UNIVERSAL HARMONIC FUNCTIONS
ON THE HYPERBOLIC SPACE

BY

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Abstract. We prove universal overconvergence phenomena for harmonic functions on the real hyperbolic space.

1. Introduction and statement of the results. Let \( B_n \) be the ball model of the \( n \)-dimensional hyperbolic space, i.e. the unit ball of \( \mathbb{R}^n \) equipped with the metric \( ds^2 = |dx|^2/(1-|x|^2)^2 \). Throughout this paper we consider the case \( n \geq 3 \). The hyperbolic Laplacian \( D \) on \( B_n \), acting on smooth functions, is given by

\[
D = (1-r^2)^2 \Delta + 2(n-2)(1-r^2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i},
\]

where \( r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2} \) and \( \Delta \) is the Euclidean Laplacian. As usual, a function \( u \) on an open set \( \Omega \) of \( B_n \) is called \( H \)-harmonic if \( Du = 0 \) on \( \Omega \).

Let us set \( F_k(x) = 2F_1(k, 1-n/2, k+n/2; x) \) where \( 2F_1 \) is the Gaussian hypergeometric function, and let us denote by \( C_k^\rho \) the Gegenbauer polynomial \([BM]\). Then, by the main result of \([BM]\), the hyperbolic Poisson kernel \( P_{h,r_0} \) on the ball \( B(0,r_0) \), where \( r_0 < 1 \), is given by

\[
P_{h,r_0}(x,\xi) = \frac{\Gamma(n/2)}{2\pi^{n/2}r_0^{n-1}} \sum_{k=0}^{\infty} \frac{\rho + k}{\rho} \frac{|x|^k}{r_0^k} \frac{F_k(|x|^2)}{F_k(r_0^2)} C_k^\rho \left( \frac{\langle x,\xi \rangle}{|x|} \right),
\]

where \( \rho = (n-2)/2, |\xi| = 1, |x| < r_0 \) and \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product. This kernel solves the Dirichlet problem on the ball \( B(0, r_0) \) with boundary data \( \varphi \in C(\mathbb{S}^{n-1}(0, r_0)) \). In fact, if \( d\sigma_{r_0} \) is the canonical measure on \( \mathbb{S}^{n-1}(0, r_0) \), we set

\[
P_{h,r_0}[\varphi](r\zeta) = \int_{\mathbb{S}^{n-1}(0, r_0)} P_{h, r_0}(r\zeta, \xi) \varphi(\xi) d\sigma_{r_0}(\xi).
\]

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Then \( \mathbb{P}_{h, r_0} \) is \( \mathcal{H} \)-harmonic on \( B(0, r_0) \) and has \( \varphi \) as boundary value (see Proposition [1.1]). Let us note that \( B(0, r_0) = \{ x \in \mathbb{R}^n : \| x \| < r_0 \} \), where \( \| \cdot \| \) is the Euclidean distance, and that \( B(0, r_0) \) is a hyperbolic ball with center at 0 and radius \( \frac{1}{2} \log \frac{1+r_0}{1-r_0} \).

Let us write \( \varphi(r_0 \xi) = \varphi_{r_0}(\xi) \) and let us recall that if \( \varphi_{r_0} \) belongs to \( L^2(S^{n-1}) \), then it admits an expansion in spherical harmonics ([StW, pp. 141–145]):

\[
\varphi_{r_0}(\xi) = \sum_{k=0}^{\infty} \varphi_{r_0,k}(\xi), \quad \xi \in S^{n-1},
\]

with

\[
\varphi_{r_0,k}(\xi) = \int_{S^{n-1}} Z_k^\xi(\eta) \varphi_{r_0}(\eta) \, d\sigma(\eta),
\]

where \( Z_k^\xi \) is the zonal polynomial of degree \( k \) and pole at \( \xi \) (see Section 2).

Using the expansion (1.2) of \( \varphi_{r_0} \) and the expression (1.1) of the Poisson kernel we prove the following proposition.

**Proposition 1.1.** The Dirichlet problem \( Du = 0 \) on \( B(0, r_0) \) with boundary data \( \varphi \in C(S^{n-1}(0, r_0)) \) has a unique solution given by

\[
(1.4) \quad u(r \xi) = \sum_{k=0}^{\infty} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0,k}(\xi), \quad r < r_0, \ \xi \in S^{n-1},
\]

where \( \varphi_{r_0,k} \) is given by (1.3).

