde{P}_m$.

It is well known that, if $(\pi_m)_m$ is a sequence of operators on a Banach space $E$ such that $\lim_{m \to \infty} \pi_m(x) = x$ for every $x \in E$, then for every
Banach space $F$ and for every $u \in E \hat{\otimes} F$ (resp. $u \in E \hat{\otimes} F$) we have \( \lim_{m \to \infty}(\pi_m \otimes I_F)(u) = u \). This remark leads to the following lemma:

**Lemma 3.1.** For every $x \in F_r(\ell_p \hat{\otimes} \ell_q)$ and every $x \in F_r(\ell_p \hat{\otimes} \ell_q)$ we have
\[
x = \lim_{m \to \infty} \tilde{P}_m(x) = \lim_{m \to \infty} \tilde{Q}_m(x).
\]

**Lemma 3.2.** For every integer $m$, $\tilde{P}_m[F_r(\ell_p \hat{\otimes} \ell_q)]$ and $\tilde{P}_m[F_r(\ell_p \hat{\otimes} \ell_q)]$ are isomorphic to $\ell_r \hat{\otimes} \ell_q$.

**Proof.** It is easy to show that
\[
\tilde{P}_m[F_r(\ell_p \hat{\otimes} \ell_q)] = F_r[(P_m \otimes I_{\ell_q})(\ell_p \hat{\otimes} \ell_q)].
\]

We have $(P_m \otimes I_{\ell_q})(\ell_p \hat{\otimes} \ell_q)$ isomorphic to $\ell_p \hat{\otimes} \ell_q$. It is well known that $\ell_p \hat{\otimes} \ell_q$ is isomorphic to the $m$-product $[\ell_q]^m$ of $\ell_q$ and so to $\ell_q$. Hence, $F_r[(P_m \otimes I_{\ell_q})(\ell_p \hat{\otimes} \ell_q)]$ is isomorphic to $\ell_r \hat{\otimes} \ell_q$. With the same argument we show that $\tilde{P}_m[F_r(\ell_p \hat{\otimes} \ell_q)]$ is isomorphic to $\ell_q \hat{\otimes} \ell_r$.

In the following we shall consider sequences of block operators. A sequence $(T_n)_{n}$ of operators from $\ell_p$ to $\ell_q$ is called a sequence of block operators if there exist two strictly increasing sequences $(i_n)_{n}$ and $(j_n)_{n}$ of integers such that $i_0 = j_0 = 0$ and, for every integer $n \geq 1$, we have
\[
T_n = (Q_{j_n}^* - Q_{j_{n-1}}^*) \circ T_n \circ (P_{i_n} - P_{i_{n-1}}).
\]

We write as lemmas the results of Tong [10] that we will use below.

**Lemma 3.3.** Let $(T_n)_{n}$ be a sequence of block operators from $\ell_p$ to $\ell_q$. Suppose that $\|T_n\| = 1$ for every $n$. Then, for every integer $N$ and for every finite sequence $(\alpha_n)_{1 \leq n \leq N}$ of scalars, we have
\[
\left\| \sum_{n=1}^{N} \alpha_n T_n \right\| = \begin{cases} 
\sum_{n=1}^{N} |\alpha_n|^{p q'} / p - q' & \text{if } 1 \leq q' < p < \infty, \\
\max_{1 \leq n \leq N} |\alpha_n| & \text{if } 1 \leq p \leq q' \leq \infty, \\
\sum_{n=1}^{N} |\alpha_n|^{q'} / q' & \text{if } 1 \leq q' < p = \infty.
\end{cases}
\]

