COLLOQUIUM MATHEMATICUM

VOL. 97 2003 NO. 1

ON SOME CLASS OF PSEUDOSYMMETRIC WARPED PRODUCTS

BY

RYSZARD DESZCZ and DOROTA KOWALCZYK (Wrocław)

Dedicated to the memory of Dr. Jan Anweiler

Abstract. We present curvature properties of pseudosymmetry type of some warped products of semi-Riemannian spaces of constant curvature.

1. Introduction. The class of warped product manifolds, for short warped products, is an extension of the class of products of semi-Riemannian manifolds. Warped products play an important role in Riemannian geometry (see e.g. [28], [29]) as well as in general relativity (see e.g. [2], [3], [29]). Many well-known spacetimes of general relativity, i.e. solutions of the Einstein equations, are warped products, e.g. the Schwarzschild spacetimes, the Kottler spacetime, the Reissner–Nordström spacetime as well as Robertson–Walker spacetimes. We recall that a warped product $\overline{M} \times_F \widetilde{M}$ of a 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = -1$, and a 3-dimensional Riemannian space $(\widetilde{M}, \widetilde{g})$ of constant curvature, with a warping function F, is said to be a *Robertson–Walker spacetime* (see e.g. [2], [3], [27], [29]). More generally, one also considers warped products $\overline{M} \times_F \widetilde{M}$ of $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1$, $\overline{g}_{11} = -1$, with a warping function F and an $(n-1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $n \geq 4$. Such warped products are called *generalized Robertson–Walker spacetimes* ([1], [21], [31]).

It is known that every Robertson–Walker spacetime is conformally flat. These manifolds also satisfy another curvature condition: the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point (see e.g. [9, Section 12.2]). For precise definitions of the symbols used, we refer to Sections 2 and 3 of this paper. In general, semi-Riemannian manifolds (M, g), $n \geq 3$, satisfying this condition are called *pseudosymmetric* ([9, Section 3.1]) A manifold (M, g) is

2000 *Mathematics Subject Classification*: 53B20, 53B30, 53B50, 53C25, 53C35, 53C80.

Key words and phrases: pseudosymmetric manifold, pseudosymmetry type manifold, warped product, spacetime, Robertson–Walker spacetime, generalized Robertson–Walker spacetime, hypersurface.

Research supported by the Agricultural University of Wrocław (Poland) grant 239/GW/2001.

pseudosymmetric if and only if on $U_R = \left\{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\right\}$ we have

$$R \cdot R = L_R Q(g, R), \tag{1}$$

where L_R is some function on U_R. It is clear that every semisymmetric manifold ($R \cdot R = 0$) is pseudosymmetric. The converse is not true (see e.g. [10], [11]). It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Recently, results on semisymmetric semi-Riemannian manifolds were obtained in [22] and [25], among others.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *Ricci-semisymmetric* if $R \cdot S = 0$ on M. The class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. Every semisymmetric manifold is Ricci-semisymmetric. The converse is not true. But under some additional assumptions the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent. For a review of recent results related to this subject see [12] and [13] and the references therein.

(1) arose from the study of totally umbilical submanifolds of semisymmetric manifolds ([9, Section 13]) as well as from considering geodesic mappings of semisymmetric manifolds (see e.g. [9, Section 10]). We mention that the Schwarzschild spacetime, the Kottler spacetime as well as the Reissner–Nordström spacetime are pseudosymmetric ([7], [20]).

In [5, Theorem 4.1] it was shown that on every 4-dimensional generalized Robertson–Walker spacetime $\overline{M} \times_F \widetilde{N}$, the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. This is equivalent on $U_C \subset \overline{M} \times \widetilde{N}$ to

$$R \cdot R - Q(S, R) = L Q(g, C), \tag{2}$$

where L is some function on U_C. The last relation is a condition of pseudosymmetry type. We refer to [4] for a review of results on semi-Riemannian manifolds satisfying such conditions. Generalized Robertson–Walker spacetimes satisfying some curvature conditions of pseudosymmetry type were considered in [6] and [17]. We also mention that the Vaidya spacetime satisfies (2) ([26], see also Example 3.2(ii)).

Investigations of generalized Robertson–Walker spacetimes as well as of other classes of spacetimes (see e.g. [23], [24], [30]) lead to the following extension of the notion of a Robertson–Walker spacetime. The warped product $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$, $\dim \overline{M} \geq 1$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} \geq 1$, $n = \dim \overline{M} + \dim \widetilde{N} \geq 4$, is said to be a *spacetime of Robertson–Walker type* if it has signature $(1, n-1)$ and at least one of the manifolds $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ is of dimension 1 or 2 or a space of constant curvature. In Section 3 we present examples of such spacetimes. Clearly, the metric of a Robertson–Walker spacetime has signature $(1, 3)$.

In Section 4 we investigate pseudosymmetric warped products of semi-Riemannian spaces of constant curvature. In particular, we obtain a curvature characterization of some class of Robertson–Walker type spacetimes. Finally, we present an example of a warped product of spaces of constant curvature which can be locally realized as a hypersurface in a space of constant curvature.

2. Preliminaries. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞. We denote by ∇, S and κ the Levi-Civita connection, Ricci tensor and scalar curvature of (M,g), respectively. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y,\\
\mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,\\
\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}\, X \wedge_g Y\right),
\end{aligned}
$$

where A is a $(0,2)$-tensor on M, $X,Y,Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M, and the *Ricci operator* $\mathcal{S}$ is defined by

$$g(X, \mathcal{S}Y) = S(X,Y).$$

The *Riemann curvature tensor* R and the *Weyl tensor* C are defined by

$$
\begin{aligned}
R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\
C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4).
\end{aligned}
$$

Further, let $\mathcal{T}(X,Y)$ be a skew symmetric endomorphism of $\Xi(M)$. For it we define a $(0,4)$-tensor T by $T(X_1,X_2,X_3,X_4) = g(\mathcal{T}(X_1,X_2)X_3,X_4)$.