Let us denote by \( \mathcal{H}_{r_0} \) the space of \( \mathcal{H} \)-harmonic functions on the ball \( B(0, r_0) \). Let \( u \in \mathcal{H}_{r_0} \) and let us assume that \( u|_{S^{n-1}(0, r_0)} = \varphi \in C(S^{n-1}(0, r_0)) \). Then by Proposition [1.1] \( u \) admits the expansion (1.4). We consider the harmonic partial sums

\[
(1.5) \quad S_N^*(u)(r \xi) = \sum_{k=0}^{N} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0,k}(\xi)
\]

and in particular for any sequence \( \{ \lambda_s \} \) of natural numbers we set

\[
(1.6) \quad S_{\lambda_s,N}^*(u)(r \xi) = \sum_{k=\lambda_1}^{\lambda_N} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0,k}(\xi).
\]

We say that a harmonic function \( u \in \mathcal{H}_{r_0} \) belongs to the class \( \mathcal{U}_{\mathcal{H}} \) of universal harmonic functions on the hyperbolic ball \( B(0, r_0) \) (see [BCNP]) if for every compact set \( K \) in \( \mathbb{B}_n \setminus B(0, r_0) \) with connected complement and for every harmonic polynomial \( P \), there exists a sequence \( \{ \lambda_s \} \) in \( \mathbb{N} \) such that

\[
\lim_{N \to \infty} \sup_{x \in K} |S_{\lambda_s,N}^*(u)(x) - P(x)| = 0.
\]
Our main result is the following theorem, which answers a question of Michel Marias.

**Theorem 1.2.** The class $\mathcal{U}_H$ is $G_\delta$-dense in $H_{r_0}$ and contains a dense vector subspace of $H_{r_0}$ except 0.

Let us say a few words about Theorem 1.2. First we recall that universal harmonic functions on the Euclidean ball have been investigated by D. H. Armitage [Ar], and in an abstract setting by F. Bayart et al. [BGNP]. It is worth mentioning that Theorem 1.2 is an analogue of the Euclidean harmonic approximation result proved in [Ar, BGNP]. Let us now present the main result of [Ar]. If $K$ is a compact set in $\mathbb{R}^n$ we denote by $\mathcal{H}(K)$ the space of functions that are harmonic on some neighborhood of $K$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ such that $0 \in \Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. In [Ar] it is shown that there exists a series $\sum H_N$, where $H_N$ is a homogeneous harmonic polynomial of degree $N$ on $\mathbb{R}^n$, such that:

(i) the series $\sum H_N$ converges on some ball of center 0 to a function that is continuous on $\Omega$ and harmonic on $\Omega$,

(ii) outside of $\Omega$, the partial sums $\sum H_N$ are dense in the space $\overline{\mathcal{HP}(\mathbb{R}^n)}$, the closure of the space of harmonic polynomials in $\mathbb{R}^n$.

But, if $K \subset \mathbb{R}^n \setminus \overline{\Omega}$ is compact with connected complement, then by Walsh’s theorem (see [G]), $\overline{\mathcal{HP}(\mathbb{R}^n)}$ is dense in $\mathcal{H}(K)$. So, the partial sums of $\sum H_N$ approach every $h \in \mathcal{H}(K)$. To prove Theorem 1.2 we use the above mentioned result of [Ar, BGNP] in the Euclidean setting; we prove in Proposition 4.2 the correspondence of $\mathcal{H}$-harmonic and Euclidean-harmonic functions on $B(0, r_0)$. This allows us to associate a universal $\mathcal{H}$-harmonic to a Euclidean-harmonic universal function. Note that in the present case of hyperbolic harmonic approximation, we can only approximate harmonic polynomials but not arbitrary harmonic functions defined in a compact set $K \subset \mathbb{B}_n \setminus B(0, r_0)$ with $K^c = \mathbb{B}_n \setminus K$ connected as is the case in the Euclidean setting. This fact is due to the absence of an analogue, in the hyperbolic setting, of the classical Walsh theorem [G].

2. Preliminaries. For the proofs we need to fix some notation. Let us recall, [J1, J2], that the hyperbolic Laplacian $D$ in polar coordinates is written as

\begin{equation}
D = D_r + D_\sigma
\end{equation}

with

\begin{equation*}
\begin{cases}
D_r = \frac{1 - r^2}{r^2} - [(1 - r^2)N^2 + (n - 2)(1 + r^2)N], \\
D_\sigma = \frac{(1 - r^2)^2}{r^2} \Delta_\sigma,
\end{cases}
\end{equation*}
where

\[ N = r \frac{d}{dr} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \]

and \( \Delta_\sigma \) is the Laplacian on the unit sphere \( S^{n-1} \). Notice that \( D_r \) and \( D_\sigma \) are the radial and the tangential parts of \( D \) respectively.

We set \( \rho = (n - 2)/2 \) and recall (see for instance [E, p. 175]) that the Gegenbauer polynomial \( C^\rho_k \) for \( k \in \mathbb{N} \) and \( \rho > 0 \) is defined as the coefficient of \( h^k \) in the Maclaurin expansion of \( (1 - 2zh + h^2)^{-\rho} \):

\[
(1 - 2zh + h^2)^{-\rho} = \sum_{k=0}^{\infty} C^\rho_k(z)h^k, \quad |z| \leq 1, |h| < 1.
\]

We denote by \( Z^k_\zeta, \zeta \in S^{n-1} \), the zonal polynomial of degree \( k \) with pole at \( \zeta \), which is given by (see [BS])

\[
Z^k_\zeta(\eta) = (C^\rho_k(1))^{-1} C^\rho_k(\langle \zeta, \eta \rangle), \quad \zeta, \eta \in S^{n-1},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product on \( S^{n-1} \).

