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AFFINE SPACES AS MODELS FOR REGULAR IDENTITIES

BY

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Abstract. In [7] and [8], two sets of regular identities without finite proper models were introduced. In this paper we show that deleting one identity from any of these sets, we obtain a set of regular identities whose models include all affine spaces over $\mathrm{GF}(p)$ for prime numbers $p \geq 5$. Moreover, we prove that this set characterizes affine spaces over $\mathrm{GF}(5)$ in the sense that each proper model of these regular identities has at least 13 ternary term functions and the number 13 is attained if and only if the model is equivalent to an affine space over $\mathrm{GF}(5)$.

1. Introduction. Axioms without finite models have attracted much attention because of their peculiar behavior, especially when the axioms are regular [1, 7, 8, 12, 15]. The characterization of affine spaces by their p_n-sequences has also been an active research area [3, 5, 14].

In this paper, we will show that a proper algebra $(A, +, \circ)$ of type $(2, 2)$ satisfying the identities $x + x = x$, $x \circ x = x$, $x + y = y + x$ and $(x + y) \circ z = (x + z) \circ y$ has at least 13 essentially ternary term functions and, moreover, it has exactly 13 essentially ternary term functions if and only if it is term equivalent to a nontrivial affine space over $\mathrm{GF}(5)$.

If we add the identity $x \circ y = y \circ x$ or $x \circ (y + z) = y \circ (x + z)$, then there exist no finite proper models for those identities and all models have infinitely many essentially n-ary term functions for all $n \geq 2$. In another respect, this result shows that affine spaces over prime fields can be defined as proper algebras of type $(2, 2)$ and as models for regular identities lying at the boundary of the finite and the infinite. This paper might be the first result characterizing affine spaces in this way.

For an algebra $\mathbf{A} = (A, F)$, we denote by $p_n(\mathbf{A})$ the number of all essentially n-ary term functions (or simply "terms") of $\mathbf{A}$ for all $n \geq 0$. By the *p_n-sequence of* $\mathbf{A}$, we mean the sequence $(p_0(\mathbf{A}), p_1(\mathbf{A}), p_2(\mathbf{A}), \ldots)$. We say that $\mathbf{A}$ is *term infinite* if $p_n(\mathbf{A})$ is infinite for all $n \geq 2$. Of course, term infinite algebras are infinite but not conversely. Two algebras are said to be

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term equivalent if they have the same set of term functions. It is clear that term equivalent algebras have the same p_n-sequence. Frequently, we regard term equivalent algebras as the same algebra.

An algebra is called *proper* if all fundamental operations are pairwise distinct and every n-ary fundamental operation is essentially n-ary for all $n \geq 1$. An identity is called *regular* if both sides of the identity involve the same variables.

Let $(A, +)$ be a vector space over a field K. For all natural numbers n and for all $\alpha_1, \ldots, \alpha_n$ in K with $\sum_{i=1}^{n} \alpha_i = 1$, define an operation f on A by $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i$ and let F be the set of all such operations; then the algebra (A, F) is called an *affine space* over K ([3]).

For other terminology, we refer the readers to [9] and [10].

The main result of this paper is the following.

THEOREM 1.1. *Let $(A, +, \circ)$ be a proper algebra of type $(2, 2)$ satisfying the following regular identities*:

(A1) $x + x = x, \quad x \circ x = x,$
(A2) $x + y = y + x,$
(A3) $(x + y) \circ z = (x + z) \circ y \quad$ (*or* $x \circ (y + z) = y \circ (x + z)$).

Then $p_3(A, +, \circ) \geq 13$ and, furthermore, $p_3(A, +, \circ) = 13$ if and only if $(A, +, \circ)$ is term equivalent to a nontrivial affine space over $\mathrm{GF}(5)$.

We prove the theorem in the next section by careful manipulation of identities and counting terms in a series of lemmas.

The following is immediate from [11] and [13].

LEMMA 1.1. *Let (A, F) be an affine space over* $\mathrm{GF}(p)$ *for a prime number p. If $F' \subseteq F$ and F' contains an essentially binary operation then (A, F') is term equivalent to (A, F).*

THEOREM 1.2. *Affine spaces over* $\mathrm{GF}(p)$ *for prime numbers $p \geq 5$ are proper models of the identities* (A1)–(A3).

