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NEW EXAMPLES OF BIHARMONIC MAPS IN SPHERES

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Abstract. We give some new methods to construct nonharmonic biharmonic maps in the unit n-dimensional sphere $\mathbb{S}^n$.

1. Introduction. It is known that a map $\phi : (M,g) \to (N,h)$ between two Riemannian manifolds is *harmonic* if it is a critical point of the *energy* $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$, and ϕ is harmonic if and only if its tension field $\tau(\phi) = \operatorname{trace} \nabla d\phi$ vanishes (see [9, 7, 15]). In the same way, as suggested by J. Eells and J. H. Sampson in [9], a map ϕ is *biharmonic* if it is a critical point of the *bienergy* $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$. G. Y. Jiang obtained in [11, 12] the first and second variation formula for the bienergy showing that the map ϕ is biharmonic if and only if

$$\tau_2(\phi) = -J(\tau(\phi)) = 0, \tag{1.1}$$

where $J = \Delta^\phi + \operatorname{trace} R^N(d\phi\cdot, \ )d\phi\cdot$ is the Jacobi operator of ϕ. The equation $\tau_2(\phi) = 0$ is called the *biharmonic equation*. Of course, any harmonic map is biharmonic, so we are interested in nonharmonic biharmonic maps. In Jiang's papers the following example was given: the generalized Clifford torus $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2})$, where $n_1 \neq n_2$, is a nonharmonic (nonminimal) biharmonic submanifold of $\mathbb{S}^{n_1+n_2+1}$.

B. Y. Chen and S. Ishikawa proved in [6] that there are no nonminimal biharmonic submanifolds of $\mathbb{R}^3$. Similarly, in [2], it was proved that there are no such submanifolds in $N^3(-1)$, where $N^3(-1)$ is a 3-dimensional manifold with negative constant sectional curvature -1.

In [1] a classification of nonminimal biharmonic submanifolds of $\mathbb{S}^3$ was given. They are: circles, spherical helices and parallel spheres. Then, in [2], two methods were presented to construct examples of nonminimal biharmonic submanifolds of the unit n-dimensional sphere $\mathbb{S}^n$ for $n > 3$. In this case the family of such submanifolds is much larger.

Biharmonic submanifolds of the Heisenberg group H_3 were studied in [4]. Examples of biharmonic helices and biharmonic integral curves were given.

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We note that H_3 has nonconstant sectional curvature, as in the previous cases.

Biharmonic Riemannian submersions were studied in [14], and biharmonic curves on surfaces in [3].

The aim of this paper is to construct some new examples of nonharmonic biharmonic maps in the sphere $\mathbb{S}^n$. First, using harmonic Riemannian submersions, we give two classes of nonharmonic biharmonic maps in $\mathbb{S}^n$ (Theorems 2.1 and 2.3). These maps have constant rank, i.e. they are subimmersions. Finally, using a particular conformal change of the canonical metric on $\mathbb{S}^n$, we get a new class of examples of biharmonic maps in $\mathbb{S}^n$ endowed with the new metric (Theorem 3.7).

NOTATION. We work in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. (M^m, g) will stand for a connected manifold of dimension m, without boundary, endowed with a Riemannian metric g. We denote by ∇ the Levi-Civita connection of (M, g). For the Riemann curvature operator we use the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. For a map $\phi : (M, g) \to (N, h)$ we denote by ∇^ϕ the connection in the pull-back bundle $\phi^{-1}TN$.

2. Biharmonic subimmersions in $\mathbb{S}^n$. Let

$$\begin{aligned}\mathbb{S}^n(a) &= \mathbb{S}^n(a) \times \{b\}\\ &= \{p = (x^1, \ldots, x^{n+1}, b) \mid (x^1)^2 + \ldots + (x^{n+1})^2 = a^2,\ a \in (0,1),\ a^2 + b^2 = 1\}\end{aligned}$$

be a parallel hypersphere of $\mathbb{S}^{n+1}$. We consider on $\mathbb{S}^{n+1}$ the canonical metric $\langle\, , \rangle$. The set of all sections of the tangent bundle of $\mathbb{S}^n(a)$ is given by

$$C(T\mathbb{S}^n(a)) = \{X = (X^1, \ldots, X^{n+1}, 0) \mid x^1X^1 + \ldots + x^{n+1}X^{n+1} = 0\}.$$

