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ON FINITELY GENERATED n-SG-PROJECTIVE MODULES

BY

DRISS BENNIS (Fez)

Abstract. We prove that finitely generated n-SG-projective modules are infinitely presented.

1. Introduction. Throughout this paper, R denotes a non-trivial associative ring with identity, and all modules are left R-modules, if not specified otherwise.

In 1967–69, Auslander and Bridger [1, 2] introduced the so called G-dimension for finitely generated modules over Noetherian rings. They proved that the G-dimension of a finitely generated module M is less than or equal to its projective dimension; and they coincide when the projective dimension of M is finite. Several decades later, Enochs and Jenda [13, 14] extended the ideas of Auslander and Bridger, and introduced the Gorenstein projective dimension, which is defined in terms of resolutions by Gorenstein projective modules: a module M is called *Gorenstein projective* (*G-projective* for short) if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and $\mathrm{Hom}(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module. This homological dimension has been extensively studied by many authors (see [10, 11, 12, 15, 18]), who proved that the Gorenstein projective dimension shares many nice properties of the classical projective dimension. Now, a guiding principle in the study of Gorenstein homological dimension has been formulated in the following meta-theorem [17, p. V]: "Every result in classical homological algebra has a counterpart in Gorenstein homological algebra."

It is well known that every finitely generated projective module M is infinitely presented; that is, M admits a free resolution

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

such that the free modules F_i are finitely generated. In this paper, we are

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concerned with the Gorenstein counterpart of this result. Namely, we investigate the following open question:

Is every finitely generated G-projective module infinitely presented?

In [19], rings which satisfy this property are called G_1-rings. In [5, Proposition 2.12], an affirmative answer is given for SG-projective modules which are particular cases of G-projective modules: a module M is called *strongly Gorenstein projective* (*SG-projective* for short) if there exists an exact sequence of projective modules of the form

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}(f)$ and $\mathrm{Hom}(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module. In [6], an extension of the notion of SG-projective module was introduced as follows: for an integer $n > 0$, a module M is called *n-strongly Gorenstein projective* (*n-SG-projective* for short) if there exists an exact sequence of modules

$$0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where each P_i is projective, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective module (equivalently, $\mathrm{Ext}^i(M, Q) = 0$ for $j + 1 \le i \le j + n$ for some positive integer j and for any projective module Q [6, Theorem 2.8]). Then 1-SG-projective modules are just SG-projective modules. In [6, Proposition 2.2], it is proved that an n-SG-projective module is projective if and only if it has finite flat dimension.

The aim of this paper is to give an affirmative answer to the question above for n-SG-projective modules (Theorem 2.6). As consequences, an extension of the characterization of finitely generated 1-SG-projective modules [5, Proposition 2.12] to finitely generated n-SG-projective modules is given (Corollary 2.7), and another relation between n-SG-projective and n-SG-flat modules is established in Corollary 2.8.

2. Main result. To show that finitely generated n-SG-projective modules are infinitely presented, we need some preparatory results.

LEMMA 2.1. *If M is an n-SG-projective module for $n \ge 2$, then there exists an exact sequence of modules*

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where each F_i is free.

Proof. Since M is n-SG-projective, there is, from the definition, an exact sequence of modules

$$(*) \qquad 0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where all P_i are projective. By Eilenberg's swindle [21, Exercise 3.13, p. 64], there exists, for each i, a free module Q_i such that $P_i \oplus Q_i = L_i$ is free. Therefore, adding to (∗) the sequences $0 \to Q_i \xrightarrow{=} Q_i \to 0$ in degrees i and $i+1$ for $i = 1, \ldots, n-2$, and the sequence $0 \to Q_{n-1} \oplus Q_n \xrightarrow{=} Q_{n-1} \oplus Q_n \to 0$ to (∗) in degrees $n-1$ and n, we get the desired exact sequence

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where $F_n = P_n \oplus Q_n \oplus Q_{n-1}$, $F_{n-1} = P_{n-1} \oplus Q_{n-1} \oplus Q_{n-2} \oplus Q_n$, $F_1 = P_1 \oplus Q_1$, and $F_i = P_i \oplus Q_i \oplus Q_{i-1}$ for $i = 2, \ldots, n-2$. ∎

Remark 2.2. If M is 1-SG-projective, then, from [6, Proposition 2.5], it is n-SG-projective for every $n \geq 2$, and thus it admits an exact sequence of modules $0 \to M \to F_n \to \cdots \to F_1 \to M \to 0$, where each F_i is free.

