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A GENERAL DIFFERENTIATION THEOREM FOR MULTIPARAMETER ADDITIVE PROCESSES

by

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To the memory of Anzelm Iwanik

Abstract. Let $(L, \|\cdot\|_L)$ be a Banach lattice of equivalence classes of real-valued measurable functions on a σ-finite measure space and $T = \{T(u) : u = (u_1, \ldots, u_d), u_i > 0, 1 \le i \le d\}$ be a strongly continuous locally bounded d-dimensional semigroup of positive linear operators on L. Under suitable conditions on the Banach lattice L we prove a general differentiation theorem for locally bounded d-dimensional processes in L which are additive with respect to the semigroup T.

1. Introduction and the results. Let (Ω, Σ, μ) be a σ-finite measure space and $(L, \|\cdot\|_L)$ a Banach lattice of equivalence classes of real-valued measurable functions on (Ω, Σ, μ) under pointwise operations. Thus we understand that if $f \in L$ then the function $f^+(\omega) = \max\{f(\omega), 0\}$ is also in L, and two functions f and g in L are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. We let $|f|(\omega) = \max\{f(\omega), -f(\omega)\}$. Hereafter all statements and relations are assumed to hold modulo sets of measure zero. By definition, the norm $\|\cdot\|_L$ has the following property (cf. p. 1 of [11]):

(I) If $f, g \in L$ and $|f(\omega)| \le |g(\omega)|$ for almost all $\omega \in \Omega$, then $\|f\|_L \le \|g\|_L$.

Moreover, in this paper, we will assume that $(L, \|\cdot\|_L)$ has the following additional properties:

(II) If g is a real-valued measurable function on Ω and if there exists an $f \in L$ such that $|g(\omega)| \le |f(\omega)|$ for almost all $\omega \in \Omega$, then $g \in L$.

(III) If $E_n \in \Sigma$, $E_n \supset E_{n+1}$ for each $n \ge 1$, and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then for all $f \in L$ we have

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$$\lim_{n\to\infty} \|f \cdot \chi_{E_n}\|_L = 0,$$

where χ_{E_n} denotes the characteristic function of E_n.

We recall that Property (II) means that L is "solid" in the vector lattice of equivalence classes of all real-valued measurable functions on (Ω, Σ, μ) (and since L is a subspace, it means that L is an "ideal" of that vector lattice), and that Property (III) is clearly implied by the "σ-order continuity" of the norm. When L is "σ-order complete", (III) implies the σ-order continuity of the norm. This follows from Theorem II.5.14 of [14], or Proposition 1.a.7 of [11], because (III) implies that L does not contain a sublattice that is vector lattice isomorphic to ℓ_∞.

An operator $S : L \to L$ is called *positive* if $Sf(\omega) \geq 0$ for almost all $\omega \in \Omega$ for every f in L^+, where $L^+ = \{f \in L : f(\omega) \geq 0$ a.e. on $\Omega\}$.

If $d \geq 1$ is an integer, then we let

$$\mathbb{P}_d = \{u = (u_1, \ldots, u_d) \in \mathbb{R}_d : u_i > 0,\ 1 \leq i \leq d\},$$
$$\mathbb{R}_d^+ = \{u = (u_1, \ldots, u_d) \in \mathbb{R}_d : u_i \geq 0,\ 1 \leq i \leq d\}.$$

Further, $\mathcal{I}_d$ denotes the class of all bounded intervals in $\mathbb{P}_d$, and λ_d is the d-dimensional Lebesgue measure. We will consider a strongly continuous d-dimensional semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ of positive linear operators on L. Thus

(i) each $T(u)$ is a positive linear operator from L to itself,
(ii) $T(u + v) = T(u)T(v)$ for $u, v \in \mathbb{P}_d$, and
(iii) $\lim_{u\to v} \|T(u)f - T(v)f\|_L = 0$ for $v \in \mathbb{P}_d$ and $f \in L$.