Let $\eta = c^{-1}(x^1, \ldots, x^{n+1}, -a^2/b)$, where $c > 0$ and $c^2 = a^2 + a^4/b^2$. Then η satisfies

$$\langle \eta, p\rangle = 0, \quad \langle \eta, X\rangle = 0, \quad |\eta| = 1,$$

i.e. η is a unit section in the normal bundle of $\mathbb{S}^n(a)$ in $\mathbb{S}^{n+1}$. By a direct computation we obtain

$$A = -\frac{1}{c}\, I, \quad B(X, Y) = -\frac{1}{c}\,\langle X, Y\rangle\eta, \quad \nabla^\perp\eta = 0, \tag{2.1}$$

where A is the shape operator, B is the second fundamental form of $\mathbb{S}^n(a)$ and $\nabla^\perp$ is the normal connection in the normal bundle of $\mathbb{S}^n(a)$ in $\mathbb{S}^{n+1}$. It was proved in [1] that $\mathbb{S}^n(a)$ is a biharmonic submanifold of $\mathbb{S}^{n+1}$, i.e. the inclusion map of $\mathbb{S}^n(a)$ in $\mathbb{S}^{n+1}$ is biharmonic, if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$.

Now, we consider a Riemannian submersion $\varphi : (M, g) \to \mathbb{S}^n(a)$, the canonical inclusion $\mathbf{i} : \mathbb{S}^n(a) \to \mathbb{S}^{n+1}$, and $\phi = \mathbf{i} \circ \varphi : (M, g) \to \mathbb{S}^{n+1}$. The rank of ϕ is constant, equal to n.

THEOREM 2.1. *Assume that* $\varphi : (M, g) \to \mathbb{S}^n(a)$ *is a harmonic Riemannian submersion. Then* $\phi : (M, g) \to \mathbb{S}^{n+1}$ *is not harmonic, and it is biharmonic if and only if* $a = 1/\sqrt{2}$ *and* $b = \pm 1/\sqrt{2}$.

Proof. Let $p \in M$. We have $T_pM = T_p^V M \oplus T_p^H M$, where $T_p^V M = \ker d\varphi_p$ and $T_p^H M$ is the orthogonal complement of $T_p^V M$ in T_pM with respect to the metric g. Let W be an open subset of $\mathbb{S}^n(a)$ such that $\varphi(p) \in W$ and let $\{Y_\alpha\}_{\alpha=1}^n$ be an orthonormal frame field of W. Set $U = \varphi^{-1}(W)$, $\{X_\alpha\} = \{Y_\alpha^H\}$, and consider an orthonormal frame field $\{X_s\}_{s=n+1}^m$ on $T^V U$. The tension field of φ is given by

$$\tau(\varphi)_p = -\sum_{s=n+1}^{m} d\varphi_p(\nabla_{X_s} X_s) \tag{2.2}$$

(see [8]). Computing the tension field of ϕ we obtain

$$\tau(\phi) = d\mathbf{i}(\tau(\varphi)) + \operatorname{trace} \nabla d\mathbf{i}(d\varphi\cdot, d\varphi\cdot) = \sum_{\alpha=1}^{n} B(Y_\alpha, Y_\alpha) = -\frac{n}{c}\,\eta,$$

i.e. ϕ is not harmonic.