These functions are eigenfunctions of the tangential part of both the Euclidean and the hyperbolic Laplacian ([T, p. 216]), that is,

\[
(2.2) \quad \Delta_\sigma(Z^k_\zeta) = -k(k + n - 2)Z^k_\zeta.
\]

Further, let us denote by \( \text{\textit{2F1}} \) the Gauss hypergeometric function

\[
\text{\textit{2F1}}(a, b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k,
\]

where \( |x| < 1, a, b, c \in \mathbb{C}, c \neq 0, -1, -2, \ldots \) and \( (a)_k = \Gamma(a+k)/\Gamma(a) \). Also, in case \( c < a < 0, a, c \in \mathbb{Z} \), we define

\[
\text{\textit{2F1}}(a, b, c; x) = \sum_{k=0}^{-a} \frac{(a)_k(b)_k}{(c)_k k!} x^k.
\]

To simplify our notation we set, for \( k = 0, 1, 2, \ldots, \)

\[
(2.3) \quad F_k(x) = \text{\textit{2F1}}(k, -\rho, k + \frac{n}{2}; x).
\]

Finally, we denote by \( \mathcal{H}_{E,r_0} \) the space of Euclidean-harmonic functions on \( B(0, r_0) \).

### 3. Expansions of \( \mathcal{H} \)-harmonic functions.

In this section we give the proof of Proposition [1.1] which gives the expansion of \( \mathcal{H} \)-harmonic functions in the ball \( B(0, r_0) \).

Let \( u \in \mathcal{H}_{r_0} \) and set

\[
u^k_\zeta(r) = \int_{S^{n-1}} Z^k_\zeta(\eta)u(r\eta) \, d\sigma(\eta), \quad r < r_0, \zeta \in S^{n-1}.
\]
To start with, we give an analogue of Theorem 6, Section 3.2 in [J2]. This is the content of the following lemma.

**Lemma 3.1.** For every $u \in \mathcal{H}_{r_0} \cap L^2(B(0,r_0))$ and $k \in \mathbb{N}$, there exists a continuous function $G_k$ on $S^{n-1}$ such that

$$u^k_\zeta(r) = G_k(\zeta)r^k F_k(r^2), \quad \zeta \in S^{n-1}, \; r < r_0,$$

where $F_k$ is defined in (2.3).

**Proof.** Let $u$ be in $\mathcal{H}_{r_0} \cap L^2(B(0,r_0))$. Using the same arguments as in [J2, Section 3.2, Theorem 6] (see also [J1]), one can obtain the following expansion of $u$ in homogeneous harmonic polynomials:

$$u(r\zeta) = \sum_{k=0}^{\infty} u^k_\zeta(r), \quad r < r_0, \; \zeta \in S^{n-1}. \quad \text{(3.1)}$$

Since $Du = 0$, or equivalently $D_r u = -D_\sigma u$, we get

$$D_r u^k_\zeta(r) = \int_{S^{n-1}} Z^k_\zeta(\eta) D_r u(r\eta) \, d\sigma(\eta) = -\int_{S^{n-1}} Z^k_\zeta(\eta) D_\sigma u(r\eta) \, d\sigma(\eta)$$

$$= -\int_{S^{n-1}} D_\sigma Z^k_\zeta(\eta) u(r\eta) \, d\sigma(\eta). \quad \text{(3.2)}$$

From (2.2) we have

$$D_\sigma Z^k_\zeta(\eta) = -\frac{(1-r^2)^2}{r^2} k(k + n - 2) Z^k_\zeta(\eta).$$

So, by (3.2),

$$D_r u^k_\zeta(r) = \frac{(1-r^2)^2}{r^2} k(k + n - 2) u^k_\zeta(r). \quad \text{(3.3)}$$

Setting $g(r^2) = u^k_\zeta(r)$ and using the expression of $D$ in polar coordinates given in (2.1), we can write (3.3) as

$$(1-z) zg''(z) + \frac{1}{2}(nz - 4z + n)g'(z) = \frac{k(k + n - 2)}{4} \frac{1-z}{z} g(z). \quad \text{(3.4)}$$

Looking for a solution of (3.4) in the form $z^a f(z)$ with $a = k/2$, we find that $f$ satisfies the hypergeometric equation

$$(1-z)zf''(z) + \left(k + \frac{n}{2} - \left(k + \frac{n}{2} + 2\right)z\right) f'(z) - k\left(1 - \frac{n}{2}\right) f(z) = 0. \quad \text{(3.5)}$$