Proof. Let $(A, +, \mathrm{GF}(p))$ be a vector space over $\mathrm{GF}(p)$ for a prime number $p \geq 5$. By elementary number theory, there is a unique $\alpha \in \mathbb{Z}_p$ such that

$$\frac{p+1}{2}\alpha = (1 - \alpha) \pmod{p},$$

and this α is different from $(p + 1)/2$. Define binary operations "$\oplus$" and "$\circ$" by

$$x \oplus y = \frac{p+1}{2}x + \frac{p+1}{2}y \quad \text{and} \quad x \circ y = \alpha x + (1 - \alpha)y.$$

Then it can be checked that the algebra $(A, \oplus, \circ)$ satisfies the identities (A1)–(A3). By Lemma 1.1, the algebras $(A, \oplus)$, $(A, \circ)$ and $(A, \oplus, \circ)$ are term equivalent to the affine space (A, F) over $\mathrm{GF}(p)$. Thus, affine spaces over GF(p) are models of the identities (A1)–(A3). Clearly $(A, \oplus, \circ)$ is a proper algebra. ■

Note that

$$p_n(\mathbf{A}) = \frac{(p-1)^n - (-1)^n}{p} \quad \text{for all } n \geq 0$$

if $\mathbf{A}$ is term equivalent to an affine space over GF(p) ([3]). Thus affine spaces over GF(5) are models of the identities (A1)–(A3) which have 13 essentially ternary terms.

We find our theorems interesting in view of the following propositions and remark.

Proposition 1.1 ([8]). *Let* $(A, +, \circ)$ *be a proper algebra of type* $(2,2)$ *satisfying*

(B1) $x + x = x, \quad x \circ x = x,$
(B2) $x + y = y + x, \quad x \circ y = y \circ x,$
(B3) $(x + y) \circ z = (x + z) \circ y.$

Then $(A, +, \circ)$ *is infinite and term infinite.*

Proposition 1.2 ([7]). *Let* $(A, +, \circ)$ *be a proper algebra of type* $(2,2)$ *satisfying*

(C1) $x + x = x, \quad x \circ x = x,$
(C2) $x + y = y + x,$
(C3) $(x + y) \circ z = (x + z) \circ y, \quad x \circ (y + z) = y \circ (x + z).$

Then $(A, +, \circ)$ *is infinite and term infinite.*

Remark 1.1. No affine space over any finite prime field GF(p), finite or infinite, is a proper model of the identities (B1)–(B3) or (C1)–(C3). In fact, if operations are defined by

$$x \oplus y = \frac{p+1}{2}x + \frac{p+1}{2}y \quad \text{and} \quad x \circ y = \alpha x + (1-\alpha)y \quad (\alpha \in \mathbb{Z}_p),$$

then the identities in (C3) would imply $\frac{p+1}{2}\alpha = 1 - \alpha$ and $\alpha = \frac{p+1}{2}(1-\alpha)$, for which no solutions exist.

Now we make a few more remarks. Firstly, the set of axioms (A1)–(A3) is weaker than the set (C1)–(C3), which in turn is weaker than (B1)–(B3). Secondly, if an algebra satisfying the identities (A1)–(A3), (B1)–(B3), or (C1)–(C3) is not proper, then it is (term equivalent to) a semilattice or even trivial. Finally, in particular, $p_3(A, +, \circ)$ is infinite for algebras satisfying the identities in (B1)–(B3) or (C1)–(C2). Thus, the propositions together with Theorem 1.1 suggest that the number 13 plays some specific role, namely it is a certain upper bound and a certain minimal number with respect to the axioms $x \circ y = y \circ x$ and $x \circ (y + z) = y \circ (x + z)$.

A groupoid $(G, \cdot)$ is said to be *medial* if it satisfies the identity $(xy)(uv) = (xu)(yv)$, and a *semilattice* if it is a commutative idempotent semigroup.

We will need the following proposition for the proof of Theorem 1.1.

PROPOSITION 1.3 ([5, 11]). *A commutative groupoid $(G, \cdot)$ is term equivalent to an affine space over* GF(5) *if and only if $(G, \cdot)$ is a medial idempotent groupoid satisfying the identity $((xy)y)x = y$.*

2. Lemmas and proof of Theorem 1.1. In this section, we prove the main theorem of this paper by a series of lemmas. Throughout the section, unless stated otherwise, $(A, +, \circ)$ will be a proper idempotent algebra of type $(2, 2)$ such that $+$ is commutative.

