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ON A CONSTRUCTION OF UNIVERSAL HEREDITARILY INDECOMPOSABLE CONTINUA BASED ON THE BAIRE CATEGORY

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Abstract. We give a proof of a theorem of Maćkowiak on the existence of universal n-dimensional hereditarily indecomposable continua, based on the Baire-category method.

1. Introduction. Our terminology follows [2]. All spaces are assumed to be normal. A subset A of the space X is *residual* if its complement $X \setminus A$ is a first category set, equivalently, if A contains a dense G_δ-subset in X. By $C(X, I^\omega)$ we denote the function space of all continuous mappings from X into the Hilbert cube I^ω, endowed with the supremum metric d_{sup}. By a *hereditarily indecomposable* (briefly, HI) compactum we mean a compact topological space X such that for any two intersecting subcontinua of X, one is contained in the other. We say that a space X has the *property* (KM) if for any two disjoint closed sets C and D in X and disjoint open sets U and V in X with $C \subset U$ and $D \subset V$ there exist closed sets X_0, X_1 and X_2 in X such that $X = X_0 \cup X_1 \cup X_2$, $C \subset X_0$, $D \subset X_2$, $X_0 \cap X_1 \subset V$, $X_1 \cap X_2 \subset U$ and $X_0 \cap X_2 = \emptyset$. We call a triple $\langle X_0, X_1, X_2 \rangle$ a *fold* of X for the quadruple $\langle C, D, U, V \rangle$. As proved by J. Krasinkiewicz and P. Minc [6], a compact space is hereditarily indecomposable iff it has the property (KM).

Let us recall that the first examples of hereditarily indecomposable n-dimensional continua were constructed by B. Knaster [5] for $n = 1$ and by R. H. Bing [1] for arbitrary $n = 2, 3, \ldots, \infty$ (for other constructions of such continua see [3] or [11, §3.8]). The existence of universal n-dimensional hereditarily indecomposable metric continua was proved by T. Maćkowiak [10], who used McCord's method of constructing universal continua, which applies inverse limits. The paper [4] of K. P. Hart and E. Pol contains another proof of Maćkowiak's theorem, which exploits a factorization theorem for HI compacta and the fact that the Čech–Stone compactification of a normal

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space with the property (KM) has itself the property (KM) (see Theorems 2.3 and 2.1 in [4]).

We will show that the proof of the theorem of Maćkowiak, as well as the proof of Theorem 1.1 from [4], can be obtained by yet another method, which uses some ideas from [4], but applies the Baire category theorem instead of Theorems 2.3 and 2.1 from [4].

Our proofs are based on the following theorem.

THEOREM 1. *If X is a normal space having the property* (KM), *then the set* $\mathcal{H} = \{f \in C(X, I^\omega) : \overline{f(X)}$ *is hereditarily indecomposable*$\}$ *is a dense* G_δ*-set in the function space* $C(X, I^\omega)$.

We will prove Theorem 1 in Section 2, and in Section 3 we will point out connections between this theorem and some results from [4].

Now we will give some corollaries and applications of Theorem 1.

COROLLARY 1. *Let* $n \in \{1, 2, \ldots, \infty\}$. *If X is a metrizable separable n-dimensional space with the property* (KM), *then the set* $\mathcal{E}$ *consisting of all mappings* $f \in C(X, I^\omega)$ *such that*

(i) f *is an embedding,*
(ii) $\dim \overline{f(X)} \le n$,
(iii) $\overline{f(X)}$ *is hereditarily indecomposable,*

is residual in the function space $C(X, I^\omega)$.

Indeed, the set of mappings satisfying each of the conditions (i), (ii) or (iii) separately is residual in $C(X, I^\omega)$ (see [8, Ch. IV, §44, VI, Theorem 2, and §45, VII, Theorem 4′]). As an immediate consequence of Corollary 1 we obtain

COROLLARY 2 (Proposition 4.4 of [4] for $\tau = \aleph_0$). *Every metrizable separable space X with the property* (KM) *has an HI metric compactification* $\tilde{X}$ *such that* $\dim \tilde{X} \le \dim X$.

In our proof of the Maćkowiak theorem and Theorem 1.1 of [4] we will also use the following proposition (needed only for metrizable separable spaces).

THEOREM 2 ([4, Theorem 3.1]). *Let $f : X \to Y$ be a perfect mapping from a space X onto a strongly zero-dimensional paracompact space Y such that for every $y \in Y$ the fiber $f^{-1}(y)$ is hereditarily indecomposable. Then X has the property* (KM).

