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n-FUNCTIONAL DIGRAPHS UNIQUELY DETERMINED BY THE SKELETON

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Abstract. We show that any total n-functional digraph D is uniquely determined by its skeleton up to the orientation of some cycles and infinite chains. Next, we characterize all graphs G such that each n-functional digraph obtained from G by directing all its edges is total. Finally, we describe finite graphs whose edges can be directed to form a total n-functional digraph without cycles.

Any total functional connected digraph is uniquely determined in the class of all functional digraphs (up to the orientation of a single cycle or a single infinite path with or without the source vertex) by its skeleton. This follows from the fact that such a digraph has exactly one loop or exactly one cycle (a loop is not considered to be a cycle, see below), or an infinite path and no cycles or loops. The *skeleton* of a digraph D is the graph obtained from D by ignoring the orientation of all the edges.

Here we generalize this result to n-functional digraphs, where n is a fixed non-negative integer. A digraph D is said to be *n-functional* (resp. *total n-functional*) if for any vertex v, its outdegree $d^D(v)$, i.e. the number of edges starting from v, is not greater than (resp. equal to) n. Further, we assume that *cycles* and *chains* (finite and infinite) have pairwise different and regular edges, whereas they may contain the same vertex more than once. In particular, a loop is not a cycle here. We also assume that a finite or infinite *path* does not encounter the same vertex twice. Besides infinite chains with source vertices, which are called $\mathbb{N}$-*chains* here, we will also use chains which are infinite in both directions. Such chains will be called $\mathbb{Z}$-*chains*.

The main aim of the present paper is to prove the following result.

THEOREM 1. *Let D be a total n-functional digraph, and H an arbitrary n-functional digraph with the same skeleton as D. Then there is a family R of pairwise disjoint cycles of D, a family S of pairwise edge-disjoint*

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$\mathbb{Z}$-chains of D and a family T of pairwise edge-disjoint $\mathbb{N}$-chains of D such that R, S, T are also pairwise edge-disjoint, and H is obtained from D by inverting the orientation of all the edges in R, S and T.

Proof. Take a vertex v of D and let D_v be the subdigraph of D consisting of v and all finite (directed) chains starting from v. Clearly, D_v is n-functional, as a subdigraph of an n-functional digraph. Moreover, exactly one of the following two possibilities holds:

$$D_v \text{ is a finite digraph} \tag{1.1}$$

or

$$\text{there is an infinite path starting from } v. \tag{1.2}$$

If D_v is infinite, then (1.2) holds by Ramsey's argument, because the out-degrees of all vertices of D_v are bounded by n, and also, for any vertex w of D_v such that $w \neq v$, there is a directed path from v to w.

Now take an n-functional digraph H having the same skeleton as D. Let F be the set of all the regular edges of D that are inversely directed in H. It is sufficient to prove that the edges in F may be divided into three pairwise disjoint sets in such a way that the edges in the first set form pairwise disjoint cycles, the edges in the second set form pairwise edge-disjoint $\mathbb{Z}$-chains, and the edges from the third set form pairwise edge-disjoint $\mathbb{N}$-chains. Of course, we may assume $F \neq \emptyset$.

Let C be the subdigraph of D consisting of F and the endpoints of all the edges in F. Then C is a non-empty n-functional digraph (as a subdigraph of an n-functional digraph) without loops and has at least one regular edge.

Take a vertex v of C. Let $e_1, \ldots, e_k$ be all the edges in F starting from v. Let $f_1, \ldots, f_l$ be all the edges in F ending at v. Let $g_1, \ldots, g_m$ be all the edges of D that start from v and are not in F. Then

$$k + m = n,$$

because D is total.

Next, $f_1, \ldots, f_l, g_1, \ldots, g_m$ are all the edges of H starting from v. Thus

$$l + m \leq n.$$

These two facts yield

$$l \leq k.$$

Hence,

$$dd^C(v) \leq d^C(v) \quad \text{for each vertex } v \text{ of } C, \tag{2}$$

where $dd^C(v)$ is the indegree of v, i.e. the number of edges of C ending at v.

Take a vertex u in C and the digraph C_u (defined at the beginning of the proof). It is easily shown (see also [2], proof of Theorem 2) that for each vertex v of C_u,

$$d^{C_u}(v) = d^C(v). \tag{3}$$

First assume that C_u is a finite digraph. Then (see e.g. [3])

$$L = \sum_{i=1}^{m} d^{C_u}(v_i) = \sum_{i=1}^{m} dd^{C_u}(v_i), \tag{Eq}$$

where L is the number of edges of C_u and $v_1, \ldots, v_m$ are all its vertices.

