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STOCHASTIC DYNAMICAL SYSTEMS WITH WEAK CONTRACTIVITY PROPERTIES I. STRONG AND LOCAL CONTRACTIVITY

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(with a chapter featuring results of Martin Benda)

Abstract. Consider a proper metric space X and a sequence $(F_n)_{n\geq 0}$ of i.i.d. random continuous mappings $\mathsf{X} \to \mathsf{X}$. It induces the stochastic dynamical system (SDS) $X_n^x = F_n \circ \cdots \circ F_1(x)$ starting at $x \in \mathsf{X}$. In this and the subsequent paper, we study existence and uniqueness of invariant measures, as well as recurrence and ergodicity of this process.

In the present first part, we elaborate, improve and complete the unpublished work of Martin Benda on local contractivity, which merits publicity and provides an important tool for studying stochastic iterations. We consider the case when the F_n are contractions and, in particular, discuss recurrence criteria and their sharpness for the reflected random walk.

1. Introduction. We start by reviewing two well known models.

First, let $(B_n)_{n\geq 0}$ be a sequence of i.i.d. real valued random variables. Then the *reflected random walk* starting at $x \geq 0$ is the stochastic dynamical system given recursively by $X_0^x = x$ and $X_n^x = |X_{n-1}^x - B_n|$. The absolute value becomes meaningful when B_n assumes positive values with positive probability; otherwise we get an ordinary random walk on $\mathbb{R}$. The reflected random walk was described and studied by Feller [20]; apparently, it was first considered by von Schelling [36] in the context of telephone networks. In the case when $B_n \geq 0$, Feller [20] and Knight [28] have computed an invariant measure for the process when the Y_n are non-lattice random variables, while Boudiba [8], [9] has provided such a measure when the Y_n are lattice variables. Leguesdron [29], Boudiba [9] and Benda [4] have also studied its uniqueness (up to constant factors). When the invariant measure has finite total mass—which holds if and only if $\mathsf{E}(B_1) < \infty$—the process is (topologically) recurrent: with probability 1, it returns infinitely often to each open set that is charged by the invariant measure. Indeed, it is positive recurrent

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in the sense that the mean return time is finite. More general recurrence criteria were provided by Smirnov [37] and Rabeherimanana [34], and also in our unpublished paper [33]: basically, recurrence holds when $\mathsf{E}\big(\sqrt{B_1}\big)$ or quantities of more or less the same order are finite. In the present paper, we shall briefly touch the situation when the B_n are not necessarily positive.

Second, let $(A_n, B_n)_{n\geq 0}$ be a sequence of i.i.d. random variables in $\mathbb{R}_*^+ \times \mathbb{R}$. (We shall always write $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}_*^+ = (0, \infty)$, the latter usually seen as a multiplicative group.) The associated *affine stochastic recursion* on $\mathbb{R}$ is given by $Y_0^x = x \in \mathbb{R}$ and $Y_n^x = A_n Y_{n-1}^x + B_n$. There is ample literature on this process, which can be interpreted in terms of a random walk on the affine group. That is, one applies products of affine matrices:

$$\begin{pmatrix} Y_n^x \\ 1 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{n-1} & B_{n-1} \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Products of affine transformations were one of the first examples of random walks on non-commutative groups (see Grenander [22]). Among the large body of further work, we mention Kesten [27], Grincevičjus [23], [24], Elie [17], [18], [19], and in particular the papers by Babillot, Bougerol and Elie [3] and Brofferio [10]. See also the more recent work of Buraczewski [11] and Buraczewski, Damek, Guivarc'h, Hulanicki and Urban [12].

The hardest and most interesting case is when A_n is *log-centered,* that is, $\mathsf{E}(\log A_n) = 0$. The development of tools for handling this case, beyond affine recursions, is the main focus of the present work. The easier and well-understood case is the *contractive* one, where $\mathsf{E}(\log A_n) < 0$.

