COLLOQUIUM MATHEMATICUM

VOL. 121 2010 NO. 1

UNIVERSAL HARMONIC FUNCTIONS ON THE HYPERBOLIC SPACE

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Abstract. We prove universal overconvergence phenomena for harmonic functions on the real hyperbolic space.

1. Introduction and statement of the results. Let $\mathbb{B}_n$ be the ball model of the n-dimensional hyperbolic space, i.e. the unit ball of $\mathbb{R}^n$ equipped with the metric $ds^2 = |dx|^2/(1-|x|^2)^2$. Throughout this paper we consider the case $n \geq 3$. The hyperbolic Laplacian D on $\mathbb{B}_n$, acting on smooth functions, is given by

$$D = (1-r^2)^2 \Delta + 2(n-2)(1-r^2) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and Δ is the Euclidean Laplacian. As usual, a function u on an open set Ω of $\mathbb{B}_n$ is called $\mathcal{H}$-*harmonic* if $Du = 0$ on Ω.

Let us set $F_k(x) = {}_2F_1(k, 1-n/2, k+n/2; x)$ where ${}_2F_1$ is the Gaussian hypergeometric function, and let us denote by C_k^ρ the Gegenbauer polynomial [BM]. Then, by the main result of [BM], the hyperbolic Poisson kernel $\mathbb{P}_{h,r_0}$ on the ball $B(0,r_0)$, where $r_0 < 1$, is given by

$$\mathbb{P}_{h,r_0}(x,\xi) = \frac{\Gamma(n/2)}{2\pi^{n/2} r_0^{n-1}} \sum_{k=0}^{\infty} \frac{\rho+k}{\rho} \frac{|x|^k}{r_0^k} \frac{F_k(|x|^2)}{F_k(r_0^2)} C_k^\rho\Big(\frac{\langle x,\xi\rangle}{|x|}\Big), \tag{1.1}$$

where $\rho = (n-2)/2$, $|\xi| = 1$, $|x| < r_0$ and $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. This kernel solves the Dirichlet problem on the ball $B(0,r_0)$ with boundary data $\varphi \in \mathcal{C}(\mathbb{S}^{n-1}(0,r_0))$. In fact, if $d\sigma_{r_0}$ is the canonical measure on $\mathbb{S}^{n-1}(0,r_0)$, we set

$$\mathbb{P}_{h,r_0}[\varphi](r\zeta) = \int_{\mathbb{S}^{n-1}(0,r_0)} \mathbb{P}_{h,r_0}(r\zeta,\xi)\varphi(\xi)\, d\sigma_{r_0}(\xi).$$

2010 *Mathematics Subject Classification*: Primary 31B05; Secondary 41A60.
Key words and phrases: hyperbolic ball, harmonic functions, Poisson kernel, universal series, overconvergence.

DOI: 10.4064/cm121-1-8

Then $\mathbb{P}_{h,r_0}[\varphi]$ is $\mathcal{H}$-harmonic on $B(0,r_0)$ and has φ as boundary value (see Proposition 1.1). Let us note that $B(0,r_0)=\{x\in\mathbb{R}^n : \|x\|<r_0\}$, where $\|\cdot\|$ is the Euclidean distance, and that $B(0,r_0)$ is a hyperbolic ball with center at 0 and radius $\frac{1}{2}\log\frac{1+r_0}{1-r_0}$.

Let us write $\varphi(r_0\zeta)=\varphi_{r_0}(\zeta)$ and let us recall that if φ_{r_0} belongs to $L^2(\mathbb{S}^{n-1})$, then it admits an expansion in spherical harmonics ([StW, pp. 141–145]):

$$\varphi_{r_0}(\zeta)=\sum_{k=0}^{\infty}\varphi_{r_0,k}(\zeta),\qquad \zeta\in\mathbb{S}^{n-1}, \tag{1.2}$$

with

$$\varphi_{r_0,k}(\zeta)=\int\limits_{\mathbb{S}^{n-1}}\mathcal{Z}_\zeta^k(\eta)\varphi_{r_0}(\eta)\,d\sigma(\eta), \tag{1.3}$$

where $\mathcal{Z}_\zeta^k$ is the zonal polynomial of degree k and pole at ζ (see Section 2).

Using the expansion (1.2) of φ_{r_0} and the expression (1.1) of the Poisson kernel we prove the following proposition.

Proposition 1.1. *The Dirichlet problem $Du=0$ on $B(0,r_0)$ with boundary data $\varphi\in\mathcal{C}(\mathbb{S}^{n-1}(0,r_0))$ has a unique solution given by*

$$u(r\zeta)=\sum_{k=0}^{\infty}\frac{F_k(r^2)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k\varphi_{r_0,k}(\zeta),\qquad r<r_0,\ \zeta\in\mathbb{S}^{n-1}, \tag{1.4}$$

where $\varphi_{r_0,k}$ is given by (1.3).