NOTATION. For simplicity, we use the notation xy for $(x+y)+y$ in this section.

Let $f = f(x_1, \ldots, x_n)$ be an n-ary term of an algebra $\mathbf{A}$ and S_n be the symmetric group of degree n. For $\sigma \in S_n$, define $f^\sigma(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$. We say that f *admits* a permutation σ in S_n if $f^\sigma = f$. The set $G(f) = \{\sigma \in S_n \mid f^\sigma = f\}$ is called the *symmetry group of f over* $\mathbf{A}$, and we say f is *symmetric* if $G(f) = S_n$. We say that two n-ary terms f and g of an algebra (A, F) are *strictly distinct* if $f^\sigma \neq g$ for all $\sigma \in S_n$. The following facts are clear.

1. $|G(f^\sigma)| = |G(f)|$ for all n-ary terms f and all $\sigma \in S_n$.
2. If $|G(f)| \neq |G(g)|$ then f and g are strictly distinct.
3. $|\{f^\sigma \mid \sigma \in S_n\}| = n!/|G(f)|$ for every n-ary term f.

LEMMA 2.1. *If $(A, +, \circ)$ satisfies $(x+y)\circ z = (x+z)\circ y$ (resp. $x\circ(y+z) = y\circ(x+z)$), then*:

1. *$(A, +, \circ)$ satisfies $(x+y)\circ x = x\circ y$ (resp. $x\circ(x+y) = y\circ x$).*
2. *The term $(x+y)\circ z$ (resp. $x\circ(y+z)$) is symmetric.*

Proof. Trivial. ∎

From now on, we will freely employ the obvious identities of the preceding lemma without referring to the lemma.

LEMMA 2.2. *Let $(A, +)$ be a commutative idempotent groupoid. Then $(A, +)$ is a semilattice if and only if both the terms $(x+y)z$ and $x(y+z)$ are symmetric.*

Proof. Let $(A, +)$ be a semilattice. Then $(x+y)z = ((x+y)+z)+z = x+y+z$ and $x(y+z) = (x+(y+z))+(y+z) = x+y+z$. Trivially, the term $x+y+z$ is symmetric for any semilattice. Conversely, suppose the terms $(x+y)z$ and $x(y+z)$ are symmetric. Then

$$\begin{aligned} xy &= (x+x)y = (y+x)x = (y+x)(x+x) = x((y+x)+x) \\ &= (x+((y+x)+x)) + ((y+x)+x) \\ &= (y+x)x + yx = (x+x)y + yx = xy + yx, \end{aligned}$$

and hence $xy = yx$. Now,

$$\begin{aligned} x+y &= (x+y)(x+y) = ((x+y)+y)x = (xy)x = (yx)x \\ &= ((y+x)+x)x = (x+x)(y+x) = x(y+x) = y(x+x) = yx. \end{aligned}$$

Consequently, $(x+y)+z = (x+y)z = (y+z)x = x(y+z) = x+(y+z)$, and this proves that $(A,+)$ is a semilattice. ∎

LEMMA 2.3 ([6]). *Let $(A,+,\circ)$ satisfy the identity $(x+y)\circ z = (x+z)\circ y$ (or $x\circ(y+z) = y\circ(x+z)$). Then xy is a noncommutative essentially binary term and, consequently, $x+y$, xy and yx are pairwise distinct essentially binary terms.*

The following lemma was proved in [6] under the additional assumption that $+$ is associative, but we prove it here without the associativity.

