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LOCAL DERIVATIONS FOR QUOTIENT AND FACTOR ALGEBRAS OF POLYNOMIALS

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Abstract. We describe all Kadison algebras of the form $S^{-1}k[t]$, where k is an algebraically closed field and S is a multiplicative subset of $k[t]$. We also describe all Kadison algebras of the form $k[t]/I$, where k is a field of characteristic zero.

1. Introduction. Let k be a field and A a k-algebra with unity. A k-linear mapping $d : A \to A$ is called a *derivation* of A if $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$. A k-linear mapping $\gamma : A \to A$ is called a *local derivation* of A if for each $a \in A$ there exists a derivation d_a of A such that $\gamma(a) = d_a(a)$.

Every derivation of A is of course a local derivation of A. We say (as in [6]) that a k-algebra A is a *Kadison algebra* if every local derivation of A is a derivation.

R. Kadison [1], in 1990, proved that polynomial rings over $\mathbb{C}$ are Kadison algebras. It was proved in [6] that any polynomial ring over k is a Kadison algebra if and only if k is infinite. J. Zieliński, in [10], gave a description of the local derivations of the polynomial ring in one variable in characteristic two. Moreover, he described all local derivations in the formal power series ring in one variable in any characteristic. Some observations on local derivations of commutative algebras are also given in [9].

There are several papers on local derivations for noncommutative algebras. Larson and Sourour [4], who introduced local derivations independently, proved that the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators on a Banach space $\mathcal{X}$ is a Kadison algebra. Shul'man [7] showed that any C^*-algebra is a Kadison algebra. Recently Wiehl [8] proved that the Weyl algebra with one pair of generators is a Kadison algebra.

In the present paper we investigate local derivations on some commutative k-algebras connected with $k[t]$, the polynomial ring in one variable over k. We describe (Theorem 2.8) all Kadison algebras of the form $S^{-1}k[t]$, where S is a multiplicative subset of $k[t]$ and the field k is algebraically closed. We also describe (Section 3) all Kadison algebras of the form $k[t]/I$,

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where I is an ideal of $k[t]$ and $\mathrm{char}(k) = 0$. Moreover, we present some new examples of nontrivial local derivations.

2. Algebras of quotients. Let $A = S^{-1}k[X]$ be the algebra of quotients ot the polynomial algebra $k[X] = k[x_1, \ldots, x_n]$ with respect to a multiplicative subset $S \subset k[X]$, where k is a field. We assume that $0 \notin S$ and we denote by $\mathcal{M}_S$ the subset of k^n defined as

$$\mathcal{M}_S = \{\lambda \in k^n \,;\, \exists_{f \in S}\, f(\lambda) = 0\}.$$

In this section we study local derivations of A. We try to determine when A is a Kadison algebra. We already know the following two partial results concerning this problem.

THEOREM 2.1 ([6]). *Let $A = S^{-1}k[X]$, where S is a multiplicative subset of $k[X]$. If k is infinite and the set $k^n \setminus \mathcal{M}_S$ is dense in the Zariski topology of k^n, then A is a Kadison algebra.* ∎

THEOREM 2.2 ([6]). *Let P be a prime ideal of $k[X]$ and let $A = S^{-1}k[X]$, where $S = k[X] \setminus P$. Then A is not a Kadison algebra.* ∎

If the multiplicative subset S is arbitrary and we cannot use the above theorems, then it is not easy to check when $A = S^{-1}k[X]$ is a Kadison algebra. This problem is not easy even for $n = 1$.

