A CONVOLUTION PROPERTY OF THE CANTOR–LEBESGUE MEASURE, II

BY

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Abstract. For $1 \leq p, q \leq \infty$, we prove that the convolution operator generated by the Cantor–Lebesgue measure on the circle $\mathbb{T}$ is a contraction whenever it is bounded from $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$. We also give a condition on $p$ which is necessary if this operator maps $L^p(\mathbb{T})$ into $L^2(\mathbb{T})$.

Let $\mathbb{T}$ be the circle group $\mathbb{R}/\mathbb{Z}$ and, for $1 \leq p \leq \infty$, write $L^p$ for the Lebesgue space formed using normalized Lebesgue measure on $\mathbb{T}$. Let $\lambda$ be the usual Cantor–Lebesgue measure on $\mathbb{T}$. We are interested in determining the $L^p$–$L^q$ mapping properties of the convolution operator defined by $\lambda$: we would like to know the indices $p, q \in [1, \infty]$ for which there is an inequality

$$||\lambda * f||_{L^q} \leq C(p, q)||f||_{L^p}$$

for $f \in L^p$. Since (1) is trivial if $q \leq p$, our interest is in the case $p < q$. The following results are in [O].

Lemma 1. Suppose $1 \leq p < q \leq \infty$. If the inequality

$$\left(\frac{1}{3} \left(\left(\frac{a+b}{2}\right)^q + \left(\frac{b+c}{2}\right)^q + \left(\frac{a+c}{2}\right)^q\right)\right)^{\frac{1}{q}} \leq \left(\frac{a^p + b^p + c^p}{3}\right)^{\frac{1}{p}}$$

holds for all $a, b, c \geq 0$, then (1) holds with $C(p, q) = 1$.

Lemma 2. Inequality (2) holds for $q = 2$ when $p \geq 2/(1 + 3^{-1/2}) \approx 1.2679$.

It follows from duality and interpolation that if $1 < p < \infty$ then there is $q$ satisfying $p < q < \infty$ and such that (1) holds with $C(p, q) = 1$. Similar results for more general measures are in [BJJ] and [R], while [C] establishes the “$L^p$-improving” property for a larger class of singular measures using a quite different method.

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The known cases of (1) are all applications of Lemma 1 and so satisfy $C(p,q) = 1$. The main result of this note is that if convolution with $\lambda$ maps $L^p$ into $L^q$, then it does so as a contraction:

**Theorem 3.** If (1) holds for $p,q \in [1,\infty]$ then (1) holds with $C(p,q) = 1$.

A more difficult and interesting problem is to determine exactly the indices for which (1) holds. Here we focus on the case $q = 2$. In addition to the information above, there are the following results.

**Proposition 4 ([B], [BJJ]).** Inequality (2) holds for $q = 2$ exactly when $p \geq \log 4/\log 3 \approx 1.2619$.

**Proposition 5.** If (1) holds then

$$\frac{1}{p} + \left(1 - \frac{\log 2}{\log 3}\right) \left(1 - \frac{1}{q}\right) \leq 1.$$  

Proposition 5 is checked by testing (1) on the indicator functions of small intervals. It shows in particular that if (1) holds with $q = 2$ then $p \geq 2(1 + \log 2/\log 3)^{-1} \approx 1.2263$, providing a necessary condition to pair with the sufficient condition provided by Lemma 1 and Proposition 4. The second result of this note narrows the gap between these two conditions.

**Proposition 6.** Suppose (1) holds with $q = 2$. Then the following inequality holds whenever $0 < a < b < 1$ and $2b < 1 + a$:

$$\left(\frac{2^a}{6a^a(b-a)^{2(b-a)}(1+a-2b)(1+a-2b)}\right)^{1/2} \leq \left(\frac{2^b}{3b^b(1-b)^{(1-b)}}\right)^{1/p}.$$  

Numerical calculations indicate that (3) fails when $p = 1.244$, $b = .0770$, and $a = .0105$. This rules out the possibility that the condition provided by Proposition 5 is sufficient as well as necessary (but leaves open the interesting possibility that the sufficient condition supplied by Beckner’s Proposition 4 is necessary). In the remainder of this note we give the proofs of Theorem 3 and Proposition 6.