By (2) and (3) we also have

$$dd^{C_u}(v_i) \le dd^C(v_i) \le d^C(v_i) = d^{C_u}(v_i).$$

This together with the fact that C_u is finite entails that for any vertex v of C_u,

$$dd^{C_u}(v) = d^{C_u}(v),$$

and thus also

$$d^C(v) = dd^C(v) \quad \text{for any vertex } v \text{ of } C_u \tag{4}$$

and

$$dd^{C_u}(v) = dd^C(v).$$

The last equality and (3) imply that for any edge f of C, if the initial or final vertex of f belongs to C_u, then C_u contains f. Using this fact we show

$$C_u = C', \tag{5}$$

where C' is the connected component of C containing u.

As C_u is connected as an undirected graph and contains u, it is contained in C'. On the other hand, for a vertex w of C' such that $w \neq u$, there is an undirected path $(f_1, \ldots, f_k)$ connecting u and w. Since the initial or final vertex of f_1 is equal to u, f_1 is contained in C_u. In particular, both endpoints of f_1 belong to C_u. Hence, the initial or final vertex of f_2 belongs to C_u. Repeating this procedure k times we deduce that $f_1, \ldots, f_k$ are contained in C_u. In particular, w belongs to C_u. This implies that C' and C_u have the same vertex set, so $C' = C_u$ (the edge sets of C' and C_u are also equal, since any edge having endpoints in C_u is contained in C_u).

By (4) and (5), we see, in particular, that for any finite connected component B of C, $d^B(w) = dd^B(w)$ for each vertex w of B. Note that C does not contain trivial (i.e. one-vertex) connected components, so B has at least one regular edge. Thus by Euler's Theorem (see e.g. [1], Chapter 11, Theorem 1), there is a (directed) cycle containing all the edges of B.

Now we remove all the finite connected components of C. (Obviously, we can assume that C has at least one infinite connected component. Otherwise we are done, because each finite connected component forms a cycle.) To simplify the notation we will also denote the resulting digraph by C. Then by (5), for each vertex v of C,

(6) C_v is an infinite digraph.

Indeed, if C_u were finite for some vertex, then by (5), C_u would be a connected component of C containing u, which is a contradiction.

First, we assume that C contains $\mathbb{Z}$-chains, as otherwise it is sufficient to take the empty family as S.

Next, take the family $\mathcal{M}$ of all sets consisting of pairwise edge-disjoint $\mathbb{Z}$-chains in C. Clearly, $\mathcal{M}$ is non-empty, since each set consisting of a single $\mathbb{Z}$-chain is in $\mathcal{M}$. Moreover, the set-theoretical union of any non-empty linearly ordered (by inclusion) subfamily of $\mathcal{M}$ also belongs to $\mathcal{M}$. Thus by Zorn's Lemma, $\mathcal{M}$ has a maximal element S (with respect to inclusion).

By the maximality of S, the digraph $\bar{C}$ obtained from C by omitting all the edges from the family S has no $\mathbb{Z}$-chain. Observe also that for any $\mathbb{Z}$-chain p and a vertex v of C, the numbers of edges of p ending at v and of those starting from v are equal. Hence by (2) we get

$$dd^{\bar{C}}(v) \leq d^{\bar{C}}(v) \quad \text{for each vertex } v \text{ of } \bar{C},$$

since any two chains in S are edge-disjoint.

Summarizing, S is the desired family of $\mathbb{Z}$-chains of C, and the digraph $\bar{C}$ satisfies (2), so we can just assume in the rest of the proof that

(A.1) C does not contain $\mathbb{Z}$-chains.

For any vertex v of C, let C^v be the subdigraph of C consisting of v and all finite (directed) chains ending at v.

Clearly, C^v can be obtained in the following three steps. First, take the digraph C^{in} obtained from C by inverting the orientation of all the edges of C. Next, take the subdigraph C^{in}_v. And finally, invert again the orientation of all the edges in C^{in}_v.

(2) implies that C^{in} is an n-functional digraph, so by (1), C^{in}_v is finite or contains an $\mathbb{N}$-path starting from v. Consequently, C^v is a finite digraph or there is an infinite path in the digraph C ending at v.