Assume now that $n = 1$. Let $p_1, \ldots, p_s \in k[t] \setminus k$ be pairwise different irreducible monic polynomials, and let S be the multiplicative subset $k[t] \setminus ((p_1) \cup \ldots \cup (p_s))$, where each (p_i) is the principal ideal of $k[t]$ generated by p_i. Consider the quotient ring

$$W = S^{-1}k[t].$$

It is clear that this ring is a unique factorization domain and that $\frac{p_1}{1}, \ldots \frac{p_s}{1}$ are all (up to association) the prime elements of W. Every derivation of W is of the form $w\frac{\partial}{\partial t}$, where $w \in W$. If $\varphi \in W$ and $n \in \mathbb{N}$, then we denote by $\varphi^{(n)}$ the nth derivative of φ.

PROPOSITION 2.3. *For any $n \in \mathbb{N}$ the mapping $\alpha_n : W \to W$ given by*

$$\alpha_n(\varphi) = (p_1 \ldots p_s)^n \varphi^{(n+1)} \quad \text{for } \varphi \in W$$

is a local derivation of W. If $n \geq 1$ and $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n + 1$, then α_n is not a derivation of W.

Proof. Let $\varphi \in W$. If $\alpha_n(\varphi) = 0$, then $\alpha_n(\varphi) = d(\varphi)$, where d is the zero derivation. If $\varphi^{(1)}$ is invertible in W, then $\alpha_n(\varphi) = d(\varphi)$, where $d : W \to W$ is the unique derivation such that $d(t) = (\varphi^{(1)})^{-1}\alpha_n(\varphi)$.

Assume now that $\alpha_n(\varphi) \neq 0$ and $\varphi^{(1)} = p_1^{r_1} \ldots p_s^{r_s} w$, where w is an invertible element of W and $r_1 + \ldots + r_s \geq 1$. Put $p := p_1 \ldots p_s$.

CASE 1. Assume that $r_1 \leq n, \ldots, r_s \leq n$. In this case consider the derivation d of W such that

$$d(t) = p_1^{n-r_1} \ldots p_s^{n-r_s} w^{-1} \varphi^{(n+1)}.$$

Then $d(\varphi) = \varphi^{(1)} d(t) = p_1^{r_1} \ldots p_s^{r_s} w p_1^{n-r_1} \ldots p_s^{n-r_s} w^{-1} \varphi^{(n+1)} = p^n \varphi^{(n+1)} = \alpha_n(\varphi)$.

CASE 2. Assume that there exists $m < s$ such that $r_1 > n, \ldots, r_m > n$ and $r_{m+1} \leq n, \ldots, r_s \leq n$. Then we have $\varphi^{(n+1)} = p_1^{r_1-n} \ldots p_m^{r_m-n} w_0$ for some $w_0 \in W$. Consider the derivation $d : W \to W$ such that

$$d(t) = p_{m+1}^{n-r_{m+1}} \ldots p_s^{n-r_s} w_0 w^{-1}.$$

Then $d(\varphi) = \varphi^{(1)} d(t) = p_1^{r_1} \ldots p_m^{r_m} p_{m+1}^{r_{m+1}} \ldots p_s^{r_s} w \cdot p_{m+1}^{n-r_{m+1}} \ldots p_s^{n-r_s} w_0 w^{-1} = p_1^n \ldots p_s^n \cdot (p_1^{r_1-n} \ldots p_m^{r_m-n} w_0) = p^n \varphi^{(n+1)} = \alpha_n(\varphi)$.

CASE 3. Let $r_1 > n, \ldots, r_s > n$. Then $\varphi^{(n+1)} = p_1^{r_1-n} \ldots p_s^{r_s-n} w_0$ for some $w_0 \in W$. Put $d(t) := w_0 w^{-1}$. Then we have $d(\varphi) = \varphi^{(1)} d(t) = p_1^{r_1} \ldots p_s^{r_s} w \cdot w_0 w^{-1} = p_1^n \ldots p_s^n \cdot (p_1^{r_1-n} \ldots p_s^{r_s-n} w_0) = p^n \varphi^{(n+1)} = \alpha_n(\varphi)$.

