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REARRANGEMENT ESTIMATES OF THE AREA INTEGRALS

BY

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Abstract. We derive weighted rearrangement estimates for a large class of area integrals. The main approach used earlier to study these questions is based on distribution function inequalities.

Introduction. For a harmonic function u on the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$, define the *Lusin area integral* and the *nontangential maximal function* by

$$A_\alpha u(x) = \Big(\int\limits_{\Gamma_\alpha(x)} t^{1-n} |\nabla u(y,t)|^2 \, dy \, dt \Big)^{1/2},$$

$$N_\alpha u(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} |u(y,t)|,$$

where $\Gamma_a(x)$ is the cone with vertex at $x \in \mathbb{R}^n$ and aperture α. That is,

$$\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}.$$

A well known and important result due to Burkholder and Gundy [2] and Fefferman and Stein [4] states that

$$\|A_a u\|_p \le c_{p,n,\alpha,\beta} \|N_\beta u\|_p \quad (0<p<\infty,\ 0<\alpha,\ \beta<\infty). \tag{1}$$

The following generalization of this inequality was proved by Kaneko [6]. Consider the differential operator L defined by

$$L = \sum_{j,k=1}^{n+1} a_{jk} \frac{\partial^2}{\partial y_j \partial y_k} + \sum_{j=1}^{n+1} b_j t^{-1} \frac{\partial}{\partial y_j},$$

where y_{n+1} denotes the $(n+1)$st variable t, and the differentiation is taken in the sense of distributions. For $v \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ such that Lv is a positive Borel measure μ_{Lv} on $\mathbb{R}^{n+1}_+$, we define the area integral $S_\alpha v$ by

$$S_\alpha v(x) = \int\limits_{\Gamma_\alpha(x)} t^{1-n} \, d\mu_{Lv}(y,t).$$

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Then, by [6],

$$\|S_a v\|_p \le c_{p,n,\alpha,\beta,L}\|N_\beta v\|_p \quad (0<p<\infty,\ 0<\alpha,\beta<\infty). \tag{2}$$

If u is harmonic on $\mathbb{R}^{n+1}_+$ and $L=\Delta$ is the Laplacian, then $L|u|^2 = 2|\nabla u|^2$. Hence, taking $L=\Delta$, $v=|u|^2$ in (2), we obtain (1). More generally, if $L=\Delta$ and v is subharmonic, then (2) gives the estimate proved by McConnell [8] for a limited range of p and by Uchiyama [10] for all $p>0$.

Note that the proofs of (1), (2) in the above-mentioned papers are based on good-λ inequalities (see also [5, 9], where the same technique was used).

In this paper we prove weighted rearrangement estimates relating $S_\alpha v$ and $N_\beta v$. The key lemma used here deals with an abstract analogue of the area integral, so our method can be applied to more general cases.

Our main result is the following.

THEOREM. *Let v be a locally integrable function on $\mathbb{R}^{n+1}_+$ such that Lv is a positive Borel measure on $\mathbb{R}^{n+1}_+$. Let ω be a weight satisfying the A_∞ condition. Then for $0<\delta\le 1$ we have*

$$((S_\alpha v)^\delta)^*_\omega(t) \le c_1 \int_{c_2 t}^{\infty} ((N_\beta v)^\delta)^*_\omega(\tau)\,\frac{d\tau}{\tau} \quad (t>0,\ 0<\alpha,\beta<\infty), \tag{3}$$

where c_1, c_2 depend only on n, α, β, L and ω.

It is clear that (3) and Hardy's inequality [1, p. 124] give a weighted version of (2).

When $\delta = 1/2$, $L=\Delta$, $v=|u|^2$ and u is harmonic, we obtain

COROLLARY. *Let u be a harmonic function on $\mathbb{R}^{n+1}_+$ and let $\omega\in A_\infty$. Then*

$$(A_\alpha u)^*_\omega(t) \le c_1 \int_{c_2 t}^{\infty} (N_\beta u)^*_\omega(\tau)\,\frac{d\tau}{\tau} \quad (t>0,\ 0<\alpha,\beta<\infty), \tag{4}$$

where c_1, c_2 depend only on n, α, β and ω.