It is known that there are two measurable decompositions $\Omega = P + N$ and $\Omega = C + D$ with respect to $T = \{T(u)\}$ (see e.g. [13]) such that

(a) if $f \in L^+$ and $\{\omega : f(\omega) > 0\} \subset N$, then $\|T(u)f\|_L = 0$ for all $u \in \mathbb{P}_d$,
(b) if $0 \neq f \in L^+$ and $\mu(P \cap \{\omega : f(\omega) > 0\}) > 0$, then $\|T(u)f\|_L > 0$ for some $u \in \mathbb{P}_d$,
(c) $T(u)f(\omega) = 0$ on D for every $u \in \mathbb{P}_d$ and $f \in L^+$,
(d) $C = \bigcup_{n=1}^\infty \{\omega : T(1/n, \ldots, 1/n)h(\omega) > 0\}$ for some $h \in L^+$.

If $\Omega = P$, then $T = \{T(u)\}$ will be called *proper*. Thus if

$$I_L = \text{strong-}\lim_{u\to 0} T(u),$$

where I_L denotes the identity operator on L, then $T = \{T(u)\}$ becomes proper. We assume below that T is locally bounded, i.e.,

$$K(T) := \sup\{\|T(u)\|_L : u \in (0, 1]^d\} < \infty, \tag{1}$$

where $\|T(u)\|_L$ denotes the operator norm of $T(u)$ on L. It then follows that for each $f \in L$ the vector-valued function $u \mapsto T(u)f$ is Bochner integrable over every $I \in \mathcal{I}_d$.

By a (d-dimensional) process F in L we mean a set function $F : \mathcal{I}_d \to L$. It is *positive* if $F(I) \in L^+$ for all $I \in \mathcal{I}_d$, and *locally bounded* if

$$K(F) := \sup\left\{\frac{\|F(I)\|_L}{\lambda_d(I)} : I \in \mathcal{I}_d,\ I \subset (0,1]^d,\ \lambda_d(I) > 0\right\} < \infty, \tag{2}$$

and *bounded* if

$$\sup\left\{\frac{\|F(I)\|_L}{\lambda_d(I)} : I \in \mathcal{I}_d,\ \lambda_d(I) > 0\right\} < \infty. \tag{3}$$

It is called *additive* (with respect to T) if the following conditions hold:

(i) $T(u)F(I) = F(u + I)$ for all $u \in \mathbb{P}_d$ and $I \in \mathcal{I}_d$.

(ii) If $I_1, \ldots, I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$, then $F(I) = \sum_{i=1}^k F(I_i)$.

Thus if $F(I) = \int_I T(u)f\,du$ for all $I \in \mathcal{I}_d$, where f is a fixed function in L^+, then $F(I)$ defines a positive locally bounded additive process in L. Following Akcoglu and del Junco [1], this process will be called *absolutely continuous.* It is known (see e.g. [1]) that there are many positive locally bounded additive processes in L which are not absolutely continuous.

In this paper we discuss d-dimensional locally bounded additive processes F in L and, in particular, study the almost everywhere convergence of the averages $\alpha^{-d}F((0,\alpha]^d)$ as α approaches zero. But, as is known, this does not make sense when the averages denote equivalence classes and not actual functions and α ranges through all positive numbers. Therefore, in this paper, we let α range through a countable dense subset $\mathbb{D}$ of the positive numbers. It may be assumed that $\mathbb{D}$ includes all positive rational numbers. Following Akcoglu and Krengel [2], we use the notation q-$\lim_{\alpha\to 0}$ and q-$\limsup_{\alpha\to 0}$, etc., to mean that these limits are taken as α tends to zero through the set $\mathbb{D}$.

We are now in a position to state our main result as follows.

THEOREM. *Let $T = \{T(u) : u \in \mathbb{P}_d\}$ be a strongly continuous locally bounded d-dimensional semigroup of positive linear operators on L, and $F : \mathcal{I}_d \to L$ be a locally bounded d-dimensional process in L which is additive with respect to T. Assume that T is proper. Then the limit*

$$f_0(\omega) := \text{q-}\lim_{\alpha\to 0} \alpha^{-d}F((0,\alpha]^d)(\omega) \tag{4}$$

exists for almost all $\omega \in \Omega$.