To simplify the notation, we denote the Levi-Civita connection $\nabla^{\mathbb{S}^n(a)}$ of $\mathbb{S}^n(a)$ by ∇^N. Computing $\Delta^\phi \tau(\phi)$ we get

$$\begin{aligned} -\Delta^\phi \tau(\phi) &= \sum_{k=1}^{m} \{\nabla^\phi_{X_k} \nabla^\phi_{X_k} \tau(\phi) - \nabla^\phi_{\nabla_{X_k} X_k} \tau(\phi)\} \\ &= \sum_{\alpha=1}^{n} \{\nabla^\phi_{X_\alpha} \nabla^\phi_{X_\alpha} \tau(\phi) - \nabla^\phi_{\nabla_{X_\alpha} X_\alpha} \tau(\phi)\} \\ &\quad + \sum_{s=n+1}^{m} \{\nabla^\phi_{X_s} \nabla^\phi_{X_s} \tau(\phi) - \nabla^\phi_{\nabla_{X_s} X_s} \tau(\phi)\}. \end{aligned} \tag{2.3}$$

But

$$\nabla^\phi_{X_\alpha} \tau(\phi) = -\frac{n}{c}\, \nabla^{\mathbb{S}^{n+1}}_{Y_\alpha} \eta = -\frac{n}{c^2}\, Y_\alpha,$$

and using (2.1) we obtain

$$\begin{aligned} \nabla^\phi_{X_\alpha} \nabla^\phi_{X_\alpha} \tau(\phi) &= -\frac{n}{c^2}\, \nabla^{\mathbb{S}^{n+1}}_{Y_\alpha} Y_\alpha = -\frac{n}{c^2} \left(\nabla^N_{Y_\alpha} Y_\alpha - \frac{1}{c}\,\eta \right) \\ &= -\frac{n}{c^2}\, \nabla^N_{Y_\alpha} Y_\alpha + \frac{n}{c^3}\,\eta. \end{aligned} \tag{2.4}$$

Further, we have

$$\nabla^\phi_{\nabla_{X_\alpha} X_\alpha} \tau(\phi) = -\frac{n}{c}\, \nabla^{\mathbb{S}^{n+1}}_{\nabla^N_{Y_\alpha} Y_\alpha} \eta = -\frac{n}{c^2}\, \nabla^N_{Y_\alpha} Y_\alpha, \tag{2.5}$$

$$\nabla^\phi_{X_s} \nabla^\phi_{X_s} \tau(\phi) = 0, \tag{2.6}$$

$$\nabla^\phi_{\nabla_{X_s} X_s} \tau(\phi) = -\frac{n}{c}\, \nabla^{\mathbb{S}^{n+1}}_{d\varphi(\nabla_{X_s} X_s)} \eta = -\frac{n}{c^2}\, d\varphi(\nabla_{X_s} X_s). \tag{2.7}$$

Inserting (2.4)–(2.7) in (2.3), and using (2.2), we obtain

$$-\Delta^{\phi}\tau(\phi) = \frac{n^2}{c^3}\,\eta. \tag{2.8}$$

A direct computation shows

$$\operatorname{trace} R^{\mathbb{S}^{n+1}}(d\phi\cdot, \tau(\phi))d\phi\cdot = \frac{n^2}{c}\,\eta. \tag{2.9}$$

Thus, (2.8), (2.9) and (1.1) give us

$$\tau_2(\phi) = \frac{n^2}{c^3}(1-c^2)\eta,$$

so ϕ is biharmonic if and only if $c = 1$, i.e. $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$. ∎

Since the radial projection

$$\mathbb{S}^n \to \mathbb{S}^n(a), \qquad x \mapsto ax,$$

is homothetic, a harmonic Riemannian submersion $\varphi : (M, g) \to \mathbb{S}^n$ becomes a harmonic Riemannian submersion $\varphi : (M, a^2 g) \to \mathbb{S}^n(a)$, and using the above theorem, we obtain a nonharmonic biharmonic subimmersion $\phi : M \to \mathbb{S}^{n+1}$. For example, the Hopf map induces a nonharmonic biharmonic map $\phi : \mathbb{S}^3(\sqrt{2}) = \{(z^1, z^2) \in \mathbb{C}^2 \mid (z^1)^2 + (z^2)^2 = 2\} \to \mathbb{S}^3$ given by

$$\phi(z^1, z^2) = \frac{1}{2\sqrt{2}}\,(2z^1\overline{z^2}, |z^1|^2 - |z^2|^2, 1).$$

We now give a converse of Theorem 2.1.