We mention that in the case when u is the Poisson integral of f, $u = f * P_t(y)$, a weaker result was proved in [7] with the Hardy–Littlewood maximal function Mf instead of $N_\beta u$ on the right hand side of (4).

1. Definitions and the main Lemma. We recall that the Hardy–Littlewood maximal function is defined by $Mf(x) = \sup |Q|^{-1}\int_Q |f(y)|\,dy$, where the supremum is taken over all cubes Q containing x.

Let ω be a non-negative, locally integrable function on $\mathbb{R}^n$. Given a measurable set E, let $\omega(E) = \int_E \omega(x)\,dx$. We say that ω satisfies *Muckenhoupt's condition* A_∞ if there exist $c,\xi>0$ so that for any cube Q and $E\subset Q$,

$$\omega(E) \le c(|E|/|Q|)^\xi \omega(Q).$$

It is well known (see, for example, [3]) that a weight ω belongs to A_∞ iff there exist $k, r \geq 1$ so that

$$\omega\{x : Mf(x) > \lambda\} \leq \frac{k}{\lambda^r} \int_{\mathbb{R}^n} |f(x)|^r \omega(x)\, dx \tag{5}$$

for all measurable functions f on $\mathbb{R}^n$ and all $\lambda > 0$.

The *non-increasing rearrangement* of f with respect to ω is defined by

$$f^*_\omega(t) = \sup_{\omega(E)=t} \inf_{x\in E} |f(x)| \qquad (0 < t < \infty).$$

Let F be an arbitrary measurable function on $\mathbb{R}^{n+1}_+$. We define its non-tangential maximal function by

$$N_\alpha F(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} |F(y,t)|.$$

It is easy to show (see, for example, [2]) that if $\beta > \alpha$, then

$$\{x : N_\beta F(x) > \lambda\} \subset \{x : M\chi_{\{N_\alpha F > \lambda\}}(x) > c_{n,\alpha,\beta}\}$$

for all $\lambda > 0$. From this and (5) we immediately get

$$(N_\beta F)^*_\omega(t) \leq (N_\alpha F)^*_\omega(ct) \qquad (t > 0,\ \alpha < \beta), \tag{6}$$

where c depends only on n, α, β and ω.

For a positive Borel measure μ on $\mathbb{R}^{n+1}_+$, define its "area integral" by

$$S_\alpha(\mu)(x) = \int_{\Gamma_\alpha(x)} d\mu(y,t).$$

Denote by $\Gamma^h_\alpha(x)$ the truncated cone, that is,

$$\Gamma^h_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < \alpha t,\ 0 < t < h\}.$$

Now we can formulate our main Lemma.

LEMMA. *Let F be an arbitrary measurable function and μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. Suppose that for any $\alpha > 0$ there exist constants $\alpha', c > 0$ so that*

$$\int_{\bigcup_{x\in E} \Gamma^{\ell_Q}_\alpha(x)} t^n\, d\mu(y,t) \leq c|Q| \sup_{x\in E} N_{\alpha'} F(x) \tag{7}$$

for each cube $Q \subset \mathbb{R}^n$ and $E \subset Q$, where ℓ_Q denotes the side length of Q. Let ω be a weight in the class $A_\infty(\mathbb{R}^n)$. Then for any $\alpha, \beta > 0$ and $0 < \delta \leq 1$ we have

$$((S_\alpha(\mu))^\delta)^*_\omega(t) \leq c_1 \int_{c_2 t}^{\infty} ((N_\beta F)^\delta)^*_\omega(s)\, \frac{ds}{s}$$

for all $t > 0$, where c_1 and c_2 depend only on n, α, β and ω.

