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TAMENESS CRITERION FOR POSETS WITH ZERO-RELATIONS AND THREE-PARTITE SUBAMALGAMS OF TILED ORDERS

BY

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Abstract. A criterion for tame prinjective type for a class of posets with zero-relations is given in terms of the associated prinjective Tits quadratic form and a list of hypercritical posets. A consequence of this result is that if $\Lambda^\bullet$ is a three-partite subamalgam of a tiled order then it is of tame lattice type if and only if the reduced Tits quadratic form $\mathbf{q}_{\Lambda^\bullet}$ associated with $\Lambda^\bullet$ in [26] is weakly non-negative. The result generalizes a criterion for tameness of such orders given by Simson [28] and gives an affirmative answer to [28, Question 4.7].

1. Introduction. Throughout this paper K is an algebraically closed field. Let us recall from [28] the notion of a poset with zero relations. If $I = (I, \preceq)$ is a partially ordered set we denote by $\max I$ the set of its maximal elements and $I^- = I \setminus \max I$. We say that I is an *r-peak poset* if $\max I$ has r elements. From now on we assume that I is finite.

The *incidence algebra* KI of I is defined as the subalgebra of the full $I\times I$-matrix algebra $\mathbb{M}_I(K)$ with coefficients in K consisting of those matrices $[\lambda_{ij}]_{i,j\in I}$ such that $\lambda_{ij} = 0$ provided $i \npreceq j$ [20].

A *poset with zero-relations* is a pair $(I, \mathfrak{Z})$, where I is a finite partially ordered set and $\mathfrak{Z}$ is a set of pairs (i, j) of elements of I satisfying the following conditions:

(Z1) if $(i_0, j_0) \in \mathfrak{Z}$ then $i_0 \prec j_0$,
(Z2) if $(i_0, j_0) \in \mathfrak{Z}$ and $i_1 \preceq i_0 \preceq j_0 \preceq j_1$ then $(i_1, j_1) \in \mathfrak{Z}$.

If the set $\mathfrak{Z}$ is empty then we identify $(I, \mathfrak{Z})$ with I.

The *incidence algebra* $K(I, \mathfrak{Z})$ is associated with a field K and a poset with zero-relations $(I, \mathfrak{Z})$. By definition it is the quotient of the incidence algebra KI of I by the ideal $Z(\mathfrak{Z})$ of KI generated by all elements e_{ij} for $(i, j) \in \mathfrak{Z}$. Here we denote by e_{ij} the elementary matrix having 1 at place (i, j). We write e_i instead of e_{ii} and we denote by the same symbols the $Z(\mathfrak{Z})$-cosets of the elements e_{ij} in $K(I, \mathfrak{Z})$.

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It is clear that the elements e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of $K(I, \mathfrak{Z})$.

It is often convenient to treat $K(I, \mathfrak{Z})$ as a K-category with the class $\{e_i : i \in I\}$ of objects; the space of morphisms from e_i to e_j is $e_i K(I, \mathfrak{Z}) e_j$ and composition is induced by multiplication in $K(I, \mathfrak{Z})$.

Assume that $(I, \mathfrak{Z})$ satisfies the condition:

(Z3) for every $i \in I$ there exists $p \in \max I$ such that $i \preceq p$ and $(i, p) \notin \mathfrak{Z}$.

If this is the case $(I, \mathfrak{Z})$ is called a *multipeak poset with zero-relations*. Thanks to the condition (Z3) the algebra $R = K(I, \mathfrak{Z})$ of such a poset is a *right multipeak algebra*, that is, the right socle of R is a projective R-module [17].

We denote by $\operatorname{mod}_{\mathrm{sp}}(K(I, \mathfrak{Z}))$ (resp. $\operatorname{prin}(K(I, \mathfrak{Z}))$) the category of right finitely generated socle projective modules (resp. prinjective modules) over $K(I, \mathfrak{Z})$ (see Section 2 for the definitions). The poset with zero-relations $(I, \mathfrak{Z})$ is said to be of *tame prinjective type* if the category $\operatorname{prin}(K(I, \mathfrak{Z}))$ has tame representation type (see [19, Chapter 14.4]).

A useful interpretation of the category $\operatorname{prin}(K(I, \mathfrak{Z}))$ in terms of matrix problems, together with the geometry of varieties of prinjective representations of posets with zero-relations and their applications are discussed by Simson in [26] and [25].

Let us restrict our attention to *subamalgam-like* posets, that is, posets $(I, \mathfrak{Z})$ with zero-relations satisfying the following conditions.

1. I is a two-peak poset with maximal elements $*, +$.
2. There is a disjoint union decomposition $I = C' \cup I_0 \cup C''$ such that
   (a) $I_0 = \{i \in *^{\nabla} \cap +^{\nabla} : (i, +) \notin \mathfrak{Z}\}$, $* \in C'$, $+ \in C''$,
   (b) there are no relations $i \preceq c'$, $c'' \preceq i$ or $c'' \preceq c'$ with $c' \in C' \setminus \{*\}$, $c'' \in C''$, $i \in I_0$ and if $c' \preceq c''$ for some $c' \in C'$, $c'' \in C''$ then there exists $i \in I_0$ such that $c' \preceq i \preceq c''$,
   (c) C' and C'' are empty or linearly ordered,
   (d) a pair (i, j) satisfying $i \prec j$ belongs to $\mathfrak{Z}$ if and only if $i \in C'$ and $j \in C''$.

EXAMPLE 1.1. Consider the six-element poset with the partial order described by the diagram

$$\begin{tikzcd}
c' \arrow[d] & & \\
\circ \arrow[d] \arrow[dr] & \circ \arrow[r] \arrow[dl] & c'' \arrow[dl] \\
\star & + &
\end{tikzcd}$$

equipped with the one-element set $\mathfrak{Z} = \{(c', +)\}$ of zero-relations. This poset is subamalgam-like.

The main result of this article is the following theorem.

THEOREM 1.2. *Let $(I, \mathfrak{Z})$ be a subamalgam-like poset with zero-relations. The following conditions are equivalent.*

(a) *The poset $(I, \mathfrak{Z})$ is of tame prinjective type.*

(b) *The category* $\mathrm{mod_{sp}}(K(I, \mathfrak{Z}))$ *is of tame representation type.*

(c) *The Tits quadratic form $\mathbf{q}_{(I,\mathfrak{Z})}$ associated with $(I, \mathfrak{Z})$ in* (3.6) *is weakly non-negative, that is, $\mathbf{q}_{(I,\mathfrak{Z})}(z) \geq 0$ for any vector $z \in \mathbb{N}^I$.*

(d) *The poset $(I, \mathfrak{Z})$ contains none of the* 13 *posets of Table* 1 *as a two-peak subposet with zero-relations*; *the dotted edge in $\widehat{\mathcal{F}}_4$ means a zero-relation.*

Throughout this paper we shall call posets from Table 1 *hypercritical posets*. The meaning of numbers at vertices of the diagrams will be explained in Lemma 3.10 below. The notion of a peak subposet is discussed in Section 3.

The importance of subamalgam-like posets comes from the fact that they are closely related to three-partite subamalgams of tiled orders introduced in [28].

If S is a (subset of a) ring and m, m' are natural numbers then $\mathbb{M}_{m\times m'}(S)$ (resp. $\mathbb{M}_m(S)$) denotes the set of all $m \times m'$ matrices (resp. $m \times m$ matrices) with coefficients in S.

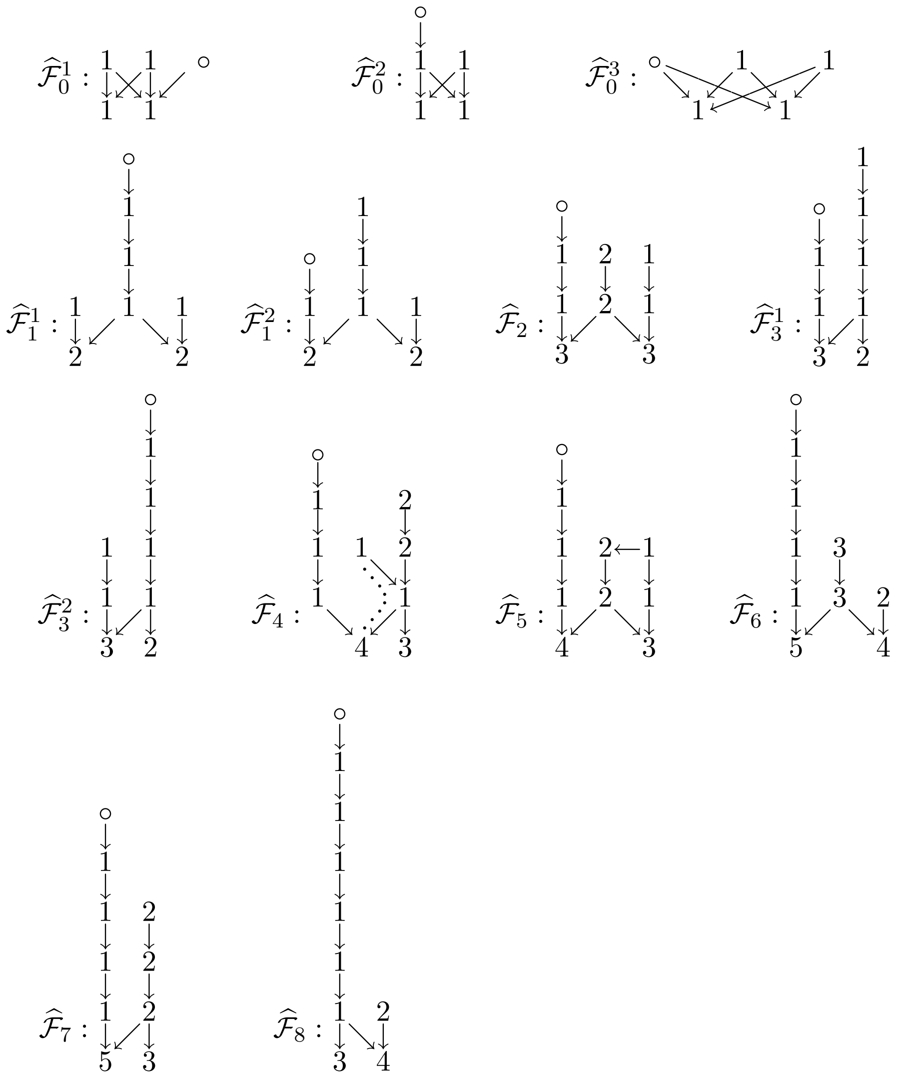
Let D be a complete discrete valuation domain over K with the unique maximal ideal $\mathfrak{p}$ such that $D/\mathfrak{p} \cong K$. Denote by $F = D_0$ the field of fractions of D.

Let C be a finite-dimensional semisimple F-algebra. Each subring Λ of C which is a finitely generated free D-module such that $\Lambda F = C$ is called a *D-order* in C.

A right Λ-module M is called a *lattice* if it is finitely generated and free as a D-module. The category of right *Λ-lattices* is denoted by $\mathrm{latt}\,\Lambda$. It is known that this category has the finite unique decomposition property [11]. The order Λ is said to be of *finite lattice type* if there are only finitely many indecomposable Λ-lattices up to isomorphism. The notions of tameness and wildness are also defined for the categories of Λ-lattices. The precise definitions can be found in [24], [1], [23], [21], [26]. Roughly speaking, Λ is of *tame lattice type* if the indecomposable Λ-lattices of fixed D-rank form a finite set of at most one-parameter families (up to isomorphism).

