COLLOQUIUM MATHEMATICUM

VOL. 91 2002 NO. 1

RELATIVE AUSLANDER–REITEN SEQUENCES FOR QUASI-HEREDITARY ALGEBRAS

BY

KARIN ERDMANN (Oxford), JOSÉ ANTONIO DE LA PEÑA (México, D.F.) and CORINA SÁENZ (Guanajuato)

Abstract. Let A be a finite-dimensional algebra which is quasi-hereditary with respect to the poset $(\Lambda, \leq)$, with standard modules $\Delta(\lambda)$ for $\lambda \in \Lambda$. Let $\mathcal{F}(\Delta)$ be the category of A-modules which have filtrations where the quotients are standard modules. We determine some inductive results on the relative Auslander–Reiten quiver of $\mathcal{F}(\Delta)$.

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott in the study of highest weight categories arising in the representation theory of semisimple Lie algebras and algebraic groups [2], and they were studied by Dlab and Ringel and others [4, 5]. Let A be quasi-hereditary. Then the simple A-modules are labelled as $L(\lambda)$ for λ in some partially ordered set $(\Lambda, \leq)$. Of central importance are the standard modules, denoted by $\Delta(\lambda)$, and the category $\mathcal{F}(\Delta)$ of modules which have a filtration where the quotients are standard modules. In [12] it was proved that this category has relative Auslander–Reiten (AR) sequences. The aim of our paper is to present some inductive results on the relative Auslander–Reiten quiver of $\mathcal{F}(\Delta)$.

For any subset Γ of Λ, let $\mathcal{F}_\Gamma(\Delta)$ be the category of modules in $\mathcal{F}(\Delta)$ where all quotients are $\Delta(\lambda)$ with $\lambda \in \Gamma$. If $\Lambda \setminus \Gamma$ is an ideal in the poset $(\Lambda, \leq)$ then there is a subalgebra A_Γ of A which is quasi-hereditary with poset $(\Gamma, \leq)$. In this case, the category $\mathcal{F}_\Gamma(\Delta)$ is equivalent to $\mathcal{F}(\Delta_{A_\Gamma})$. A main result in this paper relates, for a non-projective indecomposable module X in $\mathcal{F}_\Gamma(\Delta)$, the two relative Auslander–Reiten sequences with cokernel X in $\mathcal{F}(\Delta)$ and in $\mathcal{F}_\Gamma(\Delta)$ (Theorem 2.5). By Ringel duality, there is an equivalent version for categories of modules filtered by costandard modules. We describe this and some variations in Section 3. We continue with further results on relative Auslander–Reiten sequences. In particular, we determine certain relative Auslander–Reiten sequences where a projective-injective summand occurs; this generalizes the usual almost split sequence where a projective-injective module occurs.

2000 *Mathematics Subject Classification*: 16G70, 18G25, 20G05, 17B10.

In the last section we apply our results, together with results of [15], to determine the relative Auslander–Reiten quiver of certain algebras, called $\mathcal{D}_n$, and show that they are Δ-finite. These algebras arise as blocks of Schur algebras (see [10]).

Acknowledgements. The first and third authors thank U.N.A.M. for financial support. The second author also thanks CONACyT for support. We thank the referee for valuable comments.

1. Preliminaries

1.1. We assume throughout that k is an algebraically closed field. Let A be a finite-dimensional algebra over k; we recall the definition of a quasi-hereditary algebra. Suppose the simple modules of A are labelled as $L(\lambda)$ for $\lambda \in \Lambda$, where $(\Lambda, \leq)$ is a partially ordered set. Define the standard module $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that all composition factors are of the form $L(\mu)$ for $\mu \leq \lambda$. Dually, the costandard module $\nabla(\lambda)$ is the largest submodule of the injective hull $I(\lambda)$ of $L(\lambda)$ such that all composition factors of $\nabla(\lambda)$ are $L(\mu)$ for $\mu \leq \lambda$. Then A is *quasi-hereditary* with respect to $(\Lambda, \leq)$ provided for each $\lambda \in \Lambda$,

(i) $L(\lambda)$ occurs only once as a composition factor of $\Delta(\lambda)$,

(ii) the projective $P(\lambda)$ has a filtration where the quotients are standard modules such that $\Delta(\lambda)$ occurs once and if $\Delta(\mu)$ occurs then $\mu \geq \lambda$.

Assume now A is quasi-hereditary. Define $\mathcal{F}(\Delta)$ to be the category of all A-modules which have a Δ-*filtration*, that is, a filtration whose factor modules are of the form $\Delta(\lambda)$ with $\lambda \in \Lambda$, and define $\mathcal{F}(\nabla)$ similarly. If Γ is a subset of Λ then $\mathcal{F}_\Gamma(\Delta)$ is the full subcategory of $\mathcal{F}(\Delta)$ of those modules M where only $\Delta(\lambda)$ with $\lambda \in \Gamma$ occur as factors of a Δ-filtration of M.

We also assume that A is basic and connected. We fix a set $\{e_\lambda\}$ of orthogonal primitive idempotents such that $\sum e_\lambda = 1$ and $Ae_\lambda = P(\lambda)$.

By [5] one can always refine $(\Lambda, \leq)$ to a linear order, without changing the standard modules. We will make use of these facts in proofs.

1.2. We fix a subset Γ of Λ such that the complement $\Lambda \setminus \Gamma$ of Γ in Λ is an ideal in the poset Λ. Let e be the idempotent $e = e_\Gamma := \sum_{\lambda\in\Gamma} e_\lambda$, and $A_\Gamma := eAe$. Then A_Γ is quasi-hereditary with respect to $(\Gamma, \leq)$, with standard modules $e\Delta(\lambda)$ for $\lambda \in \Gamma$ (see [5] or [11]). Moreover, let $\overline{A} = A/AeA$. Then the A-modules $\Delta(\lambda)$ and $\nabla(\lambda)$ for $\lambda \notin \Gamma$ are $\overline{A}$-modules; moreover, $\overline{A}$ is quasi-hereditary, with standard modules $\Delta(\lambda)$ for $\lambda \notin \Gamma$. We call such algebras eAe and $\overline{A}$ *good subalgebras*, and *good quotient algebras* of A.

1.3. It is well known that $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) \neq 0$ or $\mathrm{Ext}^1_A(\nabla(\mu), \nabla(\lambda)) \neq 0$ implies $\lambda < \mu$. Moreover,

$$\operatorname{Ext}^i_A(\Delta(\lambda),\nabla(\mu))\cong\begin{cases}k & \text{for } i=0 \text{ and } \lambda=\mu,\\ 0 & \text{otherwise.}\end{cases}$$

The category $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms, and closed under taking direct summands.

1.4. Consider $\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$. By [12] this contains finitely many indecomposables and they can be labelled as $T(\lambda)$ such that $\Delta(\lambda)\subset T(\lambda)$ and $T(\lambda)/\Delta(\lambda)$ has a Δ-filtration where the occurring quotients are of the form $\Delta(\mu)$ with $\mu<\lambda$.

Let $T=\bigoplus_{\lambda\in\Lambda}T(\lambda)$ and $A':=\operatorname{End}_A(T)$. Then A' is quasi-hereditary with respect to $(\Lambda,\le^{\mathrm{op}})$ and with standard modules given by $\Delta_{A'}(\lambda)=F(\nabla_A(\lambda))$, where F is the functor $\operatorname{Hom}_A(T,-)$. It was proved by Ringel that F induces an equivalence between $\mathcal{F}(\nabla_A)$ and $\mathcal{F}(\Delta_{A'})$. We call this "Ringel equivalence".

Moreover, $(A')'$ is isomorphic to A with the same quasi-hereditary structure. Thus if $T':=\bigoplus_{\lambda\in\Lambda}T'(\lambda)$ where $T'(\lambda)$ is the indecomposable module in $\mathcal{F}(\Delta_{A'})\cap\mathcal{F}(\nabla_{A'})$ with highest weight λ, then $\operatorname{End}_{A'}(T')$ is isomorphic to A. We call A' the *Ringel dual* of A.

1.5. The category $\mathcal{F}(\Delta)$ (or $\mathcal{F}(\nabla)$) has relative Auslander–Reiten sequences; this was proved by Ringel in [12], based on work by Auslander and Smalø.

We refer to [12, 13] for some general results.

2. The category $\mathcal{F}_\Gamma(\Delta)$. Let Γ and e be as in 1.2. We will first obtain an equivalence of $\mathcal{F}_\Gamma(\Delta)$ and $\mathcal{F}(\Delta_{eAe})$. Hence the category $\mathcal{F}_\Gamma(\Delta)$ has relative Auslander–Reiten sequences; we will relate these to the relative Auslander–Reiten sequences of $\mathcal{F}(\Delta)$.