Next, if $a = -(k + n - 2)/2$, then $f$ satisfies the equation

$$(1-z)zf''(z) + \left(2 - k - \frac{n}{2} - \left(4 - k - \frac{3n}{2}\right)z\right) f'(z) - (2 - k - n)\left(1 - \frac{n}{2}\right) f(z) = 0.$$
From the equations above it follows that the independent solutions $g_1$ and $g_2$ of (3.4) are given in terms of the hypergeometric function $\, _2F_1$:

\[ g_1(x) = x^{k/2} \, _2F_1 \left( k, 1 - \frac{n}{2}, k + \frac{n}{2}; x \right), \]

\[ g_2(x) = x^{(2-k-n)/2} \, _2F_1 \left( -k - n + 2, 1 - \frac{n}{2}, -\frac{n}{2} + 2; x \right). \]

Since for $\zeta \in S^{n-1}$, $u^k_\zeta$ is regular at $r = 0$, it follows that

\[ u^k_\zeta(r) = G_k(\zeta) r^k \, _2F_1 \left( k, 1 - \frac{n}{2}, k + \frac{n}{2}; r^2 \right) = G_k(\zeta) r^k F_k(r^2). \]

Finally, since $\zeta \to u^k_\zeta(r)$ is continuous, it follows that one can choose $G_k(\zeta)$ to be a continuous function of $\zeta$. ■

**Proof of Proposition 1.1.** Let $\varphi \in C(S^{n-1}(0, r_0))$. Then $\varphi_{r_0} \in C(S^{n-1})$ and thus $\varphi_{r_0} \in L^2(S^{n-1})$, since $S^{n-1}$ is compact. Then $\varphi_{r_0}$ admits the following spherical harmonic expansion:

\[ \varphi_{r_0} = \sum_{k=0}^{\infty} \varphi_{r_0,k}, \quad \varphi_{r_0,k}(\zeta) = \int_{S^{n-1}} Z^k_\zeta(\eta) \varphi_{r_0}(\eta) \, d\sigma(\eta). \]

Let us also assume that $u$ is a solution of the Dirichlet problem in $B(0, r_0)$ with boundary data $\varphi \in C(S^{n-1}(0, r_0))$. Then $u \in C(B(0, r_0))$ and consequently $u \in L^2(B(0, r_0))$. By (3.1), we have

\[ u(r\zeta) = \sum_{k=0}^{\infty} u^k_{r\zeta}(r). \]

But, according to Lemma 3.1 $u^k_{r\zeta}(r) = G_k(\zeta) r^k F_k(r^2)$, $\zeta \in S^{n-1}$. Letting $r \to r_0$, and bearing in mind that $u(r\zeta) \to \varphi_{r_0}(\zeta)$ as $r \to r_0$, we get

\[ F_k(r_0^2) r_0^k G_k(\zeta) = \int_{S^{n-1}} Z^k_\zeta(\eta) u(r_0\eta) \, d\sigma(\eta) = \varphi_{r_0,k}(\zeta). \]

This, combined with the expansion (3.1) of $u(r\zeta)$, implies that

\[ u(r\zeta) = \sum_{k=0}^{\infty} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0,k}(\zeta), \quad r < r_0, \, \zeta \in S^{n-1}. \]

The above relation also implies that if there exists a solution of the Dirichlet problem, then the solution is unique.

It remains to prove the existence of the solution. For this we recall that the Poisson kernel $\mathbb{P}_{h,r_0}$ in the hyperbolic ball $B(0, r_0)$, computed explicitly by T. Byczkowski and J. Małecki in [BM], is given by

\[ \mathbb{P}_{h,r_0}(x, \xi) = \frac{\Gamma(n/2)}{2\pi^{n/2} r_0^{n-1}} \sum_{k=0}^{\infty} \frac{\rho + k}{\rho} \frac{|x|^k}{r_0^k} \frac{F_k(|x|^2)}{F_k(r_0^2)} C^\rho_k \left( \frac{\langle x, \xi \rangle}{|x|} \right), \]
where \( \rho = (n - 2)/2, \quad |\xi| = 1, \quad |x| < r_0 \). As stated in [BM, p. 9] the Poisson kernel is the density of the harmonic measure of the ball \( B(0, r_0) \). It is well known ([E, p. 90], [F, p. 126]) that the harmonic measure and consequently the Poisson kernel solves the Dirichlet problem on \( B(0, r_0) \), i.e. if \( \varphi \in \mathcal{C}(S^{n-1}(0, r_0)) \), then

\[
\mathbb{P}_{h, r_0}[\varphi](z) = \int_{S^{n-1}(0, r_0)} \mathbb{P}_{h, r_0}(z, r_0 \xi) \varphi(r_0 \xi) \, d\sigma_{r_0}(r_0 \xi)
\]

is an \( \mathcal{H} \)-harmonic function in \( B(0, r_0) \) and

\[
\mathbb{P}_{h, r_0}[\varphi](r \zeta) \xrightarrow{r \to r_0} \varphi_{r_0}(\zeta). \]