Let us denote by $\mathcal{H}_{r_0}$ the space of $\mathcal{H}$-harmonic functions on the ball $B(0,r_0)$. Let $u\in\mathcal{H}_{r_0}$ and let us assume that $u|_{\mathbb{S}^{n-1}(0,r_0)}=\varphi\in\mathcal{C}(\mathbb{S}^{n-1}(0,r_0))$. Then by Proposition 1.1, u admits the expansion (1.4). We consider the harmonic partial sums

$$S_N^*(u)(r\zeta)=\sum_{k=0}^{N}\frac{F_k(r^2)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k\varphi_{r_0,k}(\zeta) \tag{1.5}$$

and in particular for any sequence $\{\lambda_s\}$ of natural numbers we set

$$S_{\lambda_s,N}^*(u)(r\zeta)=\sum_{k=\lambda_1}^{\lambda_N}\frac{F_k(r^2)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k\varphi_{r_0,k}(\zeta). \tag{1.6}$$

We say that a harmonic function $u\in\mathcal{H}_{r_0}$ belongs to the class $\mathcal{U}_{\mathcal{H}}$ of *universal harmonic functions* on the hyperbolic ball $B(0,r_0)$ (see [BGNP]) if for every compact set K in $\mathbb{B}_n\setminus\overline{B(0,r_0)}$ with connected complement and for every harmonic polynomial P, there exists a sequence $\{\lambda_s\}$ in $\mathbb{N}$ such that

$$\lim_{N\to\infty}\sup_{x\in K}|S_{\lambda_s,N}^*(u)(x)-P(x)|=0.$$

Our main result is the following theorem, which answers a question of Michel Marias.

Theorem 1.2. *The class $\mathcal{U}_{\mathcal{H}}$ is G_δ-dense in $\mathcal{H}_{r_0}$ and contains a dense vector subspace of $\mathcal{H}_{r_0}$ except* 0.

Let us say a few words about Theorem 1.2. First we recall that universal harmonic functions on the Euclidean ball have been investigated by D. H. Armitage [Ar], and in an abstract setting by F. Bayart et al. [BGNP]. It is worth mentioning that Theorem 1.2 is an analogue of the Euclidean harmonic approximation result proved in [Ar, BGNP]. Let us now present the main result of [Ar]. If K is a compact set in $\mathbb{R}^n$ we denote by $\mathcal{H}(K)$ the space of functions that are harmonic on some neighborhood of K. Let Ω be an open subset of $\mathbb{R}^n$ such that $0 \in \Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. In [Ar] it is shown that there exists a series $\sum \mathcal{H}_N$, where $\mathcal{H}_N$ is a homogeneous harmonic polynomial of degree N on $\mathbb{R}^n$, such that:

(i) the series $\sum \mathcal{H}_N$ converges on some ball of center 0 to a function that is continuous on $\overline{\Omega}$ and harmonic on Ω,
(ii) outside of Ω, the partial sums $\sum \mathcal{H}_N$ are dense in the space $\overline{\mathrm{HP}}(\mathbb{R}^n)$, the closure of the space of harmonic polynomials in $\mathbb{R}^n$.

But, if $K \subset \mathbb{R}^n \setminus \overline{\Omega}$ is compact with connected complement, then by Walsh's theorem (see [G]), $\mathrm{HP}(\mathbb{R}^n)$ is dense in $\mathcal{H}(K)$. So, the partial sums of $\sum \mathcal{H}_N$ approach every $h \in \mathcal{H}(K)$. To prove Theorem 1.2 we use the above mentioned result of [Ar, BGNP] in the Euclidean setting; we prove in Proposition 4.2 the correspondence of $\mathcal{H}$-harmonic and Euclidean-harmonic functions on $B(0, r_0)$. This allows us to associate a universal $\mathcal{H}$-harmonic to a Euclidean-harmonic universal function. Note that in the present case of hyperbolic harmonic approximation, we can only approximate harmonic polynomials but not arbitrary harmonic functions defined in a compact set $K \subset \mathbb{B}_n \setminus \overline{B(0, r_0)}$ with $K^{\mathrm{c}} = \mathbb{B}_n \setminus K$ connected as is the case in the Euclidean setting. This fact is due to the absence of an analogue, in the hyperbolic setting, of the classical Walsh theorem [G].

2. Preliminaries. For the proofs we need to fix some notation. Let us recall, [J1, J2], that the hyperbolic Laplacian D in polar coordinates is written as

$$D = D_r + D_\sigma \quad \text{with} \tag{2.1}$$

$$\begin{cases} D_r = \dfrac{1-r^2}{r^2}[(1-r^2)N^2 + (n-2)(1+r^2)N], \\ D_\sigma = \dfrac{(1-r^2)^2}{r^2}\Delta_\sigma, \end{cases}$$

where

$$N = r\frac{d}{dr} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

and Δ_σ is the Laplacian on the unit sphere $\mathbb{S}^{n-1}$. Notice that D_r and D_σ are the radial and the tangential parts of D respectively.