We note that the fact that the above categories are equivalent is known. It can be derived from [2]; or from [5, Theorem 2], as was noted in [3]. Alternatively, one can prove it by using Ringel duality; see 3.2.

Let $\gamma_1,\dots,\gamma_k$ be the minimal elements in Γ and set $r(\Gamma):=\Gamma\setminus\{\gamma_1,\dots,\gamma_k\}$. This is a saturated set; note that $\operatorname{Ext}^1_A(\Delta(\gamma_i),\Delta(\gamma_j))=0$ for all i,j.

2.1. Lemma. *Let $X\in\mathcal{F}_\Gamma(\Delta)$. Then the projective cover of X is of the form $0\to\Omega(X)\to P\to X\to 0$ with $P\in\mathcal{F}_\Gamma(\Delta)$ projective and $\Omega(X)\in\mathcal{F}_{r(\Gamma)}(\Delta)$.*

Proof. The category $\mathcal{F}_\Gamma(\Delta)$ contains the projectives $P(\lambda)$ for $\lambda\in\Gamma$. By [12, Theorem 3] the category $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms, so it remains to show that all Δ-quotients of $\Omega(X)$ are of the form $\Delta(\lambda)$ for $\lambda\in r(\Gamma)$. This is true for $X\in\operatorname{add}(\Delta(\gamma_j))$, and also for $X\in\mathcal{F}_{r(\Gamma)}(\Delta)$, by 1.1(ii). In general, by 1.3, X is an extension of a module X_0 by X' where X_0 is in $\operatorname{add}(\Delta(\gamma_j))$ and X' is in $\mathcal{F}_{r(\Gamma)}(\Delta)$, and hence the projective cover $P(X)$

is a direct summand of $P(X_0) \oplus P(X')$, and $\Omega(X)$ is a direct summand of an extension of $\Omega(X_0)$ by $\Omega(X')$, and hence $\Omega(X)$ is in $\mathcal{F}_{r(\Gamma)}(\Delta)$. ∎

Note that in particular this shows that $\mathcal{F}_\Gamma(\Delta)$ is closed under projective covers and syzygies.

2.2. Definition (The trace functor $t = t_\Gamma$). For any A-module M let $t(M) := AeM$, the trace of Ae in M. If $M \in \mathcal{F}(\Delta)$ then $t(M)$ belongs to $\mathcal{F}_\Gamma(\Delta)$ and the quotient $M/t(M)$ belongs to $\mathcal{F}_{\Lambda\setminus\Gamma}(\Delta)$, by 1.3; see also [12], for example.

2.3. Lemma. *If $Y \in \mathcal{F}(\Delta)$ then $t(Y) = AeY \cong Ae \otimes_{eAe} eY$.*

This is Proposition A.3.2 of [9]. (The hypothesis on Y is necessary; for example, take the algebra $\mathcal{A}_m$ from 3.5 below, and take $e = e_m$, the primitive idempotent corresponding to the largest element in Λ. If $M = \nabla(m)$ then $t(M)$ and $Ae \otimes_{eAe} M$ are not isomorphic.)

This implies that t is exact on $\mathcal{F}(\Delta)$. It is clearly right exact; and if $\phi : X \to Y$ is mono then the map $1 \otimes \phi : Ae \otimes_{eAe} eX \to Ae \otimes_{eAe} eY$ is also mono since it can be identified with the restriction of ϕ to AeX.

2.4. Proposition. *The functors $f : A$-mod $\to eAe$-mod, $f(X) = eX$ and $g : eAe$-mod $\to A$-mod, $g(Z) = Ae \otimes_{eAe} Z$, induce inverse equivalences of the categories $\mathcal{F}_\Gamma(\Delta) \to \mathcal{F}(\Delta_{eAe})$. In particular, $\mathcal{F}_\Gamma(\Delta)$ has relative Auslander–Reiten sequences.*

Proof. We may refine the given order on Λ to a linear order, and hence we assume $\Lambda = \{1, \ldots, n\}$ and $\Gamma = \{j \in \Lambda : j \geq i\}$ for some (fixed) i. We write $\mathcal{F}_{\geq i}(\Delta)$ for $\mathcal{F}_\Gamma(\Delta)$.

It is well known that (g, f) is an adjoint pair and that $f \circ g$ is equivalent to the identity. Thus we only have to show that $g \circ f|_{\mathcal{F}_\Gamma(\Delta)}$ is also equivalent to the identity of $\mathcal{F}_\Gamma(\Delta)$.

(1) *If $X \in \mathcal{F}_{\geq i}(\Delta)$ then $eX \in \mathcal{F}(\Delta_{eAe})$*: It is known that for $m \geq i$, one has $e\Delta(m) \cong \Delta_{eAe}(m)$ (see for example [11]). Then (1) follows since the functor f is exact.

(2) *The functor f induces an isomorphism of* Hom-*spaces*: We have

$$\mathrm{Hom}_{eAe}(eX, eY) \cong \mathrm{Hom}_A(Ae \otimes_{eAe} eX, Y) \cong \mathrm{Hom}_A(AeX, Y)$$

by Lemma 2.3. For $X \in \mathcal{F}_{\geq i}(\Delta)$ we have $AeX = X$.

(3) *The functor f induces an isomorphism of* Ext-*spaces* (actually the isomorphism of Ext^1 spaces is enough for us): Let $X, Y \in \mathcal{F}_{\geq i}(\Delta)$. If $P(X)$ is a projective cover of X as an A-module then $P(X)$ and $\Omega(X)$ belong to $\mathcal{F}_{\geq i}(\Delta)$. So we get an exact sequence

$$0 \to \mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(P(X), Y) \to \mathrm{Hom}_A(\Omega(X), Y) \to \mathrm{Ext}^1_A(X, Y) \to 0$$

(and $\mathrm{Ext}^i_A(\Omega(X),Y) \cong \mathrm{Ext}^{i+1}_A(X,Y)$). Moreover, eP is a projective eAe-module, so we get an exact sequence

$$0 \to \mathrm{Hom}_{eAe}(eX,eY) \to \mathrm{Hom}_{eAe}(eP,eY) \to \mathrm{Hom}_{eAe}(e\Omega(X),eY) \to \mathrm{Ext}^1_{eAe}(eX,eY) \to 0$$

(and $\mathrm{Ext}^m_{eAe}(e\Omega(X),eY) \cong \mathrm{Ext}^{m+1}_{eAe}(eX,eY)$ for $m \geq 1$, functorially in X and in Y). By (2), the first three corresponding Hom-spaces are isomorphic. So $\mathrm{Ext}^1_A(X,Y)$ is isomorphic to $\mathrm{Ext}^1_{eAe}(eX,eY)$ for all $X,Y \in \mathcal{F}_{\geq i}(\Delta)$. For $i > 1$, the claim follows by dimension shift.

(4) *If $M \in \mathcal{F}_{eAe}(\Delta)$, then $g(M) \in \mathcal{F}_{\geq i}(\Delta)$*: We prove this by induction on the number of Δ-quotients of M. If $M = \Delta_{eAe}(m)$, then $M = e\Delta(m)$ and $Ae \otimes_{eAe} M \cong AeM$, by Lemma 2.3, and this is $\Delta(m)$. For the inductive step, let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence with M', M'' and M in $\mathcal{F}(\Delta_{eAe})$ and M' and M'' non-zero. By the inductive hypothesis $g(M')$ and $g(M'')$ are in $\mathcal{F}(\Delta)$. By (3) we know that $\mathrm{Ext}^1_A(g(M''), g(M')) \cong \mathrm{Ext}^1_{eAe}(M'', M')$. So there is an exact sequence

$$0 \to g(M') \to E \to g(M'') \to 0$$

such that the sequence $0 \to fg(M') \to f(E) \to fg(M'') \to 0$ is isomorphic to the sequence we started with, where the isomorphism is induced by f, and one deduces that $M \cong fE$. Then $g(M) \cong gf(E) = Ae \otimes_{eAe} eE \cong AeE$. By induction, $g(M')$ and $g(M'')$ are in $\mathcal{F}_{\geq i}(\Delta)$, which is extension closed, so E belongs to it as well and then $AeE = E$. ∎

2.5. Theorem. *Let $X \in \mathcal{F}_\Gamma(\Delta)$ be such that*

$$0 \to \tau_\Delta(X) \xrightarrow{\beta} E \xrightarrow{\gamma} X \to 0 \tag{1}$$

is the Auslander–Reiten sequence ending at X in $\mathcal{F}(\Delta)$. Then the exact sequence

$$0 \to t[\tau_\Delta(X)] \xrightarrow{t(\beta)} t(E) \xrightarrow{t(\gamma)} X \to 0 \tag{2}$$

is the relative Auslander–Reiten sequence ending at X in $\mathcal{F}_\Gamma(\Delta)$.