LEMMA 2.4. *If $(A,+,\circ)$ satisfies $(x+y)\circ z = (x+z)\circ y$ (or $x\circ(y+z) = y\circ(x+z)$) then $|G(xy+z)| = 1$.*

Proof. Suppose $(x+y)\circ z = (x+z)\circ y$ (if $x\circ(y+z) = y\circ(x+z)$, the proof is analogous). If $xy+z = yx+z$, then $xy = xy+xy = yx+xy = xy+yx = yx+yx = yx$, which contradicts Lemma 2.3. If $xy+z = zx+y$, then $x+y = xx+y = yx+x$ and so

$$x+y = (x+y)\circ(x+y) = (yx+x)\circ(x+y) = ((x+y)+x)\circ(yx) = yx\circ yx = yx,$$

which also contradicts Lemma 2.3. A similar argument gives the same contradiction if $xy+z = yz+x$ or $xy+z = zy+x$. Finally, if and $xy+z = xz+y$ then $x+y = xx+y = xy+x$ and so

$$\begin{aligned} x+y &= (x+y)\circ(x+y) = ((x+y)+y)\circ x = (xy)\circ x \\ &= (xy+xy)\circ x = (xy+x)\circ(xy) = (x+y)\circ(xy) \\ &= (xy+x)\circ y = (x+y)\circ y = y\circ x, \end{aligned}$$

which in turn yields the contradiction $x\circ y = y+x = x+y$. Therefore $xy+z$ admits no nontrivial permutation of its variables. ∎

LEMMA 2.5. *The following ternary terms*:

$$\begin{aligned} &s = (x+y)+z, \quad && f_1 = (x+y)\circ z, \quad && f_2 = z\circ(x+y), \\ &g_1 = (x+y)z, && g_2 = z(x+y), && q = xy+z \end{aligned}$$

are essentially ternary over $(A,+,\circ)$.

Proof. The proof follows by the binary algebra version of [4, Lemma 8.1], where a more general result is proved for algebras with a symmetric idempotent m-ary operation and an idempotent n-ary operation for arbitrary $m \geq 2$ and $n \geq 2$. ■

LEMMA 2.6. *Let* $(A, +, \circ)$ *satisfy the identity* $(x+y)\circ z = (x+z)\circ y$ (*or* $x \circ (y+z) = y \circ (x+z)$). *Then* $p_3(A, +, \circ) \geq 13$. *Moreover*, $p_3(A, +, \circ) > 13$ *if* $p_2(A, +, \circ) > 3$.

Proof. By Lemma 2.3, the terms $x+y$, xy and yx are essentially binary and pairwise distinct. If $x \circ y = y \circ x$ then, by Proposition 1.1, $(A, +, \circ)$ is term infinite and hence $p_3(A, +, \circ) > 13$. Thus, we assume $x \circ y \neq y \circ x$. The operation $+$ is not associative: otherwise, we have $xy = (x+y)+y = x+(y+y) = x+y$, a contradiction. Now consider the essentially ternary terms s, f_1, g_1, g_2 and q defined in the preceding lemma. By Lemma 2.2, g_1 and g_2 cannot be both symmetric. Note that $|G(s)| = 2$ since $+$ is commutative but not associative, $|G(f_1)| = 6$ by Lemma 2.1, and $|G(q)| = 1$ by Lemma 2.4. Also note that $|G(g_i)| = 2$ or 6 for $i = 1, 2$. Thus s, f_1, and q are pairwise strictly distinct, and g_i and q are strictly distinct for $i = 1, 2$. It can also be checked that s, g_1 and g_2 are pairwise strictly distinct. For instance, $(x+y)+z = y(x+z)$ would imply the contradiction that $(x+y)+z$ is symmetric. Now the proof splits into three cases: (1) $xy \notin \{x \circ y,\ y \circ x\}$, (2) $xy = x \circ y$ and (3) $xy = y \circ x$.

Case (1). In this case, it can be checked that the terms s, f_1, g_1, g_2 and q are pairwise strictly distinct. For instance, $f_1 = g_2$ would imply the contradiction $xy = (x+x)\circ y = y(x+x) = y \circ x$. Now, if none of g_1 and g_2 is symmetric, then $|G(g_1)| = |G(g_2)| = 2$ and so we have

$$\begin{aligned} p_3(A, +, \circ) &\geq \frac{6}{|G(s)|} + \frac{6}{|G(f_1)|} + \frac{6}{|G(g_1)|} + \frac{6}{|G(g_2)|} + \frac{6}{|G(q)|} \\ &= \frac{6}{2} + \frac{6}{6} + \frac{6}{2} + \frac{6}{2} + \frac{6}{1} = 16. \end{aligned}$$

