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ON NONSTANDARD TAME SELFINJECTIVE ALGEBRAS HAVING ONLY PERIODIC MODULES

BY

JERZY BIAŁKOWSKI (Toruń), THORSTEN HOLM (Magdeburg) and ANDRZEJ SKOWROŃSKI (Toruń)

Abstract. We investigate degenerations and derived equivalences of tame selfinjective algebras having no simply connected Galois coverings but the stable Auslander–Reiten quiver consisting only of tubes, discovered recently in [4].

Introduction. Throughout the paper, by an *algebra* we mean a basic connected, finite-dimensional associative K-algebra with an identity over a (fixed) algebraically closed field K. For such an algebra A, there exists an isomorphism $A \cong KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal in KQ. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (left) A-modules and by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$.

From Drozd's remarkable Tame and Wild Theorem [6] the class of algebras may be divided into two disjoint classes. One class consists of tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras. Accordingly we may realistically hope to classify the indecomposable finite-dimensional modules only for the tame algebras.

An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. Further, A is called *symmetric* if A and $D(A)$ are isomorphic as A-bimodules. The classical examples of selfinjective algebras are provided by the blocks of group algebras of finite groups and Hopf algebras.

An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$ where $\widehat{B}$ is the repetitive algebra [15] (locally finite-dimensional, without identity)

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$$\widehat{B} = \bigoplus_{m \in \mathbb{Z}} (B_m \oplus Q_m)$$

of an algebra B, where $B_m = B$ and $Q_m = D(B)$ for all $m \in \mathbb{Z}$, the multiplication in $\widehat{B}$ is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

for $a_m, b_m \in B_m$, $f_m, g_m \in Q_m$, and G is an admissible group of K-automorphisms of $\widehat{B}$. In particular, if $\nu_{\widehat{B}} : \widehat{B} \to \widehat{B}$ is the Nakayama automorphism of $\widehat{B}$ given by the identity shifts $B_m \to B_{m+1}$ and $Q_m \to Q_{m+1}$, then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the *trivial extension* $T(B) = B \ltimes D(B)$ of B by $D(B)$, and is a symmetric algebra.

We are concerned with the problem of classifying all tame selfinjective algebras whose stable Auslander–Reiten quiver consists only of tubes. The classification splits into two cases: the *standard* algebras, which admit simply connected Galois coverings, and the remaining *nonstandard* ones. It has been shown recently in [2] that the class of all tame standard selfinjective algebras with the stable Auslander–Reiten quiver consisting only of tubes coincides with the class of all *selfinjective algebras of tubular type*, that is, the algebras of the form $\widehat{B}/G$, where B is a tubular algebra (in the sense of Ringel [23]) and G is an admissible group of K-automorphisms of $\widehat{B}$. Moreover, it was proved in [24] that this class of algebras coincides with the class of all nondomestic (generically infinite) standard selfinjective algebras of polynomial growth. We refer to [2], [3], [17] for a complete classification of these algebras and to [18], [24] for the structure of their module categories. In the process of classifying tame blocks of group algebras of finite groups, K. Erdmann discovered various families of tame symmetric algebras (of quaternion type) having at most three simple modules, nonsingular Cartan matrices, and the stable Auslander–Reiten quiver consisting only of tubes, but only very few of them are standard (see [7]–[9]). It has been conjectured by the third named author (see [25, Section 3]) that the remaining class of nonstandard tame selfinjective algebras with the stable Auslander-Reiten quiver consisting only of tubes is formed by certain deformations of standard selfinjective algebras of tubular type.

In the recent paper [4] all selfinjective algebras socle equivalent to the (standard) selfinjective algebras of tubular type were determined. Besides the selfinjective algebras of tubular type, there are 10 types of nonstandard algebras occurring in characteristic 2 or 3 (the left column of Table 1 below), which we call *nonstandard selfinjective algebras of tubular type*. Moreover, for each nonstandard selfinjective algebra Λ of tubular type there exists a unique (up to isomorphism) standard selfinjective algebra Λ' of tubular type (the *standard form* of Λ) such that Λ is socle equivalent to Λ' but Λ and Λ' are nonisomorphic.

The main aim of this paper is to describe the basic properties of these nonstandard selfinjective algebras of tubular type. In Section 2 we show that every nonstandard selfinjective algebra Λ of tubular type degenerates to its standard form (in the affine variety of algebras of the corresponding dimension). The final Section 3 contains a derived equivalence classification of the class of nonstandard selfinjective algebras of tubular type.

For basic background on the representation theory of algebras and related topics we refer to [9], [12], [16], [23].

1. Socle equivalences. For a selfinjective algebra Λ, the left and the right socle of Λ coincide, and we denote them by $\operatorname{soc}\Lambda$. Following [26] (see also [27]) two selfinjective algebras A and B are said to be *socle equivalent* if the factor algebras $A/\operatorname{soc} A$ and $B/\operatorname{soc} B$ are isomorphic. Consider the families of bound quiver algebras listed in Table 1 (pp. 36–37).

The following fact has been established in [4, Theorem 1.1].

THEOREM 1.1. *Let Λ be a nonstandard selfinjective algebra. Then Λ is socle equivalent to a selfinjective algebra of tubular type if and only if exactly one of the following cases holds*:

(1) *K is of characteristic 3 and Λ is isomorphic to one of the bound quiver algebras Λ_1 or Λ_2.*

(2) *K is of characteristic 2 and Λ is isomorphic to one of the bound quiver algebras $\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$, Λ_4, Λ_5, Λ_6, Λ_7, Λ_8, Λ_9 or Λ_{10}.*

In fact the bound quiver algebras Λ'_1, Λ'_2, $\Lambda'_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$, Λ'_4, Λ'_5, Λ'_6, Λ'_7, Λ'_8, Λ'_9 and Λ'_{10} in the right column of Table 1 are socle equivalent to the algebras Λ_1, Λ_2, $\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$, Λ_4, Λ_5, Λ_6, Λ_7, Λ_8, Λ_9 and Λ_{10}, respectively, and hence they are their standard forms. We also note that all algebras in Table 1 except Λ_{10} and Λ'_{10} are symmetric.