4. Proof of Theorem 1.2. We need the following lemma.

**Lemma 4.1.** Let \( \varphi_{r_0} = \sum_{k=0}^{\infty} \varphi_{r_0, k} \) be the spherical harmonic expansion of \( \varphi_{r_0} \in \mathcal{C}(S^{n-1}) \). For \( 0 \leq r < r_0 \) and \( |\zeta| = 1 \), we set

\begin{equation}
V(r \zeta) := \sum_{k=1}^{\infty} \frac{\Gamma(k + n - 1)}{\Gamma(k) \Gamma(n - 1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0, k}(\zeta) \left( \frac{r}{r_0} \right)^k.
\end{equation}

Then the function \( V \) is Euclidean-harmonic and bounded on \( B(0, r_0) \).

**Proof.** It suffices to prove that the series

\begin{equation}
\sum_{k=1}^{N} \frac{\Gamma(k + n - 1)}{\Gamma(k) \Gamma(n - 1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0, k}(\zeta) \left( \frac{r}{r_0} \right)^k
\end{equation}

converges on \( B(0, r_0) \) as \( N \to \infty \). Indeed,

\[
\varphi_{r_0, k}(\zeta) = \int_{S^{n-1}} Z^k_{-}(\eta) \varphi_{r_0}(\eta) \, d\sigma(\eta),
\]

and by the definition of spherical harmonics, we have \( \Delta(r^k Z^k_{-}(\eta)) = 0 \) for all \( k \in \mathbb{N} \) and \( \zeta \in S^{n-1} \). These imply that \( \Delta(V) = 0 \).

Next we prove the convergence of the series (4.2) by using the ratio criterion for convergence of power series. First we observe that since \( |Z^k_{-}(\eta)| \leq 1 \) for any \( k \in \mathbb{N} \) and \( \zeta, \eta \in S^{n-1} \), using (1.3) we have \( \|\varphi_{r_0, k}\|_\infty \leq \|\varphi_{r_0}\|_\infty \). Thus, to prove that the series (4.2) converges, it suffices to show that

\[
\sum_{k=1}^{\infty} a_k \left( \frac{r}{r_0} \right)^k < \infty \quad \text{when} \quad r < r_0,
\]

where

\[
a_k = \frac{\Gamma(k + n - 1)}{\Gamma(k) \Gamma(n - 1)} \frac{F_k(1)}{F_k(r_0^2)}, \quad k \in \mathbb{N}.
\]
Let us recall that \( \Gamma(z + 1) = z\Gamma(z) \) for any \( z \in \mathbb{C} \). Also,

\[
F_k(1) = \frac{\Gamma(k + n/2)\Gamma(n - 1)}{\Gamma(n/2)\Gamma(k + n - 1)}
\]

(see [E, p. 61, relation (14)]). These imply that

\[
a_{k+1} \frac{a_k}{a_k} = \frac{k - n}{k + 1} \frac{k + n/2 + 1}{k + n} \frac{F_{k+1}(r_0^2)}{F_k(r_0^2)}, \quad k \geq 1, r_0 < 1.
\]

(4.3)

Bearing in mind ([BM, p. 11]) that for any \( r_0 < 1 \),

\[
\lim_{k \to \infty} F_k(r_0^2) = (1 - r_0^2)^\rho,
\]

from (4.3) we get

\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = 1.
\]

It follows that

\[
\sum_{k=1}^{\infty} a_k \left( \frac{r}{r_0} \right)^k < \infty \quad \text{when } r < r_0,
\]

and the proof of the lemma is complete. \( \blacksquare \)

For every \( t \in (0, 1) \), we set

\[
v_t(r\zeta) := \frac{V(tr\zeta)}{\sqrt{t}}
\]

(4.4)

with \( V \) defined in (4.1). By Lemma 4.1 \( v_t \) is Euclidean-harmonic in \( B(0, r_0) \).

Let us set

\[
T(v_t)(r\zeta) = \int_0^1 v_t(r\zeta)[(1 - t)(1 - tr^2)]^{n/2-1} \frac{dt}{t^{1/2}}.
\]

(4.5)

The integral above converges since \( V \) is harmonic and \( V(0) = 0 \). The following proposition gives the relation between the Euclidean and hyperbolic harmonic functions on the ball \( B(0, r_0) \).

