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ON PAWLAK'S PROBLEM CONCERNING ENTROPY OF ALMOST CONTINUOUS FUNCTIONS

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Abstract. We prove that if $f : \mathbb{I} \to \mathbb{I}$ is Darboux and has a point of prime period different from 2^i, $i = 0, 1, \ldots$, then the entropy of f is positive. On the other hand, for every set $A \subset \mathbb{N}$ with $1 \in A$ there is an almost continuous (in the sense of Stallings) function $f : \mathbb{I} \to \mathbb{I}$ with positive entropy for which the set $\mathrm{Per}(f)$ of prime periods of all periodic points is equal to A.

A classical result of Misiurewicz says that for a continuous function $f : \mathbb{I} \to \mathbb{I}$, where $\mathbb{I} = [0, 1]$, f has positive entropy iff it has a periodic point of period different from 2^n, $n = 0, 1, \ldots$ (see e.g. [1]). M. Čiklová proved an analogous result for all functions whose graph is a connected G_δ set [3]. R. Pawlak asked recently whether or not an analogous theorem holds for almost continuous functions [8].

In this note we consider both implications contained in Pawlak's problem. In Theorem 3, for a given set $A \subset \mathbb{N}$ with $1 \in A$ we construct an almost continuous function $f : \mathbb{I} \to \mathbb{I}$ such that:

(i) f maps non-degenerate intervals $J \subset \mathbb{I}$ onto the whole $\mathbb{I}$;
(ii) f has only periodic points of periods from A.

This generalizes an old result of Kellum [5]. (Recall that every almost continuous function $f : \mathbb{I} \to \mathbb{I}$ has fixed points.)

Let $A = \{1\}$. Then (i) implies that f has positive entropy and, by (ii), f has no periodic points of period different from 1, so the implication "$\Rightarrow$" in Pawlak's problem does not hold. In Theorem 2 we prove that if $f : \mathbb{I} \to \mathbb{I}$ is Darboux and has a point of prime period different from 2^i, $i = 0, 1, \ldots$, then the entropy of f is positive. This generalizes Theorem 4.8 of [3] and shows that the implication "$\Leftarrow$" in Pawlak's problem is true. The proof of Theorem 2 is based on Lemma 1, in which we employ some ideas which are abstracted from the proof of Sharkovskiĭ's theorem due to Block, Guckenheimer, Misiurewicz and Young (see [4]). These ideas were originally used

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for continuous functions, but it was observed in [10] (see Appendix there; see also proof of the first part of Lemma 4.2 in [11]) that they work also for Darboux functions with the property $\mathcal{I}$. We prove that, in fact, some parts of those proofs work for all Darboux functions.

We will use standard notions and terminology concerning dynamical systems and real functions, following [8] and [3]. A function $f : \mathbb{I} \to \mathbb{I}$ is *Darboux* ($f \in \mathrm{D}$) if f maps intervals onto intervals. f is *almost continuous in the sense of Stallings* ($f \in \mathrm{ACS}$) if every open set $U \subset \mathbb{I} \times \mathbb{I}$ containing f contains also a continuous function $g : \mathbb{I} \to \mathbb{I}$. (We identify a function with its graph.) Recall that $\mathrm{ACS} \subset \mathrm{D}$ ([9]; see also [7]). Moreover, if $f : \mathbb{I} \to \mathbb{I}$ meets each closed set $K \subset \mathbb{I}^2$ with $|\mathrm{dom}(K)| = \mathfrak{c}$ (where $\mathrm{dom}(K)$ denotes the x-projection of K, and $|X|$ denotes the cardinality of X), then $f \in \mathrm{ACS}$ ([6]; see also [7]).

For a function $f : \mathbb{I} \to \mathbb{I}$, $\mathrm{Per}(f)$ denotes the set of all prime periods of f, i.e., $n \in \mathrm{Per}(f)$ iff there is a periodic point $x \in \mathbb{I}$ such that n is the prime period of x.

Let $I, J \subset \mathbb{I}$ be closed intervals. We say that I *f-covers* J if $J \subset f(I)$. We denote this fact by $I \to_f J$ (or $I \to J$, if f is clear from context). We say that a map $f : \mathbb{I} \to \mathbb{I}$ is *turbulent* if there are compact non-overlapping intervals $I, J \subset \mathbb{I}$ such that $I \cup J \subset f(I) \cap f(J)$.