A $(0,4)$-tensor T is said to be a *generalized curvature tensor* if

$$
\begin{aligned}
&T(X_1,X_2,X_3,X_4) = T(X_3,X_4,X_1,X_2),\\
&T(X_1,X_2,X_3,X_4) + T(X_2,X_3,X_1,X_4) + T(X_3,X_1,X_2,X_4) = 0.
\end{aligned}
$$

For a generalized curvature tensor T, a symmetric $(0,2)$-tensor field A and a $(0,k)$-tensor field T_1, $k \geq 1$, we define the $(0,k+2)$-tensor fields $T \cdot T_1$, $Q(A,T)$ and $A \cdot T_1$ by

$$
\begin{aligned}
&(T \cdot T_1)(X_1,\ldots,X_k;X,Y) = (\mathcal{T}(X,Y)\cdot T_1)(X_1,\ldots X_k)\\
&\qquad = -T_1(\mathcal{T}(X,Y)X_1,X_2,\ldots,X_k) - \ldots - T_1(X_1,\ldots,X_{k-1},\mathcal{T}(X,Y)X_k),\\
&Q(A,T_1)(X_1,\ldots,X_k;X,Y) = ((X \wedge_A Y)\cdot T_1)(X_1,\ldots,X_k)\\
&\quad = -T_1((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \ldots - T_1(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k),\\
&(A \cdot T_1)(X_1,\ldots,X_k) = -T_1(\mathcal{A}X_1,X_2,\ldots,X_k) - \ldots - T_1(X_1,X_2,\ldots,\mathcal{A}X_k),
\end{aligned}
$$

where the endomorphism $\mathcal{A}$ is defined by $g(\mathcal{A}X,Y) = A(X,Y)$. Setting in the above formulas $\mathcal{T}(X,Y) = \mathcal{R}(X,Y)$ or $\mathcal{C}(X,Y)$, $T_1 = R, C$ or S, and

$A = g$ or S we obtain the following tensors, among others: $R \cdot R$, $R \cdot S$, $Q(g,R)$, $Q(g,C)$, $Q(g,S)$ and $Q(S,R)$.

Further, for $(0,2)$-tensors A and B their *Kulkarni–Nomizu product* $A \wedge B$ is given by

$$(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) \\ - A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).$$

In particular, for a $(0,2)$-tensor A we define the $(0,4)$-tensor $\overline{A}$ by $\overline{A} = \frac{1}{2} A \wedge A$. The $(0,4)$-tensor G is defined by $G = \overline{g}$. Let T_1 and T_2 be $(0,k)$-tensors on M. According to [8] the tensors T_1 and T_2 are *pseudosymmetrically related* to a generalized curvature tensor T and a symmetric $(0,2)$-tensor A if at every point of M the tensors $T \cdot T_1$ and $Q(A, T_2)$ are linearly dependent. In particular, when $T_1 = T_2$, we say that the tensor T_1 is *pseudosymmetric with respect to the tensors T and A*.

Let T_{hijk}, V_{hijk}, and A_{ij} be the local components of generalized curvature tensors T and V and a symmetric $(0,2)$-tensor A on M, respectively, where $h, i, j, k, l, m \in \{1, \dots, n\}$. The local components $(T \cdot V)_{hijklm}$ and $Q(A,V)_{hijklm}$ of the tensors $T \cdot V$ and $Q(A,V)$ are

$$(T \cdot V)_{hijklm} = g^{pq}(T_{pijk}V_{qhlm} + T_{hpjk}V_{qilm} + T_{hipk}V_{qjlm} + T_{hijp}V_{qklm}),$$

$$Q(A,V)_{hijklm} = A_{hl}V_{mijk} + A_{il}V_{hmjk} + A_{jl}V_{himk} + A_{kl}V_{hijm} \\ - A_{hm}V_{lijk} - A_{im}V_{hljk} - A_{jm}V_{hilk} - A_{km}V_{hijl}.$$

Let T be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geq 4$. We denote by $\mathrm{Ric}(T)$, $\mathrm{Weyl}(T)$ and $\kappa(T)$ the Ricci tensor, Weyl tensor and scalar curvature of T, respectively. The subsets U_T, $U_{\mathrm{Ric}(T)}$ and $U_{\mathrm{Weyl}(T)}$ of M are defined in the same manner as the subsets U_R, U_S and U_C of M, respectively. Let us consider generalized curvature tensors T having on $U = U_{\mathrm{Ric}(T)} \cap U_{\mathrm{Weyl}(T)} \subset M$ a decomposition

$$T = \frac{L_1}{2} A \wedge A + L_2 g \wedge A + L_3 G, \tag{3}$$

where L_1, L_2 and L_3 are some functions on U and A is a $(0,2)$-symmetric tensor on U; such tensors were investigated in [26].

Proposition 2.1 ([22, Lemma 3.1]). *Let B be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold (M,g), $n \geq 3$, and let $\mathcal{U}_B$ be the set of all points of M at which B is not proportional to g. If on $\mathcal{U}_B$ we have $\frac{1}{2} B \wedge B = L_2 g \wedge B + L_3 G$ then $L_3 = -L_2^2$ and $\mathrm{rank}(B - L_2 g) = 1$ on $\mathcal{U}_B$.*

Proposition 2.2 ([26, Proposition 3.3]). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T having on $U = U_{\mathrm{Ric}(T)} \cap U_{\mathrm{Weyl}(T)} \subset M$ a decomposition of the form (3). Then $T \cdot T - Q(\mathrm{Ric}(T), T) = L\, Q(g, \mathrm{Weyl}(T))$ and $L = (n-2)(L_1^{-1} L_2^2 - L_3)$ on U.*

We also have

THEOREM 2.1 ([15, Theorem 4.2]). *If the curvature tensor R of a semi-Riemannian manifold (M,g), $n \geq 4$, has on $U = U_S \cap U_C \subset M$ a decomposition of the form* (3) *with $A = S$ then on U we have*

$$R \cdot R = L_R Q(g,R), \qquad L_R = (n-2)(L_1^{-1}L_2^2 - L_3) - L_1^{-1}L_2, \tag{4}$$

$$R \cdot R - Q(S,R) = (L_R + L_1^{-1}L_2)Q(g,C). \tag{5}$$

In the same manner we can prove

PROPOSITION 2.3. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T having on $U = U_{\mathrm{Ric}(T)} \cap U_{\mathrm{Weyl}(T)} \subset M$ a decomposition of the form* (3) *with $A = \mathrm{Ric}(T)$. Then $T \cdot T = L_T Q(g,T)$ and $L_T = (n-2)(L_1^{-1}L_2^2 - L_3) - L_1^{-1}L_2$ on U.*

We also have the following converse statement.