Now we show that the second case is impossible. Assume to the contrary that $p = (\ldots, e_3, e_2, e_1)$ is an infinite path with v as its target vertex. Let B be the subdigraph of C obtained from C by removing the edges $e_1, e_2, e_3, \ldots$ For any vertex u of p other than v, the numbers of edges of p ending at u and of those starting from u are equal, whereas at v one more edge ends than starts. Hence, by (2), for any vertex u of B,

$$dd^B(u) \leq d^B(u) \quad \text{and} \quad dd^B(v) \leq d^B(v) - 1.$$

Take the digraph B_v. Since B_v is a subdigraph of B we deduce by (3) that for any vertex u of B_u,

$$dd^{B_u}(u) \leq d^{B_u}(u) \quad \text{and} \quad dd^{B_u}(v) \leq d^{B_u}(v) - 1.$$

These inequalities and the equality (Eq) imply that B_v is infinite. Thus by (1.1–2) (note that B is n-functional, as a subdigraph of C), there is an $\mathbb{N}$-path q of B starting from v. Obviously, the paths p and q together form a $\mathbb{Z}$-chain. This contradiction entails that for any vertex v of C,

(7) C^v is a finite digraph.

Assume that C has cycles, and let $\mathcal{M}$ be the family of all sets consisting of pairwise edge-disjoint cycles of C. Obviously, $\mathcal{M}$ is non-empty, because any set consisting of one cycle of C belongs to $\mathcal{M}$. It is also easy to show that the set-theoretical union of a linearly ordered (by inclusion) subfamily of $\mathcal{M}$ belongs to $\mathcal{M}$ as well. Thus using Zorn's Lemma we can take a maximal element U in $\mathcal{M}$.

Take the digraph obtained from C by omitting all the edges from the family U, and next removing all isolated vertices. Then, of course, the resulting digraph $\bar{C}$ has no cycles. Moreover, since U is a family of pairwise edge-disjoint cycles, the new digraph also satisfies (2). These two facts imply that $\bar{C}$ satisfies (6) (or is empty, but then we are done). Otherwise $\bar{C}_v$ is finite and non-trivial for some vertex v, and then by (4) and Euler's Theorem we get a cycle of $\bar{C}$, a contradiction. Therefore this digraph $\bar{C}$ will also be denoted by C. More precisely, we can assume that

(A.2) C does not contain cycles.

Observe that there is a family R_2 of pairwise disjoint cycles of C which contains all the edges of U. To see this, take the subdigraph B of C consisting of all the edges and vertices from U. Clearly,

$$d^B(w) = dd^B(w) \quad \text{for any vertex } w \text{ of } B.$$

Take a vertex v of B and the digraph B^v. Then for each vertex w of B^v,

(8) $$dd^{B^v}(w) = dd^B(w)$$

and

$$d^{B^v}(w) \leq d^B(w).$$

The proof of (8) is analogous to that of (3), and the inequality follows from the fact that B^v is a subdigraph of B.

Thus

$$d^{B^v}(w) \leq dd^{B^v}(w) \quad \text{for each vertex } w \text{ of } B^v.$$

Hence, because B^v is finite by (7) (as a subdigraph of C^v), we deduce (in exactly the same way as for (4)) that for each vertex w of B^v,

$$d^{B^v}(w) = dd^{B^v}(w),$$

and consequently,

$$d^{B^v}(w) = d^B(w). \tag{9}$$

The equalities (8) and (9) imply (see the proof of (5)) that B^v is the connected component of B containing v. Hence, since v was arbitrarily chosen, each connected component of B is finite. Note also that each connected component is non-trivial, by the definition of B. Thus by Euler's Theorem, each connected component is a cycle. Taking all these cycles we obtain the family R_2 of pairwise disjoint cycles containing all the edges of U. Note that the families R_1 and R_2 are disjoint (where R_1 is the family of cycles containing all the finite connected components of C), so their union $R = R_1 \cup R_2$ is the desired family.

It remains to show that the edges of C form pairwise edge-disjoint $\mathbb{N}$-chains. First, by (1.1–2) and (6) each edge lies on some $\mathbb{N}$-path. (In particular, C contains $\mathbb{N}$-paths, being non-empty by assumption.) Secondly, each path p can be completed to a *maximal* $\mathbb{N}$-path (i.e. to a path such that $dd^C(v) = 0$, where v is its source). Indeed, if $dd^C(v) \geq 1$, then the digraph C^v is non-trivial. Thus, since it is finite by (7), we can take a path q ending at v with maximal length. As C has no cycles, we first deduce, by the maximality of q, that $dd^C(u) = 0$, where u is the initial vertex of q, and secondly, q and p form a new path containing p.