Therefore, α_n is a local derivation of W. Obviously α_0 is a derivation, because $\alpha_0 = \frac{\partial}{\partial t}$. Observe that if $n \geq 1$, then $\alpha_n(t) = 0$ and $\alpha_n(t^{n+1}) = p^n (n+1)!$.

Let $n \geq 1$ and suppose that α_n is a derivation of W. Then, since $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n+1$, we have a contradiction: $0 \neq p^n (n+1)! = \alpha_n(t^n) = n t^{n-1} \alpha_n(t) = n t^{n-1} 0 = 0$. Hence, if $n \geq 1$, then α_n is not a derivation. ∎

COROLLARY 2.4. *If* $\mathrm{char}(k) \neq 2$, *then W is not a Kadison algebra.* ∎

Proposition 2.3 can be generalized in the following way.

PROPOSITION 2.5. *Let $n \geq 1$ and let $d_1, \ldots, d_{n+1} : W \to W$ be nonzero derivations. Let $\beta : W \to W$ be the mapping defined as*

$$\beta(\varphi) = p^n d_1 \ldots d_{n+1}(\varphi) \quad \text{for } \varphi \in W,$$

where $p = p_1 \ldots p_s$. Then β is a local derivation of W. If $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > n+1$, *then β is not a derivation of W.*

In the proof of this proposition we will use the following standard and easy lemma.

LEMMA 2.6. *Let $d_1, \ldots, d_m$ be derivations of W. Then there exist elements $w_1, \ldots, w_m \in W$ such that*

$$d_1 \ldots d_m(\varphi) = w_1 \varphi^{(1)} + w_2 \varphi^{(2)} + \ldots + w_m \varphi^{(m)}$$

for all $\varphi \in W$. ∎

Proof of Proposition 2.5. By Lemma 2.6 there exist $w_1, \ldots, w_{n+1} \in W$ such that

$$\beta(\varphi) = p^n w_1 \varphi^{(1)} + p^n w_2 \varphi^{(2)} + \ldots + p^n w_{n+1} \varphi^{(n+1)}$$

for every $\varphi \in W$. Hence $\beta = \beta_1 + \ldots + \beta_{n+1}$, where $\beta_i(\varphi) = p^n w_i \varphi^{(i)}$ for $\varphi \in W$ and $i = 1, \ldots, n+1$. Observe that if $i \in \{1, \ldots, n+1\}$, then $\beta_i = p^{n-i-1} w_i \alpha_{i-1}$, where $\alpha_{i-1}(\varphi) = p^i \varphi^{(i+1)}$. We know, by Proposition 2.3, that each α_{i-1} is a local derivation of W. This implies that each β_i is also a local derivation of W. Therefore, β is a local derivation of W as a sum of local derivations.

Suppose that β is a derivation of W. Then the composition $d_1 \ldots d_{n+1}$ is a derivation of W. But it is well known ([3], [5]) that then $d_i = 0$ for some $i \in \{1, \ldots, n+1\}$. Hence, we have a contradiction, because the derivations $d_1, \ldots, d_{n+1}$ are nonzero. ■

If the number s of irreducible prime elements in $A = S^{-1}k[t]$ is infinite, then the following example shows that Corollary 2.4 is not true in general.

EXAMPLE 2.7. Let $\mathrm{char}(k) = 0$ and let S be a multiplicative subset of $k[t]$ of the form $S = k[t] \smallsetminus \bigcup_{n \in \mathbb{N}} (t - n)$. Then $A = S^{-1}k[t]$ is a Kadison algebra.

Proof. Observe that, in this case, $k \smallsetminus \mathcal{M}_S = \{0, 1, \ldots\}$. Since every closed subset in the Zariski topology of k^1 is k^1 or finite, the set $k \smallsetminus \mathcal{M}_S$ is dense in k^1. Hence, by Theorem 2.1, A is a Kadison algebra. ■

THEOREM 2.8. *Assume that k is algebraically closed and* $\mathrm{char}(k) \neq 2$. *Let S be a multiplicative subset of $k[t]$ such that* $0 \notin S$. *Then $A = S^{-1}k[t]$ is not a Kadison algebra if and only if the set $k \smallsetminus \mathcal{M}_S$ is finite.*

Proof. The implication $\Rightarrow$ is a consequence of Theorem 2.1. Assume that the set $k \smallsetminus \mathcal{M}_S$ is finite.