Proposition 2.2. *Assume that $\varphi : (M, g) \to \mathbb{S}^n(1/\sqrt{2})$ is a Riemannian submersion with basic tension field, i.e. $\tau(\varphi)(p) = \tau(\varphi)(q)$ whenever $\varphi(p) = \varphi(q)$. Then the map ϕ is biharmonic if and only if φ is harmonic.*

Proof. From the composition law we have

$$\tau(\phi) = \tau(\varphi) - n\eta.$$

As $\tau(\varphi)$ is basic we can think of it as a vector field on $\mathbb{S}^n(1/\sqrt{2})$. Denoting $\nabla^{\mathbb{S}^n(1/\sqrt{2})}$ by ∇^N, we obtain

$$\begin{aligned}
\nabla^{\phi}_{X_\alpha}\tau(\phi) &= \nabla^N_{Y_\alpha}\tau(\varphi) - \langle Y_\alpha, \tau(\varphi)\rangle\eta - nY_\alpha,\\
\nabla^{\phi}_{X_\alpha}\nabla^{\phi}_{X_\alpha}\tau(\phi) &= \nabla^N_{Y_\alpha}\nabla^N_{Y_\alpha}\tau(\varphi) - 2\langle Y_\alpha, \nabla^N_{Y_\alpha}\tau(\varphi)\rangle\eta\\
&\quad - \langle \nabla^N_{Y_\alpha}Y_\alpha, \tau(\varphi)\rangle\eta - \langle Y_\alpha, \tau(\varphi)\rangle Y_\alpha\\
&\quad - n\nabla^N_{Y_\alpha}Y_\alpha + n\eta,\\
\nabla^{\phi}_{\nabla_{X_\alpha}X_\alpha}\tau(\phi) &= \nabla^N_{\nabla^N_{Y_\alpha}Y_\alpha}\tau(\varphi) - \langle \nabla^N_{Y_\alpha}Y_\alpha, \tau(\varphi)\rangle\eta - n\nabla^N_{Y_\alpha}Y_\alpha,\\
\nabla^{\phi}_{X_s}\nabla^{\phi}_{X_s}\tau(\phi) &= 0,\\
\nabla^{\phi}_{\nabla_{X_s}X_s}\tau(\phi) &= \nabla^N_{d\varphi(\nabla_{X_s}X_s)}\tau(\varphi) - \langle d\varphi(\nabla_{X_s}X_s), \tau(\varphi)\rangle\eta\\
&\quad - nd\varphi(\nabla_{X_s}X_s),
\end{aligned}$$

and

$$\operatorname{trace} R^{\mathbb{S}^{n+1}}(d\phi\cdot, \tau(\phi))d\phi\cdot = (1-n)\tau(\varphi) + n^2\eta.$$

It follows that the normal part of $\tau_2(\phi)$ to $\mathbb{S}^n(1/\sqrt{2})$ is

$$-(2\operatorname{div}\tau(\varphi) + |\tau(\varphi)|^2)\eta. \tag{2.10}$$

If ϕ is biharmonic, then (2.10) implies

$$\operatorname{div}\tau(\varphi) = -\frac{1}{2}\,|\tau(\varphi)|^2,$$

and using the Stokes theorem, we get $\tau(\varphi) = 0$, i.e. φ is harmonic.

The converse is immediate. ∎

Let n_1, n_2 be two positive integers such that $n = n_1 + n_2$ and let r_1, r_2 be two positive real numbers such that $r_1^2 + r_2^2 = 1$. Let $\varphi_1 : (M_1, g_1) \to \mathbb{S}^{n_1}(r_1)$ and $\varphi_2 : (M_2, g_2) \to \mathbb{S}^{n_2}(r_2)$ be harmonic Riemannian submersions, and $\phi = \mathbf{i} \circ (\varphi_1 \times \varphi_2)$, where $\mathbf{i} : \mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2) \to \mathbb{S}^{n+1}$ is the canonical inclusion.