Fix a function $f : \mathbb{I} \to \mathbb{I}$. For $\varepsilon > 0$ and a positive integer n, we say that a set $M \subset \mathbb{I}$ is *(n, ε)-separated* if for any different points $x, y \in M$ there is $i < n$ with $|f^i(x) - f^i(y)| > \varepsilon$. Let $S(n, \varepsilon)$ denote the maximal cardinality of an (n, ε)-separated set. The *topological entropy* of a map f is defined by

$$h(f) = \lim_{\varepsilon \to 0} (\limsup_{n \to \infty} \log S(n, \varepsilon)).$$

See e.g. [2]; compare [8] or [3]. In our considerations it will only be important that every turbulent Darboux map has a positive entropy [8, Proposition 2.4].

LEMMA 1. *Assume $g : \mathbb{I} \to \mathbb{I}$ is Darboux. If g has a periodic point x with odd period $k > 1$, then g^n is turbulent for some $n \in \mathbb{N}$.*

Proof. We may assume that k is the minimal odd number in $\mathrm{Per}(g) \setminus \{1\}$. Let $\{x_1, \ldots, x_k\}$ be the orbit of x with $x_1 < \cdots < x_k$, and let $J = [x_1, x_k]$. Note that g permutes the orbit. Clearly, $g(x_i) \neq x_i$ for each i, thus $g(x_1) > x_1$ and $g(x_k) < x_k$. So, we can choose the largest i for which $g(x_i) > x_i$. Set $I_1 = [x_i, x_{i+1}]$. Since $g(x_{i+1}) < x_{i+1}$, it follows that $g(x_{i+1}) \leq x_i$, and consequently $I_1 \to_g I_1$.

In what follows, by a *basic interval* we will understand any interval of the form $[x_j, x_{j+1}]$. Let $\mathcal{J}$ denote the family of all basic intervals. Set $\mathcal{O}_1 = \{I_1\}$ and for $j > 1$ let $\mathcal{O}_j$ be the family of all $J \in \mathcal{J}$ which are g-covered by some $I \in \mathcal{O}_{j-1}$.

CLAIM 1. *$I_1 \in \mathcal{O}_2$, $\mathcal{O}_2 \neq \{I_1\}$ and $I_1 \to I_2$ for every $I_2 \in \mathcal{O}_2$. Moreover, $\bigcup \mathcal{O}_2$ is an interval.*

In fact, since x has period $k > 2$, either $g(x_{i+1}) \neq x_i$ or $g(x_i) \neq x_{i+1}$. Thus $g(I_1)$ includes at least one basic interval $I_2 \neq I_1$, and for every such interval we have $I_1 \to I_2$. Finally, since g is Darboux, $\bigcup \mathcal{O}_2$ is an interval.

Claim 2. *The sequence $\{\mathcal{O}_l\}_{l\in\mathbb{N}}$ is increasing and $\bigcup \mathcal{O}_l$ is connected for each $l \in \mathbb{N}$.*

Indeed, by Claim 1, $\mathcal{O}_1 \subset \mathcal{O}_2$ and $\bigcup \mathcal{O}_2$ is an interval. Thus $\mathcal{O}_2 \subset \mathcal{O}_3$. We will show that $\bigcup \mathcal{O}_3$ is an interval. Suppose $\bigcup \mathcal{O}_3$ is not connected. Then there exist $a, b \in \mathbb{N}$ with $1 \le a < b \le k$ and $[x_a, x_b] \cap \bigcup \mathcal{O}_3 = \{x_a, x_b\}$. By the Darboux property of g, for every basic interval $[x_j, x_{j+1}] \in \mathcal{O}_2$, $g(x_j)$ and $g(x_{j+1})$ are on the same side of $[x_a, x_b]$. But $\bigcup \mathcal{O}_2$ is connected and so all images of points x_j from $\bigcup \mathcal{O}_2$ lie on the same side of $[x_a, x_b]$. This is impossible because there exist $x'_a, x'_b \in \bigcup \mathcal{O}_2$ such that $f(x'_a) = x_a$ and $f(x'_b) = x_b$.