We will prove this by deducing it from a more general result. Assume that $\mathcal{C}$ and $\mathcal{D}$ are full subcategories of some module categories, and suppose we have an adjoint pair (F,G) of (covariant) functors,

$$F : \mathcal{D} \to \mathcal{C}, \qquad G : \mathcal{C} \to \mathcal{D},$$

which preserve exact sequences. We denote the adjunction by ϕ, so that $f \mapsto \phi(f) : \mathrm{Hom}_{\mathcal{C}}(FX,Y) \to \mathrm{Hom}_{\mathcal{D}}(X,GY)$ is a bijection, for $X \in \mathcal{D}$ and $Y \in \mathcal{C}$. Let η and ι be the canonical natural transformations $\eta : \mathrm{Id}_{\mathcal{D}} \to GF$ and $\iota : FG \to \mathrm{Id}_{\mathcal{C}}$.

We apply the following to get 2.5 by taking $\mathcal{C} = \mathcal{F}(\Delta)$ and $\mathcal{D} = \mathcal{F}_\Gamma(\Delta)$. Then if F is the inclusion and $G = t$ then the hypotheses are satisfied; and for M in $\mathcal{D}$, the map η_M is an isomorphism.

2.6. We assume that for all X in $\mathcal{D}$ the map η_X is an isomorphism. First observe the following.

Suppose $0 \to Z \xrightarrow{\beta} Y \xrightarrow{\gamma} X \to 0$ *is a non-split exact sequence in* $\mathcal{C}$. *Assume that* ι_X *is an isomorphism. Then the sequence*

$$0 \to GZ \xrightarrow{G\beta} GY \xrightarrow{G\gamma} GX \to 0$$

is non-split.

Namely, suppose for a contradiction that $G\gamma \circ h = \mathrm{Id}_{GX}$ for some $h : GX \to GY$. Then $F(G\gamma \circ h) = FG(\gamma) \circ F(h) = \mathrm{Id}_{FGX}$. Since ι is a natural transformation we deduce $\gamma \circ \iota_Y \circ Fh = \iota_X \circ FG\gamma \circ Fh = \iota_X$ and $\gamma \circ (\iota_Y \circ Fh \circ (\iota_X)^{-1}) = \mathrm{Id}_X$, and γ is a split epimorphism, a contradiction.

Proposition. *Suppose that*

$$0 \to Z \xrightarrow{\beta} Y \xrightarrow{\gamma} X \to 0$$

is a relative Auslander–Reiten sequence in $\mathcal{C}$. *Assume that the canonical map* $\iota_X : FG \to X$ *is an isomorphism, and assume also that* GZ *is indecomposable. Then*

$$0 \to GZ \xrightarrow{G\beta} GY \xrightarrow{G\gamma} GX \to 0$$

is a relative Auslander–Reiten sequence in $\mathcal{D}$.

Proof. By the above observation, the last sequence does not split. So we only have to show that $G\gamma$ is a sink map.

Suppose W is in $\mathcal{D}$ and $\delta : W \to GX$ is not a split epimorphism. Let $\delta_1 : FW \to X$ with $\phi(\delta_1) = \delta$, so that $\delta_1 = \iota_X \circ F\delta$. We claim that δ_1 is not a split epimorphism. Otherwise, $F\delta$ is also a split epimorphism (since ι_X is an isomorphism). This implies that δ is a split epimorphism. Namely, suppose $F\delta \circ \nu = \mathrm{Id}_{FGX}$; then

$$\mathrm{Id}_{GFGX} = GF\delta \circ G\nu = GF\delta \circ \eta_W \circ (\eta_W)^{-1} \circ G\nu = \eta_{GX} \circ \delta \circ (\eta_W)^{-1} \circ G\nu,$$

which means that $\alpha := \delta \circ (\eta_W)^{-1} \circ G\nu$ is the inverse of η_{GX} and hence $\delta \circ \alpha = \mathrm{Id}_{GX}$, that is, δ is a split epimorphism, a contradiction.

Note that FW is in $\mathcal{C}$. Since γ is a sink map there is now $\psi : FW \to Y$ such that $\gamma \circ \psi = \delta_1$. It follows that $G\gamma \circ G\psi = G\delta_1$ and

$$G\gamma \circ G\psi \circ \eta_W = G\delta_1 \circ \eta_W = G\iota_X \circ GF\delta \circ \eta_W = G\iota_X \circ \eta_{GX} \circ \delta = \delta. \ \blacksquare$$

We will study a necessary condition which ensures that the module GZ as above is indecomposable.

2.6.1. Lemma. *Assume in addition that $\mathcal{D}$ is closed under direct summands. Suppose $0 \to Z \xrightarrow{\beta} Y \xrightarrow{\gamma} X \to 0$ is a relative Auslander–Reiten sequence in $\mathcal{C}$ and that ι_X is an isomorphism. Then GZ is indecomposable provided that whenever $h : FV \to Z$ is such that $\phi(h)$ is a split monomorphism for $V \in \mathcal{D}$ and $Z \in \mathcal{C}$ then $\mathrm{Im}(h)$ and $Z/\mathrm{Im}(h)$ belong to $\mathcal{C}$.*

The additional hypothesis holds whenever $\mathcal{C}$ is extension closed, and moreover, for all $Z \in \mathcal{C}$, the map ι_Z is one-to-one and $Z/\mathrm{Im}(\iota_Z)$ is in $\mathcal{C}$; this is easy to see.

In our application, when $\mathcal{C} = \mathcal{F}(\Delta)$ and $\mathcal{D} = \mathcal{F}_\Gamma(\Delta)$, and when $G = t$ and F is the inclusion, these are satisfied. Note that $\mathrm{Im}(\iota_Z)$ is part of a heredity chain, so $Z/\mathrm{Im}(\iota_Z)$ belongs to $\mathcal{C}$ for $Z \in \mathcal{C}$.

Proof of 2.6.1. Suppose $GZ = V_1 \oplus V_2$ where V_1 and V_2 are both nonzero, so that they are in $\mathcal{D}$, and let $\varepsilon_i : V_i \to GZ$ and $\pi_i : GZ \to V_i$ with $\pi_i \varepsilon_i = \mathrm{Id}_{V_i}$. Consider $e_i := \phi^{-1}(\varepsilon_i) : FV_i \to Z$; then we have $e_i = \iota_Z \circ F\varepsilon_i$ and $\varepsilon_i = Ge_i \circ \eta_{V_i}$. Define $\overline{Z}_1 := Z/\mathrm{Im}(e_2)$ and $\overline{Z}_2 := Z/\mathrm{Im}(e_1)$, and let $\sigma_i : Z \to \overline{Z}_i$ be the canonical epimorphism. Our hypotheses ensure that the modules $\overline{Z}_i$ are in $\mathcal{C}$. Since σ_i is not a split monomorphism, there are $\theta_i : Y \to \overline{Z}_i$ with

$$\sigma_i = \theta_i \circ \beta \qquad (i = 1, 2).$$

We will show (using the maps $G\theta_i$) that GY is isomorphic to $GX \oplus GZ$, hence the exact sequence $0 \to GZ \to GY \to GX \to 0$ splits. This will be a contradiction to the earlier observation, and hence will complete the proof.

(1) Since $0 = \sigma_2 \circ e_1 = \sigma_2 \circ \iota_Z \circ F\varepsilon_1$ there is a homomorphism $m_2 : FV_2 \to \overline{Z}_2$ with

$$m_2 \circ F\pi_2 = \sigma_2 \circ \iota_Z.$$

Similarly, there is $m_1 : FV_1 \to \overline{Z}_1$ with $m_1 \circ F\pi_1 = \sigma_1 \circ \iota_Z$. This implies that

$$\begin{aligned}\phi(m_i) \circ \pi_i = \phi(m_i \circ F\pi_i) &= \phi(\sigma_i \circ \iota_Z) = G(\sigma_i \circ \iota_Z) \circ \eta_{GZ} \\ &= G\sigma_i \circ G\iota_Z \circ \eta_{GZ} = G\sigma_i\end{aligned}$$

and therefore

(2) $\phi(m_i) \circ \pi_i = G\sigma_i = G\theta_i \circ G\beta$ for $i = 1, 2$.