COROLLARY. *Assume that $L = L_p(\mu)$ for some p with $1 \le p < \infty$. Then the above limit function f_0 is a function in $L_p(\mu)$ satisfying*

$$\lim_{u\to 0} \|T(u)f_0 - f_0\|_{L_p(\mu)} = 0. \tag{5}$$

Akcoglu and del Junco [1] (see also pp. 693–708 of [4], and [15]) proved such a differentiation theorem in the case $L = L_1(\mu)$, under the assumptions that $T = \{T(u) : u \in \mathbb{P}_d\}$ is a contraction semigroup of positive linear operators on $L_1(\mu)$ and that F is a bounded additive process in $L_1(\mu)$, and then Lin [10] used Akcoglu and del Junco's result to obtain the theorem for the case where $L = L_p(\mu)$ with $1 \leq p < \infty$, assuming only the local boundedness of the semigroup T. Thus the present theorem may be considered to be a generalization.

We note that, besides $L_p(\mu)$-spaces, there are many interesting Banach lattices of functions which share the additional properties (II) and (III). Examples are Lorentz spaces and Orlicz spaces, etc. Here the author intends to generalize the differentiation theorem to such function spaces. Since calculating norms of functions in Lorentz spaces or Orlicz spaces is somewhat complicated and vague (cf. [7] and [8]), the author thinks that it is preferable to consider a locally bounded semigroup T instead of a contraction semigroup. It is interesting to note that the hypothesis that T is a proper semigroup cannot be omitted from the Theorem. A counterexample can be found in Theorem 2 of [12]. On the other hand, when $L = L_1(\mu)$, Emilion [6] has proved the theorem for a contraction semigroup on $L_1(\mu)$ in which the operators need not be positive and the semigroup need not be proper. The necessity of positivity of the operators in the general context follows from the example of Akcoglu and Krengel in [3].

In §2 we provide some necessary lemmas, and the proofs are given in §3. In the last section an example is presented to show that the limit function f_0 does not necessarily belong to L.

Acknowledgments. In the original manuscript of the paper the author considered a uniformly bounded semigroup T and a positive bounded additive process F. The referee suggested that the theorem might hold without the positivity assumption on the process F, by modifying the proof of [1]. He also suggested that Lin's paper [10] would be useful when T is assumed to be locally bounded. These suggestions led the author to the present version, and hence he would like to express his sincere gratitude to the referee.

2. Lemmas. The next two lemmas are basic throughout the paper. Their proofs can be found in [13], and hence we omit the details here.

LEMMA 1. *There exists a strictly positive measurable function w on Ω such that $\int_\Omega fw\,d\mu < \infty$ for all $f \in L^+$.*

The function w defines by integration the positive linear functional $f \mapsto \int_\Omega fw\,d\mu$ on L, which, since L is a Banach lattice, is necessarily continuous in norm (see e.g. pp. 2–3 in [11]). Hence the functional S^*w is well defined for every bounded linear operator $S : L \to L$. The point of the next lemma is that if S is positive, then the functional corresponding to S^*w is also given by integration with respect to a non-negative measurable function.

LEMMA 2. *Let $S : L \to L$ be a positive linear operator and w be a non-negative measurable function on Ω such that $\int_\Omega fw\,d\mu < \infty$ for all $f \in L^+$. Then there exists a non-negative measurable function v on Ω, written as $v = S^*w$, such that*

$$\int_\Omega (Sf)w\,d\mu = \int_\Omega fv\,d\mu$$

for all $f \in L$.

Let $T = \{T(u) : u \in \mathbb{P}_d\}$ be a strongly continuous locally bounded d-dimensional semigroup of positive linear operators on L. Since $K(T) < \infty$, an easy computation shows that there exists a constant $\beta > 0$ such that for every $f \in L$ the vector-valued function

$$t = (t_1, \ldots, t_d) \mapsto e^{-\beta(t_1+\ldots+t_d)}T(t)f$$

is Bochner integrable on $\mathbb{P}_d$. Then we can define a positive linear operator $S_T : L \to L$ by

$$S_T f = \int_{\mathbb{P}_d} e^{-\beta(t_1+\ldots+t_d)}T(t)f\,dt. \tag{6}$$

If w denotes the strictly positive measurable function in Lemma 1, then we let

$$v_T = S_T^* w. \tag{7}$$

The following result is essential in §3.

LEMMA 3. *We have $P = \{\omega : v_T(\omega) > 0\}$, and*

$$0 \le T(t)^* v_T \le e^{\beta(t_1+\ldots+t_d)} v_T$$

on Ω for each $t \in \mathbb{P}_d$.