A family L of $\mathbb{N}$-paths is said to contain *relatively maximal* paths if for any $p \in L$ with initial vertex v, each edge of C ending at v lies on some path belonging to L.

Let $\mathcal{M}$ be the family of all sets consisting of relatively maximal and pairwise edge-disjoint $\mathbb{N}$-paths. Obviously, $\mathcal{M}$ is non-empty, because each set consisting of one maximal $\mathbb{N}$-path belongs to $\mathcal{M}$. Observe also that the set-theoretical union of any subfamily $\mathcal{N} \subseteq \mathcal{M}$ contains relatively maximal paths, and moreover, if $\mathcal{N}$ is linearly ordered by inclusion, then $\bigcup \mathcal{N}$ has pairwise edge-disjoint paths. Hence, $\bigcup \mathcal{N} \in \mathcal{M}$.

Thus, using Zorn's Lemma, we take a maximal element T in $\mathcal{M}$. Of course, we want to prove that T contains all the edges of C (which would complete the proof). Assume otherwise, take the digraph B' obtained from C by removing all the edges from T (but not the vertices), and let B be a non-trivial connected component of B'. Then by (2),

$$dd^B(w) \leq d^B(w) \quad \text{for each vertex } w \text{ of } B.$$

More precisely, if w is the source vertex of some path in T, then all edges (in C) ending at w belong to some paths in T, so $dd^B(w) = 0$. If for each path p in T, w is an inner vertex of p, then the number, say k, of edges from T that end at w is equal to that of those that start from w. Hence, $dd^B(w) = dd^C(w) - k \leq d^C(w) - k = d^B(w)$.

Secondly, for any vertex v of B, the digraph B_v is infinite. Otherwise, if B_u is finite for some vertex u (B is connected, which implies that B_u is non-trivial), then by the above inequality and the equality (Eq) we obtain

$$dd^{B_u}(w) = d^{B_u}(w) \quad \text{for any vertex } w \text{ of } B_u,$$

and consequently, by Euler's Theorem, B contains a cycle (in fact, there is a cycle containing all the edges of B; see (5)), but this is in contradiction with (A.2).

Now by (1.1–2), there is an $\mathbb{N}$-path in B (note that B is n-functional, as a subdigraph of the n-functional digraph C). Since B does not contain cycles, there is also a maximal infinite path p in B. But then $T \cup \{p\}$ is a new element of the family $\mathcal{M}$ properly containing T, a contradiction. ∎

Remark. Note that in the proof we construct the family T in such a way that each of its elements is an $\mathbb{N}$-path (not just a chain). But if we admit the weaker condition that T is a family of pairwise edge-disjoint $\mathbb{N}$-chains (instead of paths), then we can choose families R', S', T' (in Theorem 1) such that R' is disjoint (not only edge-disjoint) from S' and T'.

Indeed, let R, S, T be the families from Theorem 1, and take the families R_1 and R_2 of all cycles in R that have common vertices with S and T, respectively. Obviously, the family $R' = R \setminus (R_1 \cup R_2)$ is disjoint from $S \cup T \cup R_1 \cup R_2$. Thus it is sufficient to construct two new edge-disjoint families S' and T' of pairwise edge-disjoint $\mathbb{Z}$-chains and pairwise edge-disjoint $\mathbb{N}$-chains, respectively, containing all the edges of $S \cup T \cup R_1 \cup R_2$.

For each cycle $c \in R_1$ (resp. $c \in R_2$) we choose some $\mathbb{Z}$-chain from S (resp. $\mathbb{N}$-chain from T) which has at least one common vertex with c.

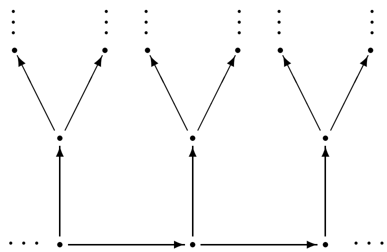
Take a chain $p \in S$ and let $a = (\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots)$ be the sequence of successive vertices of p. Next, take the family F_p of all cycles for which we have chosen p, and for each cycle c in F_p take a common vertex of c and p. Thus we obtain a subsequence $a' = (\ldots, v_{i_{-1}}, v_{i_0}, v_{i_1}, \ldots)$ of pairwise different vertices (because our cycles are pairwise disjoint). Note that we can arrange all the cycles in F_p in a sequence $(\ldots, c_{i_{-1}}, c_{i_0}, c_{i_1}, \ldots)$. Now it is sufficient to insert each cycle c_{i_j} of F_p in the corresponding place of the sequence a (i.e. vertices of c_{i_j} in place of the corresponding element v_{i_j} in a). Applying this construction to each chain from S we obtain the required family S'.