We restrict our attention to D-orders of a special form considered in [28], namely so-called three-partite subamalgams of D-orders. A criterion

TABLE 1



for finite lattice type of such orders is given in [26]. It is expressed in terms of weak positivity of the Tits quadratic form associated with the given order. The paper [28] gives a criterion for tame lattice type for a class of three-partite subamalgams of D-orders. Note also that the problem of determining whether a given order in the class being considered is of polynomial growth is solved in [27].

The second aim of the present paper is to generalize the main result of [28] to the whole class of three-partite subamalgams of tiled orders as conjectured in [28, Question 4.7].

Let us recall the basic definitions from [26] and [28].

Suppose that Λ is a lattice of the form

$$\Lambda = \left.\begin{pmatrix} D & {}_1D_2 & \dots & {}_1D_n \\ \mathfrak{p} & D & \dots & {}_2D_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & {}_{n-1}D_n \\ \mathfrak{p} & \mathfrak{p} & \dots & D \end{pmatrix}\right\} n \tag{1.3}$$

where

(a) ${}_iD_j$ is either D or $\mathfrak{p}$,

(b) Λ admits a three-partition

$$\Lambda = \left(\begin{array}{c|c|c} \Lambda_1 & \mathcal{X} & \mathbb{M}_{n_1\times n_2}(D) \\ \hline \mathbb{M}_{n_3\times n_1}(\mathfrak{p}) & \Lambda_3 & \mathcal{Y} \\ \hline \mathbb{M}_{n_2\times n_1}(\mathfrak{p}) & \mathbb{M}_{n_2\times n_3}(\mathfrak{p}) & \Lambda_2 \end{array}\right) \begin{array}{l} \}n_1 \\ \}n_3 \\ \}n_2 \end{array} \tag{1.4}$$

where $n_1 = n_2$, $\Lambda_2 = \Lambda_1$, $n_1 + n_2 + n_3 = n$ and Λ_3 is a hereditary $n_3 \times n_3$ matrix D-order

$$\Lambda_3 = \left.\begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix}\right\} n_3$$

Let ε_1, ε_3 and ε_2 be the matrix idempotents of Λ corresponding to the identity elements of Λ_1, Λ_3 and Λ_2, respectively. A *three-partite subamalgam* of Λ is by definition the D-suborder $\Lambda^\bullet$ of Λ consisting of all matrices $\lambda = [\lambda_{ij}]$ in Λ whose left upper $n_1 \times n_1$ corner $\varepsilon_1\lambda\varepsilon_1$ is congruent modulo $\mathbb{M}_{n_1}(\mathfrak{p})$ to its right lower $n_1 \times n_1$ corner $\varepsilon_2\lambda\varepsilon_2$ (see [28]). More precisely

$$\Lambda^\bullet = \big\{\lambda \in \Lambda : \varepsilon_1\lambda\varepsilon_1 - \varepsilon_2\lambda\varepsilon_2 \in \mathbb{M}_{n_1}(\mathfrak{p})\big\}. \tag{1.5}$$

The main application of Theorem 1.2 is the following refinement of Theorem 1.5 of [28] suggested in [28, Question 4.7].

Theorem 1.6. *Let K be an algebraically closed field and D a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D.*

Let Λ be a three-partite D-order of the form (1.4) *and let $\Lambda^\bullet$ be the subamalgam* (1.5) *of $\Lambda \subseteq \mathbb{M}_n(D)$, where $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 are as above.*

The following conditions are equivalent.

(a) *The D-order $\Lambda^\bullet$ is of tame lattice type.*

(b) *The integral reduced Tits quadratic form* $\mathbf{q}_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$ (see (7.1)) *is weakly non-negative.*

(c) *The two-peak poset* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *with zero-relations associated with* $\Lambda^\bullet$ *in* (7.2) *contains none of the forms listed in Table* 1 *as a two-peak subposet with zero-relations.*

The equivalence of these conditions was proved by Simson in [28] under the additional assumption that either the part $\mathcal{X}$ or $\mathcal{Y}$ of the D-order Λ consists of matrices with coefficients in $\mathfrak{p}$ only. Another condition equivalent to (a), (b) and (c) expressed in terms of minor D-suborders of $\Lambda^\bullet$ can be found in [28].

The paper is organized as follows. Section 2 is devoted to general remarks on bipartite algebras and prinjective modules. Here we follow [10]. In Section 3 we present more information on posets with zero-relations and their socle projective representations.

Section 4 contains a discussion of the most important tools used in the proof of Theorem 1.2 which is given in Sections 5 and 6.

It is shown in Section 7 how this theorem implies Theorem 1.6.

The main results of the paper were presented at the 9th International Conference on Representations of Algebras in Beijing, 2000.

2. Bipartite algebras and adjustment functors. Throughout this section let R be a finite-dimensional K-algebra. All modules considered are right finitely generated, the category formed by them is denoted by $\mathrm{mod}(R)$.

Assume that R is *bipartite*, that is, has a triangular matrix form

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \tag{2.1}$$

where A and B are K-algebras and M is an A-B-bimodule. It is well known that R-modules can be identified with triples $(X'_A,\ X''_B,\ \phi : X'_A \otimes_A M \to X''_B)$, where X'_A is an A-module, X''_B is a B-module and ϕ is a B-homomorphism. There are two functors

$$\mathbf{\Theta}_B, \mathbf{\Theta}^A : \mathrm{mod}(R) \to \mathrm{mod}(R)$$

called *adjustment functors* which are defined on objects of $\mathrm{mod}(R)$ by the formulas

$$\mathbf{\Theta}_B(X'_A, X''_B, \phi) = (X'_A, \mathrm{Im}\,\phi, \mathrm{res}\,\phi), \qquad \mathbf{\Theta}^A(X'_A, X''_B, \phi) = (\mathrm{Im}\,\bar{\phi}, X''_B, J_\phi),$$

where the map $\mathrm{res}\,\phi : X'_A \otimes_A M \to \mathrm{Im}\,\phi$ is given by $(\mathrm{res}\,\phi)(x \otimes m) = \phi(x \otimes m)$, $\bar{\phi}$ is the homomorphism adjoint to ϕ and J_ϕ is the map adjoint to the embedding $\mathrm{Im}\,\phi \to \mathrm{Hom}_B(M, X''_B)$ (see [10]).

Following [10] denote by $\mathrm{prin}(R)^A_B$ or $\mathrm{prin}(M)$ the category of *prinjective modules*, that is, the full subcategory of $\mathrm{mod}(R)$ consisting of R-modules of the form

$$X = (X'_A, X''_B, \phi : X'_A \otimes_A M \to X''_B)$$

where X'_A is a projective A-module and X''_B is an injective B-module.

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{prin}(R)^A_B & \xrightarrow{\mathbf{\Theta}_B} & \mathrm{mod}^{\mathrm{pg}}(R)^A \\ \big\downarrow{\scriptstyle \mathbf{\Theta}^A} & & \big\downarrow{\scriptstyle \mathbf{\Theta}^A} \\ \mathrm{mod}_{\mathrm{ic}}(R)_B & \xrightarrow{\mathbf{\Theta}_B} & \mathrm{adj}(R)^A_B \end{array} \tag{2.2}$$

of full subcategories of $\mathrm{mod}(R)$ and functors induced by suitable adjustment functors defined as follows. The module $X = (X'_A, X''_B, \phi)$ belongs to $\mathrm{mod}^{\mathrm{pg}}(R)^A$ if X'_A is a projective A-module and the B-homomorphism $\phi : X'_A \otimes_A M \to X''_B$ is surjective. The module $X = (X'_A, X''_B, \phi)$ belongs to $\mathrm{mod}_{\mathrm{ic}}(R)_B$ if X''_B is B-injective and the morphism $\bar{\phi} : X'_A \to \mathrm{Hom}_B(M, X''_B)$ adjoint to ϕ is injective. The category $\mathrm{adj}(R)^A_B$ consists of modules $X = (X'_A, X''_B, \phi)$ such that ϕ is surjective and $\bar{\phi}$ is injective.

Assume now that R is a right multipeak algebra. There is a canonical way of presenting R in a triangular matrix form

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \tag{2.3}$$

where B treated as an R-module is isomorphic to the direct sum of all simple projective R-modules. It is known that if R is a right multipeak algebra then

$$\mathrm{mod}_{\mathrm{ic}}(R)_B = \mathrm{mod}_{\mathrm{sp}}(R)$$

where $\mathrm{mod}_{\mathrm{sp}}(R)$ is the full subcategory of $\mathrm{mod}(R)$ formed by modules with projective right R-socle [15, 2.6′], that is, *socle projective modules.* In such a situation the functor $\mathbf{\Theta}_B$ will be denoted by $\mathbf{\Theta}$.

We refer to [24] for the definition of the representation types of the categories considered above. It is shown in [3] that the adjustment functors preserve and respect the tame representation type.

The bipartite algebra R is *of tame prinjective type* if the category $\mathrm{prin}(R)^A_B$ is of tame representation type.

3. Posets with zero-relations. In this section we give more information on posets with zero-relations which will be needed later. Let $(I, \mathfrak{z})$ be a poset with zero-relations. Given an element $x \in I$ let $x^{\nabla} = \{y \in I : y \preceq x\}$ and $x^{\Delta} = \{y \in I : y \succeq x\}$.

A poset I is a *garland* if for every $x \in I$ there exists at most one $y \in I$ incomparable with x.

By the *width* of a poset I is meant the maximal cardinality of a subset of I consisting of pairwise incomparable elements.

The reflection duality for right multipeak algebras is defined in [17, Definition 2.21]. Let us recall this construction in the context of multipeak posets with zero-relations. Assume in addition that the poset $(I, \mathfrak{Z})$ is subamalgam-like. Then the *reflection dual poset* $(I^\bullet, \mathfrak{Z}^\bullet)$ associated with $(I, \mathfrak{Z})$ can be described as follows:

- $I^\bullet = I$ as sets,
- the partial order $\preceq^\bullet$ in $I^\bullet$ is the minimal partial order relation such that
  - — for $i, j \notin \max I$, $i \preceq^\bullet j$ if and only if $j \preceq i$ in I,
  - — for $p \in \max I$, $i \preceq^\bullet p$ if and only if $i \preceq p$ in I,
- $\mathfrak{Z}^\bullet = \{(i,j) \in I^\bullet \times I^\bullet : i \prec^\bullet j,\ i \in C'' \setminus \{+\},\ j \in C' \cup \{*\}\}$

where C' and C'' are the subsets as in the definition of subamalgam-like posets.

It is easy to observe that $(I^\bullet, \mathfrak{Z}^\bullet)$ is subamalgam-like as well. Moreover, the following proposition follows from the general properties of reflection duality (see [17, Section 2]).

PROPOSITION 3.1. *Let $(I, \mathfrak{Z})$ be a multipeak poset with zero-relations and $(I^\bullet, \mathfrak{Z}^\bullet)$ its reflection dual poset. Then*

(1) *the Tits quadratic form $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative if and only if $\mathbf{q}_{(I^\bullet,\mathfrak{Z}^\bullet)}$ is weakly non-negative.*

(2) *$(I, \mathfrak{Z})$ is of tame prinjective type only if $(I^\bullet, \mathfrak{Z}^\bullet)$ is.* ∎

Given $i \in I$ let $P_i = e_i K(I, \mathfrak{Z})$. Define the modules Q_i as follows (the definition depends on whether $i \in \max I$ or not):

If $i \in \max I$ then $Q_i = \bigoplus_{j \in I} (e_j K(I, \mathfrak{Z}) e_i)^*$ as a K-vector space and the right $K(I, \mathfrak{Z})$-module structure is defined so that the map

$$(-) \cdot e_{jk} : (e_j K(I, \mathfrak{Z}) e_i)^* \to (e_k (K(I, \mathfrak{Z}) e_i)^*$$

is dual to the map

$$e_{jk} \cdot (-) : e_k (K(I, \mathfrak{Z})) e_i \to e_j (K(I, \mathfrak{Z})) e_i$$

for any j, k in I such that $j \preceq k$ and $(j, k) \notin \mathfrak{Z}$.