Lemma 2.3. *Consider a commutative diagram of modules with exact rows and an exact left column:*

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_2 & \to & B_2 & \to & C_2 & \to & 0 \end{array}$$

If A_1, B_1, B_2, C_2, and $A_3 = \operatorname{Coker}(A_1 \to A_2)$ are G-projective, then so is C_1.

Proof. Applying, for a projective module Q, the functor $\operatorname{Hom}_R(-, Q)$ to the diagram, we get, by hypotheses and [18, Proposition 2.3], the following commutative diagram with exact rows and an exact right column:

$$\begin{array}{ccccccc} \operatorname{Hom}_R(B_2,Q) & \to & \operatorname{Hom}_R(A_2,Q) & \to & \operatorname{Ext}_R(C_2,Q)=0 & & \\ \downarrow & & \downarrow & & & & \\ \operatorname{Hom}_R(B_1,Q) & \to & \operatorname{Hom}_R(A_1,Q) & \to & \operatorname{Ext}_R(C_1,Q) & \to & \operatorname{Ext}_R(B_1,Q)=0 \\ & & \downarrow & & & & \\ & & \operatorname{Ext}_R(A_3,Q)=0 & & & & \end{array}$$

Then the homomorphism $\operatorname{Hom}_R(B_1, Q) \to \operatorname{Hom}_R(A_1, Q)$ is surjective, which implies that $\operatorname{Ext}_R(C_1, Q) = 0$. Therefore, [18, Corollary 2.11] applied to the short exact sequence $0 \to A_1 \to B_1 \to C_1 \to 0$ shows that C_1 is G-projective. ∎

Corollary 2.4. *Consider a short exact sequence of G-projective modules $0 \to A \to B \to C \to 0$. Then, for every G-projective submodule A' of A and every G-projective submodule B' of B which contains A', the module $C' = B'/A'$ is also G-projective.*

Proof. By hypotheses, there exists a commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

Therefore, from Lemma 2.3, the module $C' = B'/A'$ is G-projective. ∎

For any positive integer n, a module M is said to be *n-presented* whenever there is an exact sequence of modules

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where each F_i is finitely generated and free. In particular, 0-presented and 1-presented modules are finitely generated and finitely presented modules respectively. For a finitely generated module M, we denote

$$\lambda(M) = \sup\{n : M \text{ is an } n\text{-presented module}\}.$$

Clearly, $\lambda(M) = 0$ if and only if M is finitely generated, and $\lambda(M) = 1$ if and only if M is finitely presented. If $\lambda(M) = \infty$, equivalently if M is n-presented for every positive integer n, we say that M is *infinitely presented*; then M admits a free resolution of modules

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

such that the free modules F_i are finitely generated. For example, every finitely generated projective module is infinitely presented [9, Exercise 7(a), p. 180].

Lemma 2.5 ([9, Exercise 6(c) and (d), p. 180]). *For every short exact sequence of modules $0 \to A \to B \to C \to 0$, we have*

$$\lambda(C) \geq \inf\{\lambda(B), \lambda(A) + 1\} \quad \textit{and} \quad \lambda(A) \geq \inf\{\lambda(B), \lambda(C) - 1\}.$$

In particular, if B is finitely generated and projective, then $\lambda(C)=\lambda(A)+1$.

Now, we are ready to prove our main result.

Main Theorem 2.6. *Let $n \geq 1$ be an integer. If M is a finitely generated n-SG-projective module, then it is infinitely presented.*

Proof. The case $n = 1$ is proved in [5, Proposition 2.12], so we assume that $n \geq 2$. It is sufficient to construct a family of short exact sequences of finitely generated modules

$$\begin{array}{lclllllllll} \alpha_n & : & 0 & \to & M & \to & L_n & \to & H_n & \to & 0 \\ \alpha_{i-1} & : & 0 & \to & H_i & \to & L_{i-1} & \to & H_{i-1} & \to & 0 \quad \text{for } i = n, \dots, 3, \\ \alpha_1 & : & 0 & \to & H_2 & \to & P_1 & \to & M & \to & 0 \end{array}$$

where each L_i is free and P_1 is projective. Indeed, applying successively Lemma 2.5 to these sequences α_i, we get

$$\lambda(M) = \lambda(H_2) + 1 = \lambda(H_3) + 2 = \cdots = \lambda(M) + n.$$

Therefore, $\lambda(M) = \infty$.