Let $\overline{S}$ denote the multiplicative subset of $k[t]$ generated by the set of all prime elements which appear as factors of elements from S. Then it is clear that $S^{-1}k[t] = \overline{S}^{-1}k[t]$ and $\mathcal{M}_S = \mathcal{M}_{\overline{S}}$. Therefore, we may assume that $S = \overline{S}$.

Observe that (since k is algebraically closed) $\mathcal{M}_S = \{a \in k\,;\ t - a \in S\}$. In particular, $\mathcal{M}_S = k \ \Leftrightarrow\ S^{-1}k[t] = k(t)$. Hence, if the set $k \smallsetminus \mathcal{M}_S$ is empty, then $A = k(t)$ and, by Theorem 2.2, A is not a Kadison algebra.

Assume now that $k \smallsetminus \mathcal{M}_S = \{a_1, \ldots, a_s\}$ and let $p_1 = t - a_1, \ldots, p_s = t - a_s$. Then $S = k[t] \smallsetminus ((p_1) \cup \ldots \cup (p_s))$ so, by Proposition 2.3, A is not a Kadison algebra. ■

We will show, in Example 2.12, that the assumption "k is algebraically closed" in the above theorem is important. To this end we first prove the following proposition.

PROPOSITION 2.9. *Let S be a multiplicative subset of $k[t]$ such that $A = S^{-1}k[t] \neq k(t)$. If $\alpha : A \to A$ is a local derivation such that $\alpha(k[t]) = 0$, then $\alpha = 0$.*

Proof. The assumption $A \neq k(t)$ implies that there exists at least one irreducible polynomial $p \in k[t]$ such that $(p) \cap S = \emptyset$.

Let $\varphi = \frac{f}{g} \in A$, where $f, g \in k[t]$ and $g \in S$. Then $g \notin (p)$. So $\gcd(g, p) = 1$ and moreover, $\gcd(g, p^n) = 1$ for any natural n. Let $n \geq 1$. There exist polynomials $u_n, v_n \in k[t]$ such that $1 = -u_n g + v_n p^n$. Then $f + f u_n g = f v_n p^n$ and

$$\begin{aligned}\alpha\left(\frac{f}{g}\right) &= \alpha\left(\frac{f}{g}\right) + 0 = \alpha\left(\frac{f}{g} + f u_n\right) \\ &= \alpha\left(\frac{f + f u_n g}{g}\right) = \alpha\left(\frac{f}{g v_n}\, p^n\right) = \alpha(r_n p^n),\end{aligned}$$

where $r_n = \frac{f v_n}{g} \in A$. Let $d : A \to A$ be a derivation such that $d(r_n p^n) = \alpha(r_n p^n)$. Then $\alpha\big(\frac{f}{g}\big) = d(r_n p^n) = d(r_n)p^n + n p^{n-1} r_n d(p) = p^{n-1} w_n$ for some $w_n \in A$. Since A is a unique factorization domain and p is irreducible, the element p^{n-1} divides in A the element $\alpha\big(\frac{f}{g}\big)$. Therefore, $\alpha\big(\frac{f}{g}\big)$ is divisible by p^{n-1} for all $n \geq 1$. This implies that $\alpha\big(\frac{f}{g}\big) = 0$, so $\alpha = 0$. ∎

COROLLARY 2.10. *Let S be a multiplicative subset of $k[t]$ such that $A = S^{-1}k[t] \neq k(t)$. If $\alpha : A \to A$ is a local derivation such that $\alpha|_{k[t]} : k[t] \to A$ is a derivation, then α is a derivation.*