**Lemma 3.4.** Let $(T_n)_{n}$ be a sequence of block operators from $\ell_p$ to $\ell_q$. Suppose that $\|T_n\|_{\text{max}} = 1$ for every $n$. Then, for every integer $N$ and for every finite sequence $(\alpha_n)_{1 \leq n \leq N}$ of scalars, we have
three strictly increasing sequences of integers

separable dual, hence, by \([2]\), it does not contain a subspace isomorphic to

\[
\|\sum_{n=1}^{N} \alpha_n T_n\|_{\text{nuc}} = \begin{cases}
\sum_{n=1}^{N} |\alpha_n| & \text{if } 1 \leq q' < \infty,
\max_{1 \leq n \leq N} |\alpha_n| & \text{if } p = 1 \text{ and } q' = \infty,
\sum_{n=1}^{N} |\alpha_n|^{p-1} p^{\frac{1}{p}} & \text{if } 1 < p < q' = \infty,
\sum_{n=1}^{N} |\alpha_n| & \text{if } 1 \leq q' \leq p \leq \infty.
\end{cases}
\]

The following lemma is a direct consequence of the proof of the theorem of \([9]\).

**Lemma 3.5.** Let \(X\) be an infinite-dimensional subspace of \(\ell_p \widehat{\otimes} \ell_q\). If \(q' > p\), then \(X\) contains a subspace isomorphic to \(\ell_\sigma\) where \(\sigma = p\) or \(\sigma = q\) or \(\sigma = \frac{pq}{p+q-pq} = \frac{pq'}{q'-p}\), and if \(q' \leq p\), then \(X\) contains a subspace isomorphic to \(c_0\).

**4. Operators from \(\ell_\sigma\) into \(\ell_p \widehat{\otimes} \ell_q\).** For every \(b \in B(E, F)\) we denote by \(T_b \in \mathcal{L}(E, F^*)\) the operator defined by \((T_b(x))(y) = b(x, y)\) for every \(x \in E\) and \(y \in F\). We recall that the operator \(b \mapsto T_b\) is an isometric isomorphism from \(B(E, F)\) onto \(\mathcal{L}(E, F^*)\).

**Theorem 4.1.** Let \(1 \leq p, q, r\) be real numbers such that \(1 \leq r, 1 \leq p < r'\) and \(1 \leq q < r'\). Then the space \((\ell_p \widehat{\otimes} \ell_q) \widehat{\otimes} \ell_r\) does not contain a subspace isomorphic to \(c_0\).

**Proof.** By Grothendieck’s result \([5]\) the space \(\ell_p \widehat{\otimes} \ell_q\) is the dual of \(\ell_p \widehat{\otimes} \ell_q\) (with \(c_0\) in place of \(\ell_\infty\) when \(p = 1\)). Therefore the space \(\ell_p \widehat{\otimes} \ell_q\) is a separable dual, hence, by \([2]\), it does not contain a subspace isomorphic to \(c_0\).

We assume that \(F_r(\ell_p \widehat{\otimes} \ell_q)\) contains a subspace isomorphic to \(c_0\); we shall show that this leads to a contradiction. We shall construct a normalized basic sequence \((x_n)_n\) of \(F_r(\ell_p \widehat{\otimes} \ell_q)\) equivalent to the unit vector basis of \(c_0\) and three strictly increasing sequences of integers \((i_n)_n, (j_n)_n, (k_n)_n\) such that \(i_0 = j_0 = k_0 = 0\) and, for every integer \(n \geq 1\),

\[
(4.1) \quad x_n = (R_{k_n} - R_{k_n-1})(x_n) = (\widetilde{P}_{i_n} - \widetilde{P}_{i_n-1})(x_n) = (\widetilde{Q}_{j_n} - \widetilde{Q}_{j_n-1})(x_n).
\]

This will be done in three stages. We begin with a normalized basic sequence \((u_n)_n\) of \(F_r(\ell_p \widehat{\otimes} \ell_q)\) equivalent to the unit vector basis of \(c_0\).

In the first stage we show that there exists a normalized basic sequence \((v_n)_n\) of \(F_r(\ell_p \widehat{\otimes} \ell_q)\) equivalent to the unit vector basis of \(c_0\) and a strictly
increasing sequence \((m_n)_n\) of integers such that \(m_0 = 0\) and, for every integer \(n \geq 1\),

\[
(4.2) \quad v_n = (R_{m_n} - R_{m_{n-1}})(v_n).
\]