THEOREM 2.3. *The map ϕ is a nonharmonic biharmonic subimmersion if and only if $r_1 = r_2 = 1/\sqrt{2}$ and $n_1 \neq n_2$.*

Proof. We set

$$\xi(p) = \left(\frac{r_2}{r_1}\,p_1, -\frac{r_1}{r_2}\,p_2\right),$$

where $p = (p_1, p_2) \in \mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2)$. Then ξ is a unit section in the normal bundle of $\mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2)$ in $\mathbb{S}^{n+1}$.

By a straightforward computation we obtain

$$\tau(\phi) = \frac{r_1^2 n_2 - r_2^2 n_1}{r_1 r_2}\,\xi,$$
$$\tau_2(\phi) = \frac{r_2^2 - r_1^2}{r_1 r_2}\left(\frac{r_1^2 n_2 - r_2^2 n_1}{r_1 r_2}\right)^2 \xi = \frac{r_2^2 - r_1^2}{r_1 r_2}\,|\tau(\phi)|^2 \xi.$$

Thus $\tau(\phi) \neq 0$ and $\tau_2(\phi) = 0$ if and only if $r_1 = r_2 = 1/\sqrt{2}$ and $n_1 \neq n_2$. ∎

3. Biharmonic submanifolds of $(\mathbb{S}^n, e^{2\varrho}\langle\, , \rangle)$**.** We start with the well known results about the conformal changes of the metrics.

Let (N, h) be a Riemannian manifold and let $\varrho \in C^\infty(N)$ be a smooth real map. Set $\widetilde{h} = e^{2\varrho}h$ and denote by ∇^N the Levi-Civita connection of the metric h and by $\widetilde{\nabla}^N$ the Levi-Civita connection of $\widetilde{h}$. We have

$$\widetilde{\nabla}^N_X Y = \nabla^N_X Y + P(X, Y),$$

where the tensor field P is given by

$$P(X, Y) = (X\varrho)Y + (Y\varrho)X - h(X, Y)\operatorname{grad}\varrho.$$

For the corresponding curvature tensor fields we have

$$\widetilde{R}^N(X,Y)Z = R^N(X,Y)Z + (\nabla^N_X P)(Y,Z) - (\nabla^N_Y P)(X,Z) + P(X,P(Y,Z)) - P(Y,P(X,Z)). \tag{3.1}$$

Suppose that $(N,h) = \mathbb{S}^n$ with the canonical metric $\langle\, , \rangle$ and $\varrho(x) = \langle u, x\rangle$, for $x \in \mathbb{S}^n$, where u is a constant vector in $\mathbb{R}^{n+1}$ and $u \neq 0$. Then $\nabla^{\mathbb{S}^n}_X \operatorname{grad}\varrho = -\varrho X$ and $\operatorname{grad}\varrho = u - \varrho r$, where $r = x^1 e_1 + \ldots + x^{n+1}e_{n+1}$ is the radial vector field and $\{e_1, \ldots, e_{n+1}\}$ denotes the canonical frame of $\mathbb{R}^{n+1}$. For this choice of N formula (3.1) becomes

$$\begin{aligned} \widetilde{R}^{\mathbb{S}^n}(X,Y)Z = {} & \langle Z,Y\rangle X - \langle Z,X\rangle Y \\ & + 2\varrho\{\langle Z,Y\rangle X - \langle Z,X\rangle Y\} \\ & + (Y\varrho)(Z\varrho)X - (X\varrho)(Z\varrho)Y \\ & + \{\langle Y,Z\rangle (X\varrho) - \langle X,Z\rangle (Y\varrho)\} \operatorname{grad}\varrho \\ & + |\operatorname{grad}\varrho|^2 \{\langle Z,X\rangle Y - \langle Z,Y\rangle X\}. \end{aligned} \tag{3.2}$$

Now, we consider $\mathbb{S}^{n-1} = \mathbb{S}^{n-1} \times \{0\}$ and let

$$\mathbf{i}_1 : (\mathbb{S}^{n-1}, \langle\, , \rangle) \to (\mathbb{S}^n, \langle\, , \rangle) \quad \text{and} \quad \mathbf{i}_2 : (\mathbb{S}^{n-1}, \langle\, , \rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\, , \rangle)$$

be the canonical inclusions. We have $\mathbf{i}_2 = \mathbf{1} \circ \mathbf{i}_1$, where $\mathbf{1} : (\mathbb{S}^n, \langle\, , \rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\, , \rangle)$ is the identity map. Of course, $\mathbf{i}_1$ is totally geodesic, so it is harmonic and biharmonic.

