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DIVERGENCE OF GENERAL OPERATORS ON SETS OF MEASURE ZERO

BY

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Abstract. We consider sequences of linear operators U_n with a localization property. It is proved that for any set E of measure zero there exists a set G for which $U_n\mathbb{I}_G(x)$ diverges at each point $x \in E$. This result is a generalization of analogous theorems known for the Fourier sum operators with respect to different orthogonal systems.

In 1876 P. Du Bois-Reymond [5] constructed an example of a continuous function whose trigonometric Fourier series diverges at some point. In 1923 A. N. Kolmogorov [11] proved that for a function from $L^1(\mathbb{T})$ the divergence of the Fourier series can hold everywhere. On the other hand, according to the Carleson–Hunt theorem ([4], [7]) the Fourier series of functions from $L^p(\mathbb{T})$, $p > 1$, converge a.e. A natural question is whether the Fourier series of a function from L^p $(p > 1)$ or C may diverge on an arbitrary given set of measure zero. In fact the investigation of this problem began before Carleson's theorem. First S. B. Stechkin [14] proved in 1951 that for any set $E \subset \mathbb{T}$ of measure zero there exists a function $f \in L^2(0, 2\pi)$ whose Fourier series diverges on E. Then in 1963 L. V. Taĭkov [15] showed that f can be taken from $L^p(0, 2\pi)$ for any $1 \le p < \infty$. In 1965 Kahane and Katznelson [8] proved the existence of a continuous complex valued function whose Fourier series diverges on a given set of measure zero. Essentially developing Kahane–Katznelson's approach V. V. Buzdalin [3] proved that for any set of measure zero there exists a continuous real valued function whose Fourier series diverges on that set. The same question has also been investigated for other classical orthonormal systems. Sh. V. Kheladze [9] constructed a function from $L^p(0, 1)$ $(1 < p < \infty)$ whose Fourier–Walsh series diverges on a given set of measure zero. In another paper [10] he proved the same for Vilenkin systems. Then V. M. Bugadze [1] proved that for the Walsh system the function in question can be taken from L^∞. In fact Bugadze proved the same also for Haar ([2]), Walsh–Paley and Walsh–Kaczmarz sys-

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tems ([1]). The Haar system in such problems was also considered in the papers of M. A. Lunina [12] and V. I. Prokhorenko [13]. Recently U. Goginava [6] proved that for any set of measure zero there exists a bounded function whose Walsh–Fejér means diverges on that set. For other problems concerning divergent Fourier series the reader is referred to the papers of P. L. Ul'yanov [16] and W. L. Wade [17].

In this paper we notice that this phenomenon is common for general sequences of linear operators with a localization property. We consider sequences of linear operators

$$U_n f(x) = \int_a^b K_n(x,t) f(t)\,dt, \qquad n = 1, 2, \ldots, \tag{1}$$

with

$$|K_n(x,t)| \le M_n. \tag{2}$$

We say the sequence (1) has the *localization property* (L-property) if for any $f \in L^1(a,b)$ with $f(x) = 1$ for $x \in I = (\alpha, \beta)$ $(\subset [a,b])$ we have

$$\lim_{n\to\infty} U_n f(x) = 1 \qquad \text{for } x \in I,$$

and the convergence is uniform in each closed set $A \subset I$. We prove the following

Theorem. *If the sequence of operators* (1) *has the localization property, then for any set of measure zero $E \subset [a,b]$ there exists a set $G \subset [a,b]$ such that*

$$\liminf_{n\to\infty} U_n \mathbb{I}_G(x) \le 0, \qquad \limsup_{n\to\infty} U_n \mathbb{I}_G(x) \ge 1 \qquad \textit{for any } x \in E,$$

where $\mathbb{I}_G$ denotes the characteristic function of G.

This theorem can be applied to the Fourier partial sum operators with respect to all classical orthogonal systems (trigonometric, Walsh, Haar, Franklin and Vilenkin systems). Moreover, instead of partial sums we can also discuss linear means of partial sums corresponding to an arbitrary regular summation method $T = \{a_{ij}\}$. It is well known that all these operators have the localization property. So the following corollary is an immediate consequence of the main result.