Proceeding inductively, we verify that $\mathcal{O}_l \subset \mathcal{O}_{l+1}$ and $\bigcup \mathcal{O}_{l+1}$ is connected for each $l \in \mathbb{N}$.

Claim 3. *For every interval $I_{l+1} \in \mathcal{O}_{l+1}$ there is a chain of intervals $I_2, \ldots, I_l$, with $I_j \in \mathcal{O}_j$ for each j, which satisfy $I_1 \to I_2 \to \cdots \to I_l \to I_{l+1}$.*

Claim 4. *$\mathcal{O}_{l+1} = \mathcal{O}_l$ for some $l \le k-1$. For such l the set $\mathcal{O}_l$ contains all basic intervals, and therefore $I_1 \to_{g^l} J$.*

Such an $l \le k-1$ exists because all $\mathcal{O}_l$ are included in $\mathcal{J}$, the sequence $\{\mathcal{O}_l\}_l$ is increasing and $\mathcal{J}$ has only $k-1$ elements. Now, since $x_1 = f^a(x_i)$ and $x_k = f^b(x_i)$ for some $a, b \in \mathbb{N}$, we see that $\mathcal{O}_l$ contains the basic intervals $[x_1, x_2]$ and $[x_{k-1}, x_k]$. By Claim 2, $\bigcup \mathcal{O}_l$ is connected, thus $\mathcal{O}_l$ contains all basic intervals.

Claim 5. *There is at least one basic interval I different from I_1 which g-covers I_1.*

In fact, since k is odd, more of the points x_j lie on one side of I_1, say on the left, than on the other. Hence there is a $j \le i$ such that $g(x_j) \le x_i$. Let j be the least integer with this property. Since $g(x_i) > x_i$, we have $j < i$ and $g(x_{j+1}) \ge x_{j+1}$. Set $I = [x_j, x_{j+1}]$. Then $I_1 \neq I$, and the Darboux property of g yields $I \to I_1$.

Finally, let $n = l+1$. Since J is g^l-covered by I_1, it is g^n-covered by I. Thus I_1 and I are non-overlapping, $I_1 \cup I \subset J$ and $I_1 \cup I \subset J \subset g^n(I_1) \cap g^n(I)$. ∎

Theorem 2. *Assume $f : \mathbb{I} \to \mathbb{I}$ is Darboux. If f has a periodic point x with prime period $2^n k$, where $k > 1$ is odd and $n \ge 0$, then $h(f) > 0$.*

Proof. Set $g = f^{2^n}$; then g has a cycle of prime period k. By Lemma 1, g^m is turbulent for some m, so $h(g^m) > 0$. By [3, Proposition 3.6], $h(g^m) = mh(g)$, so $h(g) > 0$, and consequently $h(f) > 0$. ∎

THEOREM 3. *For every set A of positive integers with $1 \in A$ there exists an almost continuous function $f : \mathbb{I} \to \mathbb{I}$ such that $f(J) = \mathbb{I}$ for each non-degenerate interval $J \subset \mathbb{I}$, and* $\mathrm{Per}(f) = A$.

Proof. Let $\{K_\alpha : \alpha < \mathfrak{c}\}$ be a well-ordering of the family of all closed subsets $K \subset \mathbb{I}^2$ with $|\mathrm{dom}(K)| = \mathfrak{c}$. First, choose a family $\{B_n : n \in A\}$ of pairwise disjoint subsets of $\mathbb{I}$ with $|B_n| = n$ for each $n \in A$. Let $B = \bigcup_{n\in A} B_n$. Note that B is countable.

For $n \in A$ set $B_n = \{b_0^n, b_1^n, \ldots, b_{n-1}^n\}$ and define $f_n : B_n \to B_n$ by the formula $f_n(b_i^n) = b_{i+1}^n$ for $i < n-1$, and $f_n(b_{n-1}^n) = b_0^n$. Let $f' = \bigcup_{n\in A} f_n$. Then f' is a function which permutes the set B. We will extend f' to an almost continuous function $f : \mathbb{I} \to \mathbb{I}$. For every $\alpha < \mathfrak{c}$ choose (by transfinite induction) an $x_\alpha \in \mathbb{I}$ and define $f(x_\alpha)$ such that:

(1) $x_\alpha \in \mathrm{dom}(K_\alpha) \setminus [\bigcup_{\beta<\alpha}\{x_\beta, f(x_\beta)\} \cup B]$;
(2) $(x_\alpha, f(x_\alpha)) \in K_\alpha$.