We set $\rho = (n-2)/2$ and recall (see for instance [E, p. 175]) that the Gegenbauer polynomial C_k^ρ for $k \in \mathbb{N}$ and $\rho > 0$ is defined as the coefficient of h^k in the Maclaurin expansion of $(1 - 2zh + h^2)^{-\rho}$:

$$(1 - 2zh + h^2)^{-\rho} = \sum_{k=0}^{\infty} C_k^\rho(z) h^k, \qquad |z| \le 1,\ |h| < 1.$$

We denote by $\mathcal{Z}_\zeta^k$, $\zeta \in \mathbb{S}^{n-1}$, the zonal polynomial of degree k with pole at ζ, which is given by (see [BS])

$$\mathcal{Z}_\zeta^k(\eta) = (C_k^\rho(1))^{-1} C_k^\rho(\langle \zeta, \eta \rangle), \qquad \zeta, \eta \in \mathbb{S}^{n-1},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{S}^{n-1}$.

These functions are eigenfunctions of the tangential part of both the Euclidean and the hyperbolic Laplacian ([T, p. 216]), that is,

$$\Delta_\sigma(\mathcal{Z}_\zeta^k) = -k(k+n-2)\mathcal{Z}_\zeta^k. \tag{2.2}$$

Further, let us denote by ${}_2F_1$ the Gauss hypergeometric function

$${}_2F_1(a, b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

where $|x| < 1$, $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $(a)_k = \Gamma(a+k)/\Gamma(a)$. Also, in case $c < a < 0$, $a, c \in \mathbb{Z}$, we define

$${}_2F_1(a, b, c; x) = \sum_{k=0}^{-a} \frac{(a)_k (b)_k}{(c)_k k!} x^k.$$

To simplify our notation we set, for $k = 0, 1, 2, \ldots$,

$$F_k(x) = {}_2F_1\Big(k, -\rho, k + \frac{n}{2}; x\Big). \tag{2.3}$$

Finally, we denote by $\mathcal{H}_{E,r_0}$ the space of Euclidean-harmonic functions on $B(0, r_0)$.

3. Expansions of $\mathcal{H}$-harmonic functions. In this section we give the proof of Proposition 1.1 which gives the expansion of $\mathcal{H}$-harmonic functions in the ball $B(0, r_0)$.

Let $u \in \mathcal{H}_{r_0}$ and set

$$u_\zeta^k(r) = \int_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta) u(r\eta)\, d\sigma(\eta), \qquad r < r_0,\ \zeta \in \mathbb{S}^{n-1}.$$

To start with, we give an analogue of Theorem 6, Section 3.2 in [J2]. This is the content of the following lemma.

LEMMA 3.1. *For every $u \in \mathcal{H}_{r_0} \cap L^2(B(0,r_0))$ and $k \in \mathbb{N}$, there exists a continuous function G_k on $\mathbb{S}^{n-1}$ such that*

$$u_\zeta^k(r) = G_k(\zeta) r^k F_k(r^2), \qquad \zeta \in \mathbb{S}^{n-1},\ r < r_0,$$

where F_k is defined in (2.3).

Proof. Let u be in $\mathcal{H}_{r_0} \cap L^2(B(0,r_0))$. Using the same arguments as in [J2, Section 3.2, Theorem 6] (see also [J1]), one can obtain the following expansion of u in homogeneous harmonic polynomials:

$$u(r\zeta) = \sum_{k=0}^{\infty} u_\zeta^k(r), \qquad r < r_0,\ \zeta \in \mathbb{S}^{n-1}. \tag{3.1}$$

Since $Du = 0$, or equivalently $D_r u = -D_\sigma u$, we get

$$\begin{aligned} D_r u_\zeta^k(r) &= \int_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta) D_r u(r\eta)\, d\sigma(\eta) = -\int_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta) D_\sigma u(r\eta)\, d\sigma(\eta) \\ &= -\int_{\mathbb{S}^{n-1}} D_\sigma \mathcal{Z}_\zeta^k(\eta) u(r\eta)\, d\sigma(\eta). \end{aligned} \tag{3.2}$$

From (2.2) we have

$$D_\sigma \mathcal{Z}_\zeta^k(\eta) = -\frac{(1-r^2)^2}{r^2}\, k(k+n-2)\mathcal{Z}_\zeta^k(\eta).$$

So, by (3.2),

$$D_r u_\zeta^k(r) = \frac{(1-r^2)^2}{r^2}\, k(k+n-2) u_\zeta^k(r). \tag{3.3}$$

Setting $g(r^2) = u_\zeta^k(r)$ and using the expression of D in polar coordinates given in (2.1), we can write (3.3) as

$$(1-z)zg''(z) + \frac{1}{2}(nz - 4z + n)g'(z) = \frac{k(k+n-2)}{4}\, \frac{1-z}{z}\, g(z). \tag{3.4}$$

Looking for a solution of (3.4) in the form $z^a f(z)$ with $a = k/2$, we find that f satisfies the hypergeometric equation

$$(1-z)zf''(z) + \left(k + \frac{n}{2} - \left(k + \frac{n}{2} + 2\right)z\right)f'(z) - k\left(1 - \frac{n}{2}\right)f(z) = 0.$$