COROLLARY 3 (T. Maćkowiak [10]). *For every* $n \in \{1, 2, \ldots, \infty\}$ *there exists a hereditarily indecomposable metric continuum Z_n of dimension n containing a copy of every hereditarily indecomposable metric continuum of dimension at most n.*

Proof. It suffices to modify slightly the proof of the theorem of Maćkowiak given in [4, Corollary 4.1], using Corollary 2 and Theorem 2 instead of Theorem 1.1 of [4]. For the convenience of the reader, let us describe this modification. Let $\mathcal{P}$ be the subset of the hyperspace 2^{I^ω} of the Hilbert cube consisting of all HI continua of dimension n or less. Since $\mathcal{P}$ is a G_δ-subset of 2^{I^ω} (see [8, §45, IV, Theorem 4 and §48, V, Remark 5]), there exists a continuous surjection $\varphi : Y \to \mathcal{P}$, where Y is the space of irrationals. Let $X = \{(x,t) : t \in Y \text{ and } x \in \varphi(t)\}$ be the subspace of $I^\omega \times Y$ and let $\pi : I^\omega \times Y \to Y$ be the projection. Then the restriction $f = \pi|X : X \to Y$ is a perfect map (see [7, §18] or [11, Exercise 1.11.26]) with hereditarily indecomposable fibers, hence X has the property (KM) by Theorem 2. Since the dimension of the fibers of f does not exceed n, we have $\dim X = n$ by a theorem on dimension-lowering mappings (see [2, Theorem 1.12.4]). From Corollary 2 it follows that X has an n-dimensional HI compactification $X^\star$. Now, applying the pseudosuspension method of Maćkowiak, one constructs an n-dimensional HI continuum Z_n containing $X^\star$ (see [4, proof of Corollary 4.1]). Since $X^\star$ contains a copy of every HI continuum of dimension $\leq n$, Z_n satisfies the required conditions.

Note that in the case when $n = \infty$, the above theorem states that there exists a universal hereditarily indecomposable metric continuum. The next corollary is a strengthening of Theorem 1.1 from [4].

Corollary 4. *Let $f : X \to Y$ be a perfect mapping with HI fibers from an n-dimensional metrizable separable space X onto a zero-dimensional metrizable separable space Y. Let $Y^\star$ be any 0-dimensional metric compactification of Y. Then the set $\mathcal{H}$ of all embeddings $h : X \to I^\omega$ such that $\overline{h(X)}$ is HI, $\dim \overline{h(X)} \leq n$ and the mapping $f \circ h^{-1} : h(X) \to Y$ extends to $f^\star : \overline{h(X)} \to Y^\star$ is residual in the function space $C(X, I^\omega)$.*

Proof. By Theorem 2, the space X has the property (KM), so by Corollary 1, the set $\mathcal{E}$ of all embeddings $h : X \to I^\omega$ such that $\overline{h(X)}$ is HI and $\dim \overline{h(X)} \leq n$ is residual in $C(X, I^\omega)$. By Theorem 3.4 of [13], the set $\mathcal{F}$ of all embeddings $h : X \to I^\omega$ such that the map $f \circ h^{-1} : X \to Y$ extends to $f^\star : \overline{h(X)} \to Y^\star$ is residual in the function space $C(X, I^\omega)$. Thus the set $\mathcal{H} = \mathcal{E} \cap \mathcal{F}$ is residual in $C(X, I^\omega)$.

2. Proof of Theorem 1. For the convenience of the reader, we will give all the details of the proof.

Let $\mathcal{F}$ be a countable base for closed sets in I^ω which is closed under finite intersections. Let $\mathcal{D} = \{\langle C, D, U, V\rangle : C \subset U,\ D \subset V,\ U \cap V = \emptyset$ and all $C, D, I^\omega \setminus U, I^\omega \setminus V$ belong to $\mathcal{F}\}$.