COROLLARY 2.1 ([14, Corollary 6.1]). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T and suppose*

$$T \cdot T = Q(\mathrm{Ric}(T),T) + LQ(g,\mathrm{Weyl}(T)) \quad \textit{and} \quad T \cdot T = L_T Q(g,T)$$

on $U = U_{\mathrm{Ric}(T)} \cap U_{\mathrm{Weyl}(T)} \subset M$. If at $x \in U$ the tensor $\mathrm{Ric}(T)$ has no decomposition into a metrical part and a part of rank at most one then at x we have

$$T = \frac{L_1}{2}\mathrm{Ric}(T) \wedge \mathrm{Ric}(T) + L_2 g \wedge \mathrm{Ric}(T) + L_3 G \tag{6}$$

for some $L_1, L_2, L_3 \in \mathbb{R}$.

3. Warped products. Let now $(\overline{M},\overline{g})$ and $(\widetilde{N},\widetilde{g})$, $\dim \overline{M} = p$, $\dim \widetilde{N} = n-p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$, respectively. Let $F : \overline{M} \to \mathbb{R}^+$ be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F \widetilde{N}$ of $(\overline{M},\overline{g})$ and $(\widetilde{N},\widetilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$ defined by

$$\overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1)\pi_2^* \widetilde{g},$$

where $\pi_1 : \overline{M} \times \widetilde{N} \to \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \to \widetilde{N}$ are the natural projections. Let $\{\overline{U} \times \widetilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric g in this chart are: $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F\widetilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $h, i, j, k \ldots \in \{1, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$). It is known that the local components Γ^h_{ij} of the Levi-Civita connection ∇ of $\overline{M} \times_F \widetilde{N}$ are:

$$\begin{aligned} &\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\,\overline{g}^{ab} F_b \widetilde{g}_{\alpha\beta}, \\ &\Gamma^\alpha_{a\beta} = \frac{1}{2F}\,F_a \delta^\alpha_\beta, \quad \Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \\ &F_a = \partial_a F, \quad \partial_a = \partial/\partial x^a. \end{aligned} \tag{7}$$

The local components R_{hijk} of the curvature tensor R and the local components S_{hk} of the Ricci tensor S of $\overline{M} \times_F \widetilde{N}$ which may not vanish identically are the following (see e.g. [11], [17]):

$$\begin{aligned} &R_{abcd} = \overline{R}_{abcd}, \quad R_{\alpha bc\beta} = -\frac{1}{2}\,T_{bc}\widetilde{g}_{\alpha\beta}, \\ &R_{\alpha\beta\gamma\delta} = F\left(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\,\widetilde{G}_{\alpha\beta\gamma\delta}\right), \end{aligned} \tag{8}$$

$$\begin{aligned} &S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F}\,T_{ab}, \\ &S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \left(\frac{\operatorname{tr} T}{2} + (n-p-1)\,\frac{\Delta_1 F}{4F}\right)\widetilde{g}_{\alpha\beta}, \end{aligned} \tag{9}$$

$$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F}\,F_a F_b, \quad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b, \tag{10}$$

where T denotes the $(0,2)$-tensor with local components T_{ab} and $\operatorname{tr} T = \operatorname{tr}_{\overline{g}} T = \overline{g}^{ab} T_{ab}$. The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ satisfies

$$\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + (n-p-1)\,\frac{\Delta_1 F}{4F}\right). \tag{11}$$

Let $\overline{M} \times_F \widetilde{N}$ be the warped product of semi-Riemannian spaces of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, with $T = \frac{1}{p}\operatorname{tr} T \overline{g}$ on $U = U_S \cap U_C \subset \overline{M} \times \widetilde{N}$. Examples of such warped products are given in [11] and [20]. Under the above assumptions, (8), (9) and (11) turn into

$$R_{abcd} = \varrho_1 G_{abcd}, \quad \varrho_1 = \frac{\overline{\kappa}}{(p-1)p}, \tag{12}$$

$$R_{\alpha bc\beta} = \varrho_2 G_{\alpha bc\beta}, \quad \varrho_2 = -\frac{\operatorname{tr} T}{2pF}, \tag{13}$$

$$R_{\alpha\beta\gamma\delta} = \varrho_3 G_{\alpha\beta\gamma\delta}, \quad \varrho_3 = \frac{1}{F}\left(\frac{\widetilde{\kappa}}{(n-p)(n-p-1)} - \frac{\Delta_1 F}{4F}\right), \tag{14}$$

$$S_{ab} = \mu_1 g_{ab}, \quad \mu_1 = \frac{1}{2pF}(2F\overline{\kappa} - (n-p)\operatorname{tr} T), \tag{15}$$

$$S_{\alpha\beta} = \mu_2 g_{\alpha\beta}, \quad \mu_2 = \frac{1}{F}\left(\frac{\widetilde{\kappa}}{n-p} - \frac{\operatorname{tr} T}{2} - (n-p-1)\,\frac{\Delta_1 F}{4F}\right), \tag{16}$$

$$\kappa = p\mu_1 + (n-p)\mu_2. \tag{17}$$

Evidently, if (15) and (16) hold at every point of $U_S \subset \overline{M} \times \widetilde{N}$ then $\mu_1 - \mu_2 \neq 0$ on U_S. Next, using (12)–(16), we get

$$
\begin{aligned}
C_{abcd} &= \left(\varrho_1 - \frac{2\mu_1}{n-2} + \frac{\kappa}{(n-2)(n-1)} \right) G_{abcd}, \\
C_{\alpha bc\beta} &= \left(\varrho_2 - \frac{\mu_1 + \mu_2}{n-2} + \frac{\kappa}{(n-2)(n-1)} \right) G_{\alpha bc\beta}, \\
C_{\alpha\beta\gamma\delta} &= \left(\varrho_3 - \frac{2\mu_2}{n-2} + \frac{\kappa}{(n-2)(n-1)} \right) G_{\alpha\beta\gamma\delta}.
\end{aligned} \tag{18}
$$

As a conclusion, the Weyl tensor C of $\overline{M} \times_F \widetilde{N}$ vanishes at a point if and only if

$$
\begin{aligned}
\varrho_1 &= \frac{1}{n-2} \left(2\mu_1 - \frac{\kappa}{n-1} \right), \\
\varrho_2 &= \frac{1}{n-2} \left(\mu_1 + \mu_2 - \frac{\kappa}{n-1} \right), \\
\varrho_3 &= \frac{1}{n-2} \left(2\mu_2 - \frac{\kappa}{n-1} \right).
\end{aligned} \tag{19}
$$

It follows that $\varrho_1 - 2\varrho_2 + \varrho_3 = 0$ at every point at which the tensor C vanishes. Thus if $\varrho_1 - 2\varrho_2 + \varrho_3 \neq 0$ at $x \in \overline{M} \times \widetilde{N}$ then $x \in U_C \subset \overline{M} \times \widetilde{N}$.