Next, if $a = -(k+n-2)/2$, then f satisfies the equation

$$\begin{aligned} &(1-z)zf''(z) \\ &\quad + \left(2 - k - \frac{n}{2} - \left(4 - k - \frac{3n}{2}\right)z\right)f'(z) - (2-k-n)\left(1 - \frac{n}{2}\right)f(z) = 0. \end{aligned}$$

From the equations above it follows that the independent solutions g_1 and g_2 of (3.4) are given in terms of the hypergeometric function ${}_2F_1$:

$$g_1(x) = x^{k/2}\,{}_2F_1\Big(k, 1-\frac{n}{2}, k+\frac{n}{2}; x\Big),$$
$$g_2(x) = x^{(2-k-n)/2}\,{}_2F_1\Big(-k-n+2, 1-\frac{n}{2}, -\frac{n}{2}+2; x\Big).$$

Since for $\zeta \in \mathbb{S}^{n-1}$, u_ζ^k is regular at $r=0$, it follows that

$$u_\zeta^k(r) = G_k(\zeta)\,r^k\,{}_2F_1\Big(k, 1-\frac{n}{2}, k+\frac{n}{2}; r^2\Big) = G_k(\zeta) r^k F_k(r^2).$$

Finally, since $\zeta \to u_\zeta^k(r)$ is continuous, it follows that one can choose $G_k(\zeta)$ to be a continuous function of ζ. ∎

Proof of Proposition 1.1. Let $\varphi \in \mathcal{C}(\mathbb{S}^{n-1}(0,r_0))$. Then $\varphi_{r_0} \in \mathcal{C}(\mathbb{S}^{n-1})$ and thus $\varphi_{r_0} \in L^2(\mathbb{S}^{n-1})$, since $\mathbb{S}^{n-1}$ is compact. Then φ_{r_0} admits the following spherical harmonic expansion:

$$\varphi_{r_0} = \sum_{k=0}^{\infty} \varphi_{r_0,k}, \qquad \varphi_{r_0,k}(\zeta) = \int_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta)\varphi_{r_0}(\eta)\,d\sigma(\eta).$$

Let us also assume that u is a solution of the Dirichlet problem in $B(0,r_0)$ with boundary data $\varphi \in \mathcal{C}(\mathbb{S}^{n-1}(0,r_0))$. Then $u \in \mathcal{C}(\overline{B(0,r_0)})$ and consequently $u \in L^2(B(0,r_0))$. By (3.1), we have

$$u(r\zeta) = \sum_{k=0}^{\infty} u_\zeta^k(r).$$

But, according to Lemma 3.1, $u_\zeta^k(r) = G_k(\zeta) r^k F_k(r^2)$, $\zeta \in \mathbb{S}^{n-1}$. Letting $r \to r_0$, and bearing in mind that $u(r\zeta) \to \varphi_{r_0}(\zeta)$ as $r \to r_0$, we get

$$F_k(r_0^2) r_0^k G_k(\zeta) = \int_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta) u(r_0\eta)\,d\sigma(\eta) = \varphi_{r_0,k}(\zeta).$$

This, combined with the expansion (3.1) of $u(r\zeta)$, implies that

$$u(r\zeta) = \sum_{k=0}^{\infty} \frac{F_k(r^2)}{F_k(r_0^2)} \Big(\frac{r}{r_0}\Big)^k \varphi_{r_0,k}(\zeta), \qquad r < r_0,\ \zeta \in \mathbb{S}^{n-1}. \tag{3.5}$$

The above relation also implies that if there exists a solution of the Dirichlet problem, then the solution is unique.

It remains to prove the existence of the solution. For this we recall that the Poisson kernel $\mathbb{P}_{h,r_0}$ in the hyperbolic ball $B(0,r_0)$, computed explicitly by T. Byczkowski and J. Małecki in [BM], is given by

$$\mathbb{P}_{h,r_0}(x,\xi) = \frac{\Gamma(n/2)}{2\pi^{n/2} r_0^{n-1}} \sum_{k=0}^{\infty} \frac{\rho+k}{\rho} \frac{|x|^k}{r_0^k} \frac{F_k(|x|^2)}{F_k(r_0^2)}\, C_k^\rho\Big(\frac{\langle x,\xi\rangle}{|x|}\Big),$$

where $\rho = (n-2)/2$, $|\xi| = 1$, $|x| < r_0$. As stated in [BM, p. 9] the Poisson kernel is the density of the harmonic measure of the ball $B(0, r_0)$. It is well known ([E, p. 90], [F, p. 126]) that the harmonic measure and consequently the Poisson kernel solves the Dirichlet problem on $B(0, r_0)$, i.e. if $\varphi \in \mathcal{C}(\mathbb{S}^{n-1}(0, r_0))$, then

$$\begin{aligned}\mathbb{P}_{h,r_0}[\varphi](z) &= \int\limits_{\mathbb{S}^{n-1}(0,r_0)} \mathbb{P}_{h,r_0}(z, r_0\xi)\varphi(r_0\xi)\, d\sigma_{r_0}(r_0\xi) \\ &= \int\limits_{\mathbb{S}^{n-1}} \mathbb{P}_{h,r_0}(z,\xi)\varphi_{r_0}(\xi)\, d\sigma(\xi),\end{aligned}$$

is an $\mathcal{H}$-harmonic function in $B(0, r_0)$ and

$$\mathbb{P}_{h,r_0}[\varphi](r\zeta) \xrightarrow[r\to r_0]{} \varphi_{r_0}(\zeta). \ \blacksquare$$