For $\mathbf{D} = \langle C, D, U, V\rangle \in \mathcal{D}$ let us define

$$\mathcal{G}_{\mathbf{D}} = \{f \in C(X, I^\omega) : \text{there exists a fold } \langle X_0, X_1, X_2\rangle \text{ in } \overline{f(X)} \\ \text{for } \langle C \cap \overline{f(X)}, D \cap \overline{f(X)},\ U \cap \overline{f(X)}, V \cap \overline{f(X)}\rangle\}.$$

By a theorem of Krasinkiewicz and Minc [6], for $f \in C(X, I^\omega)$, $\overline{f(X)}$ is HI iff $\overline{f(X)}$ has the property (KM), which is equivalent to $f \in \bigcap_{\mathbf{D}\in\mathcal{D}} \mathcal{G}_{\mathbf{D}}$ (as observed in [3], a compact space Z has the property (KM) iff there exists a fold in Z for every quadruple $\langle C, D, U, V\rangle$ such that $C \subset U$, $D \subset V$, $U \cap V = \emptyset$ and the sets $C, D, X \setminus U, X \setminus V$ belong to a given base for closed sets in Z that is closed under finite intersections). Thus $\mathcal{H} = \bigcap_{\mathbf{D}\in\mathcal{D}} \mathcal{G}_{\mathbf{D}}$, and therefore to prove Theorem 1 it suffices to show that for every $\mathbf{D} \in \mathcal{D}$ the set $\mathcal{G}_{\mathbf{D}}$ is open and dense in $C(X, I^\omega)$.

Fix $\mathbf{D} = \langle C, D, U, V\rangle \in \mathcal{D}$. First we will show that

(i) the set $\mathcal{G}_{\mathbf{D}}$ is open in $C(X, I^\omega)$.

Suppose that $f \in \mathcal{G}_{\mathbf{D}}$ and let $\langle X_0, X_1, X_2\rangle$ be a fold in $\overline{f(X)}$ for $\langle C \cap \overline{f(X)}, D \cap \overline{f(X)}, U \cap \overline{f(X)}, V \cap \overline{f(X)}\rangle\}$. For $\epsilon > 0$ and $A \subset I^\omega$ let $B_\epsilon(A) = \{x \in I^\omega : \operatorname{dist}(x, A) < \epsilon\}$ be a ball around A of radius ϵ. Let $\epsilon > 0$ be such that a sequence

$$\mathbf{B} = \langle \overline{B_\epsilon(C)}, \overline{B_\epsilon(D)}, \overline{B_\epsilon(I^\omega \setminus U)}, \overline{B_\epsilon(I^\omega \setminus V)}, \overline{B_\epsilon(X_0)}, \overline{B_\epsilon(X_1)}, \overline{B_\epsilon(X_2)}\rangle$$

forms a swelling of a sequence

$$\mathbf{A} = \langle C, D, I^\omega \setminus U, I^\omega \setminus V, X_0, X_1, X_2\rangle$$

(see [2, Theorem 3.1.1]), i.e. if the intersection of some elements of $\mathbf{A}$ is empty then the intersection of the corresponding elements of $\mathbf{B}$ (i.e., the closures of the ϵ-balls around these elements) is empty.

To prove (i) it suffices to check that for every $g \in C(X, I^\omega)$ such that $d_{\sup}(f, g) < \epsilon/2$, the triple $\langle \overline{B_\epsilon(X_0)} \cap \overline{g(X)}, \overline{B_\epsilon(X_1)} \cap \overline{g(X)}, \overline{B_\epsilon(X_2)} \cap \overline{g(X)}\rangle$ is a fold for the quadruple $\langle C \cap \overline{g(X)}, D \cap \overline{g(X)}, U \cap \overline{g(X)}, V \cap \overline{g(X)}\rangle$ and thus $g \in \mathcal{G}_{\mathbf{D}}$.

Indeed, since $\overline{f(X)} = X_0 \cup X_1 \cup X_2$ and $d_{\sup}(f, g) < \epsilon/2$, it follows that $\overline{g(X)} \subset \overline{B_\epsilon(X_0)} \cup \overline{B_\epsilon(X_1)} \cup \overline{B_\epsilon(X_2)}$.

Recall that $\mathbf{B}$ is a swelling of $\mathbf{A}$. Thus, since $X_0 \cap X_2 = \emptyset$, we have $\overline{B_\epsilon(X_0)} \cap \overline{B_\epsilon(X_2)} = \emptyset$. Moreover, since $C \cap X_1 = \emptyset = C \cap X_2$, it follows that $\overline{B_\epsilon(C)} \cap \overline{B_\epsilon(X_1)} = \emptyset$ and $\overline{B_\epsilon(C)} \cap \overline{B_\epsilon(X_2)} = \emptyset$, hence

$$C \cap \overline{g(X)} \subset \overline{B_\epsilon(C)} \cap \overline{g(X)} \subset \overline{B_\epsilon(X_0)} \cap \overline{g(X)}.$$

Similarly, $D \cap \overline{g(X)} \subset \overline{B_\epsilon(X_2)} \cap \overline{g(X)}$.