Proof. Put $d := \alpha|_{k[t]}$. The mapping $d : k[t] \to A$ is a derivation. Let us extend it, in a natural way, to a derivation $\overline{d} : A \to A$. Let $\beta = \alpha - \overline{d}$. Then $\beta : A \to A$ is a local derivation such that $\beta(k[t]) = 0$. By Proposition 2.9, $\beta = 0$, so $\alpha = \overline{d}$ is a derivation. ∎

Now let F be a field such that $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ (where $\mathbb{Q}$, $\mathbb{R}$ are the fields of rational and real numbers, respectively). Consider the multiplicative subset of $F[t]$ defined by

$$S_0 = F[t] \smallsetminus \Big(\bigcup_{n \in \mathbb{N}^*} (t^2 + n) \cup \bigcup_{n \in \mathbb{N}^*} (t^2 + t + n) \Big),$$

where $\mathbb{N}^* = \mathbb{N} \smallsetminus \{0\}$.

THEOREM 2.11. *Let S be a multiplicative subset of $F[t]$ such that $S \subseteq S_0$, and let $A = S^{-1}F[t]$. Then A is a Kadison algebra.*

Proof. The algebra A is a unique factorization domain ([2]) and every polynomial of the form $t^2 + n$ or $t^2 + t + n$, where $n \in \mathbb{N}^*$, is a prime element of A. Let $\alpha : A \to A$ be a local derivation. There exists a derivation $d : A \to A$ such that $\alpha(t) = d(t)$. Put $\gamma = \alpha - d$. The mapping γ is a local derivation of A and $\gamma(t) = 0$. We will show that $\gamma = 0$. The proof of this fact is divided in several steps.

(i) Let $m \in \mathbb{N}^*$. We show that if $\gamma(t^m) = 0$, then $\gamma(t^{m+2}) = 0$. Assume that $\gamma(t^m) = 0$ and let $\gamma(t^{m+2}) = \frac{a}{b}$, where $a, b \in F[t]$ with $b \in S$. Let $n \in \mathbb{N}^*$ and consider the polynomial $w_n = t^{m+2} + p_n t^m$, where $p_n = \frac{m+2}{m} n$. Then $\gamma(w_n) = \gamma(t^{m+2}) = \frac{a}{b}$ and there exists a derivation $d_n : A \to A$ such that $d_n(w_n) = \gamma(w_n)$. Let $d_n(t) = \frac{u_n}{v_n}$, where $u_n, v_n \in F[t]$, $v_n \in S$. Then

$$\frac{a}{b} = d_n(w_n) = ((m+2)t^{m+1} + m p_n t^{m-1}) \frac{u_n}{v_n} = (m+2)t^{m-1}(t^2 + n) \frac{u_n}{v_n}$$

and this implies that every prime element of the form $t^2 + n$, for any $n \in \mathbb{N}^*$, divides $\frac{a}{b}$. Hence, $\frac{a}{b} = 0$, that is, $\gamma(t^{m+2}) = 0$.

(ii) We show that $\gamma(t^2) = 0$. We already know, by (i), that $\gamma(t^3) = 0$ (because $\gamma(t) = 0$). Assume that $\gamma(t^2) = \frac{a}{b}$, where $a, b \in F[t]$ with $b \in S$. Let $n \in \mathbb{N}^*$ and consider the polynomial $w_n = t^2 + \frac{2}{3}t^3 + 2nt$. Then $\gamma(w_n) = \gamma(t^2) = \frac{a}{b}$ and there exists a derivation $d_n : A \to A$ such that $d_n(w_n) = \gamma(w_n)$. Let $d_n(t) = \frac{u_n}{v_n}$, where $u_n, v_n \in F[t]$, $v_n \in S$. Then

$$\frac{a}{b} = d_n(w_n) = (2t + 2t^2 + 2n) \frac{u_n}{v_n} = 2(t^2 + t + n) \frac{u_n}{v_n}$$

and this implies that every prime element of the form $t^2 + t + n$, for any $n \in \mathbb{N}^*$, divides $\frac{a}{b}$. Hence, $\frac{a}{b} = 0$, that is, $\gamma(t^2) = 0$.

