CONVERGENCE OF SEQUENCES OF ITERATES
OF RANDOM-VALUED VECTOR FUNCTIONS

BY

RAFAŁ KAPICA (Katowice)

Abstract. Given a probability space \((\Omega, \mathcal{A}, P)\) and a closed subset \(X\) of a Banach lattice, we consider functions \(f : X \times \Omega \to X\) and their iterates \(f^n : X \times \Omega^\mathbb{N} \to X\) defined by \(f^1(x, \omega) = f(x, \omega_1), f^{n+1}(x, \omega) = f(f^n(x, \omega), \omega_{n+1})\), and obtain theorems on the convergence (a.s. and in \(L^1\)) of the sequence \((f^n(x, \cdot))\).

It is well known that iteration processes play an important role in mathematics and they are especially important in solving equations. However, it may happen that instead of the exact value of a function at a point we know only some parameters of this value. In [1] iterates of such functions were defined and simple results on the behaviour of the iterates were obtained for scalar-valued functions. It is the aim of the present paper to consider such functions with values in Banach lattices. The basic theorem on the convergence of iterates is obtained in [1] (see also [10; Chapter 12]) by using a submartingale convergence theorem. It is well known (see e.g. [5]) that for martingales with values in a Banach space the convergence theorem holds only if the space has the Radon–Nikodym property. Hence beside a direct use of submartingale convergence theorems we also apply some other martingale methods to get the convergence of the sequence of iterates for an arbitrary \(AL\)-space. Basic notions and facts connected with lattices and used in this paper may be found in [4] and [14].

Fix a probability space \((\Omega, \mathcal{A}, P)\), a separable Banach lattice \(E\) and its closed subset \(X\). Let \(\mathcal{B}\) denote the \(\sigma\)-algebra of all Borel subsets of \(X\). We say that \(f : X \times \Omega \to X\) is a random-valued vector function if it is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B} \otimes \mathcal{A}\). The iterates of \(f\) are defined by

\[
\begin{align*}
f^1(x, \omega_1, \omega_2, \ldots) &= f(x, \omega_1), \\
f^{n+1}(x, \omega_1, \omega_2, \ldots) &= f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1}),
\end{align*}
\]

for \(x \in X\) and \((\omega_1, \omega_2, \ldots) \in \Omega^\infty := \Omega^\mathbb{N}\). Note that \(f^n : X \times \Omega^\infty \to X\) is

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a random-valued function on the product probability space \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\). More exactly, the \(n\)th iterate \(f^n\) is \(\mathcal{B} \otimes \mathcal{A}_n\)-measurable, where \(\mathcal{A}_n\) denotes the \(\sigma\)-algebra of all sets of the form
\[
\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \omega_2, \ldots, \omega_n) \in A\}
\]
with \(A\) in the product \(\sigma\)-algebra \(\mathcal{A}^n\).

In what follows, \(f : X \times \Omega \to X\) is a fixed random-valued function such that
\[
E\|f^n(x, \cdot)\| < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.
\]
We also assume that the mean \(m : X \to E\) defined by
\[
m(x) = Ef(x, \cdot)
\]
is continuous. Moreover we assume that \(x_0 \in X\) is fixed and the sequence \((f^n(x_0, \cdot))\) is \(L^1\)-bounded. Concerning this assumption consult the Remark, Proposition 1, and Example below. It is easy to check that then
\[
E(f^{n+1}(x, \cdot)|\mathcal{A}_n) = m \circ f^n(x, \cdot)
\]
for \(x \in X\) and \(n \in \mathbb{N}\).

Our first theorem shows that the limit of \((f^n(x_0, \cdot))\) is a fixed point of \(m\).

**Theorem 1.** Assume that \(E\) does not contain isomorphic copies of \(c_0\) and either
\[
m(x) \geq x \quad \text{for } x \in X\]
or
\[
m(x) \leq x \quad \text{for } x \in X.
\]
If the sequence \((f^n(x_0, \cdot))\) converges in measure to an integrable \(\xi : \Omega^\infty \to E\), then \(m \circ \xi = \xi\).

**Proof.** Applying Fatou’s lemma to a subsequence of \((\|m(f^n(x_0, \omega))\|)\) we get integrability of \(m \circ \xi\). Assume (3) and put \(g = m \circ \xi - \xi\), \(g_n = m \circ f^n(x_0, \cdot) - f^n(x_0, \cdot)\) and (pointwise) \(h_n = \inf\{g_n, g\}\) for \(n \in \mathbb{N}\). Then the sequence \((h_n)\) converges to \(g\) in measure, \(h_n \leq g_n\) and \(h_n \leq g\) for \(n \in \mathbb{N}\). Moreover, the sequence \((Ef^n(x_0, \cdot))\) is bounded and (in view of (2) and (3)) increasing, whence, according to the theorem of Tzafriri ([18], see also [12; Theorem 1.c.4]), convergent. Consequently,
\[
0 \leq Eg = \lim_{n \to \infty} Eh_n \leq \lim_{n \to \infty} Eg_n = \lim_{n \to \infty} E(f^{n+1}(x_0, \cdot) - f^n(x_0, \cdot)) = 0.
\]

In the next theorem, which is our main result, we assume additionally that the Banach lattice considered is an \(AL\)-space, i.e. \(\|x + y\| = \|x\| + \|y\|\) for all \(x, y \geq 0\) in \(E\) (cf. [14], [16]).