(3) We claim that $G\sigma_i$ is a split epimorphism and $G\overline{Z}_i \cong V_i$, and hence that $\phi(m_i)$ is an isomorphism $(i = 1, 2)$: By the hypotheses we have an exact sequence $0 \to \mathrm{Im}(e_1) \to Z \xrightarrow{\sigma_2} \overline{Z}_2 \to 0$ in $\mathcal{C}$ and hence an exact sequence

$$0 \to G(\mathrm{Im}\, e_1) \to GZ \xrightarrow{G\sigma_2} G\overline{Z}_2 \to 0.$$

Now $G(\mathrm{Im}\, e_1) = \mathrm{Im}(Ge_1) = \mathrm{Im}(Ge_1 \circ \eta_{V_1}) = \mathrm{Im}(\varepsilon_1)$, which is isomorphic to V_1. So $G\sigma_2$ induces an isomorphism from $\mathrm{Im}(\varepsilon_2)$ $(\cong V_2)$ onto $G\overline{Z}_2$. But GZ is isomorphic to the direct sum $\mathrm{Im}(\varepsilon_1) \oplus \mathrm{Im}(\varepsilon_2)$ and hence $G\sigma_2$ is a split epimorphism. Then by (2) also $\phi(m_2)$ is a split epimorphism and then an isomorphism, as a map between isomorphic modules.

(4) Let $\kappa_i = G\beta \circ \varepsilon_i \circ \phi(m_i)^{-1}$ for $i = 1, 2$. By (2) we have $G\theta_i \circ \kappa_i = \mathrm{Id}_{G\overline{Z}_i}$, hence the maps $f_i := \kappa_i \circ G\theta_i$ are projections of GY. Define submodules of GY by

$$V_i' := G\beta \circ \varepsilon_i(V_i).$$

One checks using (2) that $f_i(V_j') = 0$ if $i \neq j$ and that the restriction of f_i to V_i' is the identity, so that $V_i' \subseteq \mathrm{Im}(f_i)$. By dimensions, we must have $V_i' = \mathrm{Im}(f_i)$ for $i = 1, 2$. We deduce $f_i f_j = 0$ for $i \neq j$, hence $f := f_1 + f_2$ is a projection of GY, and moreover, $\mathrm{Im}(f) = \mathrm{Im}\, f_1 \oplus \mathrm{Im}\, f_2 = V_1' \oplus V_2'$ and $GY = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$. But $GX \cong GY/G\beta(GZ) = GY/V_1' \oplus V_2'$, and consequently, $\mathrm{Ker}(f) \cong GX$. Then the sequence must split as required, and this completes the proof. ∎

2.7. We will later use the dual of 2.6; we state it for convenience. Let $\mathcal{C}, \mathcal{D}$ and F, G be as defined after 2.5, but now assume that for all C in $\mathcal{C}$ the map $\iota_C : FGC \to C$ is an isomorphism. Then we have

PROPOSITION (∗). *Suppose*

$$0 \to M \xrightarrow{\beta} N \xrightarrow{\gamma} R \to 0$$

is a relative Auslander–Reiten sequence in $\mathcal{D}$ with η_M an isomorphism. If FR is indecomposable then

$$0 \to FM \xrightarrow{F\beta} FN \xrightarrow{F\gamma} FR \to 0$$

is a relative Auslander–Reiten sequence in $\mathcal{C}$.

LEMMA (∗). *Assume in addition that $\mathcal{C}$ is closed under direct summands. Suppose*

$$0 \to M \to N \to R \to 0$$

is a relative Auslander–Reiten sequence in $\mathcal{D}$ and that η_M is an isomorphism. Then FR is indecomposable provided that whenever $p : R \to GA$ for $R \in \mathcal{D}$ and $A \in \mathcal{C}$ with $\phi^{-1}(p)$ is a split epimorphism then $\mathrm{Ker}(p)$ *and* $\mathrm{Im}(p)$ *are in $\mathcal{D}$.*

2.8. REMARK. We continue with the quasi-hereditary algebra. Let $0 \to \tau_\Delta(X) \to E \to X \to 0$ be any AR-sequence in $\mathcal{F}(\Delta)$. By applying the trace t with respect to Γ we always get an exact sequence

$$0 \to t(\tau_\Delta(X)) \to tE \to tX \to 0.$$

If tX is a proper submodule of X then this sequence splits, by the Auslander–Reiten property.

Recall that $\mathcal{F}(\Delta)$ contains the projectives. Moreover, the injective indecomposable objects in $\mathcal{F}(\Delta)$ are precisely the $T(\lambda)$ (see [12]). For the subcategory $\mathcal{F}_\Gamma(\Delta)$ we have the following.

2.9. LEMMA. *Let $E \in \mathcal{F}_\Gamma(\Delta)$ be indecomposable. Then*

(a) *E is projective in $\mathcal{F}_\Gamma(\Delta)$ if and only if $E = P(\lambda)$ for some $\lambda \in \Gamma$.*

(b) *E is injective in $\mathcal{F}_\Gamma(\Delta)$ if and only if $E = t(T(\lambda))$ for some $\lambda \in \Gamma$.*

Proof. (a) is clear. For (b), let E be indecomposable injective in $\mathcal{F}_\Gamma(\Delta)$. Then $f(E) = eE$ is injective indecomposable in $\mathcal{F}(\Delta_{eAe})$ by 2.4, so $E \cong g \circ f(E) = g \circ f(T(\lambda)) \cong Ae \otimes_{eAe} eT(\lambda) = t(T(\lambda))$ for some $\lambda \in \Gamma$.

Now, let $E = t(T(\lambda))$. Then $E \cong g \circ f(T(\lambda))$. Therefore E is injective in $\mathcal{F}_\Gamma(\Delta)$, by 2.4. ■

2.10. REMARK. There are enough injectives in the category $\mathcal{F}_\Gamma(\Delta)$.

3. Some variations and examples

3.1. We keep the notation of 1.2, and let $t = t_\Gamma$ be as in 2.2. In addition, we set $\Theta = \Lambda \setminus \Gamma$.

If $\overline{A} := A/AeA$ then one can identify the categories $\mathcal{F}(\Delta_{\overline{A}})$ and $\mathcal{F}_\Theta(\Delta)$; in particular, $\mathcal{F}_\Theta(\Delta)$ has relative Auslander–Reiten sequences. Let $X \in \mathcal{F}(\Delta)$ and let

$$0 \to X \to H \to \tau_\Delta^{-1}(X) \to 0 \tag{3}$$

be a relative AR-sequence in $\mathcal{F}(\Delta)$. Since the functor t is exact, we get an exact sequence $0 \to tX \to tH \to t(\tau_\Delta^{-1}X) \to 0$. Taking quotients we get an exact sequence

$$0 \to \overline{X} \to \overline{H} \to \overline{\tau_\Delta^{-1}(X)} \to 0 \tag{4}$$

where $\overline{M} = M/tM$. If $\overline{X}$ is a proper quotient of X then (4) splits, by the AR-property. Otherwise:

THEOREM. *Assume X belongs to $\mathcal{F}_\Theta(\Delta)$ and has a relative Auslander–Reiten sequence* (3) *in $\mathcal{F}(\Delta)$. Then $X = \overline{X}$, and* (4) *is a relative Auslander–Reiten sequence in $\mathcal{F}_\Theta(\Delta)$.*

This is a special case of 2.7; take $\mathcal{D} = \mathcal{F}(\Delta)$ and $\mathcal{C} = \mathcal{F}(\Delta_{\overline{A}})$, and let F be the functor which takes X to X/tX and G the inclusion functor.

3.2. The equivalence of $\mathcal{F}_\Gamma(\Delta)$ with $\mathcal{F}(\Delta_{eAe})$ (which guarantees the existence of relative AR-sequences in $\mathcal{F}_\Gamma(\Delta)$) can also be seen via Ringel dual. Let $T' := \bigoplus_\lambda T'(\lambda)$ be a full tilting module for A'; then $\mathrm{End}_{A'}(T')$ can be identified with A (see 1.2); it follows that we can take for e the projection of T' with image $T'_\Gamma := \bigoplus_{\lambda\in\Gamma} T'(\lambda)$ and kernel $\bigoplus_{\lambda\in\Theta} T'(\lambda)$. This identifies eAe with the endomorphism ring of T'_Γ.

Since $(\Gamma, \le^{\mathrm{op}})$ is an ideal of $(\Lambda, \le^{\mathrm{op}})$ we know (see 1.2) that A' has a quasi-hereditary quotient B of the form $B = A'/A'fA'$ with standard modules $\Delta_{A'}(\lambda)$, $\lambda \in \Gamma$. Then T'_Γ is naturally a module for B and it is then a full tilting module for B. Hence $\mathrm{End}_B(T'_\Gamma)$ is the Ringel dual of B, but $\mathrm{End}_B(T'_\Gamma) = \mathrm{End}_{A'}(T'_\Gamma) \cong eAe$.