Proof. An easy modification of the proof of Lemma 8 of [13] yields the result.

3. Proofs

Proof of Theorem. Since $T = \{T(u)\}$ is proper by hypothesis, an easy consideration shows that it may be assumed without loss of generality that

$$C = P = \Omega. \tag{8}$$

Indeed, by Lemma 3, we have $v_T(\omega) > 0$ for almost all $\omega \in \Omega$. Choose a constant $\eta > 0$ such that

$$\Big|\int_\Omega f \cdot v_T\,d\mu\Big| \le \int_\Omega |f| \cdot v_T\,d\mu \le \eta \cdot \|f\|_L$$

for all $f \in L$. Then it follows that

$$(9) \qquad \int_\Omega |T(t)f| \cdot v_T \, d\mu \le \int_\Omega |f| \cdot T(t)^* v_T \, d\mu \le e^{\beta(t_1+\ldots+t_d)} \int_\Omega |f| \cdot v_T \, d\mu \le e^{\beta(t_1+\ldots+t_d)} \eta \cdot \|f\|_L$$

for all $t = (t_1, \ldots, t_d) \in \mathbb{P}_d$ and $f \in L$. Since L is a dense subspace of $L_1(v_T \, d\mu)$ by Property (II), $T = \{T(t) : t \in \mathbb{P}_d\}$ can be regarded as a strongly continuous semigroup of positive linear operators on $L_1(v_T \, d\mu)$ such that

$$(10) \qquad \|T(t)\|_{L_1(v_T d\mu)} \le e^{\beta(t_1+\ldots+t_d)}$$

for all $t = (t_1, \ldots, t_d) \in \mathbb{P}_d$, and the process F can be considered to be a locally bounded additive process in $L_1(v_T \, d\mu)$. Since the decomposition $\Omega = C + D$ corresponding to the original semigroup $T = \{T(u)\}$ on L is identical with the one corresponding to the new semigroup $T = \{T(u)\}$ on $L_1(v_T \, d\mu)$ given by (10), and since $T(u)L_1(C, v_T \, d\mu) \subset L_1(C, v_T \, d\mu)$ for all $u \in \mathbb{P}_d$ (cf. (c) in §1), it follows from [1] that the restriction $T_C = \{T(u)|_{L_1(C,\, v_T \, d\mu)} : u \in \mathbb{P}_d\}$ of the new semigroup $T = \{T(u)\}$ to $L_1(C,\, v_T \, d\mu)$ becomes strongly continuous at the origin. Let

$$T_0 = \text{strong-}\lim_{u \to 0} : T(u)|_{L_1(C, v_T d\mu)}.$$

Then, by an easy approximation argument, T_C can be extended continuously to $\mathbb{R}_d^+$. We will denote the extended semigroup by $T_C = \{T_u : u \in \mathbb{R}_d^+\}$.

Let $I \in \mathcal{I}_d$ with $I \subset (0,\, 1]^d$. Then, the local boundedness of F in $L_1(v_T \, d\mu)$ implies

$$(11) \qquad \lim_{n \to \infty} \|F(I \cap (1/n, 1]^d) - F(I)\|_{L_1(v_T \, d\mu)} = 0.$$

Since

$$F(I \cap (1/n, 1]^d) \in T(1/n, \ldots, 1/n) L_1(v_T \, d\mu) \subset L_1(C, v_T \, d\mu),$$

we deduce from (11) that $F(I) \in L_1(C, v_T \, d\mu)$ and that $T_0 F(I) = F(I)$. Next, let $I \in \mathcal{I}_d$ be given arbitrarily. Then we can choose a decomposition

$$\{\alpha_i + I_i : \alpha_i \in \mathbb{R}_d^+,\ I_i \in \mathcal{I}_d,\ 1 \le i \le n\}$$

of I such that $I_i \subset (0, 1]^d$ for each $i = 1, \ldots, n$. Then put

$$(12) \qquad \widehat{F}(I) := \sum_{i=1}^{n} T_{\alpha_i} F(I_i).$$

Clearly, $\widehat{F}(I)$ is well defined, and the set function $\widehat{F} : \mathcal{I}_d \to L_1(C, v_T \, d\mu)$ becomes a process in $L_1(C, v_T \, d\mu)$ which is additive with respect to the semigroup $T_C = \{T_u : u \in \mathbb{R}_d^+\}$. Since $\widehat{F}(I) = F(I)$ for all $I \in \mathcal{I}_d$ with $I \subset (0, 1]^d$, we may assume that $\widehat{F} = F$, $C = P = \Omega$, $T_C = \{T_u : u \in \mathbb{R}_d^+\}$