Thus, it remains to prove the existence of the short exact sequences α_i. Since M is n-SG-projective, there exists, by Lemma 2.1, an exact sequence

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where each F_i is free. Then we get a family of short exact sequences

$$\begin{array}{lclllllllll} \beta_n & : & 0 & \to & M & \to & F_n & \to & G_n & \to & 0 \\ \beta_{i-1} & : & 0 & \to & G_i & \to & F_{i-1} & \to & G_{i-1} & \to & 0 \quad \text{for } i = n, \dots, 3, \\ \beta_1 & : & 0 & \to & G_2 & \to & F_1 & \to & M & \to & 0 \end{array}$$

Since M is finitely generated and it embeds in the free module F_n, it embeds in a finitely generated free submodule L_n of F_n such that $F_n = L_n \oplus E_n$ where E_n is also a free module. Thus, we obtain the short exact sequence α_n which is the bottom exact sequence of the following commutative diagram with exact columns and rows:

$$(\mathbf{\Gamma}_n) \qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & E_n & \to & K_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & F_n & \to & G_n & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & L_n & \to & H_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $F_n = L_n \oplus E_n \to L_n$ is the canonical surjection, and the homomorphisms $G_n \to H_n$ and $E_n \to K_n$ follow from [21, Exercise 2.7, p. 27].

Now, we construct the short exact sequence α_{n-1}. Using the right vertical sequence in the diagram $(\mathbf{\Gamma}_n)$ and the short exact sequence β_{n-1}, we get the following pullback diagram:

$$(\mathbf{\Omega}_n)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K_n & = & K_n & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & G_n & \to & F_{n-1} & \to & G_{n-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & H_n & \to & P_{n-1} & \to & G_{n-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

From the diagram $(\mathbf{\Gamma}_n)$, $K_n \cong E_n$ is free; and from Corollary 2.4, H_n is G-projective. Then, by the bottom and middle exact sequences of the diagram $(\mathbf{\Omega}_n)$ and [18, Theorem 2.5], the module P_{n-1} is G-projective with finite projective dimension. Then P_{n-1} is projective by [18, Proposition 2.27]. Hence there exists, from Eilenberg's swindle [21, Exercise 3.13, p. 64], a free module Q_{n-1} such that $P_{n-1} \oplus Q_{n-1} = O_{n-1}$ is free. Thus, adding the short exact sequence

$$0 \to 0 \to Q_{n-1} \stackrel{=}{\longrightarrow} Q_{n-1} \to 0$$

to the bottom exact sequence of the diagram $(\mathbf{\Omega}_n)$, we get the exact sequence

$$0 \to H_n \to O_{n-1} \to N_{n-1} \to 0,$$

where $N_{n-1} = G_{n-1} \oplus Q_{n-1}$. Since H_n embeds in the free module O_{n-1}, and since H_n is finitely generated (by the bottom exact sequence of the diagram $(\mathbf{\Gamma}_n)$), it embeds in a finitely generated free submodule L_{n-1} of O_{n-1} such that $O_{n-1} = L_{n-1} \oplus E_{n-1}$ where E_{n-1} is also a free module. Then, similarly to the diagram $(\mathbf{\Gamma}_n)$, we get a diagram $(\mathbf{\Gamma}_{n-1})$ in which the bottom exact sequence is the desired short exact sequence α_{n-1}:

$$(\mathbf{\Gamma}_{n-1})\qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & E_{n-1} & \to & K_{n-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_n & \to & O_{n-1} & \to & N_{n-1} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & H_n & \to & L_{n-1} & \to & H_{n-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The short exact sequence α_{n-2} is obtained as follows. Adding the short exact sequence

$$0 \to Q_{n-1} \xrightarrow{=} Q_{n-1} \to 0 \to 0$$

to the short exact sequence β_{n-2}, we get the exact sequence

$$0 \to N_{n-1} \to F'_{n-2} \to G_{n-2} \to 0,$$

where $F'_{n-2} = F_{n-2} \oplus Q_{n-1}$. Using this short exact sequence and the right vertical sequence in the diagram $(\mathbf{\Gamma}_{n-1})$, we get the pullback diagram