4. Proof of Theorem 1.2. We need the following lemma.

LEMMA 4.1. *Let $\varphi_{r_0} = \sum_{k=0}^{\infty} \varphi_{r_0,k}$ be the spherical harmonic expansion of $\varphi_{r_0} \in \mathcal{C}(\mathbb{S}^{n-1})$. For $0 \le r < r_0$ and $|\zeta| = 1$, we set*

$$V(r\zeta) := \sum_{k=1}^{\infty} \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0,k}(\zeta)\left(\frac{r}{r_0}\right)^k. \tag{4.1}$$

Then the function V is Euclidean-harmonic and bounded on $B(0, r_0)$.

Proof. It suffices to prove that the series

$$\sum_{k=1}^{N} \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \frac{F_k(1)}{F_k(r_0^2)} \varphi_{r_0,k}(\zeta)\left(\frac{r}{r_0}\right)^k \tag{4.2}$$

converges on $B(0, r_0)$ as $N \to \infty$. Indeed,

$$\varphi_{r_0,k}(\zeta) = \int\limits_{\mathbb{S}^{n-1}} \mathcal{Z}_\zeta^k(\eta)\varphi_{r_0}(\eta)\, d\sigma(\eta),$$

and by the definition of spherical harmonics, we have $\Delta(r^k \mathcal{Z}_\zeta^k(\eta)) = 0$ for all $k \in \mathbb{N}$ and $\zeta \in \mathbb{S}^{n-1}$. These imply that $\Delta(V) = 0$.

Next we prove the convergence of the series (4.2) by using the ratio criterion for convergence of power series. First we observe that since $|\mathcal{Z}_\zeta^k(\eta)| \le 1$ for any $k \in \mathbb{N}$ and $\zeta, \eta \in \mathbb{S}^{n-1}$, using (1.3) we have $\|\varphi_{r_0,k}\|_\infty \le \|\varphi_{r_0}\|_\infty$. Thus, to prove that the series (4.2) converges, it suffices to show that

$$\sum_{k=1}^{\infty} a_k \left(\frac{r}{r_0}\right)^k < \infty \quad \text{when } r < r_0,$$

where

$$a_k = \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \frac{F_k(1)}{F_k(r_0^2)}, \qquad k \in \mathbb{N}.$$

Let us recall that $\Gamma(z+1) = z\Gamma(z)$ for any $z \in \mathbb{C}$. Also,

$$F_k(1) = \frac{\Gamma(k+n/2)\Gamma(n-1)}{\Gamma(n/2)\Gamma(k+n-1)}$$

(see [E, p. 61, relation (14)]). These imply that

$$\frac{a_{k+1}}{a_k} = \frac{k-n}{k+1}\,\frac{k+n/2+1}{k+n}\,\frac{F_{k+1}(r_0^2)}{F_k(r_0^2)}, \qquad k \geq 1,\ r_0 < 1. \tag{4.3}$$

Bearing in mind ([BM, p. 11]) that for any $r_0 < 1$,

$$\lim_{k\to\infty} F_k(r_0^2) = (1-r_0^2)^\rho,$$

from (4.3) we get

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1.$$

It follows that

$$\sum_{k=1}^{\infty} a_k \left(\frac{r}{r_0}\right)^k < \infty \quad \text{when } r < r_0,$$

and the proof of the lemma is complete. ∎

For every $t \in (0,1)$, we set

$$v_t(r\zeta) := \frac{V(tr\zeta)}{\sqrt{t}}, \tag{4.4}$$

with V defined in (4.1). By Lemma 4.1, v_t is Euclidean-harmonic in $B(0,r_0)$.

Let us set

$$T(v_t)(r\zeta) = \int_0^1 v_t(r\zeta)[(1-t)(1-tr^2)]^{n/2-1}\,\frac{dt}{t^{1/2}}. \tag{4.5}$$

The integral above converges since V is harmonic and $V(0) = 0$. The following proposition gives the relation between the Euclidean and hyperbolic harmonic functions on the ball $B(0,r_0)$.