It is easy to see that Q_i is the $K(I, \mathfrak{Z})$-injective envelope of the simple projective module P_i.

Now assume that $i \notin \max I$. Let $j \in I$ and denote by $U_i(j)$ the cokernel of the map

$$(-) \cdot e_{i,\max} : e_j K(I, \mathfrak{Z}) e_i \to e_j K(I, \mathfrak{Z}) e_{\max}$$

where $e_{\max} = \sum_{p\in\max I} e_p$ and $e_{i,\max} = \sum_{p\in\max I,\, i\preceq p,\, (i,p)\notin\mathfrak{Z}} e_{ip}$. We set $Q_i = \bigoplus_{j\in I}(U_i(j))^*$ and the right action of $K(I,\mathfrak{Z})$ on Q_i is defined as in the case $i \in \max I$.

Recall [19, Sections 5.2 and 11.9], [17], [13] that a module X in the category $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$ is called *sp-injective* if it is injective with respect to the monomorphisms f in $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$ with cokernel in $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$ or equivalently: the functor $\mathrm{Ext}^1_{K(I,\mathfrak{Z})}(-,X)$ vanishes on $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$. The module X is said to be *hereditary sp-injective* if every indecomposable socle projective $K(I,\mathfrak{Z})$-module X' with $\mathrm{Hom}_{K(I,\mathfrak{Z})}(X,X') \neq 0$ is sp-injective.

LEMMA 3.2. *The modules P_i, $i \in I$, form a complete set of pairwise non-isomorphic indecomposable projective $K(I,\mathfrak{Z})$-modules. The modules Q_i, $i \in I$, form a complete set of pairwise non-isomorphic indecomposable sp-injective $K(I,\mathfrak{Z})$-modules, that is, injective objects of* $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$.

Proof. The assertion about projectives is standard. The remaining one follows by application of the *reflection duality* functor (see [17] for the definition): one can check that the module Q_i is reflection dual to the indecomposable projective associated with i over the reflection dual algebra to $K(I,\mathfrak{Z})$. Then the lemma follows from the general properties of the reflection duality (see also [15, 2.14, 2.16] and [19, Section 5.2]). ∎

Note that the hereditary sp-injectives are reflection dual to hereditary projective modules.

We say that $(I',\mathfrak{Z}')$ is a *peak subposet* of $(I,\mathfrak{Z})$ if

- $I' \subseteq I$, $\max I' = I' \cap \max I$,
- $i \preceq j$ in I' if and only if there exists a sequence $i_0,\dots,i_r$ of elements of I' such that $i_0 = i$, $i_r = j$, $i_k \preceq i_{k+1}$ in I and $(i_k,i_{k+1}) \notin \mathfrak{Z}$ for $k = 0,\dots,r-1$,
- $\mathfrak{Z}' = \{(i,j) \in I' \times I' : i \preceq j \text{ in } I'\} \cap \mathfrak{Z}$.

It follows that in this case $K(I',\mathfrak{Z}')$ is a full subcategory of $K(I,\mathfrak{Z})$. Moreover, a peak subposet of a multipeak poset with zero-relations is a multipeak poset with zero-relations.

Now we shall recall two functors relating the categories $\mathrm{mod}_{\mathrm{sp}}(K(I',\mathfrak{Z}'))$ and $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$ when $(I,\mathfrak{Z})$ is a peak subposet of $(I',\mathfrak{Z}')$. Write $R = K(I,\mathfrak{Z})$ and $R' = K(I',\mathfrak{Z}')$ as bipartite algebras

$$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A' & M' \\ 0 & B' \end{pmatrix}$$

respectively in the canonical way as at the end of Section 2.

Assume $A' = eAe$ and $B' = fBf$ for idempotents e, f in A and B respectively.

There is a *restriction functor*

$$\mathbf{res}_{I'} : \mathrm{mod}_{\mathrm{sp}}(R) \to \mathrm{mod}_{\mathrm{sp}}(R')$$

given by $X \mapsto \mathbf{\Theta}(X(e+f))$ (see Section 2).

Define a functor

$$\widetilde{\mathbf{T}}_{I'} : \mathrm{prin}(R') \to \mathrm{prin}(R)$$

by the formula $\widetilde{\mathbf{T}}_{I'}(X'_{A'}, X''_{B'}, \phi) = (X'_{A'} \otimes_{A'} eA, X''_B, \widetilde{\phi})$, where X''_B is just $X''_{B'}$ treated as a B-module and $\widetilde{\phi}$ is induced by ϕ. We omit defining $\widetilde{\mathbf{T}}_{I'}$ on homomorphisms since it is done in a standard way.

Further, let $\widetilde{\mathbf{L}}_{I'} = \mathrm{Hom}_{R'}(R(e+f), -) : \mathrm{mod}(R') \to \mathrm{mod}(R)$. See also [19, Lemma 12.2], [5, Lemma 3], [17, 2.10].

LEMMA 3.3. (a) *There is a unique* (*up to equivalence*) *functor* $\mathbf{T}_{I'}$ *making the following diagram commutative*:

$$\begin{array}{ccc} \mathrm{prin}(R') & \xrightarrow{\widetilde{\mathbf{T}}_{I'}} & \mathrm{prin}(R) \\ \big\downarrow{\scriptstyle \mathbf{\Theta}_{B'}} & & \big\downarrow{\scriptstyle \mathbf{\Theta}_{B}} \\ \mathrm{mod}_{\mathrm{sp}}(R') & \xrightarrow{\mathbf{T}_{I'}} & \mathrm{mod}_{\mathrm{sp}}(R) \end{array}$$

(b) *The functor* $\widetilde{\mathbf{L}}_{I'}$ *maps socle projective modules to socle projective ones*, *hence it induces a functor*

$$\mathbf{L}_{I'} : \mathrm{mod}_{\mathrm{sp}}(R') \to \mathrm{mod}_{\mathrm{sp}}(R).$$

(c) *The functors* $\mathbf{T}_{I'}$ *and* $\mathbf{L}_{I'}$ *are right quasi-inverses of* $\mathbf{res}_{I'}$.

(d) *The functor* $\mathbf{L}_{I'}$ *is a right adjoint to* $\mathbf{res}_{I'}$.

(e) *The functor* $\mathbf{L}_{I'}$ *maps the sp-injective modules in* $\mathrm{mod}_{\mathrm{sp}}(R')$ *onto sp-injective modules in* $\mathrm{mod}_{\mathrm{sp}}(R)$.

(f) *The functors* $\mathbf{T}_{I'}$ *and* $\mathbf{L}_{I'}$ *are full and faithful.*

(g) *If the category* $\mathrm{mod}_{\mathrm{sp}}(R)$ *is of tame representation type then so is* $\mathrm{mod}_{\mathrm{sp}}(R')$.

Outline of proof. The assertions (a), (b) and (c) can be checked directly. The adjointness of $(\mathbf{res}_{I'}, \mathbf{L}_{I'})$ is a standard fact and (e) is its consequence. To prove (f) observe that $\mathbf{T}_{I'}$ and $\mathbf{L}_{I'}$ are faithful by (c) and $\mathbf{L}_{I'}$ is full thanks to (d). Fullness of $\mathbf{T}_{I'}$ needs to be checked directly—first note that the functor $\widetilde{\mathbf{T}}_{I'}$ is full and $\mathbf{\Theta}_B$ is full thanks to results of [10]. The assertion (g) is standard, the reader is referred to [5, Lemma 3(c)]. ∎

The functors

$$\mathbf{T}_{I'}, \mathbf{L}_{I'} : \mathrm{mod}_{\mathrm{sp}}(R') \to \mathrm{mod}_{\mathrm{sp}}(R) \tag{3.4}$$

defined in Lemma 3.3 are called the *upper* and *lower induction functors* respectively.

REMARK 3.5. The functor $\mathbf{T}_{I'}$ is not a left adjoint to $\mathbf{res}_{I'}$. To see this consider the following example of a poset and its peak subposet:

$$\begin{array}{ccc} \circ & & \circ \to \circ \\ \downarrow & \subseteq & \downarrow \\ \circ & & \circ \end{array}$$

and their indecomposable socle projective representations

$$\begin{array}{ccc} K & & K \to K \\ \downarrow & \text{and} & \downarrow \\ K & & K \end{array}$$

The crucial role in our considerations is played by the *prinjective Tits quadratic form*

$$\mathbf{q}_{(I,\mathfrak{Z})} : \mathbb{Z}^I \to \mathbb{Z} \tag{3.6}$$

of $(I,\mathfrak{Z})$ defined following [26, 2.10] by the formula

$$\mathbf{q}_{(I,\mathfrak{Z})}(x) = \langle x, x\rangle_{(I,\mathfrak{Z})}$$

where the bilinear form $\langle \cdot, -\rangle_{(I,\mathfrak{Z})}$ is given by

$$\langle x, y\rangle_{(I,\mathfrak{Z})} = \sum_{\substack{i\preceq j,\, j\notin \max I \\ (i,j)\notin\mathfrak{Z}}} x_j y_i + \sum_{p\in\max I} x_p y_p - \sum_{\substack{i\prec p,\, p\in\max I \\ (i,p)\notin\mathfrak{Z}}} x_i y_p, \tag{3.7}$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$.

Together with the form (3.7) we shall use its symmetrization $(-,-)_{(I,\mathfrak{Z})}$:

$$(x, y)_{(I,\mathfrak{Z})} = \tfrac{1}{2}(\langle x, y\rangle_{(I,\mathfrak{Z})} + \langle y, x\rangle_{(I,\mathfrak{Z})}). \tag{3.8}$$

Recall from [10], [19] that given a $K(I,\mathfrak{Z})$-module X the *coordinate vector* of X is the vector $\mathbf{cdn}\, X \in \mathbb{Z}^I$ such that $\mathbf{cdn}\, X(i)$ is the multiplicity of $e_i K(I,\mathfrak{Z})$ as a direct summand in the projective cover of X if $i \neq *, +$ and $\mathbf{cdn}\, X(i) = \dim X e_i$ otherwise. We shall use the exponential convention (see [19, Remark 11.57]) of writing coordinate vectors, that is, the vector $(i_1,\ldots,i_r) \in \mathbb{Z}^r$ is written as $1^{i_1}2^{i_2}\ldots r^{i_r}$, k^{i_k} is omitted if $i_k = 0$ and we write k instead of k^1.

LEMMA 3.9 [10]. *For any prinjective $K(I,\mathfrak{Z})$-modules X, Y,*

$$\langle \mathbf{cdn}\, X, \mathbf{cdn}\, Y\rangle_{(I,\mathfrak{Z})} = \dim_K \operatorname{Hom}_{K(I,\mathfrak{Z})}(X,Y) - \dim_K \operatorname{Ext}^1_{K(I,\mathfrak{Z})}(X,Y). \ \blacksquare$$

LEMMA 3.10. *Suppose that $(I,\mathfrak{Z})$ is a subamalgam-like multipeak poset with zero-relations. The following conditions are equivalent.*

(a) *The Tits quadratic form $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative.*
(b) *$(I,\mathfrak{Z})$ contains none of the posets of Table 1 as a peak subposet.*

Proof. The proof is analogous to the construction of the list of hypercritical two-peak posets [4, Theorem 5.5]. The list of critical subamalgam-like posets with zero-relations is given in [26] (it can also be recovered from [29]). They are just subposets consisting of elements marked by digits in Table 1. The numbers associated with the elements of such a critical poset $C = (C, \mathfrak{z}_C)$ are the coordinates of the *radical vector* μ_C of the corresponding quadratic form $\mathbf{q}_C$, that is, the integral vector μ_C generating the group $\mathrm{Rad}(\mathbf{q}_C) = \{v \in \mathbb{Z}^C : \mathbf{q}_C(v) = 0\}$.