**Proposition 4.2.** For every \( \mathcal{H} \)-harmonic function \( u \) on \( B(0, r_0) \), there exists a Euclidean-harmonic function \( v_t \) on \( B(0, r_0) \) such that

\[
u(r\zeta) = u(0) + T(v_t)(r\zeta), \quad \zeta \in S^{n-1}, r < r_0, t \in (0, 1).
\]

**Proof.** The proof of the proposition follows the steps of the corresponding result for harmonic functions on the hyperbolic ball \( \mathbb{B}_n \) proved in [J2, Section 6.1]). Let \( u \) be an \( \mathcal{H} \)-harmonic function on \( B(0, r_0) \) such that \( u(0) = 0 \). Then by Proposition 1.1 we have the following expansion of \( u \):

\[
u(r\zeta) = \sum_{k=1}^{\infty} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0, k}(\zeta), \quad 0 < r < r_0, \zeta \in S^{n-1}.
\]
Also, by [E, p. 59, relation (10)], for \( k \geq 1 \) we have

\[
\frac{F_k(r^2)}{F_k(1)} = \frac{\Gamma(k + n - 1)}{\Gamma(k)\Gamma(n - 1)} \int_0^1 t^{k-1}[(1 - t)(1 - tr^2)]^{n/2-1} dt.
\]

The relations above, (4.1) and (4.4) imply that

\[
u(r\zeta) = \sum_{k=1}^{\infty} \int_0^1 \frac{F_k(1)}{F_k(r_0^2)} \frac{r^k}{r_0^k} \frac{\Gamma(k + n - 1)}{\Gamma(k)\Gamma(n - 1)} t^{k-1} \varphi_{r_0,k}(\zeta)[(1 - t)(1 - tr^2)]^{n/2-1} dt
\]

\[= \int_0^1 \frac{V(tr\zeta)}{\sqrt{t}} [(1 - t)(1 - tr^2)]^{n/2-1} dt\]

\[= \int_0^1 v_t(r\zeta)[(1 - t)(1 - tr^2)]^{n/2-1} \frac{dt}{t^{1/2}}.
\]

Note that the interchange of the series and integral in (4.6) is possible since, as shown in Lemma 4.1, the series above are absolutely convergent.

**Remark 4.3.** Let \( u \) and \( v_t \) be as in Proposition 4.2. Let \( \{\lambda_s\} \) be a sequence of natural numbers and recall that the partial sums \( S^*_{\lambda_s,N}(u) \) of \( u \) are defined in (1.6). Let us also set

\[S_{\lambda_s,N}(v_t)(r\zeta) = \sum_{k=\lambda_1}^{\lambda_N} \frac{\Gamma(k + n - 1)}{\Gamma(k)\Gamma(n - 1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0,k}(\zeta) \left( \frac{r}{r_0} \right)^k t^{k-1/2}.
\]

Then by Proposition 4.2, we get

\[T(S_{\lambda_s,N}(v_t))(r\zeta) = T\left( \sum_{k=\lambda_1}^{\lambda_N} \frac{\Gamma(k + n - 1)}{\Gamma(k)\Gamma(n - 1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0,k}(\zeta) \left( \frac{r}{r_0} \right)^k t^{k-1/2} \right)
\]

\[= \sum_{k=\lambda_1}^{\lambda_N} \int_0^1 \frac{F_k(1)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \frac{\Gamma(k + n - 1)}{\Gamma(k)\Gamma(n - 1)} t^{k-1} \varphi_{r_0,k}(\zeta)[(1 - t)(1 - tr^2)]^{n/2-1} dt
\]

\[= \sum_{k=\lambda_1}^{\lambda_N} \frac{F_k(r^2)}{F_k(r_0^2)} \left( \frac{r}{r_0} \right)^k \varphi_{r_0,k}(\zeta) = S^*_{\lambda_s,N}(u)(r\zeta).
\]

Next, let us endow the spaces \( \mathcal{H}_{E,r_0} \) and \( \mathcal{H}_{r_0} \) with the topology of uniform convergence on compact subsets of \( B(0, r_0) \). Then \( \mathcal{H}_{E,r_0} \) and \( \mathcal{H}_{r_0} \) are Fréchet spaces.

**Lemma 4.4.** The operator \( T \) is continuous from \( \mathcal{H}_{E,r_0} \) onto \( \mathcal{H}_{r_0} \).
Proof. From Proposition 4.2 it follows that $T$ is onto. For the boundedness of $T$, we note that since

$$\frac{(1-t)(1-tr^2)}{t^{1/2}} \sim \frac{1}{t^{1/2}} \text{ as } t \to 0,$$

we have

$$|T(v_t)(r\zeta)| = \left| \int_0^1 v_t(r\zeta)[(1-t)(1-tr^2)]^{n/2-1} \frac{1}{t^{1/2}} dt \right| \leq \|v_t\|_\infty \int_0^1 [(1-t)(1-tr^2)]^{n/2-1} \frac{dt}{t^{1/2}} \leq c\|v_t\|_\infty$$

for all $r\zeta$ varying on any compact subset of $B(0,r_0)$. This shows that $T$ is continuous.

We denote by $\mathcal{H}(\mathbb{B}_n)$ the space of all $\mathcal{H}$-harmonic functions in $\mathbb{B}_n$ and by $\mathcal{H}_E(\mathbb{R}^n)$ the space of all Euclidean-harmonic functions in $\mathbb{R}^n$.