2. Degenerations. For a positive integer d, we denote by $\operatorname{alg}_d(K)$ the affine variety of associative algebra structures with identity on the affine space K^d. Then the general linear group $\mathrm{GL}_d(K)$ acts on $\operatorname{alg}_d(K)$ by transport of structure, and the $GL_d(K)$-orbits in $\operatorname{alg}_d(K)$ correspond to the isomorphism classes of d-dimensional algebras (see [16] for more details). We shall identify a d-dimensional algebra A with the corresponding point of $\operatorname{alg}_d(K)$. For two d-dimensional algebras A and B, we say that B is a *degeneration* of A (A is a *deformation* of B) if B belongs to the closure of the $\mathrm{GL}_d(K)$-orbit of A in the Zariski topology of $\operatorname{alg}_d(K)$. It follows from Geiss's Theorem [10] (see also [5]) that if A degenerates to B and B is a tame algebra, then A is also a tame algebra.

Table 1

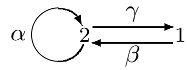
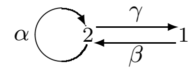
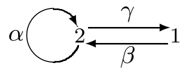
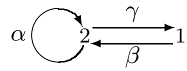
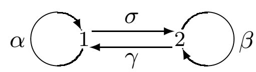
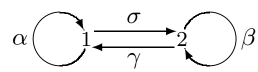
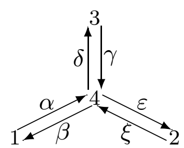
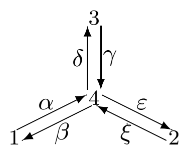
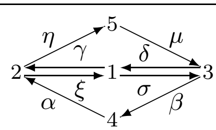
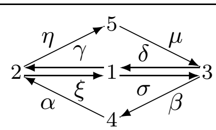
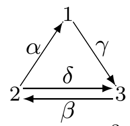
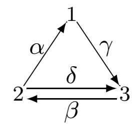
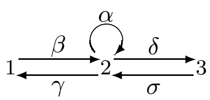
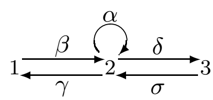
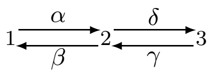
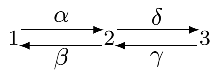
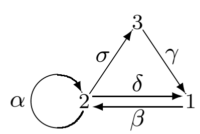
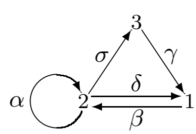
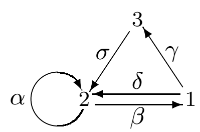
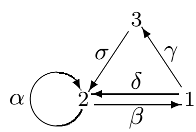
Char.	Tub. type	Nonstandard algebras	Standard algebras
3	$(3,3,3)$	 $\alpha^2 = \gamma\beta$ $\beta\alpha\gamma = \beta\alpha^2\gamma$ $\beta\alpha\gamma\beta = 0$ $\gamma\beta\alpha\gamma = 0$ Λ_1	 $\alpha^2 = \gamma\beta$ $\beta\alpha\gamma = 0$ Λ'_1
		 $\alpha^2\gamma = 0, \quad \beta\alpha^2 = 0$ $\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0$ $\beta\gamma = \beta\alpha\gamma$ $\alpha^3 = \gamma\beta$ Λ_2	 $\alpha^3 = \gamma\beta$ $\beta\gamma = 0$ $\beta\alpha^2 = 0$ $\alpha^2\gamma = 0$ Λ'_2
2	$(2,2,2,2)$	 $\alpha^4 = 0, \quad \gamma\alpha^2 = 0, \quad \alpha^2\sigma = 0$ $\alpha^2 = \sigma\gamma + \alpha^3, \quad \lambda\beta^2 = \gamma\sigma$ $\gamma\alpha = \beta\gamma, \quad \sigma\beta = \alpha\sigma$ $\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$	 $\alpha^2 = \sigma\gamma$ $\lambda\beta^2 = \gamma\sigma$ $\gamma\alpha = \beta\gamma$ $\sigma\beta = \alpha\sigma$ $\Lambda'_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$
	$(3,3,3)$	 $\beta\alpha + \delta\gamma + \varepsilon\xi = 0$ $\gamma\delta = 0, \quad \xi\varepsilon = 0, \quad \alpha\beta\alpha = 0$ $\beta\alpha\beta = 0, \quad \alpha\beta = \alpha\delta\gamma\beta$ Λ_9	 $\beta\alpha + \delta\gamma + \varepsilon\xi = 0$ $\alpha\beta = 0, \quad \xi\varepsilon = 0$ $\gamma\delta = 0$ Λ'_9
	$(2,3,6)$	 $\mu\beta = 0, \quad \alpha\eta = 0, \quad \beta\alpha = \delta\gamma$ $\xi\sigma = \eta\mu, \quad \sigma\delta = \gamma\xi + \sigma\delta\sigma\delta$ $\delta\sigma\delta\sigma = 0, \quad \xi\gamma\xi\gamma = 0$ Λ_{10}	 $\mu\beta = 0, \quad \alpha\eta = 0, \quad \sigma\delta = \gamma\xi$ $\beta\alpha = \delta\gamma, \quad \xi\sigma = \eta\mu$ Λ'_{10}

Table 1 (cont.)