(iii) The assumption $\gamma(t) = 0$ and steps (i) and (ii) imply that $\gamma(F[t]) = 0$. Hence, by Proposition 2.9, $\gamma = 0$. But $\gamma = \alpha - d$. Therefore, $\alpha = d$ is a derivation. ∎

Example 2.12. Consider the algebra $A = S^{-1}k[t]$, where $k = F = \mathbb{Q}$ and $S = S_0$. Then $k \setminus \mathcal{M}_S = \emptyset$ and, by Theorem 2.11, A is a Kadison algebra. Thus, Theorem 2.8 is not valid if the field k is not algebraically closed. ∎

There are simple examples of algebras of the form $S^{-1}k[t]$ such that we do not know if they are Kadison algebras. For instance, let $k = \mathbb{R}$ and $S = k[t] \setminus \bigcup_{n \in \mathbb{N}^*} (t^2 + n)$. In this case $k \setminus \mathcal{M}_S = \emptyset$. Is it true that $S^{-1}k[t]$ is a Kadison algebra?

Using the same arguments as in the proof of Proposition 2.3 we obtain:

Proposition 2.13. *Let $A = S^{-1}k[X]$, where $k[X] = k[x_1, \ldots, x_n]$ and $S = k[X] \setminus \bigcup_{i=1}^m (x_i)$. Let Δ be the partial derivative $\frac{\partial}{\partial x_1}$ and let $m \in \mathbb{N}$. Then the mapping $\alpha_m : A \to A$ defined by*

$$\alpha_m(\varphi) = (x_1 \ldots x_n)^m \Delta^{m+1}(\varphi) \quad \text{for } \varphi \in A$$

is a local derivation of A. If $m \geq 1$ and $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) > m + 1$, then α_m is not a derivation. ∎

Note yet the following proposition for the algebra of the form $S^{-1}k[X]$.

PROPOSITION 2.14. *Let* $A = S^{-1}k[X]$, *where* $k[X] = k[x_1, \dots, x_n]$ *and* S *is a multiplicative subset of* $k[X]$. *Denote by* M *the maximal ideal* $(x_1, \dots, x_n)$ *and assume that* $M \cap S = \emptyset$. *If* $\alpha : A \to A$ *is a local derivation such that the mapping* $\alpha|_{k[X]} : k[X] \to A$ *is a derivation, then* α *is a derivation of* A.

Proof. There exists a unique derivation $d : A \to A$ such that $d(x_i) = \alpha(x_i)$ for all $i = 1, \dots, n$. Let $\alpha_1 = \alpha - d$. Then $\alpha_1 : A \to A$ is a local derivation such that $\alpha_1(k[X]) = 0$. We will show that $\alpha_1 = 0$.

Let $\varphi = \frac{u}{v} \in A$ with $u, v \in k[X]$ and $v \in S$. If $v \in k$, then $\alpha_1(\varphi) = 0$ because $\varphi \in k[X]$. Assume that $v \notin k$. Since $v \in S$ and $S \cap M = \emptyset$, the element v is not in M, so $a := v(0)$ is a nonzero element of k, where $0 = (0, \dots, 0)$. Then $w := v - a \in M$, that is, $v = w + a$, where $w \in M$ and $a \in k \setminus \{0\}$. Therefore

$$\varphi = \frac{u}{v} = \frac{u}{w + a} = \frac{a^{-1}u}{a^{-1}w + 1}.$$

Put $f := a^{-1}u$ and $h := a^{-1}w$. Obviously $h \neq 0$ since $v \notin k$. Then $\varphi = \frac{f}{h+1}$, $f, h \in k[X]$ and $0 \neq h \in M$. So, for any odd number $s \geq 3$ we have

$$\varphi = \frac{f}{h+1} = \frac{f + fh^s - fh^s}{h+1} = f\left(\frac{h^s + 1}{h+1}\right) - h^s \frac{f}{h+1} = w + h^s r,$$

where $w \in k[X]$, $r \in A$. This implies that

$$\alpha_1(\varphi) = \alpha_1(w + h^s r) = \alpha_1(w) + \alpha_1(h^s r) = \alpha_1(h^s r).$$