$= \{T(u) : u \in \mathbb{R}_d^+\} = T$, and $L_1(v_T\, d\mu) = L$, for the proof of the Theorem. This is the reason why we may assume (8), from the beginning. We continue the proof as follows. Since F is additive with respect to $\{T(u) : u \in \mathbb{R}_d^+\}$, we immediately find that

$$\|F(I)\|_{L_1(v_T d\mu)} \le \eta \cdot K(F)\lambda_d(I)e^{\beta(b_1+\ldots+b_d)} \tag{13}$$

for all $I = (a_1, b_1] \times \ldots \times (a_d, b_d] \in \mathcal{I}_d$.

We assume for a moment that F is a positive process. Then define the process

$$G(I) = \int\limits_I e^{-\beta(t_1+\ldots+t_d)}\, dF(t) \tag{14}$$

(cf. (3.3) of [1]). As in [10], we see, by using (13), that

$$\|G(I)\|_{L_1(v_T d\mu)} = \int\limits_\Omega \Big(\int\limits_I e^{-\beta(t_1+\ldots+t_d)}\, dF(t)\Big) v_T\, d\mu \le \eta \cdot K(F)\lambda_d(I)$$

and, since F is additive with respect to T,

$$\begin{aligned} e^{-\beta(u_1+\ldots+u_d)}T(u)G(I) &= e^{-\beta(u_1+\ldots+u_d)}T(u)\Big[\int\limits_I e^{-\beta(t_1+\ldots+t_d)}\, dF(t)\Big] \\ &= \int\limits_{u+I} e^{-\beta(t_1+\ldots+t_d)}\, dF(t) = G(u+I). \end{aligned}$$

Thus G becomes a positive bounded additive process in $L_1(v_T\, d\mu)$ with respect to the strongly continuous contraction semigroup $\{e^{-\beta(u_1+\ldots+u_d)}T(u) : u \in \mathbb{P}_d\}$ on $L_1(v_T\, d\mu)$. Thus we can apply Theorem (1.11) of [1] to infer that the limit

$$g_0(\omega) := \text{q-}\lim_{\alpha\to 0} \alpha^{-d} G((0,\alpha]^d)(\omega)$$

exists for almost all $\omega \in \Omega$. This, together with the definition of G (cf. (14)), yields that the equality

$$g_0(\omega) = \text{q-}\lim_{\alpha\to 0} \alpha^{-d} F((0,\alpha]^d)(\omega)$$

holds for almost all $\omega \in \Omega$.

If F is a locally bounded non-positive additive process, then we apply the argument in (3.6) of [1] to the semigroup and process in $L_1(v_T\, d\mu)$ given by (10) and (13), and obtain that F can be written as

$$F = F_1 - F_2,$$

where F_1 and F_2 are positive locally bounded additive processes in $L_1(v_T\, d\mu)$ satisfying the inequality

$$\|F_i(I)\|_{L_1(v_T d\mu)} \le \eta \cdot K(F)\lambda_d(I)e^{\beta(b_1+\ldots+b_d)}$$

for all $I = (a_1, b_1] \times \ldots \times (a_d, b_d] \in \mathcal{I}_d$ and $i = 1, 2$. Hence the proof is complete.

Proof of Corollary. From $K(F) < \infty$ and Fatou's lemma it follows that $f_0 \in L_p(\mu)$. We may assume, as in the proof of the Theorem, that $C = P = \Omega$. Then the strong limit $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists in $L_p(\mu)$. Indeed, when $p = 1$, this follows from the proof of Theorem 3 of [12], since $C = \Omega$. When $1 < p < \infty$, it is a consequence of Theorem 7.1.11 of [9], since $L_p(\mu)$ is then a reflexive Banach space. (As is easily seen from its proof, Theorem 7.1.11 of [9] holds for the d-dimensional semigroup T.) Therefore the semigroup T can be extended continuously to $\mathbb{R}_d^+$ by an easy approximation argument, and consequently there exist constants $\delta > 0$ and $M > 0$ such that

$$\|T(t)\|_{L_p(\mu)} \le M e^{\delta(t_1+\ldots+t_d)} \tag{15}$$

for all $t = (t_1, \ldots, t_d) \in \mathbb{R}_d^+$.