Proof. Assume first that μ has compact support in $\mathbb{R}^{n+1}_+$. Let $B(x,t)$ denote the ball in $\mathbb{R}^n$ with center x and radius t. Take a C^∞ function φ on $\mathbb{R}^n$ satisfying $\chi_{B(0,\alpha)} \le \varphi \le \chi_{B(0,2\alpha)}$, and define

$$\widetilde{S}_\alpha(\mu)(x) = \int_{\mathbb{R}^{n+1}_+} \varphi\left(\frac{x-y}{t}\right) d\mu(y,t).$$

Clearly, $S_\alpha(\mu)(x) \le \widetilde{S}_\alpha(\mu)(x)$.

Let E be a set of ω-measure t such that $\widetilde{S}_\alpha(\mu)(x) \ge (\widetilde{S}_\alpha(\mu))^*_\omega(t)$ for any $x \in E$. Choose λ so that $c(2^n\lambda)^\xi = 1/3$, where c, ξ are the constants from the definition of A_∞, and apply the Calderón–Zygmund decomposition to the function χ_E and number λ. We get pairwise disjoint cubes Q_i such that $\lambda|Q_i| < |E \cap Q_i|$ and $\sum_i \omega(Q_i) \ge 3t$.

Let $\gamma = (\lambda/2)^r/(2k)$, where k, r are the constants from (5). In accordance with (7) take $\overline{\alpha} = (2\alpha + \sqrt{n})'$, and set $E_1 = \{x : \widetilde{S}_\alpha(\mu)(x) > (\widetilde{S}_\alpha(\mu))^*_\omega(2t)\}$, $E_2 = \{x : M\chi_{\{y : N_{\overline{\alpha}}F(y) > (N_{\overline{\alpha}}F)^*_\omega(\gamma t)\}}(x) > \lambda/2\}$. Observe that $\omega(E_1) \le 2t$ and, by (5), $\omega(E_2) \le t/2$. Hence, among Q_i there is a cube Q' such that $|Q' \setminus (E_1 \cup E_2)| > 0$. Take a point $x' \in Q' \setminus E_1$. We deduce that for all x, $\widetilde{S}_\alpha(\mu)(x)$ is bounded by $(\widetilde{S}_\alpha(\mu))^*_\omega(2t)$ and

$$\int_0^{\ell_{Q'}} \int_{\mathbb{R}^n} \varphi\left(\frac{x-y}{t}\right) d\mu(y,t) + \int_{\ell_{Q'}}^\infty \int_{\mathbb{R}^n} \left(\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x'-y}{t}\right)\right) d\mu(y,t).$$

By properties of φ, if $x \in Q'$, these integrals are majorized by

$$\int_{\Gamma^{\ell_{Q'}}_{2\alpha}(x)} d\mu(y,t) + c\ell_{Q'} \sum_{j=0}^\infty \int_{2^j\ell_{Q'}}^{2^{j+1}\ell_{Q'}} \int_{B(x,2\alpha t)\cup B(x',2\alpha t)} \frac{1}{t}\, d\mu(y,t). \tag{8}$$

Let $\mathfrak{F}$ be the set of all x such that $N_{\overline{\alpha}}F(x) \le (N_{\overline{\alpha}}F)^*_\omega(\gamma t)$. Since $Q' \cap E_2^c \neq \emptyset$, we have $|\mathfrak{F} \cap 2^jQ'| \ge (1-\lambda/2)|2^jQ'|$ $(j = 0, 1, \ldots)$. Note that for all $x \in Q'$, $\eta \in 2^jQ'$ and $t \ge 2^j\ell_{Q'}$ we get $|x - \eta| \le 2^j\sqrt{n}\,\ell_{Q'} \le \sqrt{n}\,t$, and hence $B(x, 2\alpha t) \cup B(x', 2\alpha t) \subset B(\eta, (2\alpha + \sqrt{n})t)$. Thus, setting $D_j = \bigcup_{\eta \in \mathfrak{F}\cap 2^jQ'} \Gamma^{2^{j+1}\ell_{Q'}}_{2\alpha+\sqrt{n}}(\eta)$, using (7) and Fubini's theorem, we see that the second term in (8) is majorized by