A case by case inspection shows that each of the posets of Table 1 is an extension of some critical poset by one point with negative index in the sense of [4, Definition 5.3] and moreover every extension of a critical poset by a point with negative index which is subamalgam-like contains one of the posets of Table 1 as a peak subposet.

For example, observe that the poset $\widehat{\mathcal{F}}_1^2$ in Table 1 is an extension of the critical poset

$$C: \quad \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[d] & \\ & 1 \arrow[d] & \\ 1 \arrow[d] & 1 \arrow[dl] \arrow[dr] & 1 \arrow[d] \\ 2 & & 2 \end{tikzcd}$$

by a point with index -1. In the above picture the elements of C are labeled by coordinates of μ_C.

Recall that $(I, \mathfrak{z})$ is an extension of a critical poset C by a point a with negative index if C is a peak subposet of $(I, \mathfrak{z})$, $I = C \cup \{a\}$ and $(\mu_C, \varepsilon_a)_{(I,\mathfrak{z})} < 0$. Here ε_a is the standard basis vector of $\mathbb{Z}^I$ associated with a.

Repeating the arguments from the proof of Theorem 5.5 of [4] we show that the posets listed in Table 1 form a complete list of minimal multipeak posets with zero-relations which are subamalgam-like and such that the associated quadratic form is not weakly non-negative. ∎

4. Right peak algebras and the upper chain reduction. In the proof of our main result we essentially use the construction of the upper chain reduction $\xi_c : I \mapsto \xi_c I$ introduced in [16]. For the convenience of the reader we briefly sketch some of the relevant ideas.

Consider a two-peak poset I with $\max I = \{*, +\}$ and assume that the poset $I_0 = *^\nabla \cap +^\nabla$ has a unique maximal element c and it is dual to the poset of socle projective modules over the incidence algebra of another poset $\bar{I}_0$ of width at most 2 (see [19, Section 2]). We consider I_0 together with the natural embedding $\bar{I}_0 \hookrightarrow I_0$ sending an element i to the element representing the indecomposable projective module corresponding to i.

The incidence algebra KI of I can be viewed in triangular matrix form as

$$\begin{pmatrix} KI_0 & N \\ 0 & KI_c \end{pmatrix}$$

where $I_c = I_* \cup I_+$ (disjoint union) and $I_* = *^\nabla \setminus c^\nabla$, $I_+ = +^\nabla \setminus c^\nabla$.

LEMMA 4.1. *Let I and I_0 be as above. Assume that I_* and I_+ are empty or linearly ordered*, *and I_0 is a garland* (*see Section* 3). *Then*

(a) *If* $\inf\{x, y\}$ *exists in I_0 for any $x, y \in I_0$ then I_0 is dual to the poset of socle projective modules over the incidence algebra of $\bar{I}_0$, where*

$$\bar{I}_0 = I_0 \setminus \{\inf\{x, y\} : x, y \in I_0 \text{ are incomparable}\}.$$

(b) *If the condition in* (a) *holds and for every $x \in I_c$ the set $\{y \in I_0 : y \preceq x\}$ has a greatest element then the right KI_0-module* $\mathrm{Hom}_{KI_c}(N, Y)$ *is projective for every socle projective KI_c-module Y.*

(c) *If* $\sup\{x, y\}$ *and* $\inf\{x, y\}$ *exist in I for any $x, y \in I_0$ and are both in I_0 then the conclusions of* (a) *and* (b) *hold.*

Proof. The statement (a) follows from the description of socle projective modules over the incidence algebra of a poset of width at most two [19, Section 2], whereas (b) and (c) are easy to check. ∎

Throughout this section we assume that the conditions (a) and (b) of Lemma 4.1 are satisfied.

Together with $\bar{I}_0$ we consider the poset $\bar{I}_0^\omega = \bar{I}_0 \setminus \{c\} \cup \{\omega\}$ with a new unique minimal element ω.

Denote by $\mathbb{H}_c$ the category $\mathrm{Hom}_{KI_c}(N, \mathrm{mod}_{\mathrm{sp}}(KI_c))$, that is, the image of the functor

$$\mathrm{Hom}_{KI_c}(N, -) : \mathrm{mod}_{\mathrm{sp}}(KI_c) \to \mathrm{mod}(K),$$

and let $\mathrm{ind}\,\mathbb{H}_c$ be a fixed set of isomorphism classes of indecomposable objects of $\mathbb{H}_c$. Let $Y = \bigoplus_{y \in \mathrm{ind}\,\mathbb{H}_c} y$ and $H = \mathbb{H}_c(Y, Y)$. Then H has a natural algebra structure induced by composition in $\mathbb{H}_c$ and Y is an H-KI_0-bimodule.

We assume that the condition (b) of Lemma 4.1 holds, hence Y is projective as a KI_0-module and it follows that it corresponds via the Yoneda functors composed with a suitable reflection to a $K\bar{I}_0^\omega$-H-bimodule M. Consider the algebra

$${}_R\Omega = \begin{pmatrix} K\bar{I}_0^\omega & M \\ 0 & H \end{pmatrix} = \begin{pmatrix} K & W \\ 0 & B \end{pmatrix}$$

where $B = \left(\begin{smallmatrix} K(\bar{I}_0 \setminus \{c\}) & \overline{M} \\ 0 & H \end{smallmatrix}\right)$, $\overline{M}$ is a $K(\bar{I}_0 \setminus \{c\})$-$H$-bimodule and W is a K-B-bimodule.

The inverse of the Yoneda functor induces a functor

$$\omega_* : \operatorname{mod}^{\mathrm{pg}} \begin{pmatrix} KI_0 & N \\ 0 & KI_c \end{pmatrix}^{KI_0} \to \operatorname{mod}^{\mathrm{pg}} \begin{pmatrix} K & W \\ 0 & B \end{pmatrix}^{K}$$

Composing ω_* with a suitable reflection (see [16], [15, Definition 2.13])

$$\nabla : \operatorname{mod}^{\mathrm{pg}} \begin{pmatrix} K & W \\ 0 & B \end{pmatrix}^{K} \to \operatorname{mod}_{\mathrm{ic}} \begin{pmatrix} B & W^* \\ 0 & K \end{pmatrix}_{K}$$

we get an equivalence

$$\operatorname{mod}^{\mathrm{pg}} \begin{pmatrix} KI_0 & N \\ 0 & KI_c \end{pmatrix}^{KI_0} \to \operatorname{mod}_{\mathrm{ic}} \begin{pmatrix} B & W^* \\ 0 & K \end{pmatrix}_{K} \tag{4.2}$$

[16, Theorem 2.22]. Define the algebra

$$\xi_c^*(KI) = \begin{pmatrix} \overline{B} & W^* \\ 0 & K \end{pmatrix}$$

where

$$\overline{B} = B/\operatorname{ann}_B W^*.$$

It is proved in [15] that

$$\operatorname{mod}_{\mathrm{ic}} \begin{pmatrix} B & W^* \\ 0 & K \end{pmatrix} \quad \text{and} \quad \operatorname{mod}_{\mathrm{sp}} \begin{pmatrix} B & W^* \\ 0 & K \end{pmatrix}$$

are equivalent categories.

This is the crucial element of the construction. Using adjustment functors and the equivalence (4.2) one can define an equivalence

$$\mathbf{\Xi} : \operatorname{mod}_{\mathrm{sp}}(KI)/[\mathbf{T}_{I_c}(\operatorname{mod}_{\mathrm{sp}}(KI_c))] \xrightarrow{\cong} \operatorname{mod}_{\mathrm{sp}}(\xi_c^* KI)/[\mathbf{L}_{\bar{I}_0}(\operatorname{mod}_{\mathrm{sp}}(K\bar{I}_0))]$$

where $\mathbf{T}_{I_c}$ and $\mathbf{L}_{\bar{I}_0}$ are defined in Lemma 3.3 (see [16, Theorem 3.4, Remark 3.15]). Given a class $\mathcal{C}$ of objects in some category we denote by $[\mathcal{C}]$ the two-sided ideal of morphisms factorizing through an object in $\mathcal{C}$.

Consider again the case when I_0 is a garland and I_*, I_+ are empty or linearly ordered. Under the additional assumption that c is maximal in $I^- = I \setminus \max I$ one can give an explicit description of the algebra $\xi_c^*(KI)$. Namely, define a poset

$$\xi_c^* I = \bar{I}_0 \cup (I_* \setminus \{*\}) \cup (I_+ \setminus \{+\}) \cup \{\overline{*}, \overline{+}\} \tag{4.3}$$

where $\overline{*}, \overline{+}$ are new elements, with partial order extending the ordering in I by the following new relations:

- $\overline{*} \prec x$ for $x \in I_* \setminus \{*\}$,
- $\overline{+} \prec y$ for $y \in I_+ \setminus \{+\}$,
- $x \preceq c$ for $x \in \xi_c^* I$.

LEMMA 4.3. *If the poset I satisfies the conditions* (a) *and* (b) *in Lemma* 4.1 *then there exists an algebra isomorphism*

$$\xi_c^*(KI) \cong K\xi_c^* I.$$

Proof. The assertion follows from the analysis of the construction of $\xi_c^*(KI)$ sketched above; see [16, Remark 3.15] for details. ∎

It is easy to check that $\xi_c^* I$ is the one-point enlargement (by the unique maximal element c) of the poset $\xi_c I$ constructed according to Remark 3.15 in [16].

5. The proof of Theorem 1.2—case A. Throughout this section we assume that $(I, \mathfrak{Z})$ is a subamalgam-like poset with zero-relations.

The aim is to prove Theorem 1.2. First we concentrate on the proof of the implication (c)⇒(a) (or (c)⇒(b)) since this is the crucial part of the whole proof.

From now on we assume that the prinjective Tits quadratic form of $(I, \mathfrak{Z})$ is weakly non-negative. It follows from Lemma 3.10 that $(I, \mathfrak{Z})$ contains no poset of Table 1 as a peak subposet. Therefore I_0 is a garland.

The case $I_0 = \emptyset$ is trivial since then $(I, \mathfrak{Z})$ decomposes into a disjoint union of two linearly ordered posets. Hence we assume that I_0 is not empty. We distinguish two cases:

A. I^0 is linearly ordered.

B. I^0 is not linearly ordered.

Case B will be treated separately in Section 6.

CASE A. We shall apply the *peak reduction* with respect to the peak $*$ (see [7]).

Let $R = K(I, \mathfrak{Z})$, $R' = K(I \setminus \{+\}, \mathfrak{Z}')$, $H'' = k\{+\}$, where $\mathfrak{Z}' = \{(i, j) \in \mathfrak{Z} : j \neq +\}$. The algebra R has a presentation

$$R \cong \begin{pmatrix} R' & {}_{R'}N_{H''} \\ 0 & H'' \end{pmatrix}$$

for an R'-H''-bimodule $N = {}_{R'}N_{H''}$.

Consider the functor

$$|-| = (-) \otimes_{R'} N : \mathrm{prin}(R') \to \mathrm{mod}(H'')$$

and denote its image by $\mathbb{K}_{H''}$.