(i) We first consider the case $p = 1$. From the proof of the theorem (cf. (9)), there exists a constant $\beta > 0$ and a strictly positive function v_T in $L_\infty(\mu)$ such that

$$\|T(t)f\|_{L_1(v_T d\mu)} \le e^{\beta(t_1+\ldots+t_d)} \|f\|_{L_1(v_T d\mu)} \tag{16}$$

for all $t = (t_1, \ldots, t_d) \in \mathbb{R}_d^+$ and $f \in L_1(\mu)$, where we may assume for later use that $\beta > \delta$. Then, since $L_1(\mu)$ is a dense subspace of $L_1(v_T\, d\mu)$, T can be regarded as a strongly continuous semigroup of positive linear operators on $L_1(v_T d\mu)$ such that

$$T(0) = \text{strong-}\lim_{t\to 0} T(t)$$

in $L_1(v_T d\mu)$, and we have

$$\|T(0)\|_{L_1(v_T d\mu)} \le 1. \tag{17}$$

To see that $T(0)$ is Markovian on $L_1(v_T\, d\mu)$, put

$$\widetilde{g} = \int\limits_{(0,1]^d} T(t) g\, dt,$$

where g is a strictly positive function in $L_1(\mu)$ ($\subset L_1(v_T\, d\mu)$). Since $\Omega = C$, it follows that $\widetilde{g}$ is a strictly positive function in $L_1(v_T\, d\mu)$. Furthermore, $T(0)\widetilde{g} = \widetilde{g}$. Thus $T(0)$ is a positive linear contraction on $L_1(v_T\, d\mu)$ having a strictly positive fixed point. Therefore $T(0)$ is conservative and Markovian (cf. pp. 116–117 in [9]). It follows that

$$\int\limits_{\Omega} (T(0)f) \cdot v_T\, d\mu = \int\limits_{\Omega} f \cdot v_T\, d\mu \tag{18}$$

for all $f \in L_1(v_T d\mu)$.

Now we apply the argument in (3.6) of [1] as before. Since $\beta > \delta$, we see, by using (15), that F can be written as

$$F = F_1 - F_2,$$

where F_1 and F_2 are positive additive processes in $L_1(\mu)$ such that

$$\|F_i(I)\|_{L_1(\mu)} \le K(F)\lambda_d(I)M \cdot e^{\beta(b_1+\ldots+b_d)} \tag{19}$$

for all $I = (a_1, b_1] \times \ldots \times (a_d, b_d] \in \mathcal{I}_d$ and $i = 1, 2$. Then define the process $G_i : \mathcal{I}_d \to L_1^+(\mu)$ by the relation

$$G_i(I) = \int_I e^{-\beta(t_1+\ldots+t_d)} dF_i(t) \tag{20}$$

for $i = 1, 2$. It follows from (19) that

$$\begin{aligned}\|G_i(I)\|_{L_1(v_T\,d\mu)} &= \int_\Omega \Big(\int_I e^{-\beta(t_1+\ldots+t_d)}\, dF_i(t)\Big) v_T\, d\mu \\ &\le K(F)\lambda_d(I)M\|v_T\|_{L_\infty(\mu)}.\end{aligned}$$

Furthermore, since F_i is additive with respect to T, we see, as in the proof of the Theorem, that G_i becomes a positive bounded additive process in $L_1(v_T\,d\mu)$ with respect to the contraction semigroup $\{e^{-\beta(u_1+\ldots+u_d)}T(u) : u \in \mathbb{P}_d\}$ on $L_1(v_T\,d\mu)$. Hence we can define

$$g_i(\omega) := \text{q-}\lim_{\alpha\to 0} \alpha^{-d} G_i((0,\alpha]^d)(\omega)$$

for almost all $\omega \in \Omega$. Then, by using (20), we find that the equality

$$g_i(\omega) = \text{q-}\lim_{\alpha\to 0} \alpha^{-d} F_i((0,\alpha]^d)(\omega)$$

holds for almost all $\omega \in \Omega$. Hence $f_0(\omega) = g_1(\omega) - g_2(\omega)$ on Ω and, by (19) and Fatou's lemma, we see that $g_i \in L_1^+(\mu)$ for each $i = 1, 2$.