Corollary. *Let $\Phi = \{\phi_n(x),\, n \in \mathbb{N}\}$, $x \in [a,b]$, be one of the above mentioned orthogonal systems and T an arbitrary regular linear summation method. Then for any set E of measure zero there exists a set $G \subset [a,b]$ such that the Fourier series of its characteristic function $f = \mathbb{I}_G$ with respect to Φ diverges at each point of E for the T-method.*

Remark. The function f in the corollary cannot be continuous in general. There are a variety of sequences of Fourier operators which converge uniformly when f is continuous.

The following lemma gives a bound for the kernels of operators (1) if the U_n have the L-property.

Lemma. *If the sequence of operators U_n has the L-property, then there exists a positive decreasing function $\phi(u)$, $u \in (0, +\infty)$, such that if $x \in [a, b]$ and $n \in \mathbb{N}$ then*

$$|K_n(x,t)| \le \phi(|x-t|) \quad \text{for almost all } t \in [a,b]. \tag{3}$$

Proof. We define

$$\phi(u) = \sup_{n\in\mathbb{N},\, x\in[a,b]} \operatorname*{ess\,sup}_{t:|t-x|\ge u} |K_n(x,t)|,$$

where $\operatorname{ess\,sup}_{t\in A} |g(t)|$ denotes $\|g\|_{L^\infty(A)}$. It is clear that ϕ is decreasing and satisfies (3) provided $\phi(u) < \infty$ for $u > 0$. To prove $\phi(u)$ is finite, suppose the converse, that is, $\phi(u_0) = \infty$ for some $u_0 > 0$. This means that for any $\gamma > 0$ there exist $l_\gamma \in \mathbb{N}$ and $c_\gamma \in [a, b]$ such that

$$|K_{l_\gamma}(c_\gamma, t)| > \gamma, \qquad t \in E_\gamma \subset [a,b] \setminus (c_\gamma - u_0, c_\gamma + u_0), \qquad |E_\gamma| > 0. \tag{4}$$

Consider the sequences c_k and l_k corresponding to the numbers $\gamma_k = k$, $k = 1, 2, \ldots$. We can fix an interval I with $|I| = u_0/3$ which contains infinitely many terms of the sequence $\{c_k\}$. Hence we can suppose that $c_\gamma \in I$ in (4) and therefore $2I \subset (c_\gamma - u_0, c_\gamma + u_0)$. So we can write

$$c_\gamma \in I, \qquad E_\gamma \subset [a,b] \setminus 2I. \tag{5}$$

Then we choose a sequence $\gamma_k \nearrow \infty$ such that for the corresponding sequences $m_k = l_{\gamma_k}$, $x_k = c_{\gamma_k}$ and $E_k = E_{\gamma_k}$ we have

$$x_k \subset I, \qquad E_k \subset (a,b) \setminus 2I, \tag{6}$$

$$|K_{m_k}(x_k, t)| \ge k^3, \qquad t \in E_k, \tag{7}$$

$$\sup_{1\le i<k} |U_{m_k} \mathbb{I}_{E_i}(x)| < 1, \qquad x \in I, \tag{8}$$

$$|E_k| \cdot \max_{1\le i<k} M_{m_i} < 1 \qquad (k > 1). \tag{9}$$

We do this by induction. Taking $\gamma_1 = 1$ we get m_1 satisfying (7). This follows from (4). Now suppose we have already chosen the numbers γ_k and m_k satisfying (6)–(9) for $k = 1, \ldots, p$. According to the L-property, $U_n \mathbb{I}_{E_i}(x)$ converges to 0 uniformly in I for any $i = 1, \ldots, p$. On the other hand, because of (2) and (4), $l_\gamma \to \infty$ as $\gamma \to \infty$. Hence we can choose $\gamma_{p+1} > (p+1)^3$ such that the corresponding m_{p+1} satisfies the inequality

$$|U_{m_{p+1}} \mathbb{I}_{E_i}(x)| < 1, \qquad x \in I,\ i = 1, \ldots, p. \tag{10}$$