EXAMPLE 3.1. (i) (see [23, (3.2)]) Let $\overline{M} \subset \{(y,t) \in \mathbb{R}^2 : y > 0\}$ be an open connected nonempty subset of $\mathbb{R}^2$ with the metric tensor $\overline{g} = dy^2 - \sinh^2 y dt^2$. Define $F(y,t) = \sinh^2 y \cosh^2 t$. Further, let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} \geq 3$, be a Riemannian space of constant curvature. Then $\overline{M} \times_F \widetilde{N}$ is a spacetime of Robertson–Walker type. We have $T_{ab} = 2F\overline{g}_{ab}$. In view of Corollary 2.1 of [11], $\overline{M} \times_F \widetilde{N}$ is a pseudosymmetric manifold.

(ii) (see [24, (2)]) Let $\overline{M}$ be an open connected nonempty subset of $\mathbb{R}^2$ with the metric tensor $\overline{g} = \exp 2f(-dy^2 + dt^2)$, where $f = f(y,t)$. Define $F(y,t) = \exp 2h$, where $h = h(y,t)$, and suppose f and h are smooth functions on $\overline{M}$. Further, let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} \geq 3$, be a Riemannian space of constant curvature. Then $\overline{M} \times_F \widetilde{N}$ is a spacetime of Robertson–Walker type.

(iii) From formulas (2.1), (2.8) and (2.9) of [30] it follows that the spacetimes considered in [30] are of Robertson–Walker type.

EXAMPLE 3.2. (i) Let $\overline{M} \subset \{(u,r) \in \mathbb{R}^2 : r > 0\}$ be an open connected nonempty subset of $\mathbb{R}^2$ with the metric tensor

$$\overline{g} = -2h du^2 - 2du dr, \tag{20}$$

where $h = h(u,r)$ is a smooth function on $\overline{M}$. Consider the warped product $\overline{M} \times_F \widetilde{N}$ with the 2-dimensional standard unit sphere $(\widetilde{N}, \widetilde{g})$ and a warping function $F = F(u,r)$.

(ii) According to [27, Section 13.4], the warped product in (i) with $F(r) = r^2$ is said to be the *Kottler spacetime*, resp., the *Schwarzschild spacetime*, if $2h(r) = 1-2m/r+\frac{1}{3}\Lambda r^2$, resp., $2h(r) = 1-2m/r$, where $m =$ const > 0 and $\Lambda =$ const $\neq 0$. It is well known that the Kottler spacetime is a non-Ricci flat Einstein manifold. The Schwarzschild spacetime is a Ricci flat manifold. The warped product $\overline{M} \times_F \widetilde{N}$ is said to be the *Reissner–Nordström spacetime* if $2h(r) = 1 - 2m/r + e^2/r^2$, where $m =$ const > 0 and $e =$ const. It is known that the spacetimes defined above are nonsemisymmetric pseudosymmetric manifolds ([20]).

(iii) The warped product in (i) is called a *Vaidya spacetime* ([27, Section 13.4]) if $2h(u, r) = 1 - 2m(u)/r$. The Ricci tensor S of a Vaidya spacetime satisfies $\operatorname{rank}(S) \leq 1$. We can check that a Vaidya spacetime is a nonpseudosymmetric manifold satisfying (2) with $L = -m(u)/r^3$ ([26]).

4. Some Robertson–Walker type spacetimes. In this section we consider warped products $\overline{M} \times_F \widetilde{N}$ such that on $U_S \subset \overline{M} \times \widetilde{N}$ the curvature tensor R has the form

$$R = \frac{L_1}{2}\, S \wedge S + L_2 g \wedge S + L_3 G, \tag{21}$$

where L_1, L_2 and L_3 are some functions on U_S. We note that L_1 is nonzero at a point of U_S if and only if the Weyl tensor C of $\overline{M} \times_F \widetilde{N}$ is nonzero at this point.

THEOREM 4.1. *Let $\overline{M} \times_F \widetilde{N}$ be the warped product of semi-Riemannian spaces of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, with $T = \frac{1}{p} \operatorname{tr} T\overline{g}$ on U_S. Define*

$$\begin{aligned} L_1 &= \mu(\varrho_1 - 2\varrho_2 + \varrho_3),\\ L_2 &= \mu((\varrho_2 - \varrho_3)\mu_1 + (\varrho_2 - \varrho_1)\mu_2),\\ L_3 &= \mu(\varrho_1\mu_2^2 - 2\varrho_2\mu_1\mu_2 + \varrho_3\mu_1^2), \qquad \mu = (\mu_1 - \mu_2)^{-2}, \end{aligned} \tag{22}$$

where ϱ_1, ϱ_2, ϱ_3, μ_1 and μ_2 are defined by (12)–(16)*. Then* (21) *is satisfied on U_S. Such a decomposition is unique on $U_S \cap U_C$.*

Proof. First of all we note that

$$\begin{aligned} \varrho_1 &= \mu_1^2 L_1 + 2\mu_1 L_2 + L_3,\\ \varrho_2 &= \mu_1\mu_2 L_1 + (\mu_1 + \mu_2) L_2 + L_3,\\ \varrho_3 &= \mu_2^2 L_1 + 2\mu_2 L_2 + L_3. \end{aligned} \tag{23}$$

Now using (12)–(16) and (23) we can easily check that $R - \frac{L_1}{2} S \wedge S - L_2 g \wedge S - L_3 G = 0$ on U_S. Lemma 3.2 of [16] implies that the decomposition (21) is unique. But this completes the proof.