Clearly, in a similar way, we can construct the family T'.

Remark. The family R' is uniquely determined (for a given digraph H), that is, for any similar family R'', there is a bijective correspondence between R' and R'' such that corresponding cycles have the same edges.

This follows from the fact that R' is a family of cycles obtained from all the finite connected components of C (we use the notation from the proof of Theorem 1). More precisely, each cycle c which is disjoint from S and T forms a connected component of C, because c is also disjoint from other cycles, and moreover, each edge of C belongs to R or S or T.

Unfortunately, the following example shows that the families S and T from Theorem 1 are not uniquely determined (for a given digraph H), even in the case of a digraph D without undirected cycles.



More precisely, this is a total 2-functional digraph, and by inverting the orientation of all its edges we also obtain a 2-functional digraph. Clearly, there are families S and T consisting of all the edges of this digraph, but it is easy to see that there are many such families. (Obviously, in each case S contains a single $\mathbb{Z}$-path, and T is a family of $\mathbb{N}$-paths.)

If an n-functional digraph has an $\mathbb{N}$-chain or a $\mathbb{Z}$-chain, then it also has an $\mathbb{N}$-path. This follows from the fact that each vertex v of an $\mathbb{N}$-chain may appear in such a chain at most n times (because at most n edges start from any vertex, in particular, from v). More precisely, let $(v_1, v_2, v_3, \ldots)$ be the sequence of vertices of an $\mathbb{N}$-chain. (Obviously, if a digraph has a $\mathbb{Z}$-chain, then it also has an $\mathbb{N}$-chain.) We first take the last occurrence of v_1 in this sequence, next we take the last occurrence of v_2, and so on. Clearly, the resulting sequence forms an $\mathbb{N}$-path.

By this fact and Theorem 1 we have

COROLLARY 2. *Let n be a positive integer and D be a total n-functional digraph without* (*directed*) $\mathbb{N}$-*paths. Let H be an arbitrary n-functional digraph with the same skeleton. Then H is obtained from D by inverting the orientation of some pairwise disjoint* (*directed*) *cycles of D. In particular, H is also total and does not contain infinite paths.*

Hence we immediately get

COROLLARY 3. *Let D be a total n-functional digraph without* (*directed*) $\mathbb{N}$-*paths and cycles. Then D is uniquely determined by its skeleton.*

Let D_1 be the digraph consisting of all the positive integers $1, 2, \ldots$ as vertices and of all the pairs $\langle k, k+1\rangle$ as directed edges. Let D_2 be the digraph obtained from D_1 by inverting the orientation of all the edges. Then D_1, D_2 are non-isomorphic functional digraphs without cycles and with the same skeleton. Note that D_1 is total, but has an $\mathbb{N}$-path; and D_2 has no $\mathbb{N}$-path, but is not total.

This example shows that the skeleton of a total n-functional digraph can be directed to form a non-total n-functional digraph. But using Corollary 2 and the results in [2] we can easily prove the following fact.

PROPOSITION 4. *Let G be an infinite graph and n a positive integer. Then the following conditions are equivalent*:

(a) *The edges of G can be directed to form a total n-functional digraph without $\mathbb{N}$-paths.*

(b) *The edges of G can be directed to form an n-functional digraph and each such n-functional digraph is total and has no $\mathbb{N}$-path.*

(c) *The edges of G can be directed to form an n-functional digraph and each such n-functional digraph is total.*

(d) *G is a graph such that*

  (d.1) *for any finite subgraph H, $m_{\mathrm{e}} \leq n \cdot m_{\mathrm{v}}$, where m_{v} and m_{e} are the numbers of vertices and edges of H, respectively*,

  (d.2) *for any vertex v, there is a finite subset W of vertices such that $v \in W$ and there are exactly $n \cdot |W|$ edges with endpoints in W.*

Proof. (a)⇒(b) follows from Corollary 2. Of course, (b) implies (a) and (c). (c)⇒(b) follows from the fact that for any n-functional digraph D with an $\mathbb{N}$-path p, we can invert the orientation of p to obtain a new n-functional digraph H such that $d^H(v) = d^D(v) - 1$, where v is the initial vertex of p in D.