Let \(\varepsilon > 0\). For every integer \(m \geq 1\), the subspace \(\text{Im} R_m\) of \(F_r(\ell_p \hat{\otimes} \ell_q)\) is isomorphic to \([\ell_p \hat{\otimes} \ell_q]^m\) so it does not contain a subspace isomorphic to \(c_0\). Due to this remark it is easy to construct by induction a normalized block basic sequence \((u'_n)_n\) of \((w_n)_n\) and a strictly increasing sequence \((m_n)_n\) of integers such that \(\|R_{m_1}(u'_1) - u'_1\| \leq \varepsilon/2\) and, for every integer \(n \geq 2\), \(\|R_{m_{n-1}}(u'_n)\| \leq \varepsilon/2^{n+1}\) and \(\|R_{m_n}(u'_n) - u'_n\| \leq \varepsilon/2^{n+1}\). For every integer \(n\) we have

\[
\|u'_n - (R_{m_n} - R_{m_{n-1}})(u'_n)\| \leq \frac{\varepsilon}{2^n}
\]

so, for \(\varepsilon > 0\) small enough, the sequence \(((R_{m_n} - R_{m_{n-1}})(u'_n))_n\) is seminormalized and equivalent to the unit vector basis of \(c_0\). For every integer \(n\) we take

\[
v_n = \frac{(R_{m_n} - R_{m_{n-1}})(u'_n)}{\|R_{m_n} - R_{m_{n-1}})(u'_n)\|}.
\]

The sequence \((v_n)_n\) is a normalized basic sequence of \(F_r(\ell_p \hat{\otimes} \ell_q)\) equivalent to the unit vector basis of \(c_0\) which satisfies condition (4.2).

In the second stage we show that there exists a normalized basic sequence \((w_n)_n\) of \(F_r(\ell_p \hat{\otimes} \ell_q)\) equivalent to the unit vector basis of \(c_0\) and two strictly increasing sequences of integers \((p_n)_n\) and \((r_n)_n\) such that \(p_0 = r_0 = 0\) and, for every integer \(n \geq 1\),

\[
(4.3) \quad w_n = (R_{r_n} - R_{r_{n-1}})(w_n) = (\tilde{P}_{p_n} - \tilde{P}_{p_{n-1}})(w_n).
\]

To do this, let \(\varepsilon_1 > 0\). By Lemma 3.2 for every integer \(p \geq 1\), the space \(\tilde{P}_p[F_r(\ell_p \hat{\otimes} \ell_q)]\) is isomorphic to \(\ell_q \hat{\otimes} \ell_r\) with \(q < r\). So, by Lemma 3.5 it does not contain a subspace isomorphic to \(c_0\). It is then easy to construct by induction a normalized block basic sequence \((u'_n)_n\) of \((w_n)_n\) and a strictly increasing sequence \((p_n)_n\) of integers such that \(\|v'_1 - \tilde{P}_{p_1}(v'_1)\| \leq \varepsilon_1/2\) and, for every integer \(n \geq 2\),

\[
\|\tilde{P}_{p_{n-1}}(v'_n)\| \leq \frac{\varepsilon_1}{2^{n+1}} \quad \text{and} \quad \|\tilde{P}_{p_n}(v'_n) - v'_n\| \leq \frac{\varepsilon_1}{2^{n+1}}.
\]

For \(\varepsilon_1 > 0\) small enough, the sequence \(((\tilde{P}_{p_n} - \tilde{P}_{p_{n-1}})(v'_n))_n\) is a seminormalized sequence equivalent to the unit vector basis of \(c_0\). For every integer \(n \geq 1\) we take

\[
w_n = \frac{(\tilde{P}_{p_n} - \tilde{P}_{p_{n-1}})(v'_n)}{\|\tilde{P}_{p_n} - \tilde{P}_{p_{n-1}})(v'_n)\|}.
\]