If one of g_1 and g_2 is symmetric, say g_1 is symmetric and g_2 is not, then $|G(g_1)| = 6$ and $|G(g_2)| = 2$, and the same argument as above gives

$$p_3(A, +, \circ) \geq \frac{6}{2} + \frac{6}{6} + \frac{6}{6} + \frac{6}{2} + \frac{6}{1} = 14.$$

Case (2). Since $xy = x \circ y$, we have $(x+y)z = (x+z)y$ from $(x+y)\circ z = (x+z)\circ y$. Thus g_1 is symmetric and so g_2 is not symmetric. Then $|G(g_1)| = 6$ and $|G(g_2)| = 2$, and hence s, g_1, g_2 and q are pairwise strictly distinct. Thus, we have

$$p_3(A, +, \circ) \geq \frac{6}{|G(s)|} + \frac{6}{|G(g_1)|} + \frac{6}{|G(g_2)|} + \frac{6}{|G(q)|} = \frac{6}{2} + \frac{6}{6} + \frac{6}{2} + \frac{6}{1} = 13.$$

Suppose that $p_2(A,+,\circ) > 3$. Then there exists another essentially binary term, say $x * y$, different from xy, yx and $x+y$. Now it can also be shown that the term $(x+y)*z$ is essentially ternary and it is equal to none of the terms obtained from s, g_1, g_2 or q by permuting variables. For instance, if $(x+y)*z = x(y+z)$ then $(x+y)*z$ is symmetric and so $x*z = (x+x)*z = (z+x)*x = z(x+x) = zx$, which is a contradiction . Therefore, we have $p_3(A,+,\circ) > 13$.

Case (3). Since $xy = y \circ x$, we have $z(x+y) = y(x+z)$ from $(x+y)\circ z = (x+z)\circ y$. Then g_2 is symmetric and so g_1 is not symmetric. The rest is similar to Case (2). ∎

Corollary 2.7. *Let* $(A,+,\circ)$ *be an idempotent algebra of type* $(2,2)$ *such that* $+$ *is a commutative and essentially binary, and let* $(A,+,\circ)$ *satisfy the identities* $(x+y)\circ z = (x+z)\circ y$ *and* $x\circ y = xy$. *Then either* $(A,+,\circ)$ *is term equivalent to a semilattice or* $p_3(A,+,\circ) \geq 13$.

Proof. Note that $(A,+,\circ)$ is term equivalent to $(A,+)$ since $x \circ y = xy = (x+y)+y$. Suppose that $(A,+,\circ)$ is not term equivalent to a semilattice. Note that $x+y = (x+y)(x+y) = ((x+y)+y)x = (xy)x$. Assume that xy is not essentially binary. Then $xy = x$ or $xy = y$. If $xy = x$ then $x+y = (xy)x = xx = x$ and if $xy = y$ then $x+y = (xy)x = yx = x$, which is a contradiction since $x+y$ is essentially binary. Therefore, xy must be essentially binary. Suppose that $x+y = xy$. Then $(x+y)+z = (x+y)z = (y+z)x = (y+z)+x$, that is, $(A,+)$ is a semilattice and hence $(A,+,\circ)$ is term equivalent to a semilattice. Suppose now $x+y \neq xy$. Then $(A,+,\circ)$ is a proper algebra, and so $p_3(A,+,\circ) \geq 13$ by Lemma 2.6. ∎

Lemma 2.8. *If* $(A,+,\circ)$ *satisfies the identity* $(x+y)\circ z = (x+z)\circ y$ *and* $p_3(A,+,\circ) = 13$, *then*:

1. $x \circ y = xy$, $x+y = (xy)x$ *and hence* $(A,+,\circ)$ *is term equivalent to* $(A,+)$ *as well as to* $(A,\circ)$.
2. *The* 13 *essentially ternary terms are obtained from the terms* s, g_1, g_2 *and* q *in Lemma* 2.5 *by permuting variables.*
3. $xy + y = yx$ *and* $xy + x = y$.
4. *The groupoid* $(A,+)$ *is medial.*

Proof. By Lemmas 2.3 and 2.6, we infer that $x+y$, xy and yx are the only essentially binary terms $(A,+,\circ)$, and they are pairwise distinct.