Let $\overline{M} \times_F \widetilde{N}$ be the warped product of semi-Riemannian spaces of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, such that

$$\frac{1}{2}T = -F L_R \overline{g} + \gamma w \otimes w \tag{24}$$

on $U_S \subset \overline{M} \times \widetilde{N}$, where γ and L_R are some functions on U_S and w is a covector field on U_S. Now from (8), (9) and (11) we obtain (12), (14), (16) and

$$R_{\alpha bc\beta} = \left(L_R g_{bc} - \frac{\gamma}{F} w_b w_c \right) g_{\alpha\beta}, \tag{25}$$

$$S_{ab} = \mu_1 g_{ab} - (n-p)\frac{\gamma}{F} w_a w_b, \qquad \mu_1 = \frac{\overline{\kappa}}{p} + (n-p)L_R. \tag{26}$$

Proposition 4.1. *Let $\overline{M} \times_F \widetilde{N}$ be the warped product of semi-Riemannian spaces of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, such that* (24) *holds on U_S, with γ and w nonzero at every point of U_S. Then* (21) *is satisfied on U_S if and only if on U_S we have*

$$\begin{aligned} &\mu_1 = \mu_2, \qquad L_2 = -\mu_1 L_1, \qquad \varrho_1 = L_3, \\ &\varrho_2 = \mu_1^2 L_1, \qquad \mu_1 L_R = -\mu_1^2 L_1 + L_3, \qquad \varrho_3 = \mu_1 L_R. \end{aligned} \tag{27}$$

Proof. Applying (12), (14), (16), (25) and (26) to (21) we find that (21) holds on U_S if and only if on U_S we have

$$\begin{aligned} (\varrho_1 - \mu_1^2 L_1 - \mu_1 L_2 - L_3) G_{abcd} &= -(n-p)\frac{\gamma}{F}(L_2 + \mu_1 L_1) \\ &\quad \times (g_{ad} w_b w_c + g_{bc} w_a w_d - g_{ac} w_b w_d - g_{bd} w_a w_c), \\ (\varrho_2 - \mu_1\mu_2 L_1 - \mu_1 L_2 - \mu_2 L_2 - L_3) G_{\alpha bc\beta} & \\ &= -(n-p)\frac{\gamma}{F}(\mu_2 L_1 + L_2) w_a w_b g_{\alpha\beta}, \\ (\varrho_3 - \mu_2^2 L_1 - 2\mu_2 L_2 - L_3) G_{\alpha\beta\delta\gamma} &= 0. \end{aligned} \tag{28}$$

From this we obtain our assertion easily.

As an immediate consequence of the above result and Lemma 3.1 of [16] we have the following

Theorem 4.2. *Let $\overline{M} \times_F \widetilde{N}$ be the warped product of semi-Riemannian spaces of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, such that* (24) *holds on U_S, with γ and w nonzero at every point of U_S. In addition, suppose that $\mu_1 = \mu_2 \neq 0$ and $\mu_1 L_R = \varrho_3$ on U_S. Define*

$$L_1 = \mu_1^{-2}(\varrho_1 - \mu_1 L_R), \qquad L_2 = -\mu_1^{-1}(\varrho_1 - \mu_1 L_R), \qquad L_3 = \varrho_1. \tag{29}$$

Then (21) *is satisfied on U_S. Such a decomposition is unique on $U_S \cap U_C$.*

From Proposition 2.3 we obtain

Corollary 4.1. *If $\overline{M} \times_F \widetilde{N}$ is the warped product of semi-Riemannian manifolds of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, satisfying* (21) *on $U_S \cap U_C$, then on $U_S \cap U_C$ we have* (4), (5) *and*

$$L_3 - L_1^{-1} L_2^2 = \frac{\varrho_1 \varrho_3 - \varrho_2^2}{\varrho_1 - 2\varrho_2 + \varrho_3}. \tag{30}$$

Applying (8) and (9) in (21) we find

$$\frac{1}{2}\left(\frac{\overline{\kappa}}{p}\,\overline{g} - \frac{n-p}{2F}\,T\right) \wedge \left(\frac{\overline{\kappa}}{p}\,\overline{g} - \frac{n-p}{2F}\,T\right) = -\frac{L_2}{L_1}\overline{g} \wedge \left(\frac{\overline{\kappa}}{p}\,\overline{g} - \frac{n-p}{2F}\,T\right) + \frac{1}{L_1}\left(\frac{\overline{\kappa}}{(p-1)p} - L_3\right)\overline{G}.$$

In view of the last relation, we now consider on $U_S \cap U_C$ the following three cases:

$$\text{(a)} \quad T = \frac{2F\overline{\kappa}}{p(n-p)}\,\overline{g}, \qquad \text{(b)} \quad L_3 = \frac{\overline{\kappa}}{(p-1)p}, \tag{31}$$

$$\text{(a)} \quad T = \frac{2F(\overline{\kappa} - p\lambda)}{p(n-p)}\,\overline{g}, \quad \lambda \in \mathbb{R} - \{0\}, \qquad \text{(b)} \quad L_3 = \frac{\overline{\kappa}}{(p-1)p} - \lambda^2 L_1 - 2\lambda L_2, \tag{32}$$

$$\text{(a)} \quad \operatorname{rank}\left(\frac{\overline{\kappa}}{p}\,\overline{g} - \frac{n-p}{2F}\,T\right) = 1, \qquad \text{(b)} \quad L_3 = \frac{\overline{\kappa}}{(p-1)p} + L_1^{-1} L_2^2. \tag{33}$$

We note that (33) is an immediate consequence of Proposition 2.1.