LEMMA 5.1. *The functor* $|-|$ *has the following properties*:

(i) *the image of every indecomposable module* X *in* $\mathrm{prin}(R')$ *is at most one-dimensional and* $\dim_K |X| = \sum_{i \in I_0} \mathbf{cdn}(X)(i)$,

(ii) *for every pair* X, Y *of indecomposable objects of* $\operatorname{prin}(R')$ *such that* $|X| \neq 0 \neq |Y|$,

$$\dim_K \operatorname{Hom}_{R'}(X,Y) - \dim_K \operatorname{Ext}^1_{R'}(Y,X) \leq \dim_K \mathbb{K}_{H''}(|X|,|Y|).$$

Proof. The category $\operatorname{prin}(K*^{\nabla})$ can be embedded into $\operatorname{prin}(R')$ via the induction functor $\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_{*^{\nabla}}$ (see Lemma 3.3). Observe that each indecomposable prinjective R'-module is either induced from a $K*^{\nabla}$-module (via the upper induction functor) or is isomorphic to $e_i R'$ for some $i \in C''$.

Moreover, the poset $*^{\nabla} = I_0 \cup C'$ has width at most 2. Denote this poset by S. It follows [19, Section 2.4] that each indecomposable prinjective KS-module is isomorphic to one of the following:

$$\begin{aligned} &P^S(i) = e_i KS, && i \in S,\\ &P^S_0(i) = P^S(i)/\operatorname{soc}(P^S(i)), && i \in S \setminus \{*\},\\ &P^S(s,t) = P^S(s) \oplus P^S(t)/\Delta(P^S(*)), && \end{aligned}$$

where $\Delta : P^S(*) \to P^S(s) \oplus P^S(t)$ is induced by the diagonal embedding of P^S_* into $\operatorname{soc}(P^S(s)) \oplus \operatorname{soc}(P^S(t))$ for any incomparable $s, t \in S$.

Denote by M the space $(N'')^*$ dual to N''. Observe that M is the restriction of the R-injective envelope E_+ of P_+ to R'.

It is easy to check that

$$\dim_K \operatorname{Hom}_{R'}(X,M) = \sum_{i \in I_0} \mathbf{cdn}\, X(i) \leq 1$$

for every indecomposable prinjective R'-module X and (i) follows since

$$\operatorname{Hom}_{R'}(X,M) \cong \operatorname{Hom}_K(|X|,K).$$

The statement (ii) follows by simple case by case inspection. It is trivial in the case when one of X, Y is either projective or of the form $P^S_0(i)$ for some i. ■

Let $\operatorname{ind}\mathbb{K}_{H''}$ be a set of representatives of isomorphism classes of indecomposable objects in $\mathbb{K}_{H'}$. Let

$$Z = \bigoplus_{z \in \operatorname{ind}\mathbb{K}_{H''}} z \quad \text{and} \quad E = \mathbb{K}_{H''}(Z,Z).$$

Then Z has an E-H''-bimodule structure and we set

$$S_* R = \begin{pmatrix} E & Z \\ 0 & H'' \end{pmatrix}$$

(compare [6, 2.6]).

There is a well-behaved functor

$$\mathbb{S}_* : \mathrm{mod}_{\mathrm{sp}}(R) \to \mathrm{mod}_{\mathrm{sp}}(S_*R)$$

defined in [6, 3.4].

It follows from Lemma 5.1(i) above that the algebra S_*R is an incidence algebra of a poset. Let us give an explicit description of that poset.

Let $\mathrm{ind}\,\mathbb{K}_{H''} = \{z_j\}_{j\in J'}$ and let Y_j be the indecomposable object of $\mathrm{prin}(R')$ corresponding to z_j for every $j \in J'$. It follows from [6, Proposition 4.5] that $S_*R \cong KJ$, where $J = J' \cup \{+\}$ is partially ordered by

$$\begin{aligned} &j_1 \preceq j_2 \Leftrightarrow \mathbb{K}(|Y_{j_2}|, |Y_{j_1}|) \neq 0 \qquad && \text{for } j_1, j_2 \in J', \\ &j \preceq + && \text{for every } j \in J. \end{aligned}$$

Following [7, 2.14] and [4] we define a map $s_*^- : \mathbb{Z}^J \to \mathbb{Z}^I$ by the formulas

$$s_*^-(\varepsilon_j) = \mathbf{cdn}(Y_j) \quad \text{for } j \in J', \qquad s_*^-(\varepsilon_+) = \varepsilon_+,$$

where ε_j denotes the jth standard basis vector of $\mathbb{Z}^J$ or $\mathbb{Z}^I$. Observe that $s_*^-(\mathbb{N}^J) \subseteq \mathbb{N}^I$.

We denote by $\mathbf{q}_J$ the Tits quadratic form associated with J, that is, the form (3.6), where J is identified with $(J, \emptyset)$.

LEMMA 5.2. *In the above notation the following hold.*

(a) $(\varepsilon_{j_1}, \varepsilon_{j_2})_J \geq (s_*^-(\varepsilon_{j_1}), s_*^-(\varepsilon_{j_2}))_{(I,3)}$ *for every* $j_1, j_2 \in J$,

(b) $\mathbf{q}_J(v) \geq \mathbf{q}_{(I,3)}(s_*^-(v))$ *for any* $v \in \mathbb{N}^J$,

(c) *if the form* $\mathbf{q}_{(I,3)}$ *is weakly non-negative then so is* $\mathbf{q}_J$.

Proof. Denote $\mathbb{K}_{H''}$ by $\mathbb{K}$.

(a) First consider the case when $j_1, j_2 \in J'$. Then

$$2(\varepsilon_{j_1}, \varepsilon_{j_2})_J = \dim_K \mathbb{K}(|Y_{j_1}|, |Y_{j_2}|) + \dim_K \mathbb{K}(|Y_{j_2}|, |Y_{j_1}|).$$

By Lemmas 5.1(ii) and 3.9 we get

$$\begin{aligned} (\varepsilon_{j_1}, \varepsilon_{j_2})_J &\geq \frac{1}{2}(\langle \mathbf{cdn}\, Y_{j_1}, \mathbf{cdn}\, Y_{j_2}\rangle_{(I,3)} + \langle \mathbf{cdn}\, Y_{j_2}, \mathbf{cdn}\, Y_{j_1}\rangle_{(I,3)}) \\ &= (s_*^-(\varepsilon_{j_1}), s_*^-(\varepsilon_{j_2}))_{(I,3)}. \end{aligned}$$

If $j_1 = +$, $j_2 \in J'$ then $(\varepsilon_{j_1}, \varepsilon_{j_2})_J = -1$ and $(s_*^-(\varepsilon_{j_1}), s_*^-(\varepsilon_{j_2}))_{(I,3)} = (\varepsilon_+, \mathbf{cdn}\, Y_{j_2})_{(I,3)}$, where $\mathbf{cdn}\, Y_{j_2}(+) = 0$ and $\mathbf{cdn}\, Y_{j_2}(i) > 0$ for at least one $i \prec +$. It is easy to observe that $(\varepsilon_+, \mathbf{cdn}\, Y_{j_2})_{(I,3)} = -\dim_K \mathrm{Ext}^1_{(I,3)}(Y_{j_2}, P_+) \leq -1$.

For $j_1 = j_2 = +$ both sides of the inequality in (a) are 1. Thus (a) follows.

The assertions (b) and (c) are direct consequences of (a). ∎

5.3. *Proof of the implication* (c)⇒(b) *of Theorem 1.2 in case A.* Since the functor $\mathbb{S}_*$ preserves and respects tame representation type (see [6, Theorem 3.3(c)]) it is enough to prove that the category $\mathrm{prin}(KJ)$ is of tame representation type. This follows from the well known Nazarova theorem [8], [19, Theorem 15.3] since by Lemma 5.2(c) the Tits quadratic form $\mathbf{q}_J$ is weakly non-negative. ■

6. Proof of Theorem 1.2 in case B. As in the previous section, $(I, \mathfrak{Z})$ is a subamalgam-like poset with zero-relations and the associated prinjective Tits quadratic form $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative. Throughout this section we assume that the poset I_0 is not linearly ordered. Let us start with a combinatorial preparation.

LEMMA 6.1. *If x and y are incomparable elements of I_0 and $z \in C' \cup C''$ then z is comparable with at least one of x and y.*

Proof. Otherwise $\mathbf{q}_{(I,\mathfrak{Z})}(v) < 0$ if $v \in \mathbb{Z}^J$ is the vector such that $v(z) = 1$, $v(x) = v(y) = v(*) = v(+) = 2$ and $v(i) = 0$ for $i \notin \{x, y, z, *, +\}$. ■

It follows that $C' \setminus \{*\} \subseteq +^{\nabla}$ and every element of C'' is comparable with at least one element of I_0.

Let c'_ω be the maximal element of $C' \setminus \{*\}$ (if $C' \setminus \{*\} \neq \emptyset$) and c''_α the minimal element of C''.

LEMMA 6.2. *Suppose that $(I, \mathfrak{Z})$ is a subamalgam-like poset with zero-relations and the Tits quadratic form $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative. There exists a subamalgam-like poset $(\widetilde{I}, \widetilde{\mathfrak{Z}})$ such that*

(a) *for any incomparable $x, y \in \widetilde{I}_0$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in $\widetilde{I}$ and are both in $\widetilde{I}_0$, where*

$$\widetilde{I}_0 = \{z \in \widetilde{I} : z \prec *,\ z \prec +,\ (x, +) \notin \widetilde{\mathfrak{Z}}\}.$$

(b) $(I, \mathfrak{Z})$ *is a peak subposet of* $(\widetilde{I}, \widetilde{\mathfrak{Z}})$,

(c) $\mathbf{q}_{(\widetilde{I},\widetilde{\mathfrak{Z}})}$ *is weakly non-negative,*

(d) *if* $\operatorname{mod}_{\mathrm{sp}}(K(\widetilde{I}, \widetilde{\mathfrak{Z}}))$ *is of tame representation type then so is* $\operatorname{mod}_{\mathrm{sp}}(K(I, \mathfrak{Z}))$.

Proof. The construction of $(\widetilde{I}, \widetilde{\mathfrak{Z}})$ goes as follows. Let $\{x_1, y_1\}, \ldots$ $\ldots, \{x_m, y_m\}$ be all two-element sets of incomparable elements of I_0 listed in such a way that $x_1 \prec \ldots \prec x_m$. Let $x_i \vee y_i$ and $x_i \wedge y_i$ for $i = 1, \ldots, m$ be new elements.

Let $L_1 = \{x_m \vee y_m\}$ if there is no $\sup\{x_m, y_m\}$ in I or $\sup\{x_m, y_m\} \notin I_0$, and $L_1 = \emptyset$ otherwise. Similarly, $L_2 = \{x_1 \wedge y_1\}$ if there is no $\inf\{x_1, y_1\}$ in I or $\inf\{x_1, y_1\} \notin I_0$, and $L_2 = \emptyset$ otherwise. Moreover, put

$$\widetilde{I} = I \cup \{x_1 \vee y_1, \ldots, x_{m-1} \vee y_{m-1}\} \cup L_1 \cup \{x_2 \wedge y_2, \ldots, x_m \wedge y_m\} \cup L_2.$$

The partial order relation in $\widetilde{I}$ extends that in I as follows:

$$\begin{aligned}
&x \wedge y \prec x, y \prec x \vee y, \\
&z \prec x \vee y \quad \text{if and only if } z \prec x \text{ or } z \prec y, \\
&x \vee y \prec z \quad \text{if and only if } x \prec z \text{ and } y \prec z, \\
&z \prec x \wedge y \quad \text{if and only if } z \prec x \text{ and } z \prec y, \\
&x \wedge y \prec z \quad \text{if and only if } x \prec z \text{ or } y \prec z
\end{aligned}$$

for every pair x, y of incomparable elements of I_0 and $z \in I$. We put $\widetilde{\mathfrak{Z}} = \mathfrak{Z}$.