We have a Ringel equivalence between $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\nabla_{A'})$ which restricts to an equivalence of $\mathcal{F}_\Gamma(\Delta)$ with $\mathcal{F}_\Gamma(\nabla_{A'})$, and also between $\mathcal{F}(\Delta_{eAe})$ and $\mathcal{F}(\nabla_B)$. Since $\mathcal{F}(\nabla_B)$ is the same as $\mathcal{F}_\Gamma(\nabla)$, we get an equivalence of $\mathcal{F}_\Gamma(\Delta)$ and $\mathcal{F}(\Delta_{eAe})$.

In principle, one may equivalently work with good quotient algebras, rather than with good subalgebras. This does not seem to be shorter; because of applications we prefer good subalgebras.

3.3. The category $\mathcal{F}_\Theta(\nabla)$ is the same as $\mathcal{F}(\nabla_{\overline{A}})$ where $\overline{A} = A/AeA$, hence it has relative AR-sequences. By 1.3, a module M with ∇-filtration has a (unique) submodule $\eta(M)$ which belongs to $\mathcal{F}_\Theta(\nabla)$ such that $\overline{\eta}(M) := M/\eta(M)$ belongs to $\mathcal{F}_\Gamma(\nabla)$. We have:

Corollary. *Assume that X in $\mathcal{F}_\Theta(\nabla)$ has a relative Auslander–Reiten sequence in $\mathcal{F}(\nabla)$ of the form*

$$0 \to \tau_\nabla(X) \to E \to X \to 0. \tag{5}$$

Then the relative Auslander–Reiten sequece of X in $\mathcal{F}_\Theta(\nabla)$ is of the form

$$0 \to \eta(\tau_\nabla(X)) \to \eta(E) \to X \to 0. \tag{6}$$

This is equivalent to 2.5 by Ringel duality; see the remark in 3.2.

3.4. Now consider $\mathcal{F}_\Gamma(\nabla)$; this also has relative AR-sequences. Namely, Ringel duality induces an equivalence

$$\mathcal{F}_\Gamma(\nabla) \cong \mathcal{F}_\Gamma(\Delta_{A'})$$

and by 3.1, this is the same as $\mathcal{F}(\Delta_B)$ and the claim follows. Now, B is the Ringel dual of eAe, and therefore by Ringel equivalence $\mathcal{F}(\Delta_B) \cong \mathcal{F}(\nabla_{eAe})$ and consequently $\mathcal{F}_\Gamma(\nabla) \cong \mathcal{F}(\nabla_{eAe})$.

By applying 3.1 with A' instead of A, and Ringel equivalence, we obtain

Corollary. *Let $X \in \mathcal{F}_\Gamma(\nabla)$ have an Auslander–Reiten sequence in $\mathcal{F}(\nabla)$ of the form*

$$0 \to X \to H \to \tau_\nabla^{-1}(X) \to 0.$$

Then the relative Auslander–Reiten sequence of X in $\mathcal{F}_\Gamma(\nabla)$ is given by

$$0 \to X \to \overline{\eta}(H) \to \overline{\eta}(\tau_\nabla^{-1}(X)) \to 0.$$

3.5. Example. Consider a block of a Schur algebra of finite type. This is Morita equivalent to an algebra $\mathcal{A}_m$, as defined in [16, 10, 7] and [14, 15]. (The quiver and the relations can be seen by 5.1 below, from the fact that $\mathcal{A}_m$ is a quotient of $\mathcal{D}_{m+1}$.) The relative AR-quiver of $\mathcal{A}_m$ is determined in [14, 15]. The simples are labelled as $L(i)$ with $i \in \Lambda$, where $\Lambda = \{1, \ldots, m\}$ with $\leq$ the natural linear order.

Let $m = 6$. Then there is an AR-sequence in $\mathcal{F}(\Delta)$

$$0 \to \begin{matrix}\Delta(2)\\ \Delta(4)\end{matrix} \to \Delta(5) \oplus P(3) \oplus \Delta(2) \to \begin{matrix}\Delta(3)\\ \Delta(5)\end{matrix} \to 0.$$

Let t_i denote the trace with respect to $\Gamma = \{j : j \geq i\}$. Since ${\scriptstyle\begin{matrix}\Delta(3)\\ \Delta(5)\end{matrix}}$ belongs to $\mathcal{F}_{\geq 3}(\Delta)$, we apply t_3 to the previous AR-sequence and get

$$0 \to \Delta(4) \to \Delta(5) \oplus P(3) \to \begin{matrix}\Delta(3)\\ \Delta(5)\end{matrix} \to 0.$$

Now there is also an AR-sequence

$$0 \to \begin{matrix}\Delta(4)\\ \Delta(6)\end{matrix} \to \begin{matrix}\Delta(2)\\ \Delta(4)\\ \Delta(6)\end{matrix} \oplus \Delta(4) \to \begin{matrix}\Delta(2)\\ \Delta(4)\end{matrix} \to 0.$$

The module ${\scriptstyle\begin{matrix}\Delta(2)\\ \Delta(4)\end{matrix}}$ belongs to $\mathcal{F}_{\geq 2}(\Delta)$; we apply t_2 to the previous AR-sequence, and we see that this is also the relative AR-sequence in $\mathcal{F}_{\geq 2}(\Delta)$.

4. Finding relative AR-sequences

4.1. In this section, we will first determine relative Auslander–Reiten sequences where certain projective-injective tilting modules occur as middle terms. Furthermore, we will show that if $\Gamma = \{\mu\}$, a maximal element, and if the tilting module $T(\mu)$ is projective-injective with only two Δ-quotients, then large parts of the relative AR-quiver of $\overline{A}$ are full subquivers of the relative AR-quiver of A. We will use the same notation as in §2 and §3, and we will tacitly apply the results from there.

4.2. If P is an indecomposable projective and injective module for an arbitrary finite-dimensional algebra then there is always the (well known) usual AR-sequence of the form

$$0 \to \mathrm{rad}(P) \to P \oplus (\mathrm{rad}(P)/\mathrm{soc}(P)) \to P/\mathrm{soc}(P) \to 0.$$

In the case when A is quasi-hereditary, such a module P belongs to $\mathcal{F}(\Delta)$ and is a tilting module. One asks whether there may be a relative analogue of the above sequence. In the case when the highest weight is maximal we have a result; the hypotheses for the following theorem are very often satisfied for Schur algebras.

Recall that if $T(\lambda)$ is an indecomposable tilting module then there is an exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0 \tag{7}$$

where $X(\lambda)$ belongs to $\mathcal{F}_{<\lambda}(\Delta)$.

THEOREM. *Let $\mu \in \Lambda$ be maximal, and assume that the tilting module $T = T(\mu)$ is projective and injective, such that $T = P(\sigma) \cong I(\sigma)$. Assume also that* $\operatorname{rad}(T)$ *has a Δ-filtration. Then there is a relative AR-sequence*

$$0 \to \operatorname{rad}(T) \to T \oplus \operatorname{rad}(X(\mu)) \to X(\mu) \to 0.$$

Proof. Clearly, such a short exact sequence exists. Let $\overline{A} = A/AeA$ where $e = e_\mu$. Then $\overline{A}$ is quasi-hereditary, and $X(\mu)$ is projective indecomposable as a module for $\overline{A}$; in fact, it is the projective cover of the simple module $L(\sigma)$ as a module for $\overline{A}$.

We have to show that for any $Y \in \mathcal{F}(\Delta)$, any homomorphism $\phi : Y \to X(\mu)$ which is not a split epimorphism, factors through $T \oplus \operatorname{rad}(X(\mu))$.

CASE 1. Assume first that Y does not have $\Delta(\mu)$ as a quotient; then Y is a module for $\overline{A}$. Since $X(\mu)$ is projective as a module for $\overline{A}$, the map ϕ is not onto, hence $\operatorname{Im}(\phi)$ is contained in the radical of $X(\mu)$ and so ϕ factors.

CASE 2. Now suppose $\Delta(\mu)$ occurs in a Δ-filtration of Y. Let $t(Y)$ be the trace of $\Delta(\mu)$ in Y. Then $t(Y)$ is a direct sum of copies of $\Delta(\mu)$ and the quotient has a filtration with $\Delta(\lambda)$'s and $\lambda < \mu$.