**Theorem 2.** Let \(E\) be an \(AL\)-space. Assume that either (3) or (4) holds. If \(m\) is a contraction, then the sequence \((f^n(x_0, \cdot))\) converges, both a.s. and in \(L^1\), to the unique fixed point of \(m\).
Proof. Assume (3) and put \( X_n = f^n(x_0, \cdot) \) for \( n \in \mathbb{N} \). Since \((X_n, \mathcal{A}_n)\) is an \( L^1 \)-bounded submartingale with values in an \( AL \)-space, we have
\[
\sum_{n=1}^{N} E\|E(X_{n+1} \mid \mathcal{A}_n) - X_n\| = \left\| \sum_{n=1}^{N} E(E(X_{n+1} \mid \mathcal{A}_n) - X_n) \right\| = \|E(X_{N+1} - X_1)\| \leq 2 \sup_{n \in \mathbb{N}} E\|X_n\|
\]
for every \( N \in \mathbb{N} \). Hence
\[
\sum_{n=1}^{\infty} E\|E(X_{n+1} \mid \mathcal{A}_n) - X_n\| \leq 2 \sup_{n \in \mathbb{N}} E\|X_n\| < \infty,
\]
which jointly with (2) shows that
\[
\lim_{n \to \infty} E\|m \circ X_n - X_n\| = 0.
\]
On the other hand, if \( L \) denotes the Lipschitz constant of \( m \), then
\[
E\|X_p - X_q\| \leq \frac{1}{1 - L} (E\|m \circ X_p - X_p\| + E\|m \circ X_q - X_q\|)
\]
for all positive integers \( p, q \). From this and (6) we infer that \((X_n)\) converges in \( L^1 \) to a \( \xi : \Omega^\infty \to E \). According to Theorem 1 (see also [14; Example 7, p. 92]) we have \( m \circ \xi = \xi \). In particular, \( m \) has a fixed point, and being a contraction, it has at most one fixed point. Consequently, \((X_n)\) converges in \( L^1 \) to the unique fixed point of \( m \). Hence, applying (5) and [11; Theorem 1.3] (cf. also [2]), we obtain the a.s. convergence of \((X_n)\) as well. \( \blacksquare \)

The following shows a possible realization of the assumptions of Theorems 1 and 2 in the simplest non-deterministic (vector) case, viz. \( \Omega = \{\omega_1, \omega_2\} \).

Example. Let \( p_1, p_2 \) be positive reals with \( p_1 + p_2 = 1 \) and \( h_1, h_2 : [0, \infty) \to [0, \infty) \) be continuous functions such that
\[
p_1 h_1(t) + p_2 h_2(t) \leq t \quad \text{for every } t \geq 0.
\]
Given a finite separable measure \( \mu \) put \( E = L^1(\mu) \), consider the subset \( X \) of \( E \) of all positive elements of \( E \) and define \( f : X \times \{\omega_1, \omega_2\} \to X \) by
\[
f(x, \omega_i) = h_i \circ x.
\]
Then
\[
m(x) = p_1 h_1 \circ x + p_2 h_2 \circ x \leq x \quad \text{and} \quad E\|f^n(x, \cdot)\| \leq \|x\|
\]
for \( x \in X \) and \( n \in \mathbb{N} \). Moreover, \( m \) is continuous. Hence all the assumptions of Theorem 1 are satisfied. If additionally \( p_1 h_1 + p_2 h_2 \) is a contraction, then so is \( m \) (with zero as its only fixed point) and all the assumptions of Theorem 2 hold.
Of course, the convergence in $L^1$ implies the uniform integrability of the sequence. Concerning the uniform integrability of $(f^n(x_0, \cdot))$ note the following simple fact.

**Proposition 1.** If there exists an integrable $\Phi : \Omega \to [0, \infty)$ such that
$$
\|f(x, \omega)\| \leq \Phi(\omega) \quad \text{for } x \in X \text{ and } \omega \in \Omega,
$$
then the sequence $(f^n(x, \cdot))$ is $L^1$-bounded and uniformly integrable for every $x \in X$.