Finally, since $X_0 \cap X_1 \cap (I^\omega \setminus V) = \emptyset$, we have

$$\overline{B_\epsilon(X_0)} \cap \overline{B_\epsilon(X_1)} \cap \overline{B_\epsilon(I^\omega \setminus V)} = \emptyset,$$

hence $\overline{B_\epsilon(X_0)} \cap \overline{B_\epsilon(X_1)} \subset V$ and $\overline{B_\epsilon(X_0)} \cap \overline{B_\epsilon(X_1)} \cap \overline{g(X)} \subset V \cap \overline{g(X)}$. Similarly, $\overline{B_\epsilon(X_1)} \cap \overline{B_\epsilon(X_2)} \cap \overline{g(X)} \subset U \cap \overline{g(X)}$. This ends the proof of (i).

Let us show now that

(ii) $\mathcal{G}_{\mathbf{D}}$ is dense in $C(X, I^\omega)$.

Take an arbitrary $f \in C(X, I^\omega)$ and $\eta > 0$. Pick $\epsilon > 0$, $\epsilon < \eta$, such that

(1) $B_{6\epsilon}(C) \subset U$ and $B_{6\epsilon}(D) \subset V$.

Since X satisfies the condition (KM), there is a fold $\langle X_0, X_1, X_2 \rangle$ in X for the quadruple $\langle f^{-1}(\overline{B_{2\epsilon}(C)}), f^{-1}(\overline{B_{2\epsilon}(D)}), f^{-1}(B_{4\epsilon}(C)), f^{-1}(B_{4\epsilon}(D)) \rangle$, i.e.,

(2) $X = X_0 \cup X_1 \cup X_2$, $f^{-1}(\overline{B_{2\epsilon}(C)}) \subset X_0$, $f^{-1}(\overline{B_{2\epsilon}(D)}) \subset X_2$, $X_0 \cap X_2 = \emptyset$, $X_0 \cap X_1 \subset f^{-1}(B_{4\epsilon}(D))$ and $X_1 \cap X_2 \subset f^{-1}(B_{4\epsilon}(C))$.

Let us note that

(3) if A and B are two closed sets in a normal space X then the set of all mappings $g : X \to I^\omega$ such that $\overline{g(A \cap B)} = \overline{g(A)} \cap \overline{g(B)}$ is a dense G_δ-set in $C(X, I^\omega)$.

For the proof of (3) see for example [9, Lemma 1.2] (where this fact is stated for a metrizable separable space X, but the proof is valid for a normal space X). Note also that the proof of (3) can be extracted from the proof of Theorem in K. Morita's paper [12], where (3) was proved for n-dimensional metrizable separable space X and I^{2n+1} instead of I^ω. For the special case of (3) when $A \cap B = \emptyset$ see also [8, Ch. IV, §45, VII, Theorem 4′].

By (3), the sets

$$\mathcal{F}_1 = \{g \in C(X, I^\omega) : \overline{g(X_0)} \cap \overline{g(X_2)} = \emptyset\},$$
$$\mathcal{F}_2 = \{g \in C(X, I^\omega) : \overline{g(X_0 \cap X_1)} = \overline{g(X_0)} \cap \overline{g(X_1)}\},$$
$$\mathcal{F}_3 = \{g \in C(X, I^\omega) : \overline{g(X_1 \cap X_2)} = \overline{g(X_1)} \cap \overline{g(X_2)}\}$$

are dense G_δ-sets in $C(X, I^\omega)$.

Thus, by the Baire theorem, the set $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ is dense in $C(X, I^\omega)$. Therefore there exists $g \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ such that $d_{\sup}(f, g) < \epsilon$. We will show that $\langle \overline{g(X_0)}, \overline{g(X_1)}, \overline{g(X_2)} \rangle$ is a fold in $\overline{g(X)}$ for $\langle C \cap \overline{g(X)}, D \cap \overline{g(X)}, U \cap \overline{g(X)}, V \cap \overline{g(X)} \rangle$, which implies that $g \in \mathcal{G}_{\mathbf{D}}$.

Obviously, $\overline{g(X)} = \overline{g(X_0)} \cup \overline{g(X_1)} \cup \overline{g(X_1)}$ and $\overline{g(X_0)} \cap \overline{g(X_2)} = \emptyset$, since $g \in \mathcal{F}_1$. It remains to show that

(4) $\overline{g(X_0)} \cap \overline{g(X_1)} \subset V$,

(5) $\overline{g(X_1)} \cap \overline{g(X_2)} \subset U$,

(6) $C \cap \overline{g(X)} \subset \overline{g(X_0)}$,

(7) $D \cap \overline{g(X)} \subset \overline{g(X_2)}$.