In this work, stochastic dynamical systems are considered in the following general setting. Let (X, d) be a *proper* metric space (i.e., closed balls are compact), and let $\mathfrak{G}$ be the monoid of all continuous mappings $\mathsf{X} \to \mathsf{X}$. It carries the topology of uniform convergence on compact sets. Now let $\widetilde{\mu}$ be a regular probability measure on $\mathfrak{G}$, and let $(F_n)_{n\geq 1}$ be a sequence of i.i.d. $\mathfrak{G}$-valued random variables (functions) with common distribution $\widetilde{\mu}$, defined on a suitable probability space $(\Omega, \mathfrak{A}, \mathsf{Pr})$. The measure $\widetilde{\mu}$ gives rise to the *stochastic dynamical system* (*SDS*) $\omega \mapsto X_n^x(\omega)$ defined by

$$X_0^x = x \in \mathsf{X}, \quad \text{and} \quad X_n^x = F_n(X_{n-1}^x), \quad n \geq 1. \tag{1.1}$$

There is ample literature on processes of this type: see e.g. Arnold [2] or Bhattacharya and Majumdar [7]. Any SDS (1.1) is a Markov chain. The transition kernel is

$$P(x, U) = \mathsf{Pr}[X_1^x \in U] = \widetilde{\mu}(\{f \in \mathfrak{G} : f(x) \in U\}),$$

where U is a Borel set in X. The associated transition operator is given by

$$P\varphi(x) = \int_{\mathsf{X}} \varphi(y)\, P(x, dy) = \mathsf{E}(\varphi(X_1^x)),$$

where $\varphi : \mathsf{X} \to \mathbb{R}$ is a measurable function for which this integral exists. The operator is Fellerian, that is, $P\varphi$ is continuous when φ is bounded and continuous. We shall write $\mathcal{C}_c(\mathsf{X})$ for the space of compactly supported continuous functions $\mathsf{X} \to \mathbb{R}$.

The SDS is called *transient* if every compact set is visited only finitely often, that is,

$$\mathsf{Pr}[d(X_n^x, x) \to \infty] = 1 \quad \text{for every } x \in \mathsf{X}.$$

We call it (*topologically*) *recurrent* if there is a non-empty, closed set $\mathsf{L} \subset \mathsf{X}$ such that for every open set U that intersects L,

$$\mathsf{Pr}[X_n^x \in U \text{ infinitely often}] = 1 \quad \text{for every } x \in \mathsf{L}.$$

In our situation, we shall even have this for every starting point $x \in \mathsf{X}$, so that L is an *attractor* for the SDS. As an intermediate notion, we call the SDS *conservative* if

$$\mathsf{Pr}[\liminf_n d(X_n^x, x) < \infty] = 1 \quad \text{for every } x \in \mathsf{X}.$$

Besides the question whether the SDS is recurrent, we shall mainly be interested in the question of existence and uniqueness (up to constant factors) of an *invariant measure.* This is a Radon measure ν on X such that for any Borel set $U \subset \mathsf{X}$,

$$\nu(U) = \int_{\mathsf{X}} \mathsf{Pr}[X_1^x \in U]\, d\nu(x).$$

We can construct the *trajectory space* of the SDS starting at x. This is

$$(\mathsf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathsf{X}^{\mathbb{N}_0}), \mathsf{Pr}_x),$$

where $\mathfrak{B}(\mathsf{X}^{\mathbb{N}_0})$ is the product Borel σ-algebra on $\mathsf{X}^{\mathbb{N}_0}$, and Pr_x is the image of the measure Pr under the mapping

$$\Omega \to \mathsf{X}^{\mathbb{N}_0}, \quad \omega \mapsto (X_n^x(\omega))_{n \geq 0}.$$

If we have an invariant Radon measure, then we can construct the measure

$$\mathsf{Pr}_\nu = \int_{\mathsf{L}} \mathsf{Pr}_x \, d\nu(x)$$

on the trajectory space. It is a probability measure only when ν is a probability measure on X. In general, it is σ-finite and invariant with respect to the time shift $T : \mathsf{X}^{\mathbb{N}_0} \to \mathsf{X}^{\mathbb{N}_0}$. Conservativity of the SDS will be used to get conservativity of the shift. We shall study ergodicity of T, which in turn will imply uniqueness of ν (up to multiplication with constants).

As often in this field, ideas that were first developed by Furstenberg, e.g. [21], play an important role at least in the background.