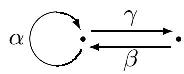
Char.	Tub. type	Nonstandard algebras	Standard algebras
2	$(2,4,4)$	 $\delta\beta\delta = \alpha\gamma,\ (\beta\delta)^3\beta = 0$ $\gamma\beta\alpha\gamma = 0,\ \alpha\gamma\beta\alpha = 0$ $\gamma\beta\alpha = \gamma\beta\delta\beta\alpha$ Λ_4	 $\delta\beta\delta = \alpha\gamma$ $\gamma\beta\alpha = 0, \quad (\beta\delta)^3\beta = 0$ Λ_4'
		 $\alpha^2 = \gamma\beta, \quad \alpha^3 = \delta\sigma, \quad \beta\delta = 0$ $\sigma\gamma = 0, \quad \alpha\delta = 0, \quad \sigma\alpha = 0$ $\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0, \quad \beta\gamma = \beta\alpha\gamma$ Λ_5	 $\alpha^2 = \gamma\beta, \quad \beta\delta = 0, \quad \beta\gamma = 0$ $\sigma\gamma = 0, \quad \alpha\delta = 0, \quad \sigma\alpha = 0$ $\alpha^3 = \delta\sigma$ Λ_5'
		 $\alpha\delta\gamma\delta = 0, \quad \gamma\delta\gamma\beta = 0$ $\alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0$ $\alpha\beta = \alpha\delta\gamma\beta$ $\beta\alpha = \delta\gamma\delta\gamma$ Λ_6	 $\beta\alpha = \delta\gamma\delta\gamma$ $\alpha\delta\gamma\delta = 0$ $\gamma\delta\gamma\beta = 0$ $\alpha\beta = 0$ Λ_6'
		 $\beta\delta = \beta\alpha\delta, \quad \alpha\sigma = 0, \quad \alpha\delta = \sigma\gamma$ $\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta, \quad \gamma\beta\delta = 0$ $\beta\delta\beta = 0, \qquad \delta\beta\delta = 0$ Λ_7	 $\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta$ $\beta\delta = 0, \quad \alpha\sigma = 0, \quad \alpha\delta = \sigma\gamma$ Λ_7'
		 $\delta\beta = \delta\alpha\beta, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma$ $\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta, \quad \delta\beta\gamma = 0$ $\beta\delta\beta = 0, \qquad \delta\beta\delta = 0$ Λ_8	 $\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta$ $\delta\beta = 0, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma$ Λ_8'

The aim of this section is to prove the following result.

THEOREM 2.1. *Let Λ be a nonstandard selfinjective algebra of tubular type and Λ' the standard selfinjective algebra of tubular type which is socle equivalent to Λ. Then Λ' is a degeneration of Λ.*

Proof. It is enough to show that there exists an algebraic family $\Lambda(a)$, $a \in K$, of algebras in $\mathrm{alg}_d(K)$ ($d = \dim_K \Lambda$) such that $\Lambda(a) \cong \Lambda$ for all $a \in K \setminus \{0\}$ and $\Lambda(0) = \Lambda'$. We have ten cases to consider.

(1) Let $\Lambda = \Lambda_1$. Consider the family $\Lambda_1(a)$, $a \in K$, of algebras in $\mathrm{alg}_{14}(K)$ given by the quiver



bound by $\alpha^2 = \gamma\beta$, $\beta\alpha\gamma = a\beta\alpha^2\gamma$, $\beta\alpha\gamma\beta = 0$ and $\gamma\beta\alpha\gamma = 0$. Clearly, $\Lambda_1(0) = \Lambda_1'$. We now prove that $\Lambda_1(a) \cong \Lambda_1$ for all $a \in K \setminus \{0\}$. In fact, we have an isomorphism $\varphi : \Lambda_1 \to \Lambda_1(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = a\gamma.$$

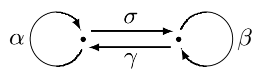
We know from [4, Theorem 1.1] that $\Lambda_1 \ncong \Lambda_1'$ for K of characteristic 3.

(2) Let $\Lambda = \Lambda_2$. Consider the family $\Lambda_2(a)$, $a \in K$, of algebras in $\mathrm{alg}_{11}(K)$ given by the quiver of (1) bound by $\alpha^3 = \gamma\beta$, $\alpha^2\gamma = 0$, $\beta\alpha^2 = 0$, $\gamma\beta\gamma = 0$, $\beta\gamma\beta = 0$ and $\beta\gamma = a\beta\alpha\gamma$. Clearly, $\Lambda_2(0) = \Lambda_2'$. For all $a \in K \setminus \{0\}$, an isomorphism $\varphi : \Lambda_2 \to \Lambda_2(a)$ is induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a^3\beta, \quad \varphi(\gamma) = \gamma.$$

Again, by [4, Theorem 1.1], $\Lambda_2 \ncong \Lambda_2'$ for K of characteristic 3.

(3) Let $\Lambda = \Lambda_3(\lambda)$, $\lambda \in K \setminus \{0, 1\}$. Consider the family $\Lambda_3(\lambda)(a)$, $a \in K$, of algebras in $\mathrm{alg}_{12}(K)$ given by the quiver

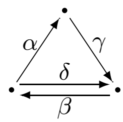


bound by $\alpha^4 = 0$, $\gamma\alpha^2 = 0$, $\alpha^2\sigma = 0$, $\alpha^2 = \sigma\gamma + a\alpha^3$, $\lambda\beta^2 = \gamma\sigma$, $\gamma\alpha = \beta\gamma$ and $\sigma\beta = \alpha\sigma$. Clearly, $\Lambda_3(\lambda)(0) = \Lambda_3'(\lambda)$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_3(\lambda) \to \Lambda_3(\lambda)(a)$ induced by the automorphism φ of the path algebra of the above quiver given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = a\gamma, \quad \varphi(\sigma) = a\sigma.$$

Moreover, by [4, Theorem 1.1], $\Lambda_3(\lambda) \ncong \Lambda_3'(\lambda)$ for K of characteristic 2.