Proposition 4.2. *Let $\overline{M} \times_F \widetilde{N}$ be a warped product of semi-Riemannian manifolds of constant curvature $(\overline{M}, \overline{g})$, $p \geq 2$, and $(\widetilde{N}, \widetilde{g})$, $n - p \geq 2$, satisfying* (21) *on U_S.*

(i) *If* (31) *holds at $x \in U_S$ then at x we have $\kappa \neq 0$ and*

$$R \cdot R = -\frac{\overline{\kappa}}{p(n-p)}\,Q(g, R), \tag{34}$$

$$L_1 = \frac{n-p}{n-p-1}\,\frac{1}{\kappa}\left(1 + \frac{(n-2)(n-1)}{(p-1)p}\,\frac{\overline{\kappa}}{\kappa}\right), \tag{35}$$

$$L_2 = -\frac{n-1}{p(p-1)}\,\frac{\overline{\kappa}}{\kappa}. \tag{36}$$

(ii) *If* (32) *holds at* $x \in U_S$ *then at* x *we have* $\kappa - n\lambda \neq 0$ *and*

$$R \cdot R = \frac{p\lambda - \overline{\kappa}}{p(n-p)}\, Q(g, R), \tag{37}$$

$$L_1 = \frac{n-p}{n-p-1}\,\frac{1}{\kappa - n\lambda}\biggl(1 - \frac{(n-2)\lambda}{\kappa - n\lambda} + \frac{(n-2)(n-1)}{(p-1)p}\,\frac{\overline{\kappa}}{\kappa - n\lambda}\biggr), \tag{38}$$

$$L_2 = \frac{n-p}{n-p-1}\,\frac{1}{(\kappa - n\lambda)^2}\biggl(\lambda(2p\lambda - \kappa) + \frac{(n-1)\overline{\kappa}((n-2p)\lambda - (n-p-1)\kappa)}{p(p-1)(n-p)}\biggr). \tag{39}$$

(iii) *If* (33) *holds at* $x \in U$ *then at* x *we have*

$$R \cdot R = 0, \quad R = \frac{L_1}{2}\, S \wedge S, \quad \operatorname{rank}(T) = 1, \quad L_2 = L_3 = \overline{\kappa} = 0.$$

Proof. (i) From (31)(a) we have

$$\operatorname{tr} T = \frac{2F\overline{\kappa}}{n-p}. \tag{40}$$

Next, applying (31)(a) and (40) to (8), (9) and (11) we find

$$R_{a\alpha\beta b} = -\frac{\overline{\kappa}}{p(n-p)}\, g_{ab} g_{\alpha\beta}, \tag{41}$$

$$\text{(a) } S_{ad} = 0, \qquad \text{(b) } S_{\alpha\beta} = \frac{\kappa}{n-p}\, g_{\alpha\beta}, \tag{42}$$

$$\kappa = \frac{1}{F}\biggl(\widetilde{\kappa} - F\overline{\kappa} - (n-p-1)(n-p)\,\frac{\Delta_1 F}{4F}\biggr). \tag{43}$$

We note that κ is nonzero at x. Indeed, $\kappa = 0$ implies $S = 0$, i.e. $x \in M - U_S$, a contradiction. Further, we set

$$H = \frac{1}{2}\, T + F L_R \overline{g}, \qquad L_R = -\frac{\overline{\kappa}}{p(n-p)}. \tag{44}$$

Evidently, $H = 0$. Now, in view of Theorem 2.1 of [11], (34) holds at x. Next, combining (40)–(43) with (22), we obtain (36). Similarly, using (22) and (40)–(43) we get (35).

(ii) From (32) we have

$$\operatorname{tr} T = \frac{2F(\overline{\kappa} - p\lambda)}{n-p}. \tag{45}$$

Substituting (32)(a) and (45) into (8), (9) and (11) we find

$$R_{a\alpha\beta b} = -\frac{\overline{\kappa} - p\lambda}{p(n-p)}\, g_{ab} g_{\alpha\beta}, \tag{46}$$

$$(47)\qquad \text{(a) } S_{ad} = \lambda g_{ad}, \qquad \text{(b) } S_{\alpha\beta} = \frac{\kappa - p\lambda}{n-p}\, g_{\alpha\beta},$$

$$(48)\qquad \kappa = \frac{1}{F}\left(\widetilde{\kappa} - F\overline{\kappa} + 2p\lambda F - (n-p-1)(n-p)\,\frac{\varDelta_1 F}{4F}\right).$$

We note that $\kappa - n\lambda$ is nonzero at x. Indeed, $\kappa - n\lambda = 0$ implies $S = \frac{\kappa}{n} g$, i.e. $x \in M - U_S$, a contradiction. Further, we set

$$(49)\qquad H = \frac{1}{2}T + FL_R\overline{g}, \qquad L_R = \frac{p\lambda - \overline{\kappa}}{p(n-p)}.$$

Evidently, $H = 0$. Now, in view of Theorem 2.1 of [11], (37) holds at x. Next, putting (45)–(48) into (22), we obtain (39). Similarly, using (22) and (45)–(48) we get (38).

(iii) From (4), by (33)(b), we obtain

$$(50)\qquad L_R = -\frac{(n-2)\overline{\kappa}}{p(p-1)} - \frac{L_2}{L_1}.$$

Further, from (33)(a) at x we have

$$(51)\qquad \frac{1}{2}T = \frac{F\overline{\kappa}}{p(n-p)}\,\overline{g} + \beta w \otimes w, \qquad \beta \in \mathbb{R},$$

where w is a covector at x. Next, we set

$$(52)\qquad H = \frac{1}{2}T + FL_R\overline{g}.$$

Applying (50) and (51) in (52) we find

$$(53)\qquad H = -\left(\frac{(n-p-1)(n-1)\overline{\kappa}}{p(p-1)(n-p)} + \frac{L_2}{L_1}\right)F\overline{g} + \beta w \otimes w.$$

From Theorem 2.2 of [11] it follows that $\operatorname{rank}(H) \leq 1$. Hence, in view of our assumptions, $\operatorname{rank}(H) = 1$. Thus at x we have

$$(54)\qquad \frac{L_2}{L_1} = -\frac{(n-p-1)(n-1)\overline{\kappa}}{p(p-1)(n-p)}.$$

Inserting now (54) in (50) we get

$$(55)\qquad L_R = -\frac{\overline{\kappa}}{p(n-p)}.$$

We also have the following relation ([11, Corollary 2.1]):

$$(56)\qquad \frac{2F\overline{\kappa}}{p(p-1)}\,(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}) = T_{ac}H_{bd} - T_{ab}H_{cd}.$$

Since $H = \beta w \otimes w$, (56) turns into

$$(57)\qquad \frac{2F\overline{\kappa}}{p(p-1)}\,(w_c\overline{g}_{ab} - w_b\overline{g}_{ac}) = w_bT_{ac} - w_cT_{ab},$$

where w_b are the local components of the covector w. This, by (51), yields

$$\overline{\kappa}(w_c\overline{g}_{ab} - w_b\overline{g}_{ac}) = 0, \tag{58}$$

and, in consequence, $\overline{\kappa} = 0$ at x. Thus (55) yields $L_R = 0$ and $R \cdot R = 0$. In addition, from (57) we get $\mathrm{rank}(T) = 1$. Further, by (54), $L_2 = 0$. Similarly, (33)(b) gives $L_3 = 0$. Now (21) reduces at x to $R = \frac{L_1}{2} S \wedge S$. Our proposition is thus proved.