Proposition 4.2. *For every $\mathcal{H}$-harmonic function u on $B(0,r_0)$, there exists a Euclidean-harmonic function v_t on $B(0,r_0)$ such that*

$$u(r\zeta) = u(0) + T(v_t)(r\zeta), \qquad \zeta \in \mathbb{S}^{n-1},\ r < r_0,\ t \in (0,1).$$

Proof. The proof of the proposition follows the steps of the corresponding result for harmonic functions on the hyperbolic ball $\mathbb{B}_n$ proved in [J2, Section 6.1]). Let u be an $\mathcal{H}$-harmonic function on $B(0,r_0)$ such that $u(0) = 0$. Then by Proposition 1.1 we have the following expansion of u:

$$u(r\zeta) = \sum_{k=1}^{\infty} \frac{F_k(r^2)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k \varphi_{r_0,k}(\zeta), \qquad 0 < r < r_0,\ \zeta \in \mathbb{S}^{n-1}.$$

Also, by [E, p. 59, relation (10)], for $k \geq 1$ we have

$$\frac{F_k(r^2)}{F_k(1)} = \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \int_0^1 t^{k-1}[(1-t)(1-tr^2)]^{n/2-1}\,dt.$$

The relations above, (4.1) and (4.4) imply that

(4.6)

$$\begin{aligned} u(r\zeta) &= \sum_{k=1}^{\infty} \int_0^1 \frac{F_k(1)}{F_k(r_0^2)} \frac{r^k}{r_0^k} \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)}\, t^{k-1}\varphi_{r_0,k}(\zeta)[(1-t)(1-tr^2)]^{n/2-1}\,dt \\ &= \int_0^1 \frac{V(tr\zeta)}{\sqrt{t}}[(1-t)(1-tr^2)]^{n/2-1}\,dt \\ &= \int_0^1 v_t(r\zeta)[(1-t)(1-tr^2)]^{n/2-1}\,\frac{dt}{t^{1/2}}. \end{aligned}$$

Note that the interchange of the series and integral in (4.6) is possible since, as shown in Lemma 4.1, the series above are absolutely convergent. ∎

REMARK 4.3. Let u and v_t be as in Proposition 4.2. Let $\{\lambda_s\}$ be a sequence of natural numbers and recall that the partial sums $S^*_{\lambda_s,N}(u)$ of u are defined in (1.6). Let us also set

$$S_{\lambda_s,N}(v_t)(r\zeta) = \sum_{k=\lambda_1}^{\lambda_N} \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \frac{F_k(1)}{F_k(r_0^2)}\, \varphi_{r_0,k}(\zeta)\left(\frac{r}{r_0}\right)^k t^{k-1/2}.$$

Then by Proposition 4.2, we get

$$\begin{aligned} &T(S_{\lambda_s,N}(v_t))(r\zeta) \\ &\quad= T\left(\sum_{k=\lambda_1}^{\lambda_N} \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)} \frac{F_k(1)}{F_k(r_0^2)}\, \varphi_{r_0,k}(\zeta)\left(\frac{r}{r_0}\right)^k t^{k-1/2}\right) \\ &\quad= \sum_{k=\lambda_1}^{\lambda_N} \int_0^1 \frac{F_k(1)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k \frac{\Gamma(k+n-1)}{\Gamma(k)\Gamma(n-1)}\, t^{k-1}\varphi_{r_0,k}(\zeta)[(1-t)(1-tr^2)]^{n/2-1}\,dt \\ &\quad= \sum_{k=\lambda_1}^{\lambda_N} \frac{F_k(r^2)}{F_k(r_0^2)}\left(\frac{r}{r_0}\right)^k \varphi_{r_0,k}(\zeta) = S^*_{\lambda_s,N}(u)(r\zeta). \end{aligned}$$

Next, let us endow the spaces $\mathcal{H}_{E,r_0}$ and $\mathcal{H}_{r_0}$ with the topology of uniform convergence on compact subsets of $B(0,r_0)$. Then $\mathcal{H}_{E,r_0}$ and $\mathcal{H}_{r_0}$ are Fréchet spaces.

LEMMA 4.4. *The operator T is continuous from $\mathcal{H}_{E,r_0}$ onto $\mathcal{H}_{r_0}$.*

Proof. From Proposition 4.2 it follows that T is onto. For the boundedness of T, we note that since

$$\frac{(1-t)(1-tr^2)}{t^{1/2}} \sim \frac{1}{t^{1/2}} \qquad \text{as } t \to 0,$$

we have

$$\begin{aligned}|T(v_t)(r\zeta)| &= \left|\int_0^1 v_t(r\zeta)[(1-t)(1-tr^2)]^{n/2-1}\frac{1}{t^{1/2}}\,dt\right| \\ &\le \|v_t\|_\infty \int_0^1 [(1-t)(1-tr^2)]^{n/2-1}\,\frac{dt}{t^{1/2}} \le c\|v_t\|_\infty\end{aligned}$$

for all $r\zeta$ varying on any compact subset of $B(0,r_0)$. This shows that T is continuous. ■

We denote by $\mathcal{H}(\mathbb{B}_n)$ the space of all $\mathcal{H}$-harmonic functions in $\mathbb{B}_n$ and by $\mathcal{H}_E(\mathbb{R}^n)$ the space of all Euclidean-harmonic functions in $\mathbb{R}^n$.