$$\begin{aligned} &c\sum_{j=0}^\infty \frac{1}{2^j} \inf_{\eta \in \mathfrak{F}\cap 2^jQ'} \int_{\Gamma^{2^{j+1}\ell_{Q'}}_{2\alpha+\sqrt{n}}(\eta)} d\mu(y,t) \\ &\qquad \le c\sum_{j=0}^\infty \frac{1}{2^j|\mathfrak{F}\cap 2^jQ'|} \int_{D_j} t^n\, d\mu(y,t) \le c(N_{\overline{\alpha}}F)^*_\omega(\gamma t). \end{aligned} \tag{9}$$

Next, observe that $|\mathfrak{F} \cap E \cap Q'| \geq \lambda|Q'|/2$ since $|E \cap Q'| > \lambda|Q'|$. Hence, exactly as above, we get

$$\inf_{x \in \mathfrak{F} \cap E \cap Q'} \int_{\Gamma_{2\alpha}^{\ell_{Q'}}(x)} d\mu(y,t) \leq c(N_{\overline{\alpha}}F)^*_\omega(\gamma t). \tag{10}$$

Combining (8)–(10), we obtain

$$\begin{aligned}((\widetilde{S}_\alpha(\mu))^\delta)^*_\omega(t) &\leq \inf_{x \in E \cap Q'} (\widetilde{S}_\alpha(\mu)(x))^\delta \leq (c(N_{\overline{\alpha}}F)^*_\omega(\gamma t) + (\widetilde{S}_\alpha(\mu))^*_\omega(2t))^\delta \\ &\leq c((N_{\overline{\alpha}}F)^\delta)^*_\omega(\gamma t) + ((\widetilde{S}_\alpha(\mu))^\delta)^*_\omega(2t).\end{aligned}$$

Notice that $\widetilde{S}_\alpha(\mu)$ has compact support, because μ does, and therefore $(\widetilde{S}_\alpha(\mu))^*_\omega(+\infty) = 0$. Thus, iterating the last estimate gives

$$((S_\alpha(\mu))^\delta)^*_\omega(t) \leq ((\widetilde{S}_\alpha(\mu))^\delta)^*_\omega(t) \leq c \int_{\gamma t/2}^\infty ((N_{\overline{\alpha}}F)^\delta)^*_\omega(s)\,\frac{ds}{s}.$$

The assumption on μ is easily removed by taking an increasing sequence $\mu_i \uparrow \mu$ with compact support, and using the fact that $|f_i| \uparrow |f|$ implies $(f_i)^*_\omega(t) \uparrow f^*_\omega(t)$ (see [1, p. 41]). Now invoke the estimate (6) to conclude the proof of the Lemma.

2. Proof of Theorem. Take $d\mu(y,t) = t^{1-n} d\mu_{Lv}(y,t)$. In view of the Lemma, to prove the Theorem, it suffices to show that for each $E \subset Q$,

$$\int_{\bigcup_{x \in E} \Gamma_\alpha^{\ell_Q}(x)} t\,d\mu_{Lv}(y,t) \leq c|Q| \sup_{x \in E} N_{\alpha'} v(x). \tag{11}$$

Essentially, this was proved in [6]. We give a sligthly different proof here.