(4) Let $\Lambda = \Lambda_4$. Consider the family $\Lambda_4(a)$, $a \in K$, of algebras in $\mathrm{alg}_{24}(K)$ given by the quiver

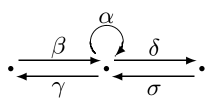


bound by $\delta\beta\delta = \alpha\gamma$, $(\beta\delta)^3\beta = 0$, $\gamma\beta\alpha\gamma = 0$, $\alpha\gamma\beta\alpha = 0$ and $\gamma\beta\alpha = a\gamma\beta\delta\beta\alpha$. Clearly, $\Lambda_4(0) = \Lambda_4'$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_4 \to \Lambda_4(a)$ induced by the path automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = \gamma, \ \varphi(\delta) = \delta.$$

We know from [4, Theorem 1.1] that $\Lambda_4 \not\cong \Lambda_4'$ for K of characteristic 2.

(5) Let $\Lambda = \Lambda_5$. Consider the family $\Lambda_5(a)$, $a \in K$, of algebras in $\mathrm{alg}_{14}(K)$ given by the quiver

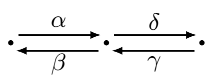


bound by $\alpha^2 = \gamma\beta$, $\alpha^3 = \delta\sigma$, $\beta\delta = 0$, $\sigma\gamma = 0$, $\alpha\delta = 0$, $\sigma\alpha = 0$, $\gamma\beta\gamma = 0$, $\beta\gamma\beta = 0$ and $\beta\gamma = a\beta\alpha\gamma$. Clearly, $\Lambda_5(0) = \Lambda_5'$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_5 \to \Lambda_5(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a^2\beta, \quad \varphi(\gamma) = \gamma, \quad \varphi(\delta) = a^3\delta, \quad \varphi(\sigma) = \sigma.$$

Again, by [4, Theorem 1.1], $\Lambda_5 \not\cong \Lambda_5'$ for K of characteristic 2.

(6) Let $\Lambda = \Lambda_6$. Consider the family $\Lambda_6(a)$, $a \in K$, of algebras in $\mathrm{alg}_{22}(K)$ given by the quiver

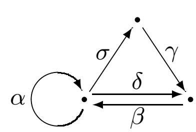


bound by $\alpha\delta\gamma\delta = 0$, $\gamma\delta\gamma\beta = 0$, $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$, $\alpha\beta = a\alpha\delta\gamma\beta$ and $\beta\alpha = \delta\gamma\delta\gamma$. Clearly, $\Lambda_6(0) = \Lambda_6'$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_6 \to \Lambda_6(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = a\gamma, \quad \varphi(\delta) = \delta.$$

It follows from [4, Theorem 1.1] that $\Lambda_6 \not\cong \Lambda_6'$ for K of characteristic 2.

(7) Let $\Lambda = \Lambda_7$. Consider the family $\Lambda_7(a)$, $a \in K$, of algebras in $\mathrm{alg}_{16}(K)$ given by the quiver



bound by $\beta\delta = a\beta\alpha\delta$, $\alpha\sigma = 0$, $\alpha\delta = \sigma\gamma$, $\gamma\beta\alpha = 0$, $\alpha^2 = \delta\beta$, $\gamma\beta\delta = 0$, $\beta\delta\beta = 0$ and $\delta\beta\delta = 0$. Clearly, $\Lambda_7(0) = \Lambda_7'$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_7 \to \Lambda_7(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = a\gamma, \quad \varphi(\delta) = a\delta, \quad \varphi(\sigma) = a\sigma.$$

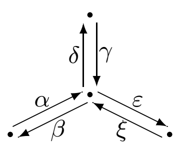
Moreover, by [4, Theorem 1.1], $\Lambda_7 \not\cong \Lambda'_7$ for K of characteristic 2.

(8) Let $\Lambda = \Lambda_8$. Consider the family $\Lambda_8(a)$, $a \in K$, of algebras in $\mathrm{alg}_{16}(K)$ given by the quiver dual to that of (7) bound by $\delta\beta = a\delta\alpha\beta$, $\sigma\alpha = 0$, $\delta\alpha = \gamma\sigma$, $\alpha\beta\gamma = 0$, $\alpha^2 = \beta\delta$, $\delta\beta\gamma = 0$, $\beta\delta\beta = 0$ and $\delta\beta\delta = 0$. Clearly, $\Lambda_8(0) = \Lambda'_8$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_8 \to \Lambda_8(a)$, induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = a\beta, \quad \varphi(\gamma) = a\gamma, \quad \varphi(\delta) = a\delta, \quad \varphi(\sigma) = a\sigma.$$

By [4, Theorem 1.1] we have $\Lambda_8 \not\cong \Lambda'_8$ for K of characteristic 2.

(9) Let $\Lambda = \Lambda_9$. Consider the family $\Lambda_9(a)$, $a \in K$, of algebras in $\mathrm{alg}_{28}(K)$ given by the quiver

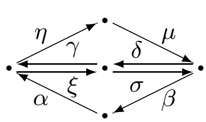


bound by $\beta\alpha + \delta\gamma + \varepsilon\xi = 0$, $\gamma\delta = 0$, $\xi\varepsilon = 0$, $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$ and $\alpha\beta = a\alpha\delta\gamma\beta$. Clearly, $\Lambda_9(0) = \Lambda'_9$. For all $a \in K \setminus \{0\}$, we have an isomorphism $\varphi : \Lambda_9 \to \Lambda_9(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\beta) = \beta, \quad \varphi(\gamma) = a\gamma,$$
$$\varphi(\delta) = \delta, \quad \varphi(\xi) = a\xi, \quad \varphi(\varepsilon) = \varepsilon.$$

Also, by [4, Theorem 1.1] we have $\Lambda_9 \not\cong \Lambda'_9$ for K of characteristic 2.