It follows from condition (4.2) that there exists a strictly increasing sequence \((r_n)_n\) of integers with \(r_0 = 0\) such that \(w_n = (R_{r_n} - R_{r_{n-1}})(w_n)\) for every
integer $n$. The sequence $(w_n)_n$ is a normalized basic sequence equivalent to
the unit vector basis of $c_0$ which satisfies condition (4.3). In the third stage we show that there exists a normalized basic sequence
$(x_n)_n$ equivalent to the unit vector basis of $c_0$ and three strictly increasing
sequences of integers $(i_n)_n$, $(j_n)_n$ and $(k_n)_n$ such that $i_0 = j_0 = k_0$ which satisfy condition (4.1).
To do this, we begin with the sequence $(w_n)_n$ satisfying condition (4.3) and we use the same method as in the second stage.
Now we show that the existence of a normalized basic sequence $(x_n)_n$ of $F_r(\ell_p \widehat{\otimes} \ell_q)$ equivalent to the unit vector basis of $c_0$ satisfying condition (4.1) leads to a contradiction.
For every integer $n$ we have $x_n = (R_{k_n} - R_{k_{n-1}})(x_n)$ so there exists a
sequence $(u_l)_l$ in $\ell_p \widehat{\otimes} \ell_q$ such that $x_1 = (u_1, \ldots, u_{k_1}, 0, 0, \ldots)$ and, for every integer $n \geq 2$, $x_n = (0, \ldots, 0, u_{k_{n-1}+1}, \ldots, u_{k_n}, 0, 0, \ldots)$.
Let us recall that $[\ell_p \widehat{\otimes} \ell_q]^*$ is isometrically isomorphic to the space $B(\ell_p, \ell_q)$ (4, 5). So, for every integer $n$, there exists $b_n \in B(\ell_p, \ell_q)$ such that $\|b_n\| = 1$ and
$$1 = \left[ \sum_{l=k_{n-1}+1}^{k_n} |b_n(u_l)|^r \right]^{1/r}.$$ It follows from condition (4.1) that for each integer $l \in \{k_{n-1} + 1, \ldots, k_n\}$ we have
$$b_n(u_l) = b_n((P_{i_n} - P_{i_{n-1}}) \otimes (Q_{j_n} - Q_{j_{n-1}}))(u_l)).$$ Condition (4.4) implies that we may suppose that $T_{b_n} = (Q_{j_n}^* - Q_{j_{n-1}}^*) \circ T_{b_n} \circ (P_{i_n} - P_{i_{n-1}})$, so $(T_{b_n})_n$ is a sequence of block operators. This last assumption implies that for $n \neq m$ and $l \in \{k_{m-1} + 1, \ldots, k_m \}$, we have $b_n(u_l) = 0$.
Let $N$ be an integer, $\alpha_1, \ldots, \alpha_N$ be scalars and let $b = \alpha_1 b_1 + \cdots + \alpha_N b_N$. We have
$$\left[ \sum_{n=1}^{k_N} |b(u_n)|^r \right]^{1/r} = [\|\alpha_1|^r + \cdots + |\alpha_N|^r]^{1/r},$$ so
$$\|x_1 + \cdots + x_N\| \geq A(N) = \sup \{[\|\alpha_1|^r + \cdots + |\alpha_N|^r]^{1/r} : \|\alpha_1 b_1 + \cdots + \alpha_N b_N\| \leq 1\}.$$ Now we compute $A(N)$.
In the case $p \leq q'$ we have, by Lemma 3.3
$$\|\alpha_1 b_1 + \cdots + \alpha_N b_N\| = \|\alpha_1 T_{b_1} + \cdots + \alpha_N T_{b_N}\| = \max_{1 \leq n \leq N} |\alpha_n|,$$ so $A(N) = N^{1/r}$. 
In the case $p > q'$, let $\sigma = pq'/(p - q')$. We also have, by Lemma 3.3

$$
\|\alpha_1 b_1 + \cdots + \alpha_N b_N\| = \|\alpha_1 T b_1 + \cdots + \alpha_N T b_N\| = \left[\sum_{n=1}^{N} |\alpha_n|^\sigma\right]^{1/\sigma}.
$$

We have

$$
\frac{1}{r} - \frac{1}{\sigma} = \frac{1}{r} - \frac{1}{q'} + \frac{1}{p}
$$

and $r < q'$, so $\sigma > r$. Therefore, $A(N) = N^{\sigma r/(\sigma - r)}$.