Remark 4.1. Necessary and sufficient conditions for a warped product to satisfy $R = \frac{L_1}{2} S \wedge S$ were found in [25, Proposition 2.2].

Example 4.1. Let $(\overline{M}, \overline{g})$, $p = \dim \overline{M} \geq 2$, be the manifold defined in Example 2.1 of [10] and let F be the function on $\overline{M}$ defined by (9) of [10]. Further, let $(\widetilde{N}, \widetilde{g})$, $n - p = \dim \widetilde{N} \geq 2$, be a semi-Riemannian space of constant curvature l. We consider the warped product $\overline{M} \times_F \widetilde{N}$ (see Example 3.2 of [10]). It satisfies the following relations, among others (see formulas (15)–(17) of [10]):

$$\begin{aligned} &R_{abcd} = \varrho_1 G_{abcd}, \quad R_{a\alpha\beta b} = \varrho_2 G_{a\alpha\beta b}, \quad R_{\alpha\beta\gamma\delta} = \varrho_3 G_{\alpha\beta\gamma\delta}, \\ &\varrho_1 = k, \quad \varrho_2 = k(1 - c\tau), \quad \varrho_3 = (l - c_1)\tau^2 - 2kc\tau + k, \end{aligned} \tag{59}$$

$$k = \frac{\overline{\kappa}}{(p-1)p} > 0, \; l = \frac{\widetilde{\kappa}}{(n-p-1)(n-p)}, \; \tau = \frac{1}{\sqrt{F}}, \; c, c_1 \in \mathbb{R}. \tag{60}$$

In the following we will assume that $l > c_1$ and $c \neq 0$. This, together with the formulas (21) and (25) of [10], implies that $U_S \cap U_C = \overline{M} \times \widetilde{N}$. Further, $T = -2k(1 - c\tau)F\overline{g}$, whence

$$T = -2FL_R\overline{g}, \quad L_R = k(1 - c\tau). \tag{61}$$

From Theorem 4.1 it follows that the curvature tensor R of $\overline{M} \times_F \widetilde{N}$ has a decomposition of the form (21), with L_1, L_2 and L_3 defined by (22). From Proposition 2.3, by making use of (4), (5), (30), (60) and (61), we obtain

$$R \cdot R = k(1 - c\tau)Q(g, R), \tag{62}$$

$$R \cdot R - Q(S, R) = -(n-2)k\left(1 - \frac{kc^2}{l - c_1}\right)Q(g, C). \tag{63}$$

We now prove that $\overline{M} \times_F \widetilde{N}$ can be (locally) realized as a hypersurface in a semi-Riemannian space of constant curvature. We set

$$\frac{\mu}{n(n+1)} = k\left(1 - \frac{kc^2}{l - c_1}\right), \quad \mu_1^2 = \frac{k^2c^2}{l - c_1}, \quad \mu_2 = \mu_1 - \frac{kc\tau}{\mu_1}, \tag{64}$$

which yields

$$\mu_1\mu_2 = \mu_1^2 - kc\tau, \quad \mu_2^2 = \mu_1^2 - 2kc\tau + (l - c_1)\tau^2. \tag{65}$$

Further, we define on $\overline{M} \times \widetilde{N}$ a symmetric $(0,2)$-tensor H by

$$H_{ab} = \mu_1 g_{ab}, \quad H_{a\alpha} = 0, \quad H_{\alpha\beta} = \mu_2 g_{\alpha\beta}. \tag{66}$$

Now, using (59), (60) and (64)–(66) we get

$$\begin{aligned}
R_{abcd} &= \left(\frac{k^2c^2}{l-c_1} + \frac{\mu}{n(n+1)}\right)G_{abcd} = \left(\mu_1^2 + \frac{\mu}{n(n+1)}\right)G_{abcd} \\
&= \frac{1}{2}\,(H \wedge H)_{abcd} + \frac{\mu}{n(n+1)}\,G_{abcd}, \\
R_{a\alpha\beta d} &= (k + \mu_1\mu_2 - \mu_1^2)G_{a\alpha\beta d} = \left(\mu_1\mu_2 + \frac{\mu}{n(n+1)}\right)G_{a\alpha\beta d} \\
&= \frac{1}{2}\,(H \wedge H)_{a\alpha\beta d} + \frac{\mu}{n(n+1)}\,G_{a\alpha\beta d}, \\
R_{\alpha\beta\gamma\delta} &= ((l-c_1)\tau^2 - 2kc\tau + k)G_{\alpha\beta\gamma\delta} = (k + \varrho_2^2 - \varrho_1^2)G_{\alpha\beta\gamma\delta} \\
&= \left(\mu_2^2 + \frac{\mu}{n(n+1)}\right)G_{\alpha\beta\gamma\delta} = \frac{1}{2}\,(H \wedge H)_{\alpha\beta\gamma\delta} + \frac{\mu}{n(n+1)}\,G_{\alpha\beta\gamma\delta}.
\end{aligned}$$

Other local components of R, $H \wedge H$ and G are zero. Thus $R = \frac{1}{2}H \wedge H + \frac{\mu}{n(n+1)}G$. In addition, using (7) and (66) we can check that H is a Codazzi tensor. Therefore $\overline{M} \times_F \widetilde{N}$ can be realized locally as a hypersurface in a semi-Riemannian space of constant curvature.