Define $d(x) = \operatorname{dist}(x, E)$. Then $d(x)$ is a Lipschitz function; moreover, $|d(x) - d(x')| \leq |x - x'|$. Let $G_\xi = \{y \in (1 + 2\alpha)Q : d(y) < \xi\}$. By Fubini's theorem,

$$\begin{aligned}\int_{\bigcup_{x \in E} \Gamma_\alpha^{\ell_Q}(x)} t\,d\mu_{Lv}(y,t) &\leq \int_0^{\ell_Q} \int_{G_{\alpha t}} t\,d\mu_{Lv}(y,t) \\ &\leq \sum_{j=0}^\infty \frac{\ell_Q}{2^j} \int_{\ell_Q/2^{j+1}}^{\ell_Q/2^j} \int_{G_{\alpha \ell_Q/2^j}} d\mu_{Lv}(y,t).\end{aligned} \tag{12}$$

Take a non-negative C^∞ function φ on $\mathbb{R}^n$ such that $\operatorname{supp}\varphi \subset B(0,1)$ and $\|\varphi\|_1 = 1$. Set $\varphi_t(x) = \varphi(x/t)t^{-n}$ and $\Phi_j(y,t) = \chi_{G_{3\alpha\ell_Q/2^{j+1}}} * \varphi_{\alpha t/4}(y)$. By the Lipschitz property of d we know that for $t \leq \ell_Q/2^{j-1}$,

$$\chi_{G_{\alpha\ell_Q/2^j}}(y) \leq \Phi_j(y,t) \leq \chi_{G_{\alpha\ell_Q/2^{j-1}}}(y).$$

Next, take a non-negative C^∞ function η on $\mathbb{R}$ such that $\chi_{(1/2,1)}(t) \le \eta(t) \le \chi_{(1/4,2)}(t)$. Since $\Phi_j(y,t)\eta(t)$ is a Schwartz function, we deduce that the right hand side of (12) is bounded by

$$\sum_{j=0}^{\infty} \frac{\ell_Q}{2^j} \int_{\mathbb{R}^{n+1}_+} \Phi_j(y,t)\eta(2^j t/\ell_Q)\, d\mu_{Lv}(y,t) \tag{13}$$

$$= \sum_{j=0}^{\infty} \frac{\ell_Q}{2^j} \int_{\mathbb{R}^{n+1}_+} L^*(\Phi_j(y,t)\eta(2^j t/\ell_Q))v(y,t)\, dy\, dt,$$

where

$$L^* = \sum_{j,k=1}^{n+1} a_{jk} \frac{\partial^2}{\partial y_j \partial y_k} - \sum_{j=1}^{n+1} b_j \frac{\partial}{\partial y_j} \frac{1}{t}.$$

It is easy to see that

$$|L^*(\Phi_j(y,t)\eta(2^j t/\ell_Q))| \le (c/t^2)\chi_{D_j}(y,t),$$

where

$$D_j = (G_{\alpha\ell_Q/2^{j-1}} \setminus G_{\alpha\ell_Q/2^j}) \times (\ell_Q/2^{j+2}, \ell_Q/2^{j-1}).$$

If $(y,t) \in D_j$, then there exist $x \in E$ so that $|y - x| \le 2d(y) < 4\alpha\ell_Q/2^j \le 16\alpha t$, and therefore $|v(y,t)| \le \sup_{x\in E} N_{16\alpha}v(x)$. Thus, the expression in (13) is majorized by

$$c \sup_{x\in E} N_{16\alpha}v(x) \sum_{j=0}^{\infty} \frac{\ell_Q}{2^j} \int_{\ell_Q/2^{j+2}}^{\ell_Q/2^{j-1}} \frac{dt}{t^2} \int_{G_{\alpha\ell_Q/2^{j-1}} \setminus G_{\alpha\ell_Q/2^j}} dy$$

$$\le c \sup_{x\in E} N_{16\alpha}v(x) \sum_{j=0}^{\infty} |G_{\alpha\ell_Q/2^{j-1}} \setminus G_{\alpha\ell_Q/2^j}| \le c|Q| \sup_{x\in E} N_{16\alpha}v(x),$$

which proves the Theorem.

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