It is clear that $(\widetilde{I}, \widetilde{\mathfrak{Z}})$ is a multipeak poset with zero-relations and it has a partition $C' \cup \widetilde{I}_0 \cup C''$ satisfying the conditions in the definition of a subamalgam-like poset. Moreover, the assertion (a) follows; (b) is obvious.

(c) Assume that $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative whereas $\mathbf{q}_{(\widetilde{I},\widetilde{\mathfrak{Z}})}$ is not. It follows that $(\widetilde{I}, \widetilde{\mathfrak{Z}})$ contains a hypercritical poset $\mathcal{H}$ containing $x_i \wedge y_i$ or $x_i \vee y_i$. Assume that $x_i \wedge y_i \in \mathcal{H}$ for some $i = 1, \ldots, m$. Observe that either $x_i \wedge y_i$ is comparable with all the remaining elements of $\widetilde{I}$ or $i = 1$, $x_1 \wedge y_1$ is minimal in $\widetilde{I}$ and the set of elements of $\widetilde{I}$ which are incomparable with $x_1 \wedge y_1$ is linearly ordered or empty. It follows by analysis of the hypercritical posets listed in Table 1 that this is impossible. We proceed analogously in the case when $x_i \vee y_i \in \mathcal{H}$ for some $i = 1, \ldots, m$.

The assertion (c) follows from Lemma 3.3(g). ∎

Let c be the maximal element of I_0. If $J'' = \{i \in I : c \prec i \preceq +\}$, D is the poset of elements of C'' incomparable with c and $J' = I \setminus (J' \cup D)$ then $J' + D + J''$ is a splitting decomposition in the sense of [17, Definition 3.3]. The Splitting Theorem [17, 3.10] implies that every indecomposable socle projective $K(I, \mathfrak{Z})$-module is induced either from a $K(D \cup J'')$-module or a $K(J' \cup D)$-module. Similarly if $c_\bullet$ is the minimal element of I_0, set $J'_\bullet = \{i \in C' : i \prec c_\bullet\}$, let $D_\bullet$ be the set of elements of $C' \setminus \{*\}$ which are incomparable with $c_\bullet$ and $J''_\bullet = I \setminus (D_\bullet \cup J'_\bullet)$. Then $J'_\bullet + D_\bullet + J''_\bullet$ is again a splitting decomposition.

Thus thanks to the Splitting Theorem and Lemma 6.2 without loss of generality we can assume that the poset I satisfies the following conditions:

(6.3)
- for any incomparable $x, y \in I_0$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in I and both lie in I_0,
- there is no $i \in I$ such that $c \prec i \prec +$,
- there is no $x \in I$ such that $x \prec i$ for every $i \in I_0$.

Observe that according to the definition of subamalgam-like posets there is no $i \in I$ such that $c \prec i \prec *$.

From now on we assume that $(I,\mathfrak{Z})$ is a subamalgam-like poset which satisfies the condition (6.3). Note also that $(I,\mathfrak{Z})$ satisfies this condition if and only if the reflection dual poset $(I^\bullet,\mathfrak{Z}^\bullet)$ does.

Proposition 6.4. *If $(I,\mathfrak{Z})$ is a subamalgam-like poset with zero-relations satisfying* (6.3) *and such that $\mathbf{q}_{(I,\mathfrak{Z})}$ is weakly non-negative then the category* $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$ *is of tame representation type.*

We precede the proof by several preparatory lemmata.

Let J be the poset $(I\setminus C')\cup\{*\}$; J is a peak subposet of $(I,\mathfrak{Z})$. Thanks to the condition (6.3), Lemma 4.1 applies to J.

Consider the one-peak poset $\xi_c^* J$ (see 4.2). Recall that there exists an equivalence of categories

$$\Xi : \mathrm{mod}_{\mathrm{sp}}(KJ)/[\mathbf{T}_{J_c}(\mathrm{mod}_{\mathrm{sp}}(KJ_c))] \cong \mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J)/[\mathbf{L}_{\bar{I}_0}(\mathrm{mod}_{\mathrm{sp}}(K\bar{I}_0))]$$

where $J_c = J\setminus c^\nabla$ and $\bar{I}_0$ is defined in the formulation of Lemma 4.1. In our case J_c is a disjoint union of two subposets $\{*\}$ and C''.

Let

$$\pi : \mathrm{mod}_{\mathrm{sp}}(KJ) \to \mathrm{mod}_{\mathrm{sp}}(KJ)/[\mathbf{T}_{J_c}(\mathrm{mod}_{\mathrm{sp}}(KJ_c))]$$

and

$$\pi' : \mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J) \to \mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J)/[\mathbf{L}_{\bar{I}_0}(\mathrm{mod}_{\mathrm{sp}}(K\bar{I}_0))]$$

be the natural projection functors.

For each object X of $\mathrm{mod}_{\mathrm{sp}}(KJ)$ having no summands in $\mathrm{mod}_{\mathrm{sp}}(KJ_c)$ there exists a unique (up to isomorphism) object Y in $\mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J)$ with no summands in $\mathbf{L}_{\bar{I}_0}(\mathrm{mod}_{\mathrm{sp}}(K\bar{I}_0))$ such that $\Xi(\pi(X)) \cong \pi'(Y)$. Denote Y by $\xi_c^*(X)$.

We shall use the following terminology: if G is a garland then a *node* is an element of G which is comparable with each element of G.

Let x_0, y_0 be incomparable elements such that all proper predecessors of x_0 and y_0 in I_0 are nodes. Given $x\in I_0$ denote by N_x the module $e_x KJ/\sum_{c''\in C''\cap x^\Delta} e_{xc''}KJ$.

Denote the set $C'\setminus\{*\}$ by C_0'. The algebra $K(I,\mathfrak{Z})$ is isomorphic to the triangular matrix algebra

$$\begin{pmatrix} KC_0' & M \\ 0 & KJ \end{pmatrix}$$

and for each $c'\in C_0'$ the right KJ-module $e_{c'}M$ is isomorphic to $N_{x_{c'}}$, where $x_{c'}$ is the (unique!) minimal element of $\{y\in I_0 : c'\preceq y\}$.

For $x \prec \inf\{x_0,y_0\}$ in I_0 or $x\in\{x_0,y_0\}$ let N_x' be the unique $K\xi_c^* J$-module with coordinate vector $x\mp c$. For $x=\inf\{x_0,y_0\}$ let N_x' be the unique $K\xi_c^* J$-module with coordinate vector $x_0y_0\mp c$.

LEMMA 6.5. *Under the above notation there is an isomorphism*

$$\xi_c^* N_x \cong N'_x$$

for all elements $x \in I_0$ such that $x \preceq x_0$ or $x \preceq y_0$ in I_0 and the module N'_x is hereditary sp-injective.

Outline of proof. The assertions follow from the observation that if $x \prec \inf\{x_0, y_0\}$ or $x = x_0$ then $x \in \bar{I}_0$ corresponds to the $K\bar{I}_0$-module $e_x K\bar{I}_0$. The element $\inf\{x_0, y_0\}$ corresponds to the unique $K\bar{I}_0$-module with coordinate vector $x_0 y_0 c$. The detailed proof requires an analysis of the ξ_c construction and is left to the reader. The remaining assertion is a consequence of the description of sp-injective modules (see [19, Section 5.2] and Lemma 3.2). ∎

LEMMA 6.6. *Under the above notation and assumptions we have*:

(1) *If X is an object of $\mathbf{T}_{J_c}(\mathrm{mod}_{\mathrm{sp}}(KJ_c))$ then $\mathrm{Hom}_{KJ}(M, X) = 0$.*

(2) *If there exists a non-zero homomorphism*

$$f : \xi_c^*(M) \to Y$$

in $[\mathbf{L}_{\bar{I}_0}(\mathrm{mod}_{\mathrm{sp}}(K\bar{I}_0))]$ then Y has a direct summand in $\mathbf{L}_{\bar{I}_0}(\mathrm{mod}_{\mathrm{sp}}(K\bar{I}_0))$.

Proof. The assertion (1) follows easily from the observation that KJ_c is hereditary and every socle projective KJ_c-module is projective and is mapped onto a projective KJ-module. To prove (2) note that by Lemma 6.5, $\xi_c^*(M)$ is hereditary sp-injective in $\mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J)$. Moreover, if Y is hereditary sp-injective in the image of the functor $\mathbf{L}_{\bar{I}_0}$ and $f : Y \to Z$ is a non-zero homomorphism in $\mathrm{mod}_{\mathrm{sp}}(K\xi_c^* J)$ with Z indecomposable then Z belongs to the image of $\mathbf{L}_{\bar{I}_0}$ as well. ∎

It follows that $\boldsymbol{\Xi}$ induces an isomorphism of the algebras $\mathrm{End}_{KJ}(M)$ and $\mathrm{End}_{K\xi_c^* J}(\xi_c^* M)$ and therefore $\xi_c^* M$ has a canonical structure of KC'_0-$K\xi_c^* J$-bimodule. Let us construct the algebra

$$\widetilde{\xi}_c K(I, \mathfrak{z}) = \begin{pmatrix} KC'_0 & \xi_c^* M \\ 0 & K\xi_c^* J' \end{pmatrix}.$$

Let $\widetilde{\xi}_c(I, \mathfrak{z})$ be the set

$$\widetilde{\xi}_c(I, \mathfrak{z}) = C'_0 \cup \xi_c^* J \tag{6.7}$$

(disjoint union) with partial order generated by the orders in C'_0 and $\xi_c^* J$ and the following new relations:

$$\begin{aligned} &c' \preceq x && \text{if } c' \preceq x,\ x \in \widetilde{I}_0 \text{ and } x \prec \inf\{x_0, y_0\}, \\ &c' \preceq x_0, y_0 && \text{if } c' \prec \inf\{x_0, y_0\}, \\ &c' \preceq \mp, && \end{aligned}$$

for $c' \in C'_0$.

EXAMPLE 6.8. Let I be the poset

$$\begin{array}{ccccccc}
c' & & 0 & & & & \\
 & \searrow & \downarrow & & & & \\
 & & 1 & & & & \\
 & \swarrow & & \searrow & & & \\
2 & & & & 3 & & \\
 & \searrow & & \swarrow & & & \\
 & & 4 & & & & \\
 & \swarrow & & \searrow & & & \\
5 & & & & 6 & & \\
 & \searrow & & \swarrow & & \searrow & \\
 & & c & & & & 7 \\
 & \swarrow & & \searrow & & \swarrow & \\
\star & & & & + & &
\end{array}$$

and let

$$\mathfrak{z} = \{(c', 7), (c', +)\}.$$

Observe that $I_0 = \{0, 1, 2, 3, 4, 5, 6, c\}$. The poset $\xi_c^*(I, \mathfrak{z})$ has the form

$$\begin{array}{ccccccc}
 & & 0 & & c' & & \\
 & & \downarrow & \swarrow\!\!\!\!\!\!\searrow & \downarrow & \searrow & \\
 & & 2 & & 3 & & \\
 & & \downarrow & \swarrow\!\!\!\!\!\!\searrow & \downarrow & & \\
 & & 5 & & 6 & & \overline{+} \\
 & & \downarrow & & \downarrow & \swarrow & \\
\overline{\ast} & \rightarrow & c & \leftarrow & 7 & &
\end{array}$$

LEMMA 6.9. *Under the above notation and assumptions*:

(1) $\widetilde{\xi}_c K(I, \mathfrak{z}) \cong K\widetilde{\xi}_c(I, \mathfrak{z})$,

(2) *if* $\mathbf{q}_{(I,\mathfrak{z})}$ *is weakly non-negative then so is either* $\mathbf{q}_{\widetilde{\xi}_c(I,\mathfrak{z})}$ *or* $\mathbf{q}_{\widetilde{\xi}_c(I^\bullet,\mathfrak{z}^\bullet)}$.