**Proof.** Clearly $\|f^n(x, \omega)\| \leq \Phi(\omega_n)$ for $x \in X$ and $\omega \in \Omega^\infty$. In particular $(f^n(x, \cdot))$ is $L^1$-bounded for $x \in X$. Moreover, if $x \in X$ and $n \in \mathbb{N}$ are fixed, then for every $A \in \mathcal{A}^\infty$ with $P^\infty(A) < N^{-1} \int_{\{\Phi > N\}} \Phi \, dP$ we have

$$
\int_A \|f^n(x, \omega)\| \, dP^\infty(\omega) \leq \int_A \Phi(\omega_n) \, dP^\infty(\omega)
\leq \int_{\{\omega \in \Omega^\infty : \Phi(\omega_n) > N\}} \Phi(\omega_n) \, dP^\infty(\omega) + N P^\infty(A)
\leq 2 \int_{\{\Phi > N\}} \Phi \, dP.
$$

In the case where the function $f$ considered has the form

$$
f(x, \omega) = x \Phi(\omega) \quad \text{for } x \in X \text{ and } \omega \in \Omega,
$$
we have the following observation.

**Proposition 2.** If $f$ has the form (7) with $\Phi : \Omega \to \mathbb{R}$ integrable, $(f^n(x_0, \cdot))$ is uniformly integrable and $x_0 \neq 0$, then either $E|\Phi| < 1$ or $|\Phi| = 1$ a.s.

**Proof.** Clearly

$$
f^n(x_0, \omega) = x_0 \prod_{k=1}^n \Phi(\omega_k)
$$
on $\Omega^\infty$, whence

$$E\|f^n(x_0, \cdot)\| = \|x_0\|(E|\Phi|)^n
$$
for every $n \in \mathbb{N}$. Consequently, $E|\Phi| \leq 1$. Assume $E|\Phi| = 1$ and define a probability measure $\mu$ on $\mathcal{A}$ by

$$
\mu(A) = \int_A |\Phi| \, dP
$$
and a sequence $(\mu_n)$ of probability measures on $\mathcal{A}^\infty$ by

$$
\mu_n(A) = \int_A \prod_{k=1}^n \Phi(\omega_k) \, dP^\infty(\omega).
$$
If $N \in \mathbb{N}$, $A \in \mathcal{A}_N$ and $n \geq N$, then $\mu_n(A) = \mu^\infty(A)$. Hence the sequence $(\mu_n)$ is pointwise convergent on $\bigcup_{n=1}^\infty \mathcal{A}_n$ to $\mu^\infty$. Applying the uniform integrability of $(f^n(x_0, \cdot))$ we get

$$\lim_{P^\infty(A) \to 0} \sup_{n \in \mathbb{N}} \mu_n(A) = 0.$$ 

This allows us to check that the union of every increasing sequence of sets of the family

$$(8) \quad \{A \in \mathcal{A}^\infty : \lim_{n \to \infty} \mu_n(A) = \mu^\infty(A)\}$$

belongs to this family. According to the Dynkin lemma ([8], see also [3; Theorem 1.3.2]), the family (8) coincides with $\mathcal{A}^\infty$. In particular, $\mu^\infty$ is absolutely continuous with respect to $P^\infty$. Hence, by the theorem of Kakutani [13; Proposition III.2.6], $E \sqrt{\mathcal{F}} \geq 1$. But $E \sqrt{\mathcal{F}} \leq \sqrt{E[\mathcal{F}]} \leq 1$, and so $E \sqrt{\mathcal{F}} = 1 = \mathcal{F}$. Consequently, $1 = \mathcal{F}$ a.s. □

Now we proceed to the case where $E$ has the Radon–Nikodym property. Since such a lattice does not contain isomorphic copies of $c_0$ (see [6]), our Theorem 1 and the theorem of Heinich [9] (cf. also [7] and [15]) imply what follows.

**Theorem 3.** Assume that $E$ has the Radon–Nikodym property. If either $f$ is lattice bounded from below and (3) holds, or $f$ is lattice bounded from above and (4) holds, then the sequence $(f^n(x_0, \cdot))$ converges a.s. to an integrable $\xi : \Omega^\infty \to E$ and $m \circ \xi = \xi$.

Note that [16; Proposition 3 and Theorem 1] and [14; Example 7, p. 92] imply the following.

**Remark.** Assume that $E$ is an $AL$-space and $f$ satisfies (1). If either $f$ is lattice bounded from above and (3) holds, or $f$ is lattice bounded from below and (4) holds, then the sequence $(f^n(x_0, \cdot))$ is $L_1$-bounded for any $x_0 \in X$.

We finish with some special cases of $E$.

**Theorem 4.** Assume that $E = l_1$ or $E$ is finite-dimensional. If (3) or (4) holds, then the sequence $(f^n(x_0, \cdot))$ converges a.s. to an integrable $\xi : \Omega^\infty \to E$ and $m \circ \xi = \xi$.

**Proof.** Assume (2) and let

$$f^n(x_0, \cdot) = M_n + A_n, \quad n \in \mathbb{N},$$

be the Doob decomposition [17]. Since $(f^n(x_0, \cdot))$ is $L_1$-bounded, it is easy to check that $\sup_{n \in \mathbb{N}} E\|M_n^\cdot\| < \infty$. Applying the theorem of J. Szulga and W. A. Woyczyński [17; Theorem 4.1] we obtain the desired limit. □
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REFERENCES


Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
E-mail: rkapica@ux2.math.us.edu.pl

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