(1.2) Proposition (Furstenberg's contraction principle). *Let $(F_n)_{n\geq 1}$ be i.i.d. continuous random mappings $\mathsf{X} \to \mathsf{X}$, and define the right process*

$$R_n^x = F_1 \circ \cdots \circ F_n(x).$$

If there is an X-valued random variable Z such that

$$\lim_{n\to\infty} R_n^x = Z \qquad \textit{almost surely for every} \;\; x \in \mathsf{X},$$

then the distribution ν of the limit Z is the unique invariant probability measure for the SDS $X_n^x = F_n \circ \cdots \circ F_1(x)$.

A proof can be found, e.g., in Letac [30] or in Diaconis and Freedman [16]. In [21], it is displayed in a more specific setting, but all ideas are clearly present. While being ideally applicable to the contractive case, this contraction principle is not the right tool for handling the log-centered case mentioned above. In the context of affine stochastic recursion, Babillot, Bougerol and Elie [3] introduced the notion of *local contractivity* (see Definition (2.1) below). This is not the same as the local contractivity property of Steinsaltz [38] and Jarner and Tweedie [25]. Local contractivitiy as defined in [3] was then exploited systematically by Benda in interesting and useful work in his PhD thesis [4] (in German) and the two subsequent preprints [5], [6] which were accepted for publication, circulated (not very widely) in preprint version but have remained unpublished. In a personal communication, Benda also gives credit to unpublished work of his late PhD advisor Kellerer (cf. the posthumous publication [26]).

We think that this material deserves to be documented in a publication, whence we include—with the consent of M. Benda whom we managed to contact—the next section on weak contractivity (§2). The proofs that we give are "streamlined", and new aspects and results are added, such as, in particular, ergodicity of the shift on the trajectory space with respect to Pr_ν (Theorem (2.13)). Ergodicity yields uniqueness of the invariant measure. Before that, we explain the alternative between recurrence and transience and the limit set (attractor) L, which is the support of the invariant measure ν.

We display briefly the classical results regarding stochastic affine recursion in §3. Then, in §4, we consider the situation when the F_n are contractions with Lipschitz constants $A_n = \mathfrak{l}(F_n) \leq 1$ (not necessarily assuming that $\mathsf{E}(\log A_n) < 0$). We provide a tool for getting strong contractivity in the recurrent case (Theorem (4.2)). A typical example is the reflected random walk. In §5, we discuss some of its properties, in particular sharpness of recurrence criteria.

This concludes Part I. In the second paper, we shall examine in detail the iteration of general Lipschitz mappings.

Since we want to present a sufficiently comprehensive picture, we have included the statements—mostly without proof—of a few known results, in particular on cases where one has strong contractivity. We also remark that the assumption of properness of the space X can be relaxed in several parts of the material presented here.

2. Local contractivity and the work of Benda

(2.1) DEFINITION.

(i) The SDS is called *strongly contractive* if for every $x \in \mathsf{X}$,

$$\mathsf{Pr}[d(X_n^x, X_n^y) \to 0 \text{ for all } y \in \mathsf{X}] = 1.$$

(ii) The SDS is called *locally contractive* if for every $x \in \mathsf{X}$ and every compact $K \subset \mathsf{X}$,

$$\mathsf{Pr}[d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^x) \to 0 \text{ for all } y \in \mathsf{X}] = 1.$$

Let $\mathsf{B}(r)$ and $\overline{\mathsf{B}}(r)$, $r \in \mathbb{N}$, be the open and closed balls in X with radius r and fixed center $o \in \mathsf{X}$, respectively. $\overline{\mathsf{B}}(r)$ is compact by properness of X.

Using Kolmogorov's 0-1 law, one gets the following alternative.