Since $L(\mu)$ does not occur as a composition factor of $X(\mu)$ and since any top composition factor of $t(Y)$ is isomorphic to $L(\mu)$, it follows that $t(Y)$ is mapped to zero by ϕ and ϕ induces a homomorphism (denote it by $\overline{\phi}$) from $Y/t(Y)$ into $X(\mu)$. If $\overline{\phi}$ is not onto then $\operatorname{Im}(\phi) = \operatorname{Im}(\overline{\phi}) \subset \operatorname{rad}(X(\mu))$ and $\overline{\phi}$ factors, hence so does ϕ.

Now suppose $\overline{\phi}$ is onto; then it splits and $Y/t(Y) = X(\mu) \oplus Z$ for some module Z. Then Y has a submodule U which is an extension of $X(\mu)$ by a direct sum of copies of $\Delta(\mu)$. Suppose we had $U = X(\mu) \oplus \Delta(\mu)^k$. Then since $\overline{\phi}$ is onto it follows that ϕ is onto, and ϕ must actually be a split epimorphism, a contradiction.

It follows that U is a non-split extension of $X(\mu)$ by $\Delta(\mu)^k$. As we will show below we have $\operatorname{Ext}^1_A(X(\mu), \Delta(\mu)) = K$. Therefore there is a unique such extension. But $T(\mu)$ is one, so $U \cong T(\mu)$ and then ϕ factors again. ∎

4.3. LEMMA. *Let $T = T(\mu)$ be as in* 4.2. *Then* $\operatorname{Ext}^1_A(X(\mu), \Delta(\mu)) \cong K$.

Proof. Let $X = X(\mu)$. Applying $\operatorname{Hom}_A(X, -)$ to (7) gives

$$0 \to \operatorname{Hom}_A(X, \Delta(\mu)) \to \operatorname{Hom}_A(X, T(\mu)) \to \operatorname{Hom}_A(X, X) \to \operatorname{Ext}^1_A(X, \Delta(\mu)) \to 0. \tag{8}$$

We consider the dimensions of the terms. First we have

$$\begin{aligned}\dim \operatorname{Hom}_A(X, \Delta(\mu)) &= \dim \operatorname{Hom}_{\overline{A}}(X, \operatorname{rad}(\Delta(\mu))) \\ &= [\operatorname{rad}(\Delta(\mu)) : L(\sigma)] = [\Delta(\mu) : L(\sigma)].\end{aligned}$$

By reciprocity (see [2, 3.11]), this number is equal to $[I(\sigma) : \Delta(\mu)]$, which is 1 since $I(\sigma) \cong T(\mu)$. Hence it suffices to show that the two middle terms in (8) have the same dimension.

Now $T(\mu) \cong I(\sigma)$ here, so we have $\dim \mathrm{Hom}_A(X, T(\mu)) = [X : L(\sigma)]$. Moreover, $X = P_{\overline{A}}(\sigma)$, which gives $\dim \mathrm{Hom}_A(X, X) = [X : L(\sigma)]$. ∎

4.4. For the rest of this section, we assume that μ is a maximal element in Λ, and that γ is maximal in $\Lambda \setminus \{\mu\}$. Take $\Gamma = \{\mu\}$, and let $e = e_\Gamma$.

Furthermore, we assume that $\mathrm{Ext}^1_A(\Delta(\gamma), \Delta(\mu)) = K$, that there are short exact sequences

$$0 \to \Delta(\mu) \to T(\mu) \to \Delta(\gamma) \to 0, \qquad 0 \to \nabla(\gamma) \to T(\mu) \to \nabla(\mu) \to 0,$$

and that $T(\mu)$ is projective-injective, hence isomorphic to $P(\gamma)$ and to $I(\gamma)$.

4.5. Lemma. *Assume that $M \in \mathcal{F}(\Delta_{\overline{A}})$ is indecomposable and $M \neq \Delta(\gamma)$. Assume also that for any Δ-quotient $\Delta(\lambda)$ occurring in M, with $\lambda \neq \gamma$, we have $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$. Then $\mathrm{Ext}^1_A(M, \Delta(\mu)) = 0$.*

Proof. Suppose we have a non-split short exact sequence

$$0 \to \Delta(\mu) \to E \to M \to 0.$$

Then $\Delta(\gamma)$ must occur in M by the hypothesis; actually, M has a submodule isomorphic to $\Delta(\gamma)$ (since γ is maximal in $\Lambda \setminus \{\mu\}$). By the hypothesis, E then has a submodule isomorphic to $T(\mu)$, containing the kernel of the exact sequence. This is a direct summand since $T(\mu)$ is injective, say $E = E_1 \oplus T(\mu)$. But then

$$M \cong E/\Delta(\mu) = E_1 \oplus T(\mu)/\Delta(\mu) \cong E_1 \oplus \Delta(\gamma)$$

and since M is indecomposable, we get $M \cong \Delta(\gamma)$. ∎

4.6. Proposition. *Suppose M is as in* 4.5. *Suppose that*

$$0 \to W = \tau_\Delta(M) \to X \to M \to 0 \tag{9}$$

is the relative AR-sequence of M in $\mathcal{F}(\Delta_{\overline{A}})$. Then this is already the relative AR-sequence of M in $\mathcal{F}(\Delta_A)$.

Proof. By the hypothesis, $\Delta(\mu)$ does not occur in W, and W is not injective in $\mathcal{F}(\Delta)$. We apply now the result in §3. Let

$$0 \to W \to U \to Z = \tau_\Delta^{-1} W \to 0 \tag{10}$$

be the relative AR-sequence in $\mathcal{F}(\Delta)$, and denote by $\overline{X}$ the module $X/t(X)$. Then by 3.1 the relative AR-sequence beginning at W in $\mathcal{F}(\Delta_{\overline{A}})$ is of the form

$$0 \to W \to \overline{U} \to \overline{Z} \to 0.$$

This is the same as (9), so $M \cong \overline{Z}$. But by 4.5 we know that $\mathrm{Ext}^1(M, \Delta(\mu)) = 0$ and hence $t(Z) = 0$ and $Z = M$. By exactness, also $t(U) = 0$ and (9) and (10) are the same, as required. ∎

5. The relative AR-quiver of $\mathcal{D}_n$

5.1.1. The algebra $\mathcal{D}_n$ for $n \geq 3$ is the algebra with quiver shown in Fig. 1 subject to the relations

$$\begin{aligned} &\alpha_1\beta_1 = 0 = \alpha_2\beta_2, && \alpha_1\beta_2\alpha_2 = 0 = \beta_2\alpha_2\beta_1, && \\ &\delta_3\beta_2 = 0 = \delta_3\beta_1, && \alpha_2\gamma_3 = 0 = \alpha_1\gamma_3 && (\text{for } n \geq 4), \\ &\gamma_3\delta_3 = \beta_2\alpha_2, && \gamma_i\gamma_{i+1} = 0 = \delta_{i+1}\delta_i, && \\ &\delta_i\gamma_i = \gamma_{i+1}\delta_{i+1} && && (3 \leq i \leq n-2). \end{aligned}$$

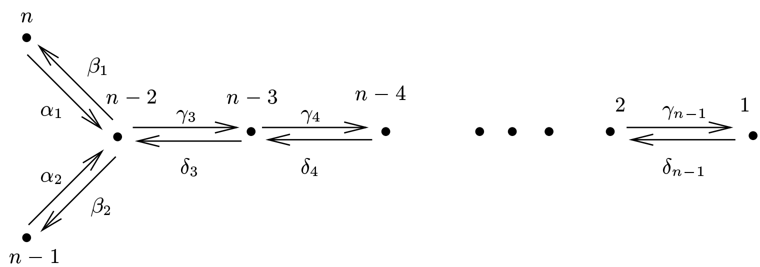


Fig. 1. The quiver of $\mathcal{D}_n$

These algebras are quasi-hereditary, with respect to the natural order on the integers. They occur as blocks of Schur algebras (see [10]); note also that they have a duality fixing the simple modules, and hence $\Delta(m)$ is dual to $\nabla(m)$ for any m. We will determine the relative AR-quiver of $\mathcal{D}_n$. It turns out to be finite, hence $\mathcal{D}_n$ is Δ-finite. Note that the algebra as such is of infinite representation type, actually wild for $n \geq 5$ (see [8, 3.4]).

Let $e = e_n$ and $\overline{A} = A/AeA$; this algebra is isomorphic to the algebra called $\mathcal{A}_{n-1}$ in [10, 15]. Its relative Auslander–Reiten quiver and the indecomposables in $\mathcal{F}(\Delta)$ were determined in [15]. A complete list of indecomposables in $\mathcal{F}(\Delta)$ is given as follows.