Since α_1 is a local derivation, there exists a derivation $\delta : A \to A$ such that $\alpha_1(h^s r) = \delta(h^s r)$. Hence $\alpha_1(h^s r) = h^{s-1}w_1$ for some $w_1 \in A$, so $\alpha_1(\varphi) = h^{s-1}w_1$. This means that, for any odd number s, the element $\alpha_1(\varphi)$ is divisible by h^{s-1}, where h is a nonzero element belonging to M. But A is a unique factorization domain, so $\alpha_1(\varphi) = 0$. Hence $\alpha_1 = \alpha - d = 0$, that is, $\alpha = d$ is a derivation. ∎

Let us end this section with the following question.

QUESTION 2.15. Let $S = k[x, y] \setminus (M_0 \cup M_1)$, where $M_0 = (x, y)$ and $M_1 = (x-1, y-1)$ are the maximal ideals of $k[x, y]$. Is it true that $S^{-1}k[x, y]$ is not a Kadison algebra?

3. Factor algebras. It is known ([6]) that the class of Kadison algebras is not closed with respect to homomorphic images. This fact is a consequence of the following proposition.

PROPOSITION 3.16. *Let* $A = k[t]/(t^n)$, *where* k *is a field of characteristic zero. Then* A *is a Kadison algebra if and only if* $n \leq 2$. ∎

In this section we describe all Kadison algebras of the form $k[t]/(f)$, where k is a field of characteristic zero and (f) is the principal ideal generated by a polynomial $f \in k[t] \setminus k$. The above proposition is the only fact, known to the authors, concerning this problem.

Assume that $f \in k[t] \setminus k$ and let $f = cf_1^{n_1} \ldots f_r^{n_r}$ be the decomposition of f into irreducible polynomials. Here $0 \neq c \in k$, $f_1, \ldots, f_r$ are pairwise different (up to association) prime polynomials from $k[t]$, and $n_1, \ldots, n_r$ are positive integers. Then, by the Chinese Remainder Theorem, the algebra $k[t]/(f)$ is isomorphic to the product algebra $k[t]/(f_1^{n_1}) \times \ldots \times k[t]/(f_r^{n_r})$. The following lemma is easy to prove.

Lemma 3.17. *Let $A_1, \ldots, A_r$ be k-algebras. Then the product $A_1 \times \ldots \times A_r$ is a Kadison algebra if and only if all $A_1, \ldots, A_r$ are Kadison algebras.* ∎

By the above lemma we can restrict our investigations to the case when the polynomial f is a power of an irreducible polynomial in $k[t]$.

Let $n \geq 1$ and let $A = k[t]/(f^n)$, where f is an irreducible polynomial from $k[t]$. Put $m := \deg f$.

Consider a field extension $k \subset L = k[\xi]$, where $\xi \in L \setminus k$ is a root of f. Every element $v \in L$ has a unique representation of the form

$$v = a_{m-1}\xi^{m-1} + \ldots + a_1\xi + a_0$$

for some $a_0, \ldots, a_{m-1} \in k$. Let $\gamma : L \to k$ be the mapping defined by $\gamma(v) = a_0$ for all $v \in L$. This mapping is of course k-linear.