(1) We first claim that $x\circ y \neq yx$. Assume that $x \circ y = yx$. Then $(A,+,\circ)$ satisfies $x(y+z) = y(x+z)$. Hence $x+y = (x+y)(x+y) = x((x+y)+y) = x(xy) = (xy+x)+xy$. Consider the term $xy+x$. If $xy+x = x$, then $x+y = (xy+x)+xy = x+xy = x$, a contradiction. If $xy+x = y$, then $xy = x(xy+x) = (xy)(x+x) = (xy)x = (xy+x)+x = y+x$, a contradiction

again. Thus $xy+x$ is essentially binary. Now, $xy+x=x+y$ would imply

$$x+y=(xy+x)+xy=(x+y)+((x+y)+y)=y(x+y)=x(y+y)=xy,$$

which is a contradiction, while $xy+x=xy$ would imply the same contradiction since $x+y=(xy+x)+xy=xy+xy=xy$. If $xy+x=yx$, then we would have

$$\begin{aligned} x+y=x(xy)=x(xy+xy)=(xy)(xy+x)=(xy+(xy+x))+(xy+x)\\ =(x+y)+yx=(x+y)+((y+x)+x)=x(x+y)=yx, \end{aligned}$$

which is also a contradiction. Thus, $xy+x$ is another essentially binary term different from $x+y$, xy and yx, and this is a contradiction. Therefore, we have $x\circ y\neq yx$ as claimed.

If we also have $x\circ y\neq xy$ then, since $x\circ y\neq x+y$, we would have the contradiction of having four essentially binary terms, namely $x+y$, xy, yx and $x\circ y$. Therefore, $x\circ y=xy$. Using this, we have $x+y=(x+y)(x+y)=((x+y)+y)x=(xy)x$. Consequently, the algebras $(A,+,\circ)$, $(A,+)$ and $(A,\circ)$ are term equivalent.

(2) Since $x\circ y=xy$ by (1), Case (2) in the proof of Lemma 2.6 applies here. Recall that s, g_1, g_2 and q are strictly distinct and $|G(s)|=2$, $|G(g_1)|=6$, $|G(g_2)|=2$ and $|G(q)|=1$. We obtain exactly 13 pairwise distinct essentially ternary terms from these terms by permuting variables.

(3) Using (1), we obtain $xy+y=((x+y)+y)+y=(x+y)y=yx$ and $x+y=(xy)x=(xy+x)+x$. Consider the term $xy+x$. If $xy+x=x+y$ then $x+y=(xy+x)+x=(x+y)+x=yx$, a contradiction. If $xy+x=xy$ then $x+y=(xy+x)+x=xy+x=xy$, another contradiction. If $xy+x=yx$ then $x+y=(xy+x)+x=yx+x=xy$, which is the same contradiction again. Thus, $xy+x$ is not essentially binary and so $xy+x=x$ or $xy+x=y$. However, $xy+x=x$ would imply the contradiction $x+y=(xy+x)+x=x+x=x$, and hence $xy+x=y$.

(4) Consider the term $x(yz)$. Using (3), we have

$$\begin{aligned} x(xy)&=(x+xy)+xy=y+xy=y+((x+y)+y)=(x+y)y=yx,\\ x(yx)&=(yx)x+x=((yx+x)+x)+x=(xy+x)+x=y+x=x+y. \end{aligned}$$

With these identities, it can be shown that $x(yz)$ is essentially ternary. By (2), $x(yz)$ is equal to one of the 13 essentially ternary terms obtained from s,g_1,g_2 and q by permuting variables. Assume that the symmetry group of $x(yz)$ is trivial. Then $x(yz)$ is equal to a term obtained from q by permuting variables. Equivalently, $\sigma x(\sigma y\sigma z)=q=xy+z$ for some permutation σ of x,y and z. Letting $x=z$ in this identity, we can show that $xy+x=yx$ or $xy+x=x+y$ using the identities $x(xy)=yx$ and $x(yx)=x+y$. However, this contradicts the identity $xy+x=y$ in (3). Thus, $x(yz)$ admits a nontrivial permutation. If $x(yz)\in\{x(zy),y(xz),y(zx),z(xy)\}$ then, by letting $x = y$, we obtain the contradiction $x + y = yx$ from the identities $x(xy) = yx$ and $x(yx) = x + y$ again. Thus, we must have $x(yz) = z(yx)$. Then $(xy)(uv) = v(u(xy)) = v(y(xu)) = (xu)(yv)$, and hence

$$\begin{aligned}(x + y) + (u + v) &= (x(yx))((u(vu))(x(yx)) = (x(yx))((ux)((vu)(yx)))\\ &= (x(ux))((yx)((vy)(ux))) = (x(ux))((y(vy))(x(ux))\\ &= (x + u) + (y + v).\end{aligned}$$