Here $(I^\bullet, \mathfrak{z}^\bullet)$ is the poset reflection dual to $(I, \mathfrak{z})$ (see Proposition 3.1).

Proof. The assertion (1) follows immediately from the definition of the poset $\widetilde{\xi}_c(I, \mathfrak{z})$ and Lemma 6.5.

(2) Let $a : \mathbb{Z}^{\widetilde{\xi}_c(I,\mathfrak{z})} \to \mathbb{Z}^I$ be a $\mathbb{Z}$-linear map acting in the following way

on the standard basis vectors ε_i, $i \in \xi_c^* I$, of $\mathbb{Z}^{\widetilde{\xi}_c(I,\mathfrak{Z})}$:

$$\begin{aligned}
\varepsilon_i &\mapsto \varepsilon_i' - \varepsilon_c' - \varepsilon_+' && \text{for } i \in C_0',\\
\varepsilon_i &\mapsto \varepsilon_i' && \text{for } i \in C'' \setminus \{+\},\\
\varepsilon_j &\mapsto \varepsilon_j' - \varepsilon_c' && \text{for } j \in \bar{I}_0 \setminus \{c\},\\
\varepsilon_c &\mapsto \varepsilon_c' + \varepsilon_*' + \varepsilon_+',\\
\varepsilon_{\overline{*}} &\mapsto -\varepsilon_*',\\
\varepsilon_{\overline{+}} &\mapsto -\varepsilon_+',
\end{aligned}$$

Here we denote by ε_x' the standard basis vector of $\mathbb{Z}^I$ corresponding to x for $x \in I$. A direct calculation shows that

$$(\varepsilon_x, \varepsilon_y)_{\widetilde{\xi}_c(I,\mathfrak{Z})} = (a(\varepsilon_x), a(\varepsilon_y))_{(I,\mathfrak{Z})}$$

for every $x, y \in \xi_c^* I$ (compare [5]). It follows that

$$\mathbf{q}_{\xi_c^*(I,\mathfrak{Z})}(v) = \mathbf{q}_{(I,\mathfrak{Z})}(a(v))$$

for any vector $v \in \mathbb{Z}^{\xi_c^* I}$.

Now assume that $\mathbf{q}_{\xi_c^*(I,\mathfrak{Z})}$ is not weakly non-negative. The Nazarova theorem (see [19, 15.3]) shows that $\xi_c^*(I,\mathfrak{Z})$ contains a peak subposet L isomorphic to a one-peak enlargement $\mathcal{N}_i^*$ of a Nazarova poset for some $i = 1, \dots, 6$.

Let S_1 (resp. S_2) be the set of nodes in I_0 which are incomparable with c_ω' (resp. c_α''). The posets S_1 and S_2 are linearly ordered or empty. Since $\bar{I}_0$ contains a pair of incomparable elements it follows that $S_1 \cap S_2 = \emptyset$.

Let s_1^+ be the maximal element of S_1. Then $C_0', S_1 \setminus \{s_1^+\}, S_2 \subseteq \xi_c^*(I,\mathfrak{Z})$ by definition of $\xi_c^*(I,\mathfrak{Z})$. Note that $c' \preceq s_2$ and $s_1 \preceq s_2$ for every $c' \in C_0'$, $s_1 \in S_1$, $s_2 \in S_2$.

Assume that all the sets $L \cap C_0'$, $L \cap S_1 \setminus \{s_1^+\}$, $L \cap S_2 \setminus \{c\}$ are non-empty. It follows from the analysis of shapes of Nazarova hypercritical posets that then all those sets have exactly one element.

It follows that without loss of generality we can assume that one of the following conditions is satisfied:

(1) $|C_0'| = 1$, $|S_1|, |S_2| \leq 2$,
(2) $C_0' = \emptyset$,
(3) $|S_1| \leq 1$,
(4) $|S_2| \leq 1$.

In case (4), after applying the reflection duality to $(I,\mathfrak{Z})$, we obtain a poset with zero-relations satisfying (3). Thus without loss of generality we can assume that $(I,\mathfrak{Z})$ satisfies one of the conditions (1)–(3) above.

Therefore $L \cap (C_0' \cup \bar{I}_0)$ contains at most one element x such that there are at least two elements incomparable with x in $L \cap (C_0' \cup \bar{I}_0)$. Moreover, the poset $L \cap (C_0' \cup \bar{I}_0)$ has width 2.

There exists a vector $v \in \mathbb{N}^{\xi_c^* I}$ such that $\mathbf{q}_{\xi_c^* I}(v) < 0$ and, in addition,

$$v(c) \geq \sum_{x \in U} v(x)$$

whenever U is a subposet of $\xi_c^*(I, \mathfrak{Z})$ of width at most 2 and such that there is at most one element in U incomparable with at least two elements of U (see [19, 15.24]).

The above remarks imply that $a(v)$ has non-negative coordinates; indeed, this follows now immediately from the formula

$$a(v)(i) = \begin{cases} 0 & \text{if } i \in I \setminus (C' \cup C'' \cup \bar{I}_0), \\ v(i) & \text{if } i \in C' \cup C'' \cup \bar{I}_0 \setminus \{c, *, +\}, \\ v(c) - \sum\limits_{x \in \bar{I}_0 \setminus \{c\} \cup C_0'} v(x) & \text{if } i = c, \\ v(c) - v(\overline{*}) & \text{if } i = *, \\ v(c) - v(\overline{+}) - \sum\limits_{i \in C_0'} v(i) & \text{if } i = +, \end{cases}$$

for any $v \in \mathbb{Z}^{\xi_c^* I}$.

Since $\mathbf{q}_{(I,\mathfrak{Z})}(a(v)) < 0$ the form $\mathbf{q}_{(I,\mathfrak{Z})}$ is not weakly non-negative, a contradiction. ∎

Consider two socle projective algebras

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}, \quad R = \begin{pmatrix} A & M' \\ 0 & B' \end{pmatrix}$$

such that M and M' are faithful as left A-modules. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be a class of socle projective B-modules (resp. B'-modules) such that $\mathcal{C} = \mathrm{add}(\mathcal{C})$ and $\mathcal{C}' = \mathrm{add}(\mathcal{C}')$. Let

$$\pi : \mathrm{mod}_{\mathrm{sp}}(B) \to \mathrm{mod}_{\mathrm{sp}}(B)/[\mathcal{C}],$$
$$\pi' : \mathrm{mod}_{\mathrm{sp}}(B') \to \mathrm{mod}_{\mathrm{sp}}(B')/[\mathcal{C}']$$

be the canonical projection functors.

Assume moreover that

(1) $\mathrm{Hom}_B(M, -)$ annihilates $[\mathcal{C}]$,

(2) if a B'-homomorphism $f : M' \to X$ belongs to $[\mathcal{C}']$ for a socle projective B'-module X then X has a direct summand in $\mathcal{C}'$,

(3) M (resp. M') has no direct summand in $\mathcal{C}$ (resp. $\mathcal{C}'$),

(4) there exists a fully faithful additive functor

$$\mathbf{S} : \mathrm{mod}_{\mathrm{sp}}(B)/[\mathcal{C}] \to \mathrm{mod}_{\mathrm{sp}}(B')/[\mathcal{C}']$$

and an isomorphism

$$\sigma : \mathbf{S}(\pi(M)) \to \pi'(M')$$

such that for every $a \in A$ the diagram

$$\begin{tikzcd} \mathbf{S}(\pi(M)) \arrow[r, "\sigma"] \arrow[d, "\mathbf{S}(\pi(\ell_a))"] & \pi'(M') \arrow[d, "\pi'(\ell_a)"] \\ \mathbf{S}(\pi(M)) \arrow[r, "\sigma"] & \pi'(M') \end{tikzcd}$$

is commutative, where $\ell_a : M \to M$ and $\ell_a : M' \to M'$ is the left multiplication by a.

Lemma 6.10. *Under the above notation and assumptions* (1)–(4) *there exists a fully faithful functor*

$$\overline{\mathbf{S}} : \mathrm{mod}_{\mathrm{sp}}(R)/[\mathrm{mod}_{\mathrm{sp}}(B)] \to \mathrm{mod}_{\mathrm{sp}}(R')/[\mathrm{mod}_{\mathrm{sp}}(B')]$$

where $\mathrm{mod}_{\mathrm{sp}}(B)$ *and* $\mathrm{mod}_{\mathrm{sp}}(B')$ *are treated as subcategories of* $\mathrm{mod}_{\mathrm{sp}}(R)$ *and* $\mathrm{mod}_{\mathrm{sp}}(R')$ *respectively. The image of the functor* $\overline{\mathbf{S}}$ *is the full subcategory of* $\mathrm{mod}_{\mathrm{sp}}(R')/[\mathrm{mod}_{\mathrm{sp}}(B')]$ *formed by cosets of* R'*-modules* $(Y'_A, Y''_{B'}, \phi)$ *such that* $Y''_{B'}$ *has no direct summands in* $\mathcal{C}'$.

Proof. We begin with two observations.

1) For every socle projective B-module X without direct summands in $\mathcal{C}$ there exists a unique (up to isomorphism) socle projective B'-module Y without direct summands in $\mathcal{C}'$ such that $\mathbf{S}(\pi(X)) \cong \pi'(Y)$. Fix a representative $s(X)$ of the isomorphism class of such Y's together with an isomorphism $\sigma_X : \mathbf{S}(\pi(X)) \to \pi'(s(X))$. Without loss of generality we can assume that $s(M) = M'$ and $\sigma_M = \sigma$.

2) If X is a socle projective B-module without direct summands in $\mathcal{C}$ then there is a bijection

$$s_X : \mathrm{Hom}_B(M, X) \to \mathrm{Hom}_{B'}(s(M), s(X))$$

such that for any homomorphism $f : M \to X$ the diagram

$$\begin{tikzcd} \mathbf{S}(\pi(M)) \arrow[r, "\mathbf{S}(\pi(f))"] \arrow[d, "\sigma_M"] & \mathbf{S}(\pi(X)) \arrow[d, "\sigma_X"] \\ \pi'(s(M)) \arrow[r, "\pi'(s_X(f))"] & \pi'(s(X)) \end{tikzcd}$$

is commutative. This is a consequence of the fact that $\mathrm{Hom}_B(M, X) \cap [\mathcal{C}] = 0$ and $\mathrm{Hom}_{B'}(s(M), s(X)) \cap [\mathcal{C}'] = 0$.

Let us construct a map

$$\widetilde{s} : (\mathrm{Ob}(\mathrm{mod}_{\mathrm{sp}}(R)))^\circ \to \mathrm{Ob}(\mathrm{mod}_{\mathrm{sp}}(R'))$$

where $(\mathrm{Ob}(\mathrm{mod}_{\mathrm{sp}}(R)))^\circ$ denotes the class of objects in $\mathrm{mod}_{\mathrm{sp}}(R)$ having no non-zero direct summands in $\mathrm{mod}_{\mathrm{sp}}(B)$.

Let $X = (X'_A, X''_B, \phi : X'_A \otimes_A M \to X''_B)$ be such an object. It follows that X''_B has no direct summands in $\mathcal{C}$.