We finish the paper with some corrections to [10]. Namely, formula (34) of [10] should have the form

$$\begin{aligned}
Q(S,R)_{\alpha abcd\beta} &= -(S_{d\alpha}R_{\beta abc} + S_{da}R_{\alpha\beta bc} + S_{db}R_{\alpha a\beta c} + S_{dc}R_{\alpha ab\beta} \\
&\quad - S_{\beta\alpha}R_{dabc} - S_{\beta a}R_{\alpha dbc} - S_{\beta b}R_{\alpha adc} - S_{\beta c}R_{\alpha abd}) \\
&= S_{db}R_{a\alpha\beta c} - S_{dc}R_{a\alpha\beta b} + S_{\beta\alpha}R_{dabc} \\
&= k\tau^2(kc - (n-p)kc^2 + (n-p-1)(l-c_1))g_{\alpha\beta}G_{abcd}.
\end{aligned} \tag{67}$$

We note that the definitions of $R \cdot R$, $Q(S,R)$ and $Q(g,C)$ and of other similar tensors in [10] and in the present paper differ in sign. Using [10, (26), (27), (32), (33), (35)] and (67) we obtain (63). Therefore assertion (v) of Theorem 4.1 of [10] should read: (v) $R \cdot R - Q(S,R)$ and $Q(g,C)$ are linearly dependent on N. This statement, together with Example 4.1, leads to the corrected version of the second part of Corollary 4.1 of [10]: the warped product $S^p(1/\sqrt{k}) \times_F S^{n-p}(1/\sqrt{l})$, $p \geq 2$, $n-p \geq 2$, $k > 0$, $l > 0$, can be locally realized as a hypersurface in a space of constant curvature.

REFERENCES

[1] L. Alias, A. Romero and M. Sánchez, *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson–Walker spacetimes*, in: Geometry and Topology of Submanifolds, VII, World Sci., River Edge, NJ, 1995, 67–70.

[2] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Dekker, New York, 1981.

[3] J. K. Beem, P. E. Ehrlich and T. G. Powell, *Warped product manifolds in relativity*, in: Selected Studies, Physics-Astrophysics, Mathematics, History of Sciences, a volume dedicated to the memory of Albert Einstein, North-Holland, Amsterdam, 1982, 41–56.

[4] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk and L. Verstraelen, *On some type of curvature conditions*, in: Banach Center Publ. 57, Inst. Math., Polish Acad. Sci., 2002, 179–194.

[5] F. Defever, R. Deszcz and M. Prvanović, *On warped product manifolds satisfying some curvature condition of pseudosymmetry type*, Bull. Greek Math. Soc. 36 (1994), 43–67.

[6] F. Defever, R. Deszcz, M. Hotloś, M. Kucharski and Z. Ṣentürk, *Generalisations of Robertson–Walker spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 43 (2000), 13–24.

[7] F. Defever, R. Deszcz, L. Verstraelen and L. Vrancken, *On pseudosymmetric spacetimes*, J. Math. Phys. 35 (1994), 5908–5921.

[8] R. Deszcz, *Certain curvature characterizations of affine hypersurfaces*, Colloq. Math. 63 (1992), 21–39.

[9] —, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A 44 (1992), 1–34.

[10] —, *Curvature properties of certain compact pseudosymmetric manifolds*, Colloq. Math. 65 (1993), 139–147.

[11] —, *On pseudosymmetric warped product manifolds*, in: Geometry and Topology of Submanifolds, V, World Sci., River Edge, NJ, 1993, 132–146.

[12] R. Deszcz and M. Głogowska, *Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces*, Colloq. Math. 94 (2002), 87–101.

[13] —, —, *Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces*, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002), 81–94.

[14] R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk and L. Verstraelen, *A review on pseudosymmetry type manifolds*, Dept. Math., Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 84, 2000.

[15] R. Deszcz and M. Hotloś, *On a certain subclass of pseudosymmetric manifolds*, Publ. Math. Debrecen 53 (1998), 29–48.

[16] —, —, *On hypersurfaces in space forms with type number two*, Dept. Math., Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 102, 2002.

[17] R. Deszcz and M. Kucharski, *On curvature properties of certain generalized Robertson–Walker spacetimes*, Tsukuba J. Math. 23 (1999), 113–130.

[18] R. Deszcz, P. Verheyen and L. Verstraelen, *On some generalized Einstein metric conditions*, Publ. Inst. Math. (Beograd) (N.S.) 60 (74) (1996), 108–120.

[19] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, in: Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ, 1991, 131–147.

[20] R. Deszcz, L. Verstraelen and L. Vrancken, *On the symmetry of warped product spacetimes*, Gen. Relativity Gravitation 23 (1991), 671–681.

[21] P. E. Ehrlich, Y.-T. Jung and S.-B. Kim, *Constant scalar curvatures on warped product manifolds*, Tsukuba J. Math. 20 (1996), 239–256.

[22] M. Głogowska, *On some class of semisymmetric manifolds*, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002), 95–106.

[23] S. W. Hawking, T. Hertog and H. S. Reall, *Brane new world*, Phys. Rev. D 62 (2000), 04351.

[24] J. Khoury, P. J. Steinhardt and D. Waldram, *Inflationary solutions in the brane world and their geometrical interpretation*, ibid. 63 (2001), 103505.

[25] D. Kowalczyk, *On semi-Riemannian manifolds satisfying some curvature conditions*, Soochow J. Math. 27 (2001), 445–461.

[26] —, *On Schwarzschild type spacetimes*, Dept. Math., Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 83, 2000.

[27] D. Kramer, H. Stephani, E. Herlt, M. MacCallum and E. Schmutzer, *Exact Solutions of Einstein's Field Equations*, Cambridge Univ. Press, Cambridge, 1980.

[28] G. I. Kruchkovich, *On some class of Riemannian spaces*, Trudy Sem. po Vekt. i Tenz. Analizu 11 (1961), 103–128 (in Russian).

[29] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[30] E. Papantonopoulos and I. Pappa, *Cosmological evolution of a brane universe in a type* 0 *string background*, Phys. Rev. D 63 (2001), 103506.

[31] M. Sánchez, *On the geometry of generalized Robertson–Walker spacetimes*: *geodesics*, Gen. Relativity Gravitation 30 (1998), 915–932.

Department of Mathematics
Agricultural University of Wrocław
Grunwaldzka 53
50-357 Wrocław, Poland
E-mail: rysz@ozi.ar.wroc.pl
dorotka@ozi.ar.wroc.pl

Received 7 October 2002 (4275)