(10) Let $\Lambda = \Lambda_{10}$. Consider the family $\Lambda_{10}(a)$, $a \in K$, of algebras in $\mathrm{alg}_{35}(K)$ given by the quiver



bound by $\beta\alpha = \delta\gamma$, $\xi\sigma = \eta\mu$, $\alpha\eta = 0$, $\mu\beta = 0$, $\sigma\delta = \gamma\xi + a\sigma\delta\sigma\delta$, $\delta\sigma\delta\sigma = 0$, $\xi\gamma\xi\gamma = 0$. Clearly, $\Lambda_{10}(0) = \Lambda'_{10}$. For all $a \in K\setminus\{0\}$, we have an isomorphism $\varphi : \Lambda_{10} \to \Lambda_{10}(a)$ induced by the path algebra automorphism φ given by

$$\varphi(\alpha) = a\alpha, \quad \varphi(\gamma) = a\gamma, \quad \varphi(\sigma) = a\sigma, \quad \varphi(\mu) = a\mu,$$
$$\varphi(\beta) = \beta, \quad \varphi(\delta) = \delta, \quad \varphi(\xi) = \xi, \quad \varphi(\eta) = \eta.$$

Finally, by [4, Theorem 1.1], $\Lambda_{10} \not\cong \Lambda'_{10}$ for K of characteristic 2. ■

3. Derived equivalences. For an algebra A, we denote by $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ the *derived category* of bounded complexes of modules from $\mathrm{mod}\, A$, which is in fact a triangulated category (see [12]). Two algebras A and B are said to be *derived equivalent* if the derived categories $D^b(\mathrm{mod}\, A)$ and $D^b(\mathrm{mod}\, B)$ are equivalent as triangulated categories.

Since Happel's work [11], interpreting tilting theory in terms of equivalences of derived categories, the machinery of derived categories has been of interest to representation theorists. J. Rickard proved in [20] his remarkable criterion: two algebras A and B are derived equivalent if and only if B is the endomorphism algebra of a tilting complex of projective A-modules. We refer to the fundamental paper [20] for definitions and details. Since a lot of interesting properties are preserved by derived equivalence of algebras, it is for many purposes important to classify algebras up to derived equivalence, instead of Morita equivalence. For instance, derived equivalent selfinjective algebras are stably equivalent [21]. Further, an algebra derived equivalent to a symmetric algebra is again symmetric [22]. Finally, we note that derived equivalent algebras have the same number of simple modules.

The derived equivalence classification has been established for some classes of tame selfinjective algebras, for example for the (standard) weakly symmetric algebras of tubular type [1] (see also [13], [19] for the derived equivalence classification of the trivial extensions of tubular algebras) and the symmetric algebras of quaternion type [14]. The aim of this section is to describe derived equivalences for all nonstandard selfinjective algebras of tubular type.

THEOREM 3.1. *Let Λ be a nonstandard selfinjective algebra of tubular type. Then Λ is derived equivalent to one of the following algebras*:

- *two simple modules*: Λ_1 *and* $\Lambda_3(a)$, $a \in K \setminus \{0,1\}$,
- *three simple modules*: Λ_4,
- *four simple modules*: Λ_9,
- *five simple modules*: Λ_{10}.

The proof of the above theorem will be a combination of several propositions. We will often need to compute the Cartan invariants of the endomorphism algebras of tilting complexes. This can be done conveniently by the following alternating sum formula due to Happel (see [12, III.1.3 and III.1.4]). For an algebra A, let $K^b(A)$ denote the *homotopy category* of bounded complexes of projective A-modules, and let $[\cdot]$ denote the shift operator. If $Q = (Q^r)_{r\in\mathbb{Z}}$ and $R = (R^s)_{s\in\mathbb{Z}}$ are bounded complexes of projective A-modules, then

$$\sum_j (-1)^i \dim_K \mathrm{Hom}_{K^b(A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim_K \mathrm{Hom}_A(Q^r, R^s).$$

Note that if $\mathrm{Hom}_{K^b(A)}(Q, R[i]) = 0$ for $i \neq 0$ (for example, for direct summands of tilting complexes) then the left-hand side reduces to

$\dim_K \operatorname{Hom}_{K^{\mathrm{b}}(A)}(Q,R)$ and the right-hand side can be easily computed using the Cartan matrix of A.

To prove the assertion of Theorem 3.1 concerning nonstandard algebras of tubular type with two simple modules, it suffices to show the following result.

PROPOSITION 3.2. *The algebras Λ_1 and Λ_2 are derived equivalent.*

Proof. We consider the bounded complex $T = T_1 \oplus T_2$ of projective Λ_1-modules, where $T_2 : 0 \to P_2 \to 0$ (concentrated in degree 0) and $T_1 : 0 \to P_2 \xrightarrow{\gamma} P_1 \to 0$ in degrees 0 and -1 (we always index the degrees in a complex decreasingly from left to right). Here and in what follows, a map denoted by a path in the quiver of an algebra always means right multiplication with this element.

Then T is a tilting complex for Λ_1. In fact, $\operatorname{add}(T)$ generates the homotopy category $K^{\mathrm{b}}(\Lambda_1)$ since the stalk complex $0 \to P_1 \to 0$ is homotopy equivalent to the mapping cone of the map of complexes $T_1 \to T_2$ given by the identity map in degree 0. (This argument for generation works for all the complexes constructed in this paper.) Moreover, from the explicit description of the algebra Λ_1 by a quiver with relations it is easy to check that $\operatorname{Hom}_{K^{\mathrm{b}}}(T_i, T_j[k]) = 0$ for all $i, j \in \{1,2\}$ and $k \neq 0$. We leave the details to the reader.

Hence, by Rickard's criterion, Λ_1 is derived equivalent to the endomorphism ring of T (in the homotopy category). From the Cartan matrix of Λ_1 we can compute (using Happel's formula) the Cartan matrix of $\operatorname{End}_{K^{\mathrm{b}}}(T)$:

$$C_{\Lambda_1} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}, \qquad C_{\operatorname{End}_{K^{\mathrm{b}}}(T)} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix},$$

and the latter is indeed equal to the Cartan matrix of Λ_2 (the first row corresponds to the vertex 1, the second row to the vertex 2 in the quivers). So it remains to describe morphisms in $\operatorname{End}_{K^{\mathrm{b}}}(T)$ (generating the radical) satisfying the relations of Λ_2 (up to homotopy). Let $\widetilde{\alpha} : T_2 \to T_2$ be given by (right) multiplication with α on P_2. Let $\widetilde{\beta} : T_1 \to T_2$ be defined by the identity map on P_2 in degree 0. Finally, let $\widetilde{\gamma} : T_2 \to T_1$ be given by (right) multiplication with α^3 on P_2 (note that this is indeed a homomorphism of complexes since $\alpha^3\gamma = \gamma\beta\alpha\gamma = 0$ in Λ_1).