(a)⇒(d). Direct all the edges of G to obtain a total n-functional digraph D without $\mathbb{N}$-paths. (d.1) holds by Corollary 6 in [2].

Take a vertex v of D and the digraph D_v from the proof of Theorem 1. By (1.1–2) and (3) in that proof, D_v is finite and total. In particular, the number of its edges is equal to $n \cdot m$, where m is the number of its vertices.

(d)⇒(a). By (d.1) and Corollary 6 in [2], all the edges of G can be directed to form an n-functional digraph D. Let v be a vertex of G and take the set W from (d.2) for v. Next, let H be the subdigraph of D spanned on W. Then H is finite n-functional and has $n \cdot |W|$ edges. These two facts imply that H is total. Hence we infer $d^D(v) = n$, so D is total.

Finally, observe that any regular edge f starting from H is contained in H; this follows from the equality $d^D(w) = n = d^H(w)$ for any vertex w of H. Hence, by simple induction, any path in D starting from v is contained in H. This fact implies that there is no $\mathbb{N}$-path starting from v, because H is finite. Thus D does not contain $\mathbb{N}$-paths. ∎

By Corollary 3 we deduce, in particular, that any finite total n-functional digraph without cycles is uniquely determined by its skeleton. Now we characterize those finite graphs whose edges can be directed to form such a digraph.

Let G be a graph (or digraph) and v its vertex. Then $G - v$ is the graph obtained from G in the following way (see e.g. [3]): first, any regular edge with one endpoint v is replaced by a loop at the other endpoint; next, v and all the loops at v are removed.

A graph (or digraph) H is said to be an *n-reduct* of G iff there is a sequence $G_0, \ldots, G_k$ of graphs such that $G_0 = G$ and $G_k = H$ and for each $i = 0, \ldots, k-1$ we have $G_{i+1} = G_i - v_i$, where v_i is a vertex of G_i with exactly n loops and the connected component of G_i containing v_i is non-trivial.

An n-reduct H of G is *maximal* if each connected component of H is trivial, or H does not contain vertices with exactly n loops.

Proposition 5. *Let G be a finite graph and n be a positive integer. Then the following conditions are equivalent*:

(a) *The edges of G can be directed to form a total n-functional digraph without* (*directed*) *cycles.*
(b) *Some maximal n-reduct of G contains only loops and each of its vertices has exactly n loops.*
(c) *G has exactly one maximal n-reduct H, and each connected component of H contains exactly one vertex and exactly n loops.*

Proof. The implication (c)⇒(b) is trivial.

(b)⇒(a). Let H be a maximal n-reduct of G. This can be witnessed by suitable sequences of graphs $G_0, \ldots, G_k$ and of their vertices $v_0, \ldots, v_{k-1}$ connecting G and H by n-reductions. First, $H = G_k$ can be regarded as a total n-functional digraph without cycles. Now, assume that all the edges of G_i (where $1 \le i \le k$) can be directed to form a total n-functional digraph without cycles. Observe that G_{i-1} is obtained from G_i by adding v_{i-1} with n loops and replacing some loops of G_{i-1} by regular edges with v_{i-1} as one of endpoints. Thus we can direct all these regular edges towards v_{i-1} and obtain a total n-functional digraph without cycles. Hence by simple induction all the edges of $G_0 = G$ can be directed in the desired way.

(a)⇒(c). Let D be a total n-functional digraph without cycles obtained from G. Then D, and also each of its connected components, contains a vertex u with exactly n loops. To see this take the very last vertex u of a path with maximal length. This vertex u has n loops, as otherwise the path can be extended by a regular edge starting from u. Obviously, no regular edge starts from u. Hence, first, $D - u$ and D have the same outdegrees of vertices. Next, for any vertex w of $D-u$, w is a root in $D-v$ iff w is a root in D (w is said to be a *root* if no regular edge ends at w). Moreover, if u is contained in a non-trivial connected component of D, then u is not a root. Observe also that $D - u$ has no cycles, because D has no cycles.

Take a maximal n-reduct K of D. Then by the above facts and simple induction, K is a total n-functional digraph having the same set of roots as D. Further, K contains only loops, since otherwise K would have to contain a non-trivial connected component with some vertex with n loops. Thus each vertex of K is a root. Summarizing, K consists of all the roots of D and each of its vertices has exactly n loops. In particular, all maximal n-reducts of D are equal. ∎

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