Let $\bar{\phi} : X'_A \to \mathrm{Hom}_B(M, X''_B)$ be the map adjoint to ϕ and

$$\widetilde{s}(X) = (X'_A, s(X''_B), \psi_\phi : X'_A \otimes_A M' \to s(X''_B))$$

where ψ_ϕ is the homomorphism adjoint to the map

$$X'_A \to \mathrm{Hom}_{B'}(M', s(X''_B)), \quad x \mapsto s_{X''_B}(\bar{\phi}(x)).$$

Thanks to the commutativity of the diagram in condition (4) above, the latter map is an A-homomorphism.

Now we shall extend $\widetilde{s}$ to a functor as required. Consider an R-homomorphism

$$f = (f', f'') : X = (X'_A, X''_B, \phi) \to Y = (Y'_A, Y''_B, \eta)$$

and assume that X and Y do not have non-zero direct summands in $\mathrm{mod}_{\mathrm{sp}}(B)$. Since the functor π' is full there exists a B'-homomorphism $g'' : s(X''_B) \to s(Y''_B)$ making the diagram

$$\begin{array}{ccc} \mathbf{S}(\pi(X''_B)) & \xrightarrow{\mathbf{S}(\pi(f''))} & \mathbf{S}(\pi(Y''_B)) \\ \downarrow{\scriptstyle \sigma_{X''_B}} & & \downarrow{\scriptstyle \sigma_{Y''_B}} \\ \pi'(s(X''_B)) & \xrightarrow{\pi'(g'')} & \pi'(s(Y''_B)) \end{array}$$

commutative. Let us prove that (f', g'') defines an R'-homomorphism from $\widetilde{s}(X)$ to $\widetilde{s}(Y)$. We need to prove that $\psi_\eta \circ (f' \otimes \mathrm{id}_{M'}) = g'' \circ \psi_\phi$. The homomorphism g'' is chosen in such a way that

$$\pi'(g'') \circ \pi'(s_{X''_B}(\bar{\phi}(x))) = \pi'(s_{Y''_B}(\overline{\eta}(f'(x))))$$

for every $x \in X'_A$. It follows that the map

$$g'' \circ s_{X''_B}(\bar{\phi}(x)) - s_{Y''_B}(\overline{\eta}(f'(x))) = g'' \circ \overline{\psi_\phi}(x) - \overline{\psi_\eta}(f'(x)) : s(M) \to s(Y''_B)$$

belongs to $[\mathcal{C}']$ and since $s(Y''_B)$ has no direct summands in $\mathcal{C}'$ it is the zero map thanks to our assumptions. This proves our claim.

Observe that the $[\mathcal{C}']$-coset of the homomorphism g'' constructed above is uniquely determined.

It is now clear that the map $\widetilde{s}$ together with the map $(f', f'') \mapsto (f', g'')$ defined above induce a functor

$$\overline{\mathbf{S}} : \mathrm{mod}_{\mathrm{sp}}(R)/[\mathrm{mod}_{\mathrm{sp}}(B)] \to \mathrm{mod}_{\mathrm{sp}}(R')/[\mathrm{mod}_{\mathrm{sp}}(B')].$$

It is easy to check that this functor is faithful and its image is as described in the lemma. We leave to the reader checking that $\overline{\mathbf{S}}$ is full. ■

COROLLARY 6.11. *The map*

$$\widetilde{s} : (\mathrm{Ob}(\mathrm{mod}_{\mathrm{sp}}(R)))^\circ \to \mathrm{Ob}(\mathrm{mod}_{\mathrm{sp}}(R'))$$

preserves indecomposability and sends non-isomorphic modules to non-isomorphic ones. ■

LEMMA 6.12. (1) *There exists an equivalence of categories*

$$\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))/[\mathbf{T}_J(\mathrm{mod}_{\mathrm{sp}}(KJ))] \xrightarrow{\simeq} \mathrm{mod}_{\mathrm{sp}}(K\xi_c^*(I,\mathfrak{Z}))/[\mathbf{T}_{\xi_c^*J}(\mathrm{mod}_{\mathrm{sp}}(K\xi_c^*J))].$$

(2) *If the categories* $\mathrm{mod}_{\mathrm{sp}}(KJ)$, $\mathrm{mod}_{\mathrm{sp}}(K(\xi_c^*J))$ *and* $\mathrm{mod}_{\mathrm{sp}}(K\xi_c^*(I,\mathfrak{Z}))$ *are of tame representation type then so is* $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$.

Proof. It is easy to see that Lemma 6.10 applies here. Thus assertion (1) follows. We only sketch the main arguments for (2). We observe that the map on objects defined in (1) is constructible in the spirit of [9], that is, it can be represented by a family of regular maps between suitable algebraic varieties of socle projective modules. Thanks to Corollary 6.11 this map preserves indecomposability. Therefore parameterizations for indecomposable modules in $\mathrm{mod}_{\mathrm{sp}}(K(\xi_c^*(I,\mathfrak{Z})))$ induce suitable parameterizations for indecomposable modules in $\mathrm{mod}_{\mathrm{sp}}(K(I,\mathfrak{Z}))$.

Then the preservation of tameness follows by standard arguments as in [9, 4.3]. ∎

Proof of Proposition 6.4. Let $\widetilde{\xi}_c(I,\mathfrak{Z})$ be the poset constructed in 6.7. Thanks to Lemma 6.9 we can assume the Tits quadratic forms of $\widetilde{\xi}_c(I,\mathfrak{Z})$ and ξ_c^*J are weakly non-negative and thus by the Nazarova theorem the categories $\mathrm{mod}_{\mathrm{sp}}(K\widetilde{\xi}_c(I,\mathfrak{Z}))$ and $\mathrm{mod}_{\mathrm{sp}}(K(\xi_c^*J)))$ are of tame representation type. The results of [28] apply to the poset J and therefore $\mathrm{mod}_{\mathrm{sp}}(KJ)$ is of tame representation type. Then the assertion follows from Lemma 6.12(2). ∎

Now we are able to finish the proof of Theorem 1.2.

6.13. *Proof of Theorem 1.2.* The equivalence of (a) and (b) is a special case of a general fact (see e.g. [3, Theorem 3.10]).

The implication (a)⇒(c) follows by standard algebraic geometry arguments; this is done in [28, Theorem 2.14].

The equivalence of (c) and (d) is the content of Lemma 3.10.

(c)⇒(b). This implication in case A is proved in 5.3. In case B we can assume thanks to Lemma 6.2 that the poset $(I,\mathfrak{Z})$ satisfies (6.3). Thus the implication (c)⇒(b) follows from Proposition 6.4. ∎

7. Applications to three-partite subamalgams of tiled D-orders. Let $\Lambda^\bullet$ be the three-partite subamalgam of the tiled order Λ as in the introduction.

A *reduced Tits quadratic form*

$$\mathbf{q}_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z} \tag{7.1}$$

is associated with $\Lambda^\bullet$ in [26], [28]. It is defined by the formula

$$
\begin{aligned}
&\mathbf{q}_{\Lambda^\bullet}(x_1,\dots,x_{n_1+n_3},\overline{x}_{n_1+1},\dots,\overline{x}_{n_1+n_3},x_*,x_+)\\
&\qquad = x_*^2+x_+^2+\sum_{j=1}^{n_1+n_3} x_j^2+\sum_{j=n_1+1}^{n_1+n_3}\overline{x}_j^2\\
&\qquad\quad + \sum_{\substack{{}_iD_j=D\\ 1\le i<j\le n_1+n_3}} x_ix_j+\sum_{s<t}\overline{x}_s\overline{x}_t+\sum_{\substack{{}_tD_s=D\\ n_1<t\le n_1+n_3<s}} x_{s-n_1-n_3}\overline{x}_t\\
&\qquad\quad - x_+\Big(\sum_{j=1}^{n_1+n_3} x_j\Big)-x_*\Big(\sum_{j=1}^{n_1}x_j+\sum_{j=n_1+1}^{n_1+n_3}\overline{x}_j\Big).
\end{aligned}
$$

Following the idea of [28] we shall consider the poset with zero-relations $(I^{*+}_{\Lambda^\bullet},\mathfrak{z}_{\Lambda^\bullet})$ associated with $\Lambda^\bullet$ in [26], [28] (see also [22]). It is the result of a two-step procedure: the first step is a reduction of the infinite-dimensional problem of lattices over $\Lambda^\bullet$ to a finite-dimensional matrix problem (see [2], [12], [22]) and the next one is to apply the covering technique to the latter problem [18], [14]. The construction can be summarized as follows. Set

$$I^{*+}_{\Lambda^\bullet}=\{1,\dots,n_1+2n_3,*,+\},\tag{7.2}$$

with partial order generated by all relations $i\preceq j$, where i,j satisfy one of the following conditions:

- ${}_iD_j=D$ and $1\le i,j\le n_1+n_3$ or $n_1+n_3<i,j\le n_1+2n_3$,
- ${}_{i-n_3}D_{j+n_1+n_3}=D$ and $1\le j\le n_1$, $n_1+n_3<i\le n_1+2n_3$,
- $1\le i\le n_1+n_3$ and $j=+$,
- $1\le i\le n_1$ or $n_1+n_3<i\le n_1+2n_3$ and $j=*$.

Finally, $\mathfrak{z}_{\Lambda^\bullet}=\{(i,j): i\preceq j,\ n_1+n_3<i\le n_1+2n_3,\ n_1<j\le n_1+n_3 \text{ or } j=+\}$.

See [31] and [32] for other reduction techniques for orders.

The importance of the above construction is established by the following assertion.

Proposition 7.3 [28, Theorem 3.4]. (1) *There exists a full additive functor*

$$\mathbb{H}:\operatorname{latt}\Lambda^\bullet\to \operatorname{mod}_{\rm sp}(K(I^{*+}_{\Lambda^\bullet},\mathfrak{z}_{\Lambda^\bullet}))$$

which reflects isomorphisms, preserves indecomposability and preserves and reflects the tame representation type. In particular the order $\Lambda^\bullet$ is of tame lattice type if and only if the poset $(I^{*+}_{\Lambda^\bullet},\mathfrak{z}_{\Lambda^\bullet})$ with zero-relations is of tame prinjective type.

(2) *The Tits quadratic forms $\mathbf{q}_{\Lambda^\bullet}$ and $\mathbf{q}_{(I^{*+}_{\Lambda^\bullet},\mathfrak{z}_{\Lambda^\bullet})}$, defined in* (7.1) *and* (3.6) *respectively, coincide.* ■

Lemma 7.4. *Let $\Lambda^{\bullet}$ be a three-partite subamalgam of a tiled order. The poset* $(I^{*+}_{\Lambda^{\bullet}}, \mathfrak{z}_{\Lambda^{\bullet}})$ *defined in* (7.2) *is subamalgam-like.*

Proof. It is enough to put $I_0 = \{1, \ldots, n_1\}$, $C' = \{n_1 + n_3 + 1, \ldots, n_1 + 2n_3, *\}$ and $C'' = \{n_1 + 1, \ldots, n_1 + n_3, +\}$. ∎

7.5. *Proof of Theorem 1.6.* In view of the results of [28] it is enough to prove the implication (b)⇒(a). Thanks to Lemma 7.4 one can apply Theorem 1.2 to the poset $(I^{*+}_{\Lambda^{\bullet}}, \mathfrak{z}_{\Lambda^{\bullet}})$ to prove it is of tame prinjective type provided its prinjective Tits quadratic form is weakly non-negative. Then the order $\Lambda^{\bullet}$ is of tame lattice type by Proposition 7.3. ∎

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