To show (4) let us note that since $X_0 \cap X_1 \subset f^{-1}(B_{4\epsilon}(D))$, we have $f(X_0 \cap X_1) \subset B_{4\epsilon}(D)$, hence $g(X_0 \cap X_1) \subset B_\epsilon(f(X_0 \cap X_1)) \subset B_{5\epsilon}(D)$, and thus, by $g \in \mathcal{F}_2$ and (1),

$$\overline{g(X_0)} \cap \overline{g(X_1)} = \overline{g(X_0 \cap X_1)} \subset \overline{B_{5\epsilon}(D)} \subset B_{6\epsilon}(D) \subset V.$$

Similarly, replacing X_0 by X_2, D by C and V by U, one proves (5).

Now, let us check (6). Observe that $f^{-1}(\overline{B_{2\epsilon}(C)}) \cap X_1 = \emptyset$, so $\overline{B_{2\epsilon}(C)} \cap f(X_1) = \emptyset$. Since $g(X_1) \subset B_\epsilon(f(X_1))$, we have $g(X_1) \subset B_\epsilon(I^\omega \setminus \overline{B_{2\epsilon}(C)})$. Obviously, $\overline{B_\epsilon(I^\omega \setminus \overline{B_{2\epsilon}(C)})} \cap C = \emptyset$, hence $\overline{g(X_1)} \cap C = \emptyset$. Similarly, since $f^{-1}(\overline{B_{2\epsilon}(C)}) \cap X_2 = \emptyset$, replacing X_1 by X_2 we get $\overline{g(X_2)} \cap C = \emptyset$, so $C \cap \overline{g(X)} \subset \overline{g(X_0)}$, which is (6).

To prove (7) we proceed similarly, replacing X_0 by X_2 and C by D.

3. Comments. The following remarks show that the method of constructing universal hereditarily indecomposable n-dimensional continua presented in this paper and the approach from [4] are closely related and in some sense equivalent.

REMARK 1. Let us show how to obtain Theorem 1 using the results of [4].

Suppose that X is a normal space satisfying the condition (KM) and let $\mathcal{H} = \{f \in C(X, I^\omega) : \overline{f(X)}$ is hereditarily indecomposable$\}$. Since the set of all HI compacta in the hyperspace 2^{I^ω} of the Hilbert cube is a G_δ-set, $\mathcal{H}$ is a G_δ-set in $C(X, I^\omega)$ (cf. [8, §44, V, Theorem 4]). Thus to prove Theorem 1 it suffices to show that $\mathcal{H}$ is dense in $C(X, I^\omega)$. This follows from Theorems 2.1 and 2.3 in [4]. Indeed, take any $f \in C(X, I^\omega)$ and $\epsilon > 0$. Suppose that $X \subset \beta X$ and f^β is an extension of f onto βX. By [4, Theorem 2.1], βX is HI and by [4, Theorem 2.3] (for $Y = I^\omega$) there exists an HI metric compactum Z and mappings $g : \beta X \to Z$ and $h : Z \to I^\omega$ such that $f^\beta = h \circ g$. Let $h' : Z \to I^\omega$ be an embedding such that $d_{\sup}(h', h) < \epsilon$. Then $f' = h' \circ f^\beta | X$ is a mapping which is ϵ-close to f and $f' \in \mathcal{H}$.

REMARK 2. Note that the special case of Theorem 2.3 of [4] when Y is a compact metric space can be strengthened in the following way:

Let $f : X \to Y$ be a continuous surjection of a hereditarily indecomposable compact space onto a compact metric space Y. Then the set $\mathcal{H}$ of mappings $g : X \to I^\omega$ such that

(i) *$g(X)$ is hereditarily indecomposable,*
(ii) $\dim g(X) \le \dim X$,
(iii) *there exists a continuous map $h : g(X) \to Y$ such that $f = h \circ g$,*

is residual in $C(X, I^\omega)$.

Indeed, the residuality of the set of mappings $g : X \to I^\omega$ satisfying (i) follows from Theorem 1, and the residuality of the set of mappings satisfying (ii) and (iii) follows from Corollary 4.3 in [14] (where we put $\tau = \aleph_0$ and observe that in this case the space $J(\aleph_0)^\omega$ can be replaced by I^ω).

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