Lemma 3.18. *For any polynomial $w \in k[t]$ there exist polynomials $h, g \in k[t]$ such that $\gamma(w(\xi)) - hw = gf^{n-1}$.*

Proof. If $\gamma(w(\xi)) = 0$, then we put $h := f^{n-1}$ and $g := -w$. Assume now that $\gamma(w(\xi)) \neq 0$. Then it is clear that $\gcd(w, f^{n-1}) = 1$. Hence, there exist polynomials $a, b \in k[t]$ such that $1 = aw + bf^{n-1}$. Multiplying this equality by $\gamma(w(\xi))$ we obtain $\gamma(w(\xi)) - hw = gf^{n-1}$, where $h = \gamma(w(\xi))a$ and $g = \gamma(w(\xi))b$. ∎

Theorem 3.19. *Let $A = k[t]/(f^n)$, where k is a field of characteristic zero, $n \geq 1$, and $f \in k[t]$ is an irreducible polynomial of degree $m \geq 1$. Then:*

(1) *if $n = 1$, then A is a Kadison algebra;*
(2) *if $m = 1$, then A is a Kadison algebra if and only if $n \leq 2$;*
(3) *if $m \geq 2$ and $n \geq 2$, then A is not a Kadison algebra.*

Proof. If $n = 1$, then the algebra A is a field and the field extension $k \subseteq A$ is algebraic. Since $\operatorname{char}(k) = 0$, A has no nonzero derivations and this implies that the zero mapping is a unique local derivation of A. So, in this case, A is a Kadison algebra.

If $m = 1$, then A is isomorphic to the algebra $k[t]/(t^n)$, so (2) is a consequence of Proposition 3.16.

Assume now that $m \geq 2$ and $n \geq 2$. Consider the mapping $\alpha : A \to A$ defined by

$$\alpha(u + (f^n)) = \gamma(u'(\xi))f + (f^n)$$

for all $u \in k[t]$, where u' denotes the derivative of u.

Note that α is well defined. In fact, if $u+(f^n) = v+(f^n)$, where $u, v \in k[t]$, then $u-v = f^n h$ for some $h \in k[t]$ and then $u'-v' = nf^{n-1}f'h+f^n h' = fh_1$, where $h_1 \in k[t]$. Hence, $u'(\xi) - v'(\xi) = f(\xi)h_1(\xi) = 0h_1(\xi) = 0$, that is, $u'(\xi) = v'(\xi)$, and this implies that $\gamma(u'(\xi))f + (f^n) = \gamma(v'(\xi))f + (f^n)$.

It is clear that α is k-linear. We will show that α is a local derivation of A. Let $u \in k[t]$. If $\alpha(u+(f^n)) = 0$, then $\alpha(u+(f^n)) = d(u+(f^n))$, where d is the zero derivation of A. Assume now that $\alpha(u + (f^n)) \neq 0$. Then, by Lemma 3.18 applied to the polynomial u',

$$\gamma(u'(\xi)) - hu' = gf^{n-1}$$

for some $h, g \in k[t]$. Multiplying this equality by f we deduce that the element $\alpha(u + (f^n)) - hfu'$ belongs to (f^n).

Let $d : k[t] \to k[t]$ be a derivation such that $d(t) = hf$. Since $d((f^n)) \subseteq (f^n)$, this derivation induces a derivation $\overline{d} : A \to A$ such that $\overline{d}(t+(f^n)) = hf + (f^n)$. Then $\overline{d}(u + (f^n)) = u'hf + (f^n) = \alpha(u + (f^n))$. Therefore, α is a local derivation.

Suppose that α is a derivation of A. Then

$$\begin{aligned}\alpha(t^2 + (f^n)) &= (2t + (f^n))\alpha(t + (f^n)) = (2t + (f^n))(f + (f^n)) \\ &= 2tf + (f^n).\end{aligned}$$

But, by the definition of α, $\alpha(t^2 + (f^n)) = 0 + (f^n)$. So, we have a contradiction: $2tf \in (f^n)$. Thus, α is not a derivation. ∎

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