This proves that $(A, +)$ is medial, and the proof is now complete. ■

Proof of Theorem 1.1. The first assertion that $p_3(A, +, \circ) \geq 13$ is proved in Lemma 2.6. To prove the second assertion, suppose that $p_3(A, +, \circ) = 13$. By Lemma 2.8, $(A, +, \circ)$ and $(A, +)$ are term equivalent, and $(A, +)$ is a medial commutative idempotent groupoid satisfying $((x + y) + y) + x = xy+x = y$. By Proposition 1.3, we deduce that $(A, +)$ and hence $(A, +, \circ)$ as well is term equivalent to a nontrivial affine space over GF(5). The converse is obvious since $p_3(\mathbf{A}) = 13$ for any nontrivial affine space $\mathbf{A}$ over GF(5) ([3]). This completes the proof. ■

Remark 2.9. If $(A, +, \circ)$ satisfies the identity $(x + y) \circ z = (x + z) \circ y$ then, by Lemma 2.3, the term xy is essentially binary. By [2], we deduce that $p_n(A, +, \circ) \geq 3^{n-1}$ for all $n \geq 1$.

Problem 2.10. Does there exist a model $(A, +, \circ)$ for the identities (A1)–(A3) with $p_n(A, +, \circ) = 3^{n-1}$ for some $n \geq 4$? Note that $p_3(A, +, \circ) \geq 13 > 3^{3-1}$ by Theorem 1.1, and $p_2(A, +, \circ) = 3^{2-1}$ if $(A, +, \circ)$ is term equivalent to an affine space over GF(5). Which varieties are subvarieties of the variety defined by (A1)–(A3) and which finite algebras can be models of the subvarieties, besides affine spaces?

REFERENCES

[1] A. K. Austin, *A note on models of identities*, Proc. Amer. Math. Soc. 16 (1965), 522–523.
[2] J. R. Cho and J. Dudek, *Medial idempotent groupoids III*, J. Austral. Math. Soc. 68 (2000), 312–320.
[3] B. Csákány, *On affine spaces over prime fields*, Acta Sci. Math. (Szeged) 37 (1975), 33–36.
[4] J. Dudek, *Polynomials in idempotent commutative groupoids*, Dissertationes Math. 286 (1989).
[5] —, *Medial idempotent groupoids I*, Czechoslovak Math. J. 41 (1991), 249–259.
[6] —, *A characterization of modular lattices*, Colloq. Math. 64 (1993), 193–201.
[7] J. Dudek and J. Gałuszka, *A characterization of distributive Steiner quasigroups and semilattices*, Discuss. Math. Algebra Stochastic Methods 15 (1995), 101–119.
[8] J. Dudek and A. Kisielewicz, *On finite models of regular identities*, Notre Dame J. Formal Logic 30 (1989), 624–628.

[9] G. Grätzer, *Universal Algebra*, 2nd ed., Springer, New York, 1979.

[10] G. Grätzer and A. Kisielewicz, *A survey of some open problems on p_n-sequences and free spectra of algebras and varieties*, in: Universal Algebra and Quasigroup Theory, A. Romanowska and J. D. H. Smith (eds.), Heldermann, Berlin, 1992, 57–88.

[11] G. Grätzer and R. Padmanabhan, *On commutative, idempotent and nonassociative groupoids*, Proc. Amer. Math. Soc. 28 (1971), 75–78.

[12] R. McKenzie, *On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model*, J. Symbolic Logic 40 (1975), 186–196.

[13] J. Płonka, *R-prime idempotent reducts of groups*, Arch. Math. (Basel) 24 (1973), 129–132.

[14] A. Romanowska and J. D. H. Smith, *Modal Theory. An Algebraic Approach to Order, Geometry and Convexity*, Heldermann, Berlin, 1985.

[15] W. Taylor, *Equational logic*, Houston J. Math., special volume, 1979, 1–83.

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