This gives (8) in the case $k = p+1$. According to (4) and the bound $\gamma_{p+1} > (p+1)^3$ we also have (7). Finally, taking E_{p+1} with small enough measure we can guarantee (9) for $k = p+1$. So the construction of the sequence γ_k satisfying (6)–(9) is complete. Now consider the function

$$g(x) = \sum_{i=1}^{\infty} \frac{\mathbb{I}_{E_i}(x)}{k^2}. \tag{11}$$

We have $g \in L^1$ and $\operatorname{supp} g \subset [a,b] \setminus 2I$. Since $x_k \in I$, using the relations (6)–(9), we obtain

$$\begin{aligned} |U_{m_k} g(x_k)| &\geq \frac{|U_{m_k}\mathbb{I}_{E_k}(x_k)|}{k^2} - \sum_{i=1}^{k-1} \frac{|U_{m_k}\mathbb{I}_{E_i}(x_k)|}{i^2} - \sum_{i=k+1}^{\infty} \frac{|U_{m_k}\mathbb{I}_{E_i}(x_k)|}{i^2} \\ &\geq k - \sum_{i=1}^{k-1} \frac{1}{i^2} - M_{m_k} \sum_{i=k+1}^{\infty} \frac{|E_i|}{i^2} \geq k - 2. \end{aligned}$$

This is a contradiction, because the convergence $U_n g(x) \to 0$ is uniform on I according to the L-property. ■

We say a family $\mathcal{I}$ of mutually disjoint semi-open intervals is a *regular partition* of an open set $G \subset (a,b)$ if $G = \bigcup_{I\in\mathcal{I}} I$ and each interval $I \in \mathcal{I}$ has two adjacent intervals $I^+, I^- \in \mathcal{I}$ with

$$2I \subset I^* = I \cup I^+ \cup I^-. \tag{12}$$

It is clear that any open set has a regular partition.

Proof of Theorem. For a given set E of measure zero we will construct a specific sequence of open sets G_k with regular partitions $\mathcal{I}_k$, $k = 1, 2, \ldots$. They will satisfy the conditions

1) if $I \in \mathcal{I}_k$ and $I = [\alpha, \beta)$ then $\alpha, \beta \notin E$,
2) if $I, J \in \bigcup_{j=1}^{k} \mathcal{I}_j$ then $J \cap I \in \{\emptyset, I, J\}$,
3) $E \subset G_k \subset G_{k-1}$ $(G_0 = [a,b])$.

In addition, for any interval $I \in \mathcal{I}$ we fix a number $\nu(I) \in \mathbb{N}$ such that

4) if $I, J \in \bigcup_{j=1}^{k} \mathcal{I}_j$ and $I \subset J$ then $\nu(I) \geq \nu(J)$,
5) $\sup_{x\in I} |U_{\nu(I)}\mathbb{I}_{G_l}(x) - 1| < 1/k^2$ if $I \in \mathcal{I}_k$ and $l \leq k$,
6) $\sup_{x\in I} |U_{\nu(I)}\mathbb{I}_{G_k}(x)| < 1/k^2$ if $I \in \mathcal{I}_l$ and $l < k$.

We define G_1 and its partition $\mathcal{I}_1$ arbitrarily, just ensuring condition 1). This can be done because $|E| = 0$ and so E^c is everywhere dense in $[a,b]$. Then using the L-property for any interval $I \in \mathcal{I}_1$ we can find $\nu(I) \in \mathbb{N}$ satisfying 5) for $k = 1$. Now suppose we have already chosen G_k and $\mathcal{I}_k$ satisfying 1)–6) for all $k \leq p$. Obviously we can choose an open set G_{p+1},

$E \subset G_{p+1} \subset G_p$, satisfying 1), 2) and the bound

$$|G_{p+1} \cap I| < \delta(I), \qquad I \in \bigcup_{k=1}^{p} \mathcal{I}_k,$$

where

$$\delta(I) = \frac{1}{6(p+1)^2 \max\{M_{\nu(I)}, M_{\nu(I^+)}, M_{\nu(I^-)}, \phi(|I|/2)/|I|\}},$$

and the function $\phi(u)$ is taken from the lemma. Suppose $I \in \mathcal{I}_l$ and $l < p+1$. We have

$$|U_{\nu(I)}\mathbb{I}_{G_{p+1}}(x)| \leq |U_{\nu(I)}\mathbb{I}_{G_{p+1}\cap I^*}(x)| + |U_{\nu(I)}\mathbb{I}_{G_{p+1}\cap (I^*)^c}(x)|. \tag{13}$$