LEMMA 4.5. *For every $N \in \mathbb{N}$ the operator $S_N^* : \mathcal{H}_{r_0} \to \mathcal{H}(\mathbb{B}_n)$ is continuous.*

Proof. Indeed, by Remark 4.3, we have $S_N^*(u) = T(S_N(v_t))$. Observe also that the correspondence $\mathcal{H}_{r_0} \ni u \mapsto v_t \in \mathcal{H}_{E,r_0}$ is continuous as can be seen by the expansion (1.4) of u and (4.1) of V.

We denote by $A \subset \mathcal{H}_{E,r_0}$ the image of the correspondence $\mathcal{H}_{r_0} \ni u \mapsto v_t \in \mathcal{H}_{E,r_0}$. Since T is continuous it suffices to show that $S_N : A \to \mathcal{H}_E(\mathbb{R}^n)$ is continuous. But if $V \in A$ is the function corresponding to u, then for $t \in (0,1)$,

$$v_t(r\zeta) := \frac{V(tr\zeta)}{\sqrt{t}} \in \mathcal{H}_{E,r_0}.$$

So,

$$S_N(v_t)(x) = \sum_{k=0}^{N} \mathcal{H}_k(v_t)(x) \tag{4.7}$$

are the partial sums of the expansion of v_t in homogeneous harmonic polynomials. Now, let us recall that v_t, as a harmonic function, is also real-analytic. So it admits a Taylor expansion

$$v_t(x) = \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{(D^m v_t)(0)}{m!} x^m, \tag{4.8}$$

where $m! = m_1! \dots m_k!$ and $x^m = x_1^{m_1} \dots x_k^{m_k}$. Using the Cauchy estimates for the Taylor coefficients $(D^m v_t)(0)$, obtained in [ABR, p. 33], one can prove that the correspondence

$$R_k : v_t \mapsto \sum_{\ell=0}^{k} \sum_{|m|=\ell} \frac{(D^m v_t)(0)}{m!} x^m$$

is bounded from $\mathcal{H}_{E,r_0}$ to the space of homogeneous polynomials of degree k. This combined with (4.7) and (4.8) implies that

$$v_t(x) = \sum_{N=0}^{\infty} \mathcal{H}_N(v_t)(x) = \sum_{N=0}^{\infty} R_N(v_t)(x).$$

Further, by [ABR, p. 24] we have $\mathcal{H}_N(v_t) = R_N(v_t)$. Thus, since $v_t \mapsto \mathcal{R}_k(v_t)$ is bounded it follows that the correspondence $v_t \mapsto \mathcal{H}_k(v_t)$ is also bounded. This yields the continuity of S_N and consequently the continuity of S_N^*. ■

Proof of Theorem 1.2. The proof is in several steps.

STEP 1. $\mathcal{U}_{\mathcal{H}} \neq \emptyset$.

For this we shall show that if V is a Euclidean universal function in $B(0, r_0)$ then $u(r\zeta) := T(V(tr\zeta)/\sqrt{t})$ is an $\mathcal{H}$-universal function. Indeed, without any loss of generality we may assume that $V(0) = 0$. Let P be an $\mathcal{H}$-harmonic polynomial. Since $T : \mathcal{H}_{E,r_0} \to \mathcal{H}_{r_0}$ is onto, there exists a Euclidean-harmonic polynomial h such that

$$T\left(\frac{h(tr\zeta)}{\sqrt{t}}\right) = P(r\zeta), \qquad r < r_0,\ |\zeta| = 1. \tag{4.9}$$

Since V is a Euclidean universal function, by [Ar, BGNP] for every compact set K in $\mathbb{R}^n$ with K^c connected and $K \subset \overline{B(0, r_0)}^c$, there exists a sequence $\{\lambda_s\}_{s\in\mathbb{N}}$ of integers such that

$$\sup_{x\in K} |S_{\lambda_s,N}(V)(x) - h(x)| < \varepsilon. \tag{4.10}$$

Recall that for $t \in (0,1)$, $v_t(r\zeta) := V(tr\zeta)/\sqrt{t}$. So, (4.10) implies that

$$\sup_{r\zeta\in K} |S_{\lambda_s,N}(v_t)(r\zeta) - h_t(r\zeta)| = \sup_{tr\zeta\in K} \left|\frac{1}{\sqrt{t}}\, S_{\lambda_s,N}(V)(tr\zeta) - \frac{1}{\sqrt{t}} h(tr\zeta)\right|.$$

But T is continuous, so by the definition of S_N^* and (4.9), it follows that

$$\sup_{r\zeta\in K} |S^*_{\lambda_s,N}(u)(r\zeta) - P(r\zeta)| = \sup_{tr\zeta\in K} |T(S_{\lambda_s,N}(v_t))(tr\zeta) - T(h_t)(tr\zeta)| < \varepsilon.$$

Therefore $u \in \mathcal{U}_{\mathcal{H}}$ and $T(\mathcal{U}_{E,r_0}) \subset \mathcal{U}_{\mathcal{H}}$. This completes the proof of Step 1.