We finally prove that

$$\lim_{u\to 0} \|T(u)g_i - g_i\|_{L_1(\mu)} = 0$$

for $i = 1, 2$. To do so, let $k \ge 1$ be an integer and define $\widetilde{f}_k$ in $L_1^+(\mu)$ by

$$\widetilde{f}_k(\omega) = \inf\{m^d F_1((0, m^{-1}]^d)(\omega) : m \ge k\}.$$

Since $0 \le \widetilde{f}_1(\omega) \le \widetilde{f}_2(\omega) \le \ldots \to g_1(\omega)$ for almost all $\omega \in \Omega$, and since $g_1 \in L_1^+(\mu)$, it follows that $\lim_{k\to\infty} \|g_1 - \widetilde{f}_k\|_{L_1(\mu)} = 0$. Then for $I \in \mathcal{I}_d$ we have

$$\begin{aligned}F_1(I) &= \text{strong-}\lim_{k\to\infty} \int_I T(u)[k^d F_1((0,k^{-1}]^d)]\,du \qquad \text{(cf. Lemma 3.2 of [1])} \\ &\ge \text{strong-}\lim_{k\to\infty} \int_I T(u)\widetilde{f}_k\,du = \int_I T(u)g_1\,du\end{aligned}$$

in $L_1(\mu)$. Therefore

$$g_1(\omega) = \text{q-}\lim_{k\to\infty} k^d F_1((0,k^{-1}]^d)(\omega)$$
$$\geq \text{q-}\lim_{k\to\infty} k^d \Big(\int_{(0,k^{-1}]^d} T(u)g_1\,du\Big)(\omega) = T(0)g_1(\omega)$$

for almost all $\omega \in \Omega$, where the last equality comes from the strong continuity at the origin of the semigroup T on $L_1(\mu)$. Hence (18) yields $g_1 = T(0)g_1$. Consequently,

$$\|T(u)g_1 - g_1\|_{L_1(\mu)} = \|T(u)g_1 - T(0)g_1\|_{L_1(\mu)} \to 0$$

as $u \in \mathbb{R}_d^+$ approaches $0 \in \mathbb{R}_d^+$. Similarly we get $\lim_{u\to 0}\|T(u)g_2 - g_2\|_{L_1(\mu)} = 0$, and hence the proof is complete for the case $p = 1$.

(ii) We next consider the case $1 < p < \infty$. Since $L_p(\mu)$ is a reflexive Banach space, a remark following Theorem 4.3 of Emilion [5] implies that there exists a function $f \in L_p(\mu)$ for which

$$F(I) = \int_I T(t)f\,dt$$

holds for all $I \in \mathcal{I}_d$. Then the strong continuity at the origin of the semigroup T yields

$$\lim_{\alpha\to 0}\|T(0)f - \alpha^{-d}F((0,\alpha]^d)\|_{L_p(\mu)} = 0,$$

whence $f_0 = T(0)f = T(0)^2 f = T(0)f_0$, and this completes the proof.

Remark. The proof of the Corollary shows that if L is reflexive, then any locally bounded additive process $F : \mathcal{I}_d \to L$ with respect to the locally bounded semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ has the form $F(I) = \int_I T(t)f\,dt$ for some function f in L. Then, since the strong limit $T(0) = \text{strong-}\lim_{u\to 0} T(u)$ exists, we get $f_0 = T(0)f$ by Lemma 2 of [13], and hence

$$\lim_{\alpha\to 0}\|\alpha^{-d}F((0,\alpha]^d) - f_0\|_L = 0. \tag{21}$$

This convergence result does not hold in general when L is not reflexive. A counterexample can be found in [1] when $L = L_1(\mu)$.

4. An example. In this section we give an example to show that the limit function f_0 in the Theorem need not belong to L, even in dimension 1.