(i) For each a, b with $1 \leq a \leq b \leq n-1$ and $b-a$ even, there is a unique indecomposable module with Δ-quotients

$$\Delta(a), \Delta(a+2), \ldots, \Delta(b-2), \Delta(b),$$

each occurring once. We call this module $W(a,b)$; note that $W(a,a) = \Delta(a)$. For convenience, we set $W(c,d) = 0$ if $c \not\leq d$.

(ii) For each $1 \leq a < n-1$, the projective $P(a)$ has Δ-quotients $\Delta(a)$, $\Delta(a+1)$, and $P(a) \cong T(a+1)$. These are therefore also injective.

5.1.2. Now consider indecomposable modules for $\mathcal{D}_n$. We have moreover $T(n) = P(n-1) \cong I(n-1)$ and it has Δ-quotients $\Delta(n)$ and $\Delta(n-1)$. Hence we can apply 4.5 and 4.6. It is easy to see that

$$\operatorname{Ext}^1_A(\Delta(j), \Delta(n)) = \begin{cases} K & \text{for } j = n-1, n-2, \\ 0 & \text{otherwise.} \end{cases}$$

One observes that there is an indecomposable module which is an extension of $W(a, n-2)$ by $\Delta(n)$, for any $a \equiv n \pmod 2$. Moreover, this module is unique up to isomorphism with this property, by the dimensions of Ext-spaces. We denote it by $W(a, n)$.

5.2. By 4.5 and 4.6, most of the relative AR-quiver of $\overline{A}$ is a full subquiver of that of A. The only modules in $\mathcal{F}(\Delta_{\overline{A}})$ whose relative AR-sequence in A can be different from that for $\overline{A}$ are those where $\Delta(n-2)$ occurs as a quotient.

First, $P_{\overline{A}}(n-2) \cong T(n-1)$ is not projective as an A-module, so it has a relative AR-sequence.

Lemma. *The relative AR-sequence in $\mathcal{F}(\Delta)$ for $T(n-1)$ is of the form*

$$0 \to \Delta(n) \to P_A(n-2) \to T(n-1) \to 0.$$

Proof. One calculates that the usual AR-translate of $T(n-1)$ is $L(n)$, the simple module. The middle of the usual AR-sequence has $\mathcal{F}(\Delta)$-approximation $P(n-2)$ (the kernel of a surjection of $P(n-2)$ onto the middle has kernel $\nabla(n-1)$, so by [12, Lemma 1] this is the approximation). This is indecomposable, so the relative AR-sequence is as stated. ∎

From the results in [15] we see which part of the relative AR-quiver of $\mathcal{A}_{n-1}$ can change for A; it is given as follows.

(a) For $n > 3$ there is a relative irreducible map $W(1, n-2) \to \Delta(n-1)$ or $W(2, n-2) \to \Delta(n-1)$ when n is odd or even, respectively.

(b) For n odd, the relative AR-sequences ending with $W(a, n-2)$ are:

$$(\zeta_0) \quad 0 \to \Delta(2) \to P(1) \oplus W(3, n-2) \to W(1, n-2) \to 0,$$

$$(\zeta_j) \quad 0 \to W(2, 2(j+1)) \to W(2, 2j) \oplus P(1+2j) \oplus W(1+2(j+1), n-2) \to W(1+2j, n-2) \to 0$$

for $1 \le j < (n-3)/2$. For $n > 3$ the last one is

$$(\zeta_{(n-3)/2}) \quad 0 \to W(2, n-1) \to W(2, n-3) \oplus P(n-2) \to \Delta(n-2) \to 0,$$

(c) For n even, the relative AR-sequences ending with $W(a, n-2)$ are, when $n > 4$:

$$(\zeta_1) \quad 0 \to W(1,3) \to \Delta(1) \oplus P(2) \oplus W(4, n-2) \to W(2, n-2) \to 0,$$

$$(\zeta_j) \quad 0 \to W(1, 1+2j) \to W(1, 1+2(j-1)) \oplus P(2j) \oplus W(2(j+1), n-2) \to W(2j, n-2) \to 0$$

for $1 \le j < (n-2)/2$, and the last for $n \ge 4$ is

$$(\zeta_{(n-2)/2}) \quad 0 \to W(1, n-1) \to W(1, n-3) \oplus P(n-2) \to \Delta(n-2) \to 0.$$

5.3. Theorem. *For n odd, the relative AR-quiver of $\mathcal{D}_n$ is obtained from that of $\mathcal{A}_{n-1}$ by adding the sequence of $T(n-1)$, and by the following operations*:

(i) *Replace* (a) *by the relative AR-sequence*

$$0 \to W(1,n) \to W(1,n-2) \oplus P(n-1) \to \Delta(n-1) \to 0.$$

(ii) *Replace* (ζ_j) *for* $0 \leq j < (n-3)/2$ *and* $n \geq 3$ *by two relative AR-sequences*

$$0 \to W(1+2(j+1),n) \to W(1+2j,n) \oplus W(1+2(j+1),n-2) \to W(1+2j,n-2) \to 0,$$
$$0 \to W(2,2(j+1)) \to W(2,2j) \oplus P(1+2j) \oplus W(1+2(j+1),n) \to W(1+2j,n) \to 0.$$

(iii) *Let* $j = (n-3)/2$ *and* $n > 3$. *Replace* (ζ_j) *by two relative AR-sequences*

$$0 \to W(2,n-1) \to P(n-2) \oplus W(2,n-3) \to W(n-2,n) \to 0,$$
$$0 \to P(n-2) \to T(n-1) \oplus W(n-2,n) \to \Delta(n-2) \to 0.$$

5.3.1. Theorem. *For n even, the relative AR-quiver of $\mathcal{D}_n$ is obtained from that of $\mathcal{A}_{n-1}$ by adding the sequence of $T(n-1)$, and by the following operations*:

(i) *Replace* (a) *by the relative AR-sequence*

$$0 \to W(2,n) \to W(2,n-2) \oplus P(n-1) \to \Delta(n-1) \to 0.$$

(ii) *For* $1 \leq j < (n-2)/2$ *and* $n > 4$, *replace* (ζ_j) *by two relative AR-sequences*

$$0 \to W(2(j+1),n)) \to W(2j,n) \oplus W(2(j+1),n-2) \to W(2j,n-2) \to 0,$$
$$0 \to W(1,2j+1) \to W(1,2(j-1)+1) \oplus P(2j) \oplus W(2(j+1),n) \to W(2j,n) \to 0.$$

(iii) *For* $j = (n-2)/2$ *and* $n \geq 4$ *replace* (ζ_j) *by two relative AR-sequences*

$$0 \to P(n-2) \to T(n-1) \oplus W(n-2,n) \to \Delta(n-2) \to 0,$$
$$0 \to W(1,n-1) \to P(n-2) \oplus W(1,n-3) \to W(n-2,n) \to 0.$$

This gives the complete relative AR-quiver (for an example see below). Namely, we have the relative AR-sequence ending and starting with $W(a,n-2)$ and $W(2,b)$ (or $W(1,b)$), and we see that $\tau_\Delta(W(a,n-2))$ is the τ_Δ^{-1}-translate of a module $W(2,b)$ (or $W(1,b)$ for n even). Hence we have found a component and then by the analogue of Auslander's Theorem (see [13]), this must be the complete relative AR-quiver, and in particular we have

Corollary. *The algebras $\mathcal{D}_n$ are Δ-finite.*

The proof of the theorems will take the rest of the paper. The following gives evidence that $\mathcal{D}_n$ does not have too many new indecomposable modules.

5.4. Lemma. *Suppose $M = W(a, n-2)$ and*

$$0 \to \Delta(n)^s \to N \to M \to 0$$

is exact. If $s \geq 2$ then N is decomposable

Proof. First, $\mathrm{Hom}_A(M, \Delta(n)) = 0$, since $\Delta(n)$ has a simple socle isomorphic to $L(n-1)$ and $L(n-1)$ is not a composition factor of M.

To prove the lemma, it suffices to show that $\mathrm{Hom}_A(N, \Delta(n)) \neq 0$ for $s \geq 2$. Then a non-zero map $0 \neq \phi : N \to \Delta(n)$ cannot factor through M and therefore it must be non-zero on restriction to some copy of $\Delta(n)$. This restriction is necessarily an isomorphism, and ϕ splits.