Assume that $\varrho(x) = x^{n+1} = \langle e_{n+1}, x\rangle$. Concerning the biharmonicity of $\mathbf{i}_2$ we obtain

PROPOSITION 3.1. *The inclusion map* $\mathbf{i}_2 : (\mathbb{S}^{n-1}, \langle\, , \rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\, , \rangle)$ *is nonharmonic biharmonic.*

Proof. From the composition law we get

$$\begin{aligned} \tau(\mathbf{i}_2) &= d\mathbf{1}(\tau(\mathbf{i}_1)) + \operatorname{trace}\nabla d\mathbf{1}(d\mathbf{i}_1\cdot, d\mathbf{i}_1\cdot) = \operatorname{trace}\nabla d\mathbf{1}(d\mathbf{i}_1\cdot, d\mathbf{i}_1\cdot) \\ &= \sum_{k=1}^{n-1} (\widetilde{\nabla}^{\mathbb{S}^n} - \nabla^{\mathbb{S}^n})(X_k, X_k) = \sum_k P(X_k, X_k) \\ &= \sum_k \{2(X_k\varrho)X_k - \operatorname{grad}\varrho\} = -(n-1)\operatorname{grad}\varrho \\ &= -(n-1)e_{n+1}, \end{aligned} \tag{3.3}$$

where $\{X_k\}_{k=1}^{n-1}$ is a local orthonormal frame field on $\mathbb{S}^{n-1}$. Thus $\mathbf{i}_2$ is not harmonic.

To compute $-\Delta^{\mathbf{i}_2}\tau(\mathbf{i}_2)$, let $p \in M$ and let $\{X_k\}_{k=1}^{n-1}$ be a geodesic frame at $p \in \mathbb{S}^{n-1}$. At p we have

$$-\Delta^{\mathbf{i}_2}\tau(\mathbf{i}_2) = \sum_k \widetilde{\nabla}^{\mathbb{S}^n}_{X_k}\widetilde{\nabla}^{\mathbb{S}^n}_{X_k}\tau(\mathbf{i}_2) = -(n-1)\sum_k \widetilde{\nabla}^{\mathbb{S}^n}_{X_k}\widetilde{\nabla}^{\mathbb{S}^n}_{X_k} e_{n+1}.$$

As

$$\begin{aligned}\widetilde{\nabla}^{\mathbb{S}^n}_{X_k} e_{n+1} &= \nabla^{\mathbb{S}^n}_{X_k} e_{n+1} + (X_k\varrho)e_{n+1} + (e_{n+1}\varrho)X_k - \langle X_k, e_{n+1}\rangle \operatorname{grad}\varrho \\ &= \nabla^{\mathbb{S}^n}_{X_k} e_{n+1} + X_k = \nabla^{\mathbb{R}^{n+1}}_{X_k} e_{n+1} + \langle X_k, e_{n+1}\rangle r + X_k = X_k,\end{aligned}$$

it follows that

$$\begin{aligned}-\Delta^{\mathbf{i}_2}\tau(\mathbf{i}_2) &= -(n-1)\sum_k \widetilde{\nabla}^{\mathbb{S}^n}_{X_k} X_k = -(n-1)\tau(\mathbf{i}_2) \\ &= (n-1)^2 e_{n+1}.\end{aligned} \tag{3.4}$$

Using (3.2) we get

$$\operatorname{trace} \widetilde{R}^{\mathbb{S}^n}(d\mathbf{i}_2\cdot, \tau(\mathbf{i}_2))d\mathbf{i}_2\cdot = (n-1)^2 e_{n+1}. \tag{3.5}$$

Inserting (3.4) and (3.5) in the biharmonic equation we deduce that $\mathbf{i}_2$ is biharmonic. ∎