It remains to check the relations (where we read compositions of maps also from left to right, just as relations in quivers). By definition we have $\widetilde{\alpha}^2\widetilde{\gamma} = 0$ and $\widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma} = 0$ (since $\alpha^5 = 0$ in Λ_1). Also by definition, $\widetilde{\alpha}^3 = \widetilde{\gamma}\widetilde{\beta}$. The other relations will hold up to homotopy. In fact, the morphisms $\widetilde{\beta}\widetilde{\alpha}^2$ and $\widetilde{\beta}\widetilde{\gamma}\widetilde{\beta}$ from T_1 to T_2 are homotopic to zero via the homotopy maps $\beta : P_1 \to P_2$ and $\beta\alpha : P_1 \to P_2$, respectively (use $\alpha^2 = \gamma\beta$ in Λ_1). Finally, the morphism

$\widetilde{\beta}\widetilde{\gamma} - \widetilde{\beta}\widetilde{\alpha}\widetilde{\gamma} : T_1 \to T_1$ is given by $\alpha^3 - \alpha^4 : P_2 \to P_2$ in degree 0 (and the zero map in degree -1). This is homotopic to zero via the homotopy map $\beta\alpha - \beta\alpha^2 : P_1 \to P_2$. In fact, use the relations $\gamma\beta\alpha = \alpha^3$ (for degree 0) and $\beta\alpha\gamma = \beta\alpha^2\gamma$ (for degree -1) in Λ_1.

Hence, $\mathrm{End}_{K^{\mathrm{b}}}(T) \cong \Lambda_2$ and therefore the algebras Λ_1 and Λ_2 are derived equivalent, by Rickard's criterion. ∎

We now prove the assertion of Theorem 3.1 in the case of three simple modules. According to the list of nonstandard algebras of tubular type (given in Section 1) it suffices to prove the following result.

PROPOSITION 3.3. (1) *Λ_4 is derived equivalent to Λ_5.*
(2) *Λ_5 is derived equivalent to Λ_6.*
(3) *Λ_5 is derived equivalent to Λ_7.*
(4) *Λ_4 is derived equivalent to Λ_8.*

In particular, any nonstandard algebra of tubular type with three simple modules is derived equivalent to the algebra Λ_4.

Proof. We deal with the assertions (1)–(4) separately. The last assertion then follows directly from the classification of nonstandard algebras of tubular type summarized in Table 1.

(1) We consider the complex $T = T_1 \oplus T_2 \oplus T_3$ of projective Λ_5-modules with $T_1 : 0 \to P_1 \to 0$, $T_2 : 0 \to P_2 \to 0$ (concentrated in degree 0), $T_3 : 0 \to P_2 \xrightarrow{\delta} P_3 \to 0$ (in degrees 0 and -1). Then T is a tilting complex for Λ_5 (we leave the details of the usual verification to the reader). Our aim is to show that the endomorphism ring $\mathrm{End}_{K^{\mathrm{b}}}(T)$ is isomorphic to the algebra Λ_4. From the Cartan matrix of Λ_5 we compute (using Happel's formula) the Cartan matrix of $\mathrm{End}_{K^{\mathrm{b}}}(T)$:

$$C_{\Lambda_5} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad C_{\mathrm{End}_{K^{\mathrm{b}}}(T)} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix},$$

and the latter indeed equals the Cartan matrix of Λ_4. So it remains to find suitable morphisms between the summands of T (corresponding to the arrows of the quiver of Λ_4). We set $\widetilde{\alpha} : T_2 \to T_1$ to be (right) multiplication by γ, and $\widetilde{\delta} : T_2 \to T_3$ multiplication by α on P_2 (note this is a homomorphism of complexes since $\alpha\delta = 0$ in Λ_5). Moreover, define $\widetilde{\beta} : T_3 \to T_2$ to be given by the identity map on P_2, and $\widetilde{\gamma} : T_1 \to T_3$ by (right) multiplication with $\beta : P_1 \to P_2$ (which is a homomorphism of complexes since $\beta\delta = 0$).

We have to check the relations. By definition, $\widetilde{\delta}\widetilde{\beta}\widetilde{\delta} = \widetilde{\alpha}\widetilde{\gamma}$ (use $\alpha^2 = \gamma\beta$ in Λ_5), and $\widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha} = \widetilde{\gamma}\widetilde{\beta}\widetilde{\delta}\widetilde{\beta}\widetilde{\alpha}$ (use $\beta\gamma = \beta\alpha\gamma$ in Λ_5). Also $\widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha}\widetilde{\gamma} = 0$ (since $\beta\gamma\beta = 0$) and $\widetilde{\alpha}\widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha} = 0$ (since $\gamma\beta\gamma = 0$). The final relation will hold up to homotopy. In fact, the morphism $(\widetilde{\beta}\widetilde{\delta})^3\widetilde{\beta} : T_3 \to T_2$ is given by α^3 (in degree 0). Since $\alpha^3 = \delta\sigma$ in Λ_5, this morphism factors over the homotopy map $\sigma : P_3 \to P_2$. Hence, $(\widetilde{\beta}\widetilde{\delta})^3\widetilde{\beta}$ is homotopic to zero.

Altogether we have shown that $\mathrm{End}_{K^{\mathrm{b}}}(T) \cong \Lambda_4$, thus by Rickard's criterion, Λ_4 and Λ_5 are derived equivalent.