In both cases, $\lim_{N \to \infty} \|x_1 + \cdots + x_N\| = \infty$ so $(x_n)_n$ is not equivalent to the unit vector basis of $c_0$, in contradiction with our construction.

**Theorem 4.2.** Let $1 \leq p, q, s$ be real numbers such that $1 \leq p < s$ and $1 \leq q < s$. Then every operator from $\ell_s$ into $\ell_p \hat{\otimes} \ell_q$ is compact. The same is true for every operator from $c_0$ into $\ell_p \hat{\otimes} \ell_q$.

**Proof.** The conclusions follow directly from Corollary 14 of [11].

**5. Operators from $\ell_s$ into $\ell_p \hat{\otimes} \ell_q$.** We recall that if $E^*$ or $F^*$ has the Radon–Nikodym property and one of $E^*$ or $F^*$ has the approximation property then, for every $b \in J(E, F)$, we have $T b \in \mathcal{N}(E, F^*)$ and the operator $b \mapsto T b$ is an isometric isomorphism from $J(E, F)$ onto $\mathcal{N}(E, F^*)$ [5].

**Theorem 5.1.** Let $1 \leq p, q, r < \infty$. The space $\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r$ contains a subspace isomorphic to $c_0$ if, and only if, $1/p + 1/q + 1/r \leq 2$.

**Proof.** Suppose there is no subspace isomorphic to $c_0$ in $\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r$. Therefore there is no subspace isomorphic to $c_0$ in $\ell_p \hat{\otimes} \ell_q$, hence we have $1/p + 1/q > 1$. The space $\ell_p \hat{\otimes} \ell_q$ contains a subspace isomorphic to $\ell_\sigma$ with $1/\sigma = 1/p + 1/q - 1$. The space $\ell_\sigma \hat{\otimes} \ell_r$ does not contain a subspace isomorphic to $c_0$ so we have

$$
\frac{1}{\sigma} + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 > 1.
$$

Conversely we suppose that $1/p + 1/q + 1/r > 2$ and that $\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r$ contains a subspace isomorphic to $c_0$. We consider $\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r$ as the space $F_r(\ell_p \hat{\otimes} \ell_q)$. We observe that none of the spaces $\ell_p \hat{\otimes} \ell_q$, $\ell_p \hat{\otimes} \ell_r$ or $\ell_q \hat{\otimes} \ell_r$ contain a subspace isomorphic to $c_0$. Proceeding as in the proof of Theorem 4.1 we can find a normalized basic sequence $(x_n)_n$ of $\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r$ equivalent to the unit basis of $c_0$ and three strictly increasing sequences of integers $(i_n)_n, (j_n)_n$ and $(k_n)_n$ such that $i_0 = j_0 = k_0$ and satisfying, for $n = 1, 2, \ldots$, condition (4.1).

Now we show that the existence of these sequences leads to a contradiction. We proceed as in the $\ell_p \hat{\otimes} \ell_q$ case. For every integer $n$ we have
\[ x_n = (R_{k_n} - R_{k_{n-1}})(x_n) \]

so there exists a sequence \((u_l)\) in \(\ell_p \hat{\otimes} \ell_q\) such that

\[ x_n = (0, \ldots, 0, u_{k_{n-1}+1}, \ldots, u_{k_n}, 0, 0, \ldots). \]

For every integer \(n\), there exists \(b_n \in [\ell_p \hat{\otimes} \ell_q]^* = \mathcal{J}(\ell_p, \ell_q)\) such that \(\|b_n\|_{\text{int}} = 1\) and

\[ \|x_n\| = \left[ \sum_{l=k_{n-1}+1}^{k_n} |b_n(u_l)|^r \right]^{1/r} = 1. \]

For \(n = 1, 2, \ldots\) and \(k_{n-1} + 1 \leq l \leq k_n\) we have

\[ b_n(u_l) = b_n((P_{l_n} - P_{l_{n-1}}) \otimes (Q_{l_n} - Q_{l_{n-1}}))(u_l). \]

It follows from condition (5.1) that we may suppose \(T_{b_n} = (Q_{l_n} - Q_{l_{n-1}}) \circ T_{b_n} \circ (P_{l_n} - P_{l_{n-1}})\). This last assumption implies that for \(n \neq m\) and \(l \in \{k_{m-1} + 1, \ldots, k_m\}\), we have \(b_n(u_l) = 0\).