**Lemma 4.5.** For every $N \in \mathbb{N}$ the operator $S_N^*: \mathcal{H}_{r_0} \to \mathcal{H}(\mathbb{B}_n)$ is continuous.

Proof. Indeed, by Remark 4.3, we have $S_N^*(u) = T(S_N(v_t))$. Observe also that the correspondence $\mathcal{H}_{r_0} \ni u \mapsto v_t \in \mathcal{H}_{E,r_0}$ is continuous as can be seen by the expansion (1.4) of $u$ and (4.1) of $V$.

We denote by $A \subset \mathcal{H}_{E,r_0}$ the image of the correspondence $\mathcal{H}_{r_0} \ni u \mapsto v_t \in \mathcal{H}_{E,r_0}$. Since $T$ is continuous it suffices to show that $S_N : A \to \mathcal{H}_E(\mathbb{R}^n)$ is continuous. But if $V \in A$ is the function corresponding to $u$, then for $t \in (0,1)$,

$$v_t(r\zeta) := \frac{V(tr\zeta)}{\sqrt{t}} \in \mathcal{H}_{E,r_0}.$$ 

So,

$$S_N(v_t)(x) = \sum_{k=0}^N \mathcal{H}_k(v_t)(x)$$

are the partial sums of the expansion of $v_t$ in homogeneous harmonic polynomials. Now, let us recall that $v_t$, as a harmonic function, is also real-analytic. So it admits a Taylor expansion

$$v_t(x) = \sum_{k=0}^\infty \sum_{|m|=k} (D^m v_t)(0) \frac{m!}{m!} x^m,$$

where $m! = m_1! \ldots m_k!$ and $x^m = x_1^{m_1} \ldots x_k^{m_k}$. Using the Cauchy estimates for the Taylor coefficients $(D^m v_t)(0)$, obtained in [ABR, p. 33], one can prove
that the correspondence

$$R_k : v_t \mapsto \sum_{\ell=0}^{k} \sum_{|m| = \ell} \frac{(D^m v_t)(0)}{m!} x^m$$

is bounded from $\mathcal{H}_{E,r_0}$ to the space of homogeneous polynomials of degree $k$. This combined with (4.7) and (4.8) implies that

$$v_t(x) = \sum_{N=0}^{\infty} \mathcal{H}_N(v_t)(x) = \sum_{N=0}^{\infty} R_N(v_t)(x).$$

Further, by [ABR, p. 24] we have $\mathcal{H}_N(v_t) = R_N(v_t)$. Thus, since $v_t \mapsto R_k(v_t)$ is bounded it follows that the correspondence $v_t \mapsto \mathcal{H}_k(v_t)$ is also bounded. This yields the continuity of $S_N$ and consequently the continuity of $S^*_N$. □

Proof of Theorem 1.2. The proof is in several steps.

Step 1. $\mathcal{U}_H \neq \emptyset$.

For this we shall show that if $V$ is a Euclidean universal function in $B(0, r_0)$ then $u(r\zeta) := T(V(tr\zeta)/\sqrt{t})$ is an $\mathcal{H}$-universal function. Indeed, without any loss of generality we may assume that $V(0) = 0$. Let $P$ be an $\mathcal{H}$-harmonic polynomial. Since $T : \mathcal{H}_{E,r_0} \to \mathcal{H}_{r_0}$ is onto, there exists a Euclidean-harmonic polynomial $h$ such that

$$T\left(\frac{h(tr\zeta)}{\sqrt{t}}\right) = P(r\zeta), \quad r < r_0, \ |\zeta| = 1. \quad (4.9)$$

Since $V$ is a Euclidean universal function, by [Ar] for every compact set $K$ in $\mathbb{R}^n$ with $K^c$ connected and $K \subset B(0, r_0)^c$, there exists a sequence $\{\lambda_s\}_{s \in \mathbb{N}}$ of integers such that

$$\sup_{x \in K} |S_{\lambda_s,N}(V)(x) - h(x)| < \varepsilon. \quad (4.10)$$

Recall that for $t \in (0, 1)$, $v_t(r\zeta) := V(tr\zeta)/\sqrt{t}$. So, (4.10) implies that

$$\sup_{r\zeta \in K} |S_{\lambda_s,N}(v_t)(r\zeta) - h_t(r\zeta)| = \sup_{tr\zeta \in K} \left| \frac{1}{\sqrt{t}} S_{\lambda_s,N}(V)(tr\zeta) - \frac{1}{\sqrt{t}} h(tr\zeta) \right|.$$

But $T$ is continuous, so by the definition of $S^*_N$ and (4.9), it follows that

$$\sup_{r\zeta \in K} |S^*_{\lambda_s,N}(u)(r\zeta) - P(r\zeta)| = \sup_{tr\zeta \in K} |T(S_{\lambda_s,N}(v_t))(tr\zeta) - T(h_t)(tr\zeta)| < \varepsilon.$$

Therefore $u \in \mathcal{U}_H$ and $T(\mathcal{U}_{E,r_0}) \subset \mathcal{U}_H$. This completes the proof of Step 1.