To do this, let $\Omega = [0,1]$ with the Lebesgue measure λ_1. Let Φ be an N-function, i.e., a function on the interval $(-\infty,\infty)$ which has the form

$$\Phi(u) = \int_0^{|u|} p(t)\,dt,$$

where the function $p(t)$ is right-continuous for $t \geq 0$, strictly positive for $t > 0$, and non-decreasing with

$$p(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} p(t) = \infty.$$

For a Lebesgue measurable function f on Ω, we define

$$M_\Phi(f) = \int_\Omega \Phi(|f(\omega)|)\, d\lambda_1(\omega).$$

Then define the Orlicz function space

$$L_\Phi = \{f : M_\Phi(f/a) < \infty \text{ for some } a > 0\},$$

and the Luxemburg norm

$$\|f\|_\Phi = \inf\{a > 0 : M_\Phi(f/a) \leq 1\}$$

for $f \in L_\Phi$. It follows (see e.g. Chapter II of [8]) that $(L_\Phi, \|\cdot\|_\Phi)$ becomes a Banach lattice under pointwise operations and that the set

$$H_\Phi = \{f : M_\Phi(f/a) < \infty \text{ for all } a > 0\}$$

is a separable closed sublattice with Properties (II) and (III) for $L = H_\Phi$ and $\|\cdot\|_L = \|\cdot\|_\Phi$. If there exist constants $k > 0$ and $u_0 \geq 0$ such that

$$\Phi(2u) \leq k\Phi(u) \quad \text{ for } u \geq u_0,$$

then Φ is said to *satisfy the Δ_2-condition*. It is known that the Δ_2-condition is equivalent to $L_\Phi = H_\Phi$ (cf. pp. 80–88 in [8]).

For a real number $t \geq 0$, define a measure preserving transformation $\phi_t : \Omega \to \Omega$ by

$$\phi_t(\omega) = t + \omega \pmod 1.$$

Clearly, $\{\phi_t : t \geq 1\}$ becomes a measurable semiflow, and hence it induces in an obvious manner a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of positive linear isometries on H_Φ by using the separability of H_Φ and the measurability of $\{T(t) : t \geq 0\}$. We also note that each $T(t)$ is an invertible isometry on L_Φ. To give the desired example we must assume $H_\Phi \neq L_\Phi$. Then there exists a non-negative function f in $L_\Phi \setminus H_\Phi$. Since $L_\Phi \subset L_1 = \{f : \int_\Omega |f(\omega)|\, d\lambda_1(\omega) < \infty\}$, we can define a function $F(I)$ in L_1^+ for an interval $I = (a, b] \in \mathcal{I}_1$ by the relation

$$F(I)(\omega) = \int_a^b f(\phi_t(\omega))\, dt.$$

The continuity in $L_1(\mathbb{R})$ of the translations yields that $F(I)(\omega)$ is a continuous and hence bounded function on $\Omega = [0, 1]$. Thus it follows that $F(I) \in H_\Phi^+$, and hence the function $I \mapsto F(I)$ defines a one-dimensional positive bounded additive process in H_Φ with respect to the semigroup

$\{T(t) : t \geq 0\}$ such that

$$\|F(I)\|_{\Phi} \leq \int_a^b \|T(t)f\|_{\Phi}\, dt = (b-a)\|f\|_{\Phi}$$

for all $I = (a, b] \in \mathcal{I}_1$. Thus by the Theorem the limit

$$f_0(\omega) := \text{q-}\lim_{\alpha \to 0} \alpha^{-1} F((0, \alpha])(\omega)$$

exists for almost all $\omega \in \Omega$. But we must have $f_0(\omega) = f(\omega)$ for almost all $\omega \in \Omega$ by Wiener's classical local ergodic theorem (see e.g. Theorem 1.2.4 of [9]), whence $f_0 \notin H_{\Phi}$. This completes our argument.

What makes the example work is that L_{Φ} is not separable when it is different from H_{Φ} (cf. p. 85 in [8]), and we do not obtain the strong continuity of $\{T(t)\}$ in L_{Φ}. Indeed, if we had the continuity, then in the example we would have

$$\frac{1}{\alpha} F((0, \alpha]) = \frac{1}{\alpha} \int_0^{\alpha} T(t) f\, dt \to f$$

in the norm $\|\cdot\|_{\Phi}$, and then $f \in H_{\Phi}$ since H_{Φ} is a closed subspace. This is a contradiction.

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