Let \(N\) be an integer, \(\alpha_1, \ldots, \alpha_N\) be scalars and let \(b = \alpha_1 b_1 + \cdots + \alpha_N b_N\). We have \([\sum_{l=1}^{k_n} |b(u_l)|^r]^{1/r} = [\sum_{n=1}^{N} |\alpha_n|^r]^{1/r}\), so

\[ \|x_1 + \cdots + x_N\| \geq \Theta(N) = \sup \left\{ \left[ \sum_{n=1}^{N} |\alpha_n|^r \right]^{1/r} ; \left[ \sum_{n=1}^{N} |\alpha_n b_n| \right]_{\text{int}} \leq 1 \right\}. \]

The integral forms \(b_1, \ldots, b_N\) may be considered as integral forms on \(\ell_p^{i_N} \times \ell_q^{j_N}\). In this case, \(\mathcal{J}(\ell_p^{i_N}, \ell_q^{j_N}) = \mathcal{N}(\ell_p^{i_N}, \ell_q^{j_N})\), so \((T_{b_n})_{1 \leq n \leq N}\) is a finite sequence of nuclear block operators from \(\ell_p^{i_N}\) to \(\ell_q^{j_N}\).

The assumption \(1/p + 1/q + 1/r > 2\) implies \(1/p + 1/q > 1\), hence \(q' > p\).

In the cases \(q' < \infty\) or \(q' = \infty\) and \(1 < p\) we let

\[ \sigma = \begin{cases} \frac{pq'}{pq' + p - q'} & \text{if } q' < \infty, \\ \frac{p}{p - 1} & \text{if } 1 < p < q' = \infty. \end{cases} \]

We observe that always \(\sigma > r\). By Lemma 3.4 we have

\[ \left\| \sum_{n=1}^{N} \alpha_n b_n \right\|_{\text{int}} = \left\| \sum_{n=1}^{N} \alpha_n T_{b_n} \right\|_{\text{nuc}} = \left[ \sum_{n=1}^{N} |\alpha_n|^\sigma \right]^{1/\sigma}, \]

and by Lemma 3.3 we have

\[ \Theta(N) = \sup \left\{ \left[ \sum_{n=1}^{N} |\alpha_n|^r \right]^{1/r} ; \left[ \sum_{n=1}^{N} |\alpha_n|^\sigma \right]^{1/\sigma} \leq 1 \right\} = N^{(\sigma - r)/\sigma r}. \]

We deduce that \(\|x_1 + \cdots + x_N\| \geq N^{(\sigma - r)/\sigma r}\), in contradiction with \((x_n)_n\) being equivalent to the unit vector basis of \(c_0\).

In the case \(p = 1\) and \(q' = \infty\), we have

\[ \Theta(N) = \left\| \sum_{n=1}^{N} \alpha_n b_n \right\|_{\text{int}} = \left\| \sum_{n=1}^{N} \alpha_n T_{b_n} \right\|_{\text{nuc}} = \max_{1 \leq n \leq N} |\alpha_n|. \]
In this case, by Lemma 3.3, \( \|x_1 + \cdots + x_N\| \geq N^{1/r} \) so the sequence \((x_n)_n\) is not equivalent to the unit vector basis of \(c_0\).

The assumptions that \(1/p + 1/q + 1/r > 2\) and that \(\ell_p \hat{\otimes} \ell_q \hat{\otimes} \ell_r\) contains a subspace isomorphic to \(c_0\) lead to a contradiction. The theorem is proved.

Corollary 14 of [11] implies:

**Theorem 5.2.** Let \(1 \leq p, q, s\) be real numbers such that \(1/p + 1/q > 1 + 1/s\). Then every operator from \(\ell_s\) into \(\ell_p \hat{\otimes} \ell_q\) is compact. The same is true for every operator from \(c_0\) into \(\ell_p \hat{\otimes} \ell_q\).

**REFERENCES**


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