Step 2. The class $\mathcal{U}_H$ is dense in $\mathcal{H}_{r_0}$ and contains a dense vector subspace of $\mathcal{H}_{r_0}$ except 0.

Recall that, by Step 1, $T(\mathcal{U}_{E,r_0}) \subset \mathcal{U}_H$. By [BGNP] the class $\mathcal{U}_{E,r_0}$ is dense in $\mathcal{H}_{E,r_0}$ and there exists a dense vector subspace $M \setminus \{0\}$ of $\mathcal{H}_{E,r_0}$.
Using the fact that $T$ is linear, continuous and onto we deduce that $\mathcal{U}_H$ is a dense set in $\mathcal{H}_{r_0}$ and $T(M) \setminus \{0\}$ is a dense vector subspace of $\mathcal{H}_{r_0}$. This completes the proof of Step 2.

**Step 3.** The class $\mathcal{U}_H$ is a $G_\delta$-set in $\mathcal{H}_{r_0}$.

Let us denote by $\mathcal{H}\mathcal{P}_N$ the space of homogeneous $\mathcal{H}$-harmonic polynomials of degree $N$. This space is finite-dimensional. Let $\mathcal{B}_N$ be a basis of $\mathcal{H}\mathcal{P}_N$ and set $\mathcal{B} = \bigcup_{N \in \mathbb{N}} \mathcal{B}_N$. Then $\mathcal{B}$ is a countable basis of $\mathcal{H}\mathcal{P}$, the space of all $\mathcal{H}$-harmonic polynomials. We set $\mathcal{B} = \{f_j\}_{j \in \mathbb{N}}$ and

$$\mathcal{L} = \{a_1 f_1 + \cdots + a_k f_k : k \in \mathbb{N}, a_i \in \mathbb{Q}\}.$$  

The set $\mathcal{L}$ is countable, so $\mathcal{L} = \{P_j\}_{j \in \mathbb{N}}$, where $P_j$ are $\mathcal{H}$-harmonic polynomials. It is obvious that for every $\mathcal{H}$-harmonic polynomial $h$ in $\mathbb{B}_n$ and every $\varepsilon > 0$, there exists $P_j \in \mathcal{L}$ such that

$$\sup_{x \in \mathbb{B}_n} |h(x) - P_j(x)| < \varepsilon. \tag{4.11}$$

As in [BGNP], we consider a sequence of compact subsets $\{K_m\}_{m \in \mathbb{N}}$ of $\mathbb{B}_n \cup \{\infty\}$, with $K_m \cap \overline{B(0, r_0)} = \emptyset$ and $(\mathbb{B}_n \cup \{\infty\}) \setminus K_m$ connected with the following property: every compact set $K \subset (\mathbb{B}_n \cup \{\infty\}) \setminus B(0, r_0)$ with $K \cap \overline{B(0, r_0)} = \emptyset$ and $(\mathbb{B}_n \cup \{\infty\}) \setminus K$ connected is contained in some $K_m$.

Next, for every $j, s, m, N \in \mathbb{N}$ we consider the sets

$$G(m, j, s, N) = \{u \in \mathcal{H}_{r_0} : \sup_{x \in K_m} |S^*_N(u)(x) - P_j(x)| < 1/s\}.$$  

By Lemma 4.5, the operator $S^*_N$ is continuous and it follows easily that the sets $G(m, j, s, N)$ are open subsets of $\mathcal{H}_{r_0}$.

It remains to show that $\mathcal{U}_H = \bigcap_{m, j, s} \bigcup_n G(m, j, s, N)$. In fact, it is clear that $\mathcal{U}_H \subset \bigcap_{m, j, s} \bigcup_n G(m, j, s, N)$. In order to prove the reverse inclusion, let $u \in \bigcap_{m, j, s} \bigcup_n G(m, j, s, N)$ and $h$ be an $\mathcal{H}$-harmonic polynomial. For any $m \in \mathbb{N}$ and for each $s \in \mathbb{N}$, there exists a polynomial $P_{js}$ in $\mathcal{L}$ such that

$$\sup_{x \in K_m} |h(x) - P_{js}(x)| < 1/s. \tag{4.12}$$

But, since $u \in \bigcup_n G(m, js, s, N)$, there exists a sequence $\{N_s\}$ of nonnegative integers such that for every $s \in \mathbb{N}$,

$$\sup_{x \in K_m} |S^*_{N_s}(u)(x) - P_{js}(x)| < 1/s. \tag{4.13}$$

From (4.12) and (4.13), it follows that

$$\sup_{x \in K_m} |S^*_{N_s}(u)(x) - h(x)| < 2/s.$$  

Letting $s \to \infty$ we find that $u \in \mathcal{U}_H$. This completes the proof of Step 3 and the proof of Theorem 1.2. ■
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