**Proof of Theorem 3.** We will show that if (1) holds for $C(p,q) \in [1,\infty)$ then (1) also holds when $C(p,q)$ is replaced by $\sqrt{C(p,q)}$. It is convenient to replace $\lambda$ with its translate by $1/2$. We will need the facts that then the Fourier transform of $\lambda$ is given by

$$\hat{\lambda}(n) = \prod_{j=0}^{\infty} \cos(2\pi 3^{-j} n)$$

and that when $f$ is 1-periodic and continuous on $\mathbb{R}$ we have, for integral $M$,

$$\lim_{M \to \infty} \int_0^1 f(\theta) f(M\theta) d\theta = \left(\int_0^1 f(\theta) d\theta\right)^2.$$
Fix a trigonometric polynomial

\[ t(\theta) = \sum_{n=-L}^{L} \hat{t}(n)e^{2\pi in\theta}. \]

Then, for positive integers \( N \),

\[ \lambda * t(\theta) \lambda * t(3^N \theta) = \sum_{n_1, n_2} \prod_{j=0}^{\infty} (\cos(2\pi 3^{-j} n_1) \cos(2\pi 3^{-j} n_2)) \hat{t}(n_1) \hat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2) \theta}. \]

Also, \( t(\theta) t(3^N \theta) = \sum_{n_1, n_2} \hat{t}(n_1) \hat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2) \theta} \) and so

\[ \lambda*(t(\cdot) t(3^N\cdot))(\theta) = \sum_{n_1, n_2} \prod_{j=0}^{\infty} \cos(2\pi 3^{-j} (n_1 + 3^N n_2)) \hat{t}(n_1) \hat{t}(n_2) e^{2\pi i (n_1 + 3^N n_2) \theta}. \]

Now

\[ \prod_{j=0}^{\infty} \cos(2\pi 3^{-j} (n_1 + 3^N n_2)) \]

\[ = \prod_{j=0}^{N} \cos(2\pi 3^{-j} n_1) \prod_{j=N+1}^{\infty} \cos(2\pi [3^{-j} n_1 + 3^N - j n_2]) \]

\[ = \prod_{j=0}^{N} \cos(2\pi 3^{-j} n_1) \]

\[ \times \prod_{j=N+1}^{\infty} [\cos(2\pi 3^{-j} n_1) \cos(2\pi 3^N - j n_2) - \sin(2\pi 3^{-j} n_1) \sin(2\pi 3^N - j n_2)]. \]

For \( M \geq N + 1 \),

\[ \prod_{j=N+1}^{M} [\cos(2\pi 3^{-j} n_1) \cos(2\pi 3^N - j n_2) - \sin(2\pi 3^{-j} n_1) \sin(2\pi 3^N - j n_2)] \]

\[ = \prod_{j=N+1}^{M} \cos(2\pi 3^{-j} n_1) \cos(2\pi 3^N - j n_2) + e \]

where the error term \( e = e(n_1, n_2, N, M) \) satisfies

\[ |e| \leq \prod_{j=N+1}^{M} [1 + |\sin(2\pi 3^{-j} n_1)|] - 1 = O(3^{-N} L) \]
since $|n_1| \leq L$. Then
\[
\left| \prod_{j=0}^{\infty} \cos(2\pi 3^{-j}(n_1 + 3^N n_2)) - \prod_{j=0}^{\infty} (\cos(2\pi 3^{-j} n_1) \cos(2\pi 3^{-j} n_2)) \right| = O(3^{-N} L)
\]
and it follows that
\[
|\lambda \ast t(\theta) \lambda \ast t(3^N \theta) - \lambda \ast (t(\cdot) t(3^N \cdot))(\theta)| \leq C(t) \cdot 3^{-N},
\]
where $C(t)$ is a positive constant depending on the trigonometric polynomial $t$.

Thus
\[
\| \lambda \ast t(\theta) \lambda \ast t(3^N \theta) \|_{L^q} - \| \lambda \ast (t(\cdot) t(3^N \cdot))(\theta) \|_{L^q} \to 0
\]
as $N \to \infty$. Since
\[
\| \lambda \ast t(\theta) \lambda \ast t(3^N \theta) \|_{L^q} \to \| \lambda \ast t \|_{L^q}^2
\]
by (4), and also
\[
\| \lambda \ast (t(\cdot) t(3^N \cdot))(\theta) \|_{L^q} \leq C(p, q) \| t(\theta) t(3^N \theta) \|_{L^p} \to C(p, q) \| t \|_{L^p}^2,
\]
it follows that
\[
\| \lambda \ast t \|_{L^q} \leq \sqrt{C(p, q)} \| t \|_{L^p}
\]
as desired. Thus the proof of Theorem 3 is complete.