To generalize the above result we consider a minimal submanifold $(M, \langle\,,\,\rangle)$ of $(\mathbb{S}^{n-1}, \langle\,,\,\rangle)$. Let $\mathbf{i} : M \to \mathbb{S}^{n-1}$, $\mathbf{j}_1 = \mathbf{i}_1 \circ \mathbf{i} : (M, \langle\,,\,\rangle) \to (\mathbb{S}^n, \langle\,,\,\rangle)$ and $\mathbf{j}_2 = \mathbf{1} \circ \mathbf{j}_1 : (M, \langle\,,\,\rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\,,\,\rangle)$ be the canonical inclusions. Again ϱ is given by $\varrho(x) = x^{n+1}$.

The map $\mathbf{j}_1$ is harmonic, and following the same steps as in the proof of Proposition 3.1, we get

- $\tau(\mathbf{j}_2) = -m e_{n+1}$,
- $-\Delta^{\mathbf{j}_2}\tau(\mathbf{j}_2) = m^2 e_{n+1}$,
- $\operatorname{trace} \widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot, \tau(\mathbf{j}_2))d\mathbf{j}_2\cdot = m^2 e_{n+1}$.

Thus we get

THEOREM 3.2. *The inclusion map* $\mathbf{j}_2 : (M, \langle\,,\,\rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\,,\,\rangle)$ *is nonharmonic biharmonic.*

REMARK 3.3. We note that:

(1) $(\mathbb{S}^n, e^{2\varrho}\langle\,,\,\rangle)$ has nonconstant sectional curvature;

(2) $\mathbf{j}_2 : (M, \langle\,,\,\rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\,,\,\rangle)$ is a Riemannian immersion;

(3) M is a pseudo-umbilical submanifold of $(\mathbb{S}^n, e^{2\varrho}\langle\,,\,\rangle)$ and its mean curvature vector field is parallel and of norm 1. This result is similar to Theorem 3.4 in [2].

Theorem 3.2 allows us to construct new examples of nonminimal (nonharmonic) biharmonic submanifolds in spaces of nonconstant sectional curvature. For example, using a well known result of H. B. Lawson (see [13]), we get

THEOREM 3.4. *There exist closed orientable embedded nonminimal biharmonic surfaces of arbitrary genus in* $(\mathbb{S}^4, e^{2\varrho}\langle\,,\,\rangle)$.

Proposition 3.5. *Let M be a submanifold of $\mathbb{S}^{n-1}$. Then $\mathbf{j}_2$ is not harmonic, and it is biharmonic if and only if $\mathbf{i}$ is harmonic, i.e. $(M,\langle\,,\,\rangle)$ is minimal in $(\mathbb{S}^{n-1},\langle\,,\,\rangle)$.*

Proof. We have

$$\begin{aligned}\tau(\mathbf{j}_2)&=\tau(\mathbf{1}\circ\mathbf{j}_1)=\tau(\mathbf{j}_1)+\operatorname{trace}\nabla d\mathbf{1}(d\mathbf{j}_1\cdot,d\mathbf{j}_1\cdot)\\&=\tau(\mathbf{1})+\operatorname{trace}\nabla d\mathbf{1}(d\mathbf{j}_1\cdot,d\mathbf{j}_1\cdot)=\tau(\mathbf{1})-m\operatorname{grad}\varrho\\&=\tau(\mathbf{i})-me_{n+1},\end{aligned}$$

so $\mathbf{j}_2$ is not harmonic. The biharmonic equation can be written as

$$\begin{aligned}\tau_2(\mathbf{j}_2)&=-\Delta^{\mathbf{j}_2}\tau(\mathbf{j}_2)-\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot,\tau(\mathbf{j}_2))d\mathbf{j}_2\,\cdot\\&=-\Delta^{\mathbf{j}_2}\tau(\mathbf{i})-\Delta^{\mathbf{j}_2}(-me_{n+1})\\&\quad-\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot,\tau(\mathbf{i}))d\mathbf{j}_2\cdot-\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot,-me_{n+1})d\mathbf{j}_2\cdot\,.\end{aligned}$$