(2) We define the following bounded complex $T = T_1 \oplus T_2 \oplus T_3$ of projective Λ_5-modules. Let $T_1 : 0 \to P_2 \xrightarrow{\gamma} P_1 \to 0$ and $T_3 : 0 \to P_2 \xrightarrow{\delta} P_3 \to 0$ (in degrees 0 and -1). Let $T_2 : 0 \to P_2 \to 0$ be the stalk complex in degree 0. Then T is a tilting complex for Λ_5. (We leave the routine verification to the reader.) We have to determine the endomorphism ring of T. Using Happel's alternating sum formula we compute the Cartan matrix of $\mathrm{End}_{K^{\mathrm{b}}}(T)$:

$$C_{\Lambda_5} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad C_{\mathrm{End}_{K^{\mathrm{b}}}(T)} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix},$$

and the latter is equal to the Cartan matrix of Λ_6. So we have to find suitable morphisms between the summands of T. Let $\widetilde{\alpha} : T_1 \to T_2$ and $\widetilde{\gamma} : T_3 \to T_2$ be both given by the identity map on P_2 (in degree 0). Let $\widetilde{\beta} : T_2 \to T_1$ be defined by (right) multiplication with α^2 in degree 0 (a map of complexes since $\alpha^2\gamma = \gamma\beta\gamma = 0$ in Λ_5). Finally, let $\widetilde{\delta} : T_2 \to T_3$ be defined by (right) multiplication with α in degree 0 (a map of complexes since $\alpha^2\delta = 0$ in Λ_5). It remains to check the relations of Λ_6. Directly from the definition we get $\widetilde{\beta}\widetilde{\alpha}\widetilde{\beta} = 0$ (since $\alpha^4 = 0$ in Λ_5) and $\widetilde{\beta}\widetilde{\alpha} = \widetilde{\delta}\widetilde{\gamma}\widetilde{\delta}\widetilde{\gamma}$ (both equal to α^2). All the other relations hold up to homotopy. In fact, the morphism $\widetilde{\alpha}\widetilde{\delta}\widetilde{\gamma}\widetilde{\delta} : T_1 \to T_3$ is given by α^2 in degree 0 and the zero map in degree -1. Hence it is homotopic to zero via the map $\beta : P_1 \to P_2$ (use $\alpha^2 = \gamma\beta$ and $\beta\delta = 0$ in Λ_5). Similarly, $\widetilde{\gamma}\widetilde{\delta}\widetilde{\gamma}\widetilde{\beta}$ is homotopic to zero via $\sigma : P_3 \to P_2$, and $\widetilde{\alpha}\widetilde{\beta}\widetilde{\alpha}$ is homotopic to zero via $\beta : P_1 \to P_2$. Finally, the morphism $\widetilde{\alpha}\widetilde{\beta} - \widetilde{\alpha}\widetilde{\delta}\widetilde{\gamma}\widetilde{\beta}$ on T_1 is given by $\alpha^2 - \alpha^3 = \gamma\beta - \gamma\beta\alpha$ in degree 0 (and the zero map in degree -1). It is homotopic to zero via $\beta - \beta\alpha : P_1 \to P_2$ (use $\beta\gamma - \beta\alpha\gamma = 0$ in Λ_5).

We have shown that $\mathrm{End}_{K^{\mathrm{b}}}(T) \cong \Lambda_6$ and thus, by Rickard's criterion, Λ_5 and Λ_6 are derived equivalent.

(3) In order to prove that Λ_5 is derived equivalent to Λ_7 we define the following complex $T = T_1 \oplus T_2 \oplus T_3$ of projective Λ_5-modules. Set $T_1 : 0 \to P_2 \xrightarrow{\gamma} P_1 \to 0$ (in degrees 0 and -1). Moreover, let $T_2 : 0 \to P_2 \to 0$ and $T_3 : 0 \to P_3 \to 0$ be the stalk complexes concentrated in degree 0. Then T is a tilting complex for Λ_5. (Again, we leave the verification to the reader.) According to Rickard's criterion, Λ_5 is derived equivalent to the endomorphism ring $\mathrm{End}_{K^{\mathrm{b}}}(T)$. The Cartan matrix of $\mathrm{End}_{K^{\mathrm{b}}}(T)$ can be

computed from the Cartan matrix of Λ_5:

$$C_{\Lambda_5} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad C_{\mathrm{End}_{K^{\mathrm{b}}}(T)} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

and the latter is indeed equal to the Cartan matrix of Λ_7. So it suffices to find suitable morphisms between the summands of T. Let $\widetilde{\alpha} : T_2 \to T_2$ be given by multiplication with α on P_2, and let $\widetilde{\sigma} : T_2 \to T_3$ be given by multiplication with $\delta : P_2 \to P_3$. Moreover, let $\widetilde{\gamma} : T_3 \to T_1$ be given by multiplication with $\sigma : P_3 \to P_2$ (which is in fact a homomorphism of complexes since $\sigma\gamma = 0$). Finally, define morphisms $\widetilde{\beta} : T_1 \to T_2$ by the identity on P_2 (in degree 0) and $\widetilde{\delta} : T_2 \to T_1$ by multiplication with α^2 on P_2 (a map of complexes since $\alpha^2\gamma = \gamma\beta\gamma = 0$ in Λ_5).