Using the lemma and the bound

$$\delta(J) \leq \frac{|J|}{6\phi(|J|/2)(p+1)^2}, \qquad J \in \mathcal{I}_l,$$

for any $x \in I$ we get

$$\begin{aligned} |U_{\nu(I)}\mathbb{I}_{G_{p+1}\cap (I^*)^c}(x)| &\leq \sum_{J\in\mathcal{I}_l\,:\,J\neq I, I^+, I^-} \int_{G_{p+1}\cap J} \phi(|x-t|)\,dt \\ &\leq \sum_{J\in\mathcal{I}_l\,:\,J\neq I, I^+, I^-} \int_{G_{p+1}\cap J} \phi(|J|/2)\,dt \\ &\leq \sum_{J\in\mathcal{I}_l\,:\,J\neq I, I^+, I^-} |G_{p+1}\cap J|\phi(|J|/2) \\ &\leq \sum_{J\in\mathcal{I}_l\,:\,J\neq I, I^+, I^-} \delta(J)\phi(|J|/2) \\ &\leq \frac{1}{6(p+1)^2} \sum_{J\in\mathcal{I}_l\,:\,J\neq I, I^+, I^-} |J| \\ &< \frac{1}{6(p+1)^2}. \end{aligned} \tag{14}$$

On the other hand, we have

$$\delta(I), \delta(I^+), \delta(I^-) \leq \frac{1}{6(p+1)^2 M_{\nu(I)}},$$

and therefore

$$\begin{aligned} |U_{\nu(I)}\mathbb{I}_{G_{p+1}\cap I^*}(x)| &\leq M_{\nu(I)}|G_{p+1}\cap I^*| \\ &\leq M_{\nu(I)}(\delta(I) + \delta(I^+) + \delta(I^-)) \leq \frac{1}{2(p+1)^2}, \qquad x \in [a,b]. \end{aligned} \tag{15}$$

Combining (13)–(15) we get 6) in the case $k = p+1$. Now we choose a partition $\mathcal{I}_{p+1}$ satisfying just conditions 1) and 2). Using the L-property we may define numbers $\nu(I)$ for $I \in \mathcal{I}_{p+1}$ satisfying condition 5) with $k = p+1$. Hence the construction of the sets G_k is complete. Now denote

$$G = \bigcup_{i=1}^{\infty} (G_{2i-1} \setminus G_{2i}).$$

We have

$$U_n \mathbb{I}_G(x) = \sum_{k=1}^{\infty} (-1)^{k+1} U_n \mathbb{I}_{G_k}(x).$$

For any $x \in E$ there exists a unique sequence $I_1 \supset I_2 \supset \cdots$, $I_k \in \mathcal{I}_k$, such that $x \in I_k$, $k = 1, 2, \ldots$. According to 6) we have

$$|U_{\nu(I_k)} \mathbb{I}_{G_l}(x)| \le 1/l^2, \quad l > k.$$

From 5) it follows that

$$|U_{\nu(I_k)} \mathbb{I}_{G_l}(x) - 1| \le 1/k^2, \quad l \le k.$$

Thus we obtain

$$\left| U_{\nu(I_k)} \mathbb{I}_G(x) - \sum_{l=1}^{k} (-1)^{l+1} \right| \le \sum_{l=1}^{k} |U_{\nu(I_k)} \mathbb{I}_{G_l}(x) - 1| + \sum_{l=k+1}^{\infty} |U_{\nu(I_k)} \mathbb{I}_{G_l}(x)|$$

$$\le k \cdot \frac{1}{k^2} + \sum_{l=k+1}^{\infty} \frac{1}{l^2} < \frac{2}{k}.$$

Since the sum $\sum_{i=1}^{k} (-1)^{k+1}$ takes values 0 and 1 alternately we get

$$\lim_{t \to \infty} U_{\nu(I_{2t})} \mathbb{I}_G(x) = 0, \quad \lim_{t \to \infty} U_{\nu(I_{2t+1})} \mathbb{I}_G(x) = 1$$

for any $x \in E$. The proof of the Theorem is complete. ■

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