STEP 2. *The class $\mathcal{U}_{\mathcal{H}}$ is dense in $\mathcal{H}_{r_0}$ and contains a dense vector subspace of $\mathcal{H}_{r_0}$ except* 0.

Recall that, by Step 1, $T(\mathcal{U}_{E,r_0}) \subset \mathcal{U}_{\mathcal{H}}$. By [BGNP] the class $\mathcal{U}_{E,r_0}$ is dense in $\mathcal{H}_{E,r_0}$ and there exists a dense vector subspace $M \setminus \{0\}$ of $\mathcal{H}_{E,r_0}$.

Using the fact that T is linear, continuous and onto we deduce that $\mathcal{U}_{\mathcal{H}}$ is a dense set in $\mathcal{H}_{r_0}$ and $T(M)\setminus\{0\}$ is a dense vector subspace of $\mathcal{H}_{r_0}$. This completes the proof of Step 2.

STEP 3. *The class $\mathcal{U}_{\mathcal{H}}$ is a G_δ-set in $\mathcal{H}_{r_0}$.*

Let us denote by $\mathcal{HP}_N$ the space of homogeneous $\mathcal{H}$-harmonic polynomials of degree N. This space is finite-dimensional. Let $\mathcal{B}_N$ be a basis of $\mathcal{HP}_N$ and set $\mathcal{B}=\bigcup_{N\in\mathbb{N}}\mathcal{B}_N$. Then $\mathcal{B}$ is a countable basis of $\mathcal{HP}$, the space of all $\mathcal{H}$-harmonic polynomials. We set $\mathcal{B}=\{f_j\}_{j\in\mathbb{N}}$ and

$$\mathcal{L}=\{a_1f_1+\cdots+a_kf_k : k\in\mathbb{N},\, a_i\in\mathbb{Q}\}.$$

The set $\mathcal{L}$ is countable, so $\mathcal{L}=\{P_j\}_{j\in\mathbb{N}}$, where P_j are $\mathcal{H}$-harmonic polynomials. It is obvious that for every $\mathcal{H}$-harmonic polynomial h in $\mathbb{B}_n$ and every $\varepsilon>0$, there exists $P_j\in\mathcal{L}$ such that

$$\sup_{x\in\mathbb{B}_n}|h(x)-P_j(x)|<\varepsilon. \tag{4.11}$$

As in [BGNP], we consider a sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}}$ of $\mathbb{B}_n\cup\{\infty\}$, with $K_m\cap\overline{B(0,r_0)}=\emptyset$ and $(\mathbb{B}_n\cup\{\infty\})\setminus K_m$ connected with the following property: every compact set $K\subset(\mathbb{B}_n\cup\{\infty\})\setminus\overline{B(0,r_0)}$ with $K\cap\overline{B(0,r_0)}=\emptyset$ and $(\mathbb{B}_n\cup\{\infty\})\setminus K$ connected is contained in some K_m.

Next, for every $j,s,m,N\in\mathbb{N}$ we consider the sets

$$G(m,j,s,N)=\{u\in\mathcal{H}_{r_0} : \sup_{x\in K_m}|S_N^*(u)(x)-P_j(x)|<1/s\}.$$

By Lemma 4.5, the operator S_N^* is continuous and it follows easily that the sets $G(m,j,s,N)$ are open subsets of $\mathcal{H}_{r_0}$.

It remains to show that $\mathcal{U}_{\mathcal{H}}=\bigcap_{m,j,s}\bigcup_n G(m,j,s,N)$. In fact, it is clear that $\mathcal{U}_{\mathcal{H}}\subset\bigcap_{m,j,s}\bigcup_n G(m,j,s,N)$. In order to prove the reverse inclusion, let $u\in\bigcap_{m,j,s}\bigcup_n G(m,j,s,N)$ and h be an $\mathcal{H}$-harmonic polynomial. For any $m\in\mathbb{N}$ and for each $s\in\mathbb{N}$, there exists a polynomial P_{j_s} in $\mathcal{L}$ such that

$$\sup_{x\in K_m}|h(x)-P_{j_s}(x)|<1/s. \tag{4.12}$$

But, since $u\in\bigcup_n G(m,j_s,s,N)$, there exists a sequence $\{N_\delta\}$ of nonnegative integers such that for every $s\in\mathbb{N}$,

$$\sup_{x\in K_m}|S^*_{N_\delta,s}(u)(x)-P_{j_s}(x)|<1/s. \tag{4.13}$$

From (4.12) and (4.13), it follows that

$$\sup_{x\in K_m}|S^*_{N_\delta,s}(u)(x)-h(x)|<2/s.$$

Letting $s\to\infty$ we find that $u\in\mathcal{U}_{\mathcal{H}}$. This completes the proof of Step 3 and the proof of Theorem 1.2. ∎

Acknowledgements. We would like to thank Michel Marias and Vassili Nestoridis for their constant help. We would also like to thank the referee for his valuable suggestions.

Both of the authors were supported by the State Scholarships Foundation of Greece (I K Y).

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Received 2 October 2009;
revised 18 February 2010 (5279)