It remains to check that these morphisms satisfy the relations of Λ_7. Immediately from the definition we see that $\widetilde{\alpha}\widetilde{\sigma} = 0$ (since $\alpha\delta = 0$), $\widetilde{\alpha}\widetilde{\delta} = \widetilde{\sigma}\widetilde{\gamma}$ (both equal to $\alpha^3 = \delta\sigma$), $\widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha} = 0$ and $\widetilde{\gamma}\widetilde{\beta}\widetilde{\delta} = 0$ (the last two since $\sigma\alpha = 0$). Moreover, $\widetilde{\alpha}^2 = \widetilde{\delta}\widetilde{\beta}$ and $\widetilde{\delta}\widetilde{\beta}\widetilde{\delta} = 0$ (since $\alpha^4 = \delta\sigma\alpha = 0$ in Λ_5). The remaining two relations hold up to homotopy. In fact, $\widetilde{\beta}\widetilde{\delta}\widetilde{\beta} : T_1 \to T_2$ is given by $\alpha^2 = \gamma\beta$ (in degree 0), so it is homotopic to zero via the map $\beta : P_1 \to P_2$. Finally, the morphism $\widetilde{\beta}\widetilde{\delta} - \widetilde{\beta}\widetilde{\alpha}\widetilde{\delta}$ on T_2 is given by $\alpha^2 - \alpha^3$ in degree 0 and the zero map in degree -1. It is homotopic to zero via the map $\beta - \beta\alpha : P_1 \to P_2$ (use $\alpha^2 - \alpha^3 = \gamma\beta - \gamma\beta\alpha$ and $\beta\gamma = \beta\alpha\gamma$ in Λ_5).

We have shown that $\mathrm{End}_{K^{\mathrm{b}}}(T) \cong \Lambda_7$ and hence Λ_5 and Λ_7 are derived equivalent.

(4) We consider the following complex $T = T_1 \oplus T_2 \oplus T_3$ of projective Λ_4-modules. We set $T_1 : 0 \to P_2 \xrightarrow{\alpha} P_1 \to 0$ and $T_3 : 0 \to P_2 \xrightarrow{\delta} P_3 \to 0$ (in degrees 0 and -1). Let $T_2 : 0 \to P_2 \to 0$ be the stalk complex in degree 0. Then T is a tilting complex for Λ_4 (verification left to the reader). The Cartan matrix of $\mathrm{End}_{K^{\mathrm{b}}}(T)$ can be computed from the Cartan matrix of Λ_4:

$$C_{\Lambda_4} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \quad C_{\mathrm{End}_{K^{\mathrm{b}}}(T)} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

and the latter is equal to the Cartan matrix of Λ_8, as desired. We now have to define suitable morphisms between the summands of T. We let $\widetilde{\alpha}$ on T_2 be given by (right) multiplication with $\delta\beta$, and $\widetilde{\beta} : T_2 \to T_1$ by $(\delta\beta)^2$ in degree 0 (a map of complexes since $(\delta\beta)^2\alpha = \alpha\gamma\beta\alpha = 0$ in Λ_4). Let $\widetilde{\delta} : T_1 \to T_2$ and $\widetilde{\sigma} : T_3 \to T_2$ be given by the identity map on P_2. Finally, let $\widetilde{\gamma} : T_1 \to T_3$ be given by (right) multiplication with $\delta\beta$ on P_2 in degree

0 and multiplication with $\gamma : P_1 \to P_3$ in degree -1 (a map of complexes since $\delta\beta\delta = \alpha\gamma$ in Λ_4). It remains to show that these morphisms satisfy the defining relations of Λ_8. Immediately from the definitions we conclude that $\widetilde{\delta}\widetilde{\alpha} = \widetilde{\gamma}\widetilde{\sigma}$ (both equal to $\delta\sigma$ in degree 0), $\widetilde{\alpha}^2 = \widetilde{\beta}\widetilde{\delta}$ (both equal to $(\delta\beta)^2$ in degree 0), $\widetilde{\alpha}\widetilde{\beta}\widetilde{\gamma} = 0$ and $\widetilde{\beta}\widetilde{\delta}\widetilde{\beta} = 0$ (since $(\delta\beta)^4 = 0$ in Λ_4). All remaining relations will be shown to hold up to homotopy. In fact, $\widetilde{\sigma}\widetilde{\alpha} : T_3 \to T_2$ is homotopic to zero via $\beta : P_3 \to P_2$, and $\widetilde{\delta}\widetilde{\beta}\widetilde{\delta} : T_1 \to T_2$ is homotopic to zero via $\gamma\beta : P_1 \to P_2$ (since $(\delta\beta)^2 = \alpha\gamma\beta$ in Λ_4). Moreover, the morphism $\widetilde{\delta}\widetilde{\beta}\widetilde{\gamma} : T_1 \to T_3$ is homotopic to zero via $\gamma\beta\delta\beta : P_1 \to P_2$ (use $(\delta\beta)^3 = \alpha\gamma\beta\delta\beta$ and $\gamma\beta\delta\beta\delta = \gamma\beta\alpha\gamma = 0$ in Λ_4). Finally, the morphism $\widetilde{\delta}\widetilde{\beta} - \widetilde{\delta}\widetilde{\alpha}\widetilde{\beta}$ on T_1 is given by $(\delta\beta)^2 - (\delta\beta)^3 = \alpha\gamma\beta - \alpha\gamma\beta\delta\beta$ in degree 0 and the zero map in degree -1. Hence it is homotopic to zero via the homotopy map $\gamma\beta - \gamma\beta\delta\beta : P_1 \to P_2$ (use $\gamma\beta\alpha = \gamma\beta\delta\beta\alpha$ in Λ_4).

Altogether we have shown that the endomorphism ring of T is isomorphic to Λ_8. Hence, Λ_4 and Λ_8 are derived equivalent, by Rickard's criterion. ∎

In [1], we have studied derived equivalences among standard algebras of tubular type. In particular, we have proved that Λ_1' and Λ_2' are derived equivalent and that Λ_4', Λ_5', Λ_6', Λ_7' and Λ_8' are derived equivalent. It is an interesting open question whether a nonstandard algebra can be derived equivalent to the corresponding standard algebra.

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Faculty of Mathematics
and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: jb@mat.uni.torun.pl
skowron@mat.uni.torun.pl

Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg
Postfach 4120
39016 Magdeburg, Germany
E-mail: thorsten.holm@mathematik.uni-magdeburg.de

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