Proof of Proposition 6. If (1) holds it is easy to see that convolution with $\lambda$ yields a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. If $q = 2$ it follows that
\[
\langle \lambda \ast \tilde{\lambda}, \chi_E \ast \chi_{-E} \rangle \leq C(p) \| E \|_{2/p}
\]
for Borel $E \subseteq \mathbb{R}$. To discretize (6) define
\[
C_N = \left\{ \sum_{j=0}^{N-1} \varepsilon_j 3^j : j \in \{0, 2\} \right\}.
\]
With “*” now representing the usual convolution on the group of integers and “| |” standing for cardinality, (6) implies that
\[
\frac{1}{12^N} \langle \chi_{C_N} \ast \chi_{-C_N}, \chi_F \ast \chi_{-F} \rangle \leq C(p) \left( \frac{|F|}{3^N} \right)^{2/p}
\]
whenever $F \subseteq \mathbb{Z}$. We will establish (3) by applying (7) to certain sets $F_{N,k}$.

Fix a positive integer $N$. For $J \subseteq \{0, 1, \ldots, N - 1\}$ put
\[
F_J = \left\{ \sum_{j \in J} \varepsilon_j 3^j : j \in \{-2, 2\} \right\}
\]
so that
\[
\chi_{F_J} = \ast \left( \delta_{-2,3j} + \delta_{2,3j} \right).
\]
For $1 \leq k \leq N - 1$ define
\[
F_{N,k} = \bigcup \{F_J : J \subseteq \{0, 1, \ldots, N - 1\}, |J| = k\}.
\]
Note that if $J_1$ and $J_2$ are disjoint then
\[
\chi_{F_{J_1}} \ast \chi_{F_{J_2}} = \chi_{F_{J_1 \cup J_2}},
\]
and that, in general,
\[
\chi_{F_{J_1}} \ast \chi_{F_{J_2}} = \sum_{j \in J_1 \cap J_2} \left( \delta_{-4.3j} + 2\delta_0 + \delta_{4.3j} \right) \ast \chi_{F_{\{j \in J_1 \cup J_2 \} \setminus (J_1 \cap J_2)}}.
\]
It follows that
\[
\chi_{F_{J_1}} \ast \chi_{F_{J_2}} \geq 2^{|J_1 \cap J_2|} \chi_{F_{\{j \in J_1 \cup J_2 \} \setminus (J_1 \cap J_2)}}.
\]
Thus, since $F_{J_1}$ and $F_{J_2}$ are disjoint if $J_1 \neq J_2$,
\[
\langle \chi_{C_N} \ast \chi_{-C_N}, \chi_{F_{N,k} \ast \chi_{F_{N,k}}} \rangle = \sum_{|J_1|=|J_2|=k} \langle \chi_{C_N} \ast \chi_{-C_N}, \chi_{F_{J_1}} \ast \chi_{F_{J_2}} \rangle \geq \sum_{|J_1|=|J_2|=k} 2^{|J_1 \cap J_2|} \langle \chi_{C_N} \ast \chi_{-C_N}, \chi_{F_{(J_1 \cup J_2) \setminus (J_1 \cap J_2)}} \rangle.
\]
Now
\[
\langle \chi_{C_N} \ast \chi_{-C_N}, \chi_{F_J} \rangle = \sum_{f \in F_J} |f + C_N \cap C_N| = 2^{|J|} 2^N - |J| = 2^N
\]
so
\[
\langle \chi_{C_N} \ast \chi_{-C_N}, \chi_{F_{N,k} \ast \chi_{F_{N,k}}} \rangle \geq 2^N \sum_{|J_1|=|J_2|=k} 2^{|J_1 \cap J_2|} = 2^N \binom{N}{k} \sum_{l=0}^{k} 2^l \binom{k}{l} \binom{N-k}{k-l}.
\]
Thus (7) implies that for $l = 0, \ldots, k$ there is the inequality
\[
\frac{2^l}{6^N} \binom{N}{k} \binom{k}{l} \binom{N-k}{k-l} \leq C(p) \left( \frac{N}{k} \right)^{2/3N}.
\]
By continuity, it is enough to establish (3) when $a$ and $b$ are rational. With such $a$ and $b$ fixed, $N$ will now stand for a positive integer such that both $aN$ and $bN$ are integers. Take $k = bN$ and $l = aN$ in (8), estimate the binomial coefficients using Stirling’s formula, take $N$th roots of both sides of the resulting inequality, and then let $N \to \infty$. This gives
\[
\frac{2^a}{6a^a(b-a)^{2(b-a)}(1+a-2b)(1+a-2b)} \leq \left( \frac{2^b}{3 \cdot b^b(1-b)(1-b)} \right)^{2/p},
\]
the conclusion of Proposition 6.
REFERENCES


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