$$(\mathbf{\Omega}_{n-1}) \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K_{n-1} & = & K_{n-1} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N_{n-1} & \to & F'_{n-2} & \to & G_{n-2} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & H_{n-1} & \to & P_{n-2} & \to & G_{n-2} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Then, similarly to $(\mathbf{\Gamma}_{n-1})$, we get a diagram $(\mathbf{\Gamma}_{n-2})$ in which the bottom exact sequence is the desired short exact sequence α_{n-2}:

$$(\mathbf{\Gamma}_{n-2}) \qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & E_{n-2} & \to & K_{n-2} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_{n-1} & \to & O_{n-2} & \to & N_{n-2} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & H_{n-1} & \to & L_{n-2} & \to & H_{n-2} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

So, similarly to the previous arguments, the short exact sequences α_{i-1} for $i = n, \ldots, 3$ are constructed recursively. In the nth step, we obtain the pullback diagram $(\mathbf{\Omega}_2)$:

$$(\mathbf{\Omega}_2)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K_2 & = & K_2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N_2 & \to & F_1' & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & H_2 & \to & P_1 & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since H_2 and M are finitely generated modules, the projective module P_1 is also finitely generated. Therefore, the bottom sequence in the diagram $(\mathbf{\Omega}_2)$ is the last desired short exact sequence α_1. ∎

Theorem 2.6 allows us to extend the characterization of finitely generated 1-SG-projective modules [5, Proposition 2.12] to finitely generated n-SG-projective modules.

COROLLARY 2.7. *For an integer $n \geq 1$ and a finitely generated module M, the following are equivalent:*

(1) *M is n-SG-projective,*

(2) *There exists an exact sequence of finitely generated modules*

$$0 \to M \to F_n \to \cdots \to F_2 \to P_1 \to M \to 0,$$

where each F_i is free and P_1 is projective, such that $\mathrm{Ext}^i(M, R) = 0$ *for every $i > 0$,*

(3) *There exists an exact sequence of finitely generated modules*

$$0 \to M \to F_n \to \cdots \to F_2 \to P_1 \to M \to 0,$$

where each F_i is free and P_1 is projective, such that $\mathrm{Ext}^i(M, F) = 0$ *for every $i > 0$ and every flat R-module F,*

(4) *There exists an exact sequence of finitely generated modules*

$$0 \to M \to F_n \to \cdots \to F_2 \to P_1 \to M \to 0,$$

where each F_i is free and P_1 is projective, such that $\mathrm{Ext}^i(M, F') = 0$ *for every $i > 0$ and every R-module F' with finite flat dimension.*

Proof. Use Theorem 2.6 and [4, proof of Lemma 3.4] (see also [4, proof of Corollary 3.5]). ∎

To complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [16] introduced the Gorenstein flat modules as follows: a module M is called *Gorenstein flat* (*G-flat* for short) if there exists an exact sequence of flat modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and $I \otimes_R -$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective right module. The G-flat modules were investigated by Holm [18] over coherent rings, and, recently, in a more general context in [3]. The relationships between G-projective and G-flat modules were investigated in many works (see, for instance, [5, Proposition 1.3 and 3.9], [4, Theorem 3.3], [18, Proposition 3.4], and [12, Corollary 4.2]). The following establishes another relation between n-SG-projective and n-SG-flat modules: for an integer $n > 0$, a module M is called *n-strongly Gorenstein flat* (*n-SG-flat* for short) if there exists an exact sequence of modules

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where each F_i is flat, such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective right module (see the note at the end of [6]).

COROLLARY 2.8. *Every finitely generated n-SG-projective module is n-SG-flat.*

Proof. Use Theorem 2.6 and [4, Theorem 3.3]. ∎

The problems we investigate in the paper are related to some problems on periodic resolutions of flat modules studied by Benson and Goodearl [7], Simson [20], and recently generalized by Bouchiba and Khaloui [8].

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Driss Bennis
Department of Mathematics
Faculty of Science and Technology of Fez
Box 2202, University S. M. Ben Abdellah
Fez, Morocco
E-mail: driss_bennis@hotmail.com

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