Applying the functor $\mathrm{Hom}(-, \Delta(n))$ to the given exact sequence gives

$$0 \to \mathrm{Hom}(N, \Delta(n)) \to \mathrm{Hom}(\Delta(n)^s, \Delta(n)) = K^s \to \mathrm{Ext}^1(M, \Delta(n)).$$

By 5.1.2 we get $\mathrm{Ext}^1(M, \Delta(n)) = K$, and so $\mathrm{Hom}(N, \Delta(n)) \neq 0$ if $s \geq 2$. ∎

5.5. Lemma. *The relative AR-sequence of $\Delta(n-1)$ is as follows*:

(a) (n *odd*)

$$0 \to W(1, n) \to W(1, n-2) \oplus P(n-1) \to \Delta(n-1) \to 0.$$

(b) (n *even*)

$$0 \to W(2, n) \to W(2, n-2) \oplus P(n-1) \to \Delta(n-1) \to 0.$$

Proof. The standard module $\Delta(n-1)$ has length 2 and socle $L(n-2)$. We find that the usual AR-translate of $\Delta(n-1)$ is $\mathrm{rad}(\nabla(n))$, which has length two, socle $L(n)$ and top $L(n-2)$. It follows that the middle of the usual AR-sequence is the direct sum of $L(n-2)$ with $\nabla(n)$. This has $\mathcal{F}(\Delta)$-approximation

(a) (n odd) $W(1, n-2) \oplus T(n)$,
(b) (n even) $W(2, n-2) \oplus T(n)$

(see [15]), and $T(n) = P(n-1)$. This is the middle term of the relative AR-sequence and the lemma follows. ∎

5.6. The previous lemma gives the first part of Theorem 5.3, and we will now finish the proof. The proof of 5.3.1 is analogous and is omitted.

(a) We determine the relative AR-sequence of $W(1, n)$; the statement is that it ends with $\Delta(2)$. Consider therefore the relative AR-sequence which starts with $\Delta(2)$, say it is

$$0 \to \Delta(2) \to U \to V \to 0. \qquad (11)$$

By 3.1, the relative AR-sequence starting with $\Delta(2)$ over $\bar{A}$ is of the form

$$0 \to \Delta(2) \to \overline{U} \to \overline{V} \to 0 \tag{12}$$

where $\overline{X} = X/t(X)$. By §3, this is the same as (ζ_0) in 5.2, hence $\overline{V} = W(1, n-2)$. By 5.4, either $V = W(1,n)$ or $V = W(1, n-2)$. Assume, for a contradiction, that $V = W(1, n-2)$. Then by 5.5, the module $W(1,n)$ must occur as a summand of U, but then $\Delta(n)$ occurs only in the middle of (11) and not at the ends, i.e. (11) cannot be exact, a contradiction. Hence $V = W(1,n)$.

Assume first that $n > 3$. Then the middle of (12) is an extension of $P(1) \oplus W(3, n-2)$ by one copy of $\Delta(n)$ (see 5.2). Since $P(1)$ does not have $\Delta(n-2)$ as a quotient, it is a direct summand of U and since there is an epimorphism from U onto $W(1,n)$ it follows that $U = P(1) \oplus W(3,n)$ (by 5.1.2).

Now assume $n = 3$. Then the middle of (12) is an extension of $P(1)$ by $\Delta(3)$, and it is a direct sum since $P(1)$ is projective and injective for $\mathcal{D}_3$.

(b) We will now determine the relative AR-sequence of $W(1, n-2)$. We know from 5.5 that there is a relative irreducible map $W(1,n) \to W(1, n-2)$, hence $\tau_\Delta(W(1, n-2))$ must be a summand of the middle of the relative AR-sequence of $W(1,n)$. We also know that there is a relative irreducible map $W(3, n-2) \to W(1, n-2)$, from the relative AR-sequences of $\bar{A}$ (if $n > 3$). The only indecomposable summand in the middle of the AR-sequence of $W(1,n)$ which can be $\tau_\Delta(W(1, n-2))$ is $W(3,n)$. Then by exactness, the RAS must be as stated.

If $n = 3$ then the proof is complete. Otherwise, one repeats these arguments for $j = 1, \ldots, (n-3)/2 - 1$. To replace (ζ_j) we use the argument of (a). This requires an irreducible map from $W(1+2j, n)$ to $W(1+2j, n-2)$, and this exists as one sees from the analysis of the step from (ζ_{j-1}). This shows that the relative AR-sequence starting with $W(2, 2(j+1))$ must end with $W(1+2j, n)$ and is of the form stated. Next, the argument of (b), together with the information from the step for $j-1$, gives the relative AR-sequence of $W(1+2j, n-2)$.

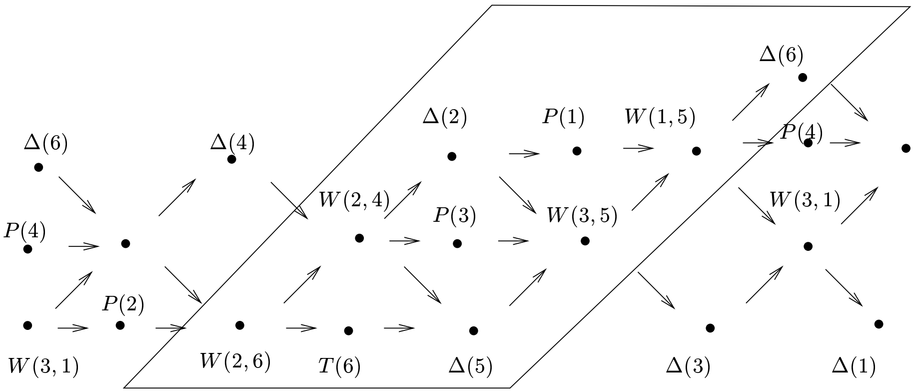
To finish, let $j = (n-3)/2$. By the arguments of (a), one proves that the relative AR-sequence starting with $W(2, n-1)$ is

$$0 \to W(2, n-1) \to P(n-2) \oplus W(2, n-3) \to W(n-2, n) \to 0. \tag{$*$}$$

It remains to find the relative AR-sequence starting with $\Delta(n-2)$. By the argument of (b), one concludes that $\tau_\Delta(\Delta(n-2)) = P(n-2)$. By 5.2, the module $T(n-1)$ must be a summand of the middle of the relative AR-sequence of $\Delta(n-2)$, and so is $W(n-2, n)$ (because of $(*)$) and by exactness the relative AR-sequence of $\Delta(n-2)$ is of the form

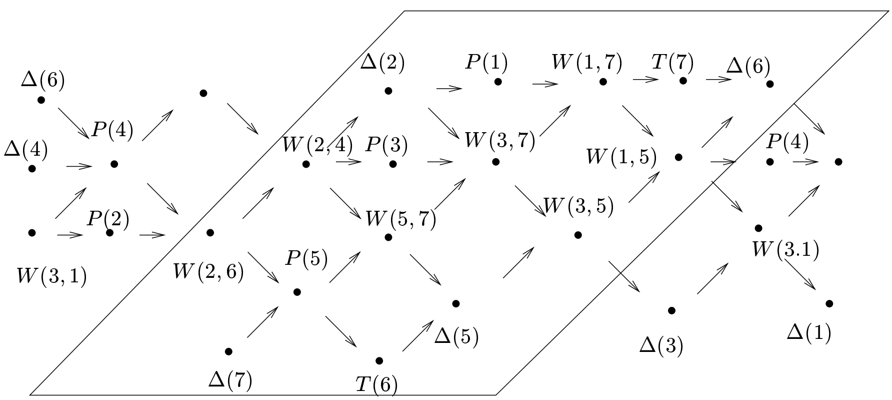
$$0 \to P(n-2) \to T(n-1) \oplus W(n-2, n) \to \Delta(n-2) \to 0. \blacksquare$$

Example. The relative AR-quivers of $\mathcal{A}_6$ and $\mathcal{D}_7$ are shown in Figures 2 and 3, respectively. Repetitions of vertices must be identified. The parts which are different are the ones in boxes. These are obtained by applying Lemma 5.2 and Theorem 5.3.



$P(5) = T(6)$

Fig. 2. AR-quiver for $\mathcal{A}_6$



$P(6) = T(7)$

Fig. 3. AR-quiver for $\mathcal{D}_7$

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Mathematical Institute
24-29 St. Giles
Oxford OX1 3LB, UK
E-mail: erdmann@maths.ox.ac.uk

Facultad de Matemáticas
Apdo. Postal 402
Guanajuato GTO 36000, Mexico
E-mail: corina@cimat.mx

Instituto de Matemáticas
Circuito Exterior
Ciudad Universitaria
04510 México D.F., Mexico
E-mail: jap@penelope.matem.unam.mx

Current address of C. Sáenz:
Instituto de Matemáticas
Circuito Exterior
Ciudad Universitaria
04510 México D.F., Mexico
E-mail: ecsv@lya.fciencias.unam.mx

Received 25 June 2001;
revised 18 August 2001 (4086)