By a straightforward computation we obtain

$$\begin{aligned}-\Delta^{\mathbf{j}_2}\tau(\mathbf{i})&=-\Delta^{\mathbf{i}}\tau(\mathbf{i})+|\tau(\mathbf{i})|^2e_{n+1},\\-\Delta^{\mathbf{j}_2}(-me_{n+1})&=-m\tau(\mathbf{j}_2)=-m\tau(\mathbf{i})+m^2e_{n+1},\end{aligned}$$

and

$$\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot,\tau(\mathbf{i}))d\mathbf{j}_2\cdot=0,\qquad\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{j}_2\cdot,-me_{n+1})d\mathbf{j}_2\cdot=m^2e_{n+1}.$$

Thus we get $\tau_2(\mathbf{j}_2)=-\Delta^{\mathbf{i}}\tau(\mathbf{i})-m\tau(\mathbf{i})+|\tau(\mathbf{i})|^2e_{n+1}$, which proves the proposition. ■

More generally, we consider $\mathbb{S}^{m_1}=\mathbb{S}^{m_1}\times\{0\}$, $0\in\mathbb{R}^{n-m_1}$, $m_1<n-1$, and let

$$\mathbf{i}_1:(\mathbb{S}^{m_1},\langle\,,\,\rangle)\to(\mathbb{S}^n,\langle\,,\,\rangle)\quad\text{and}\quad\mathbf{i}_2:(\mathbb{S}^{m_1},\langle\,,\,\rangle)\to(\mathbb{S}^n,e^{2\varrho}\langle\,,\,\rangle)$$

be the canonical inclusions. Assume that

$$\varrho(x)=\langle u,x\rangle=u^{m_1+2}x^{m_1+2}+\ldots+u^{n+1}x^{n+1},\qquad\forall x\in\mathbb{S}^n,\tag{3.6}$$

where $u=(0,\ldots,0,u^{m_1+2},\ldots,u^{n+1})\in\mathbb{R}^{n+1}$ and $u\neq0$.

Proposition 3.6. *The inclusion map $\mathbf{i}_2:(\mathbb{S}^{m_1},\langle\,,\,\rangle)\to(\mathbb{S}^n,e^{2\varrho}\langle\,,\,\rangle)$ is not harmonic, and it is biharmonic if and only if $|u|=1$.*

Proof. In a similar way we obtain

- $\tau(\mathbf{i}_2)=-m_1u\neq0$,
- $-\Delta^{\mathbf{i}_2}\tau(\mathbf{i}_2)=m_1^2|u|^2u$,
- $\operatorname{trace}\widetilde{R}^{\mathbb{S}^n}(d\mathbf{i}_2\cdot,\tau(\mathbf{i}_2))d\mathbf{i}_2\cdot=m_1^2u$.

Consequently, $\tau_2(\mathbf{i}_2)=m_1^2(|u|^2-1)u$, i.e. the map $\mathbf{i}_2$ is biharmonic if and only if $|u|=1$. ■

Next, let $(M, \langle\, ,\, \rangle)$ be a minimal submanifold of $(\mathbb{S}^{m_1}, \langle\, ,\, \rangle)$ and $\mathbf{i} : M \to \mathbb{S}^{m_1}$ the canonical inclusion. We denote by

$$\mathbf{j}_1 = \mathbf{i}_1 \circ \mathbf{i} : (M, \langle\, ,\, \rangle) \to (\mathbb{S}^n, \langle\, ,\, \rangle) \quad \text{and} \quad \mathbf{j}_2 = \mathbf{1} \circ \mathbf{j}_1 : (M, \langle\, ,\, \rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\, ,\, \rangle)$$

the canonical inclusions, where ϱ is given by (3.6). Then the map $\mathbf{j}_1$ is harmonic, and concerning $\mathbf{j}_2$ we obtain

THEOREM 3.7. *The inclusion map* $\mathbf{j}_2 : (M, \langle\, ,\, \rangle) \to (\mathbb{S}^n, e^{2\varrho}\langle\, ,\, \rangle)$ *is not harmonic, and it is biharmonic if and only if* $|u| = 1$.

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