Let $C = \mathbb{I} \setminus [\{x_\alpha : \alpha < \mathfrak{c}\} \cup B]$. Define $f'' : C \to C$ to be the identity function on C.

Clearly, the relation $f = f' \cup f'' \cup \{(x_\alpha, f(x_\alpha)) : \alpha < \mathfrak{c}\}$ is a function which maps $\mathbb{I}$ into $\mathbb{I}$. By (1) and (2), $f \in \mathrm{ACS}$, and $f(J) = \mathbb{I}$ for each non-degenerate interval $J \subset \mathbb{I}$ (observe that for each closed non-degenerate interval J and every $y \in \mathbb{I}$ the set $J \times \{y\}$ belongs to the family $\{K_\alpha : \alpha < \mathfrak{c}\}$). Since $f' \subset f$, $A \subset \mathrm{Per}(f)$. We will verify that $\mathrm{Per}(f) \subset A$. It is enough to prove that $f{\restriction}(\mathbb{I} \setminus (B \cup C))$ has no periodic points with period $k > 1$. So, suppose that $a \in \mathbb{I} \setminus B$ is a periodic point of f with period $k > 1$. Let $a_0 = a$ and $a_i = f^i(a)$ for $i = 1, \ldots, k-1$, and let $\alpha_i < \mathfrak{c}$ be the ordinal such that $a_i = x_{\alpha_i}$. Observe that $\alpha_0 > \alpha_1 > \cdots > \alpha_{k-1}$, so $\alpha_0 > \alpha_{k-1}$. But $f(a_{k-1}) = a_0 = x_{\alpha_0} \in \{x_\beta : \beta < \alpha_{k-1}\} \subset \{x_\beta : \beta < \alpha_0\}$, in contradiction with (1). ∎

Obviously, if $f : \mathbb{I} \to \mathbb{I}$ maps every non-degenerate interval $J \subset \mathbb{I}$ onto $\mathbb{I}$, then it is turbulent, hence $h(f) > 0$. Thus f constructed above for $A = \{1\}$ solves Pawlak's problem in the negative.

REMARK. Note that for the function f constructed in the proof of Theorem 3 each point $x \in \mathbb{I}$ is *eventually periodic*, i.e., for each x there exists $n \in \mathbb{N}$ such that $f^n(x)$ is a periodic point for f. (See [4].)

Indeed, clearly each $x \in B \cup C$ is a periodic point of f. Now, fix $\alpha < \mathfrak{c}$ and assume that $f(x_\alpha) \neq x_\alpha$. We will show that $f^n(x_\alpha) \in B \cup C$ for some $n < \omega$. Suppose otherwise: $f^n(x_\alpha) \in \mathbb{I} \setminus (B \cup C) = \{x_\beta : \beta < \mathfrak{c}\}$ for all n. For each n let α_n be the ordinal such that $f^n(x_\alpha) = x_{\alpha_n}$. Then $f(x_{\alpha_n}) = x_{\alpha_{n+1}}$ and (1) yields $\alpha_{n+1} < \alpha_n$, thus we obtain an infinite strictly decreasing sequence of ordinals, a contradiction.

Finally, note that the function constructed in the proof of Theorem 3 can be both Lebesgue and Baire measurable. (In fact, it is enough to choose points x_α from some meager and null set which is c-dense in $\mathbb{I}$.) However, such a construction does not work in the case of Borel measurability. Hence we pose the following question.

PROBLEM 4. *Does there exist, for every set* $A \subset \mathbb{N}$ *with* $1 \in A$*, a Borel measurable function* $f : \mathbb{I} \to \mathbb{I}$ *which satisfies all conditions of Theorem* 3 (*i.e.,* $f \in \mathrm{ACS}$*,* $h(f) > 0$*, and* $\mathrm{Per}(f) = A$)?

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