(2.2) LEMMA. *For a locally contractive SDS,*

$$\begin{array}{lll} \textit{either} & \mathsf{Pr}[d(X_n^x, x) \to \infty] = 0 & \textit{for all } x \in \mathsf{X}, \\ \textit{or} & \mathsf{Pr}[d(X_n^x, x) \to \infty] = 1 & \textit{for all } x \in \mathsf{X}. \end{array}$$

Proof. Consider

$$X_{m,m}^x = x \quad \text{and} \quad X_{m,n}^x = F_n \circ F_{n-1} \circ \cdots \circ F_{m+1}(x) \quad \text{for } n > m, \tag{2.3}$$

so that $X_n^x = X_{0,n}^x$. Then local contractivity implies that for each $x \in \mathsf{X}$, we have $\mathsf{Pr}(\Omega_0) = 1$ for the event Ω_0 consisting of all $\omega \in \Omega$ with

$$\lim_{n\to\infty} \mathbf{1}_{\mathsf{B}(r)}(X_{m,n}^x(\omega)) \cdot d(X_{m,n}^x(\omega), X_{m,n}^y(\omega)) = 0 \tag{2.4}$$

$$\text{for each } r \in \mathbb{N},\ m \in \mathbb{N}_0,\ y \in \mathsf{X}.$$

Clearly, Ω_0 is invariant with respect to the shift of the sequence (F_n).

Let $\omega \in \Omega_0$ be such that the sequence $(X_n^x(\omega))_{n\ge 0}$ accumulates at some $z \in \mathsf{X}$. Fix m and set $v = X_m^x(\omega)$. Then also $(X_{m,n}^v(\omega))_{n\ge m}$ accumulates at z. Now let $y \in \mathsf{X}$ be arbitrary. Then there is r such that $v, y, z \in \mathsf{B}(r)$. Therefore also $(X_{m,n}^y(\omega))_{n\ge m}$ accumulates at z. In particular, the fact that $(X_n^x(\omega))_{n\ge 0}$ accumulates at some point does not depend on the initial trajectory, i.e., on the specific realization of $F_1, \dots, F_m$. We infer that the set

$$\{\omega \in \Omega_0 : (X_n^x(\omega))_{n\ge 0} \text{ accumulates in } \mathsf{X}\}$$

is a tail event of $(F_n)_{n\ge 1}$. On its complement in Ω_0, we have $d(X_n^x, x) \to \infty$. ∎

If $d(X_n^x, x) \to \infty$ almost surely, then we call the SDS *transient.*

For $\omega \in \Omega$, let $\mathsf{L}^x(\omega)$ be the set of accumulation points of $(X_n^x(\omega))$ in X. The following proof is much simpler than the one in [5].

(2.5) LEMMA. *For any conservative, locally contractive SDS, there is a set* $\mathsf{L} \subset \mathsf{X}$—*the* attractor *or* limit set—*such that*

$$\mathsf{Pr}[\mathsf{L}^x(\cdot) = \mathsf{L} \text{ for all } x \in \mathsf{X}] = 1,$$

Proof. The argument of the proof of Lemma (2.2) also shows the following. For every open $U \subset \mathsf{X}$,

$$\mathsf{Pr}[X_n^x \text{ accumulates in } U \text{ for all } x \in \mathsf{X}] \in \{0,1\}.$$

X being proper, we can find a countable basis $\{U_k : k \in \mathbb{N}\}$ of the topology of X, where each U_k is an open ball. Let $\mathbb{K} \subset \mathbb{N}$ be the (deterministic) set of all k such that the above probability is 1 for $U = U_k$. Then there is $\Omega_0 \subset \Omega$ such that $\mathsf{Pr}(\Omega_0) = 1$, and for every $\omega \in \Omega_0$, the sequence $(X_n^x(\omega))_{n\geq 0}$ accumulates in U_k for some (and equivalently all) x precisely when $k \in \mathbb{K}$. Now, if $\omega \in \Omega_0$, then $y \in \mathsf{L}^x(\omega)$ if and only if $k \in \mathbb{K}$ for every k with $U_k \ni y$. We see that $\mathsf{L}^x(\omega)$ is the same set for every $\omega \in \Omega_0$. ∎

Thus, (X_n^x) is (*topologically*) *recurrent* on L when $\mathsf{Pr}[d(X_n^x, x) \to \infty] = 0$, that is, every open set that intersects L is visited infinitely often with probability 1.

For a Radon measure ν on X, its transform under P is written as νP, that is, for any Borel set $U \subset \mathsf{X}$,

$$\nu P(U) = \int_{\mathsf{X}} P(x,U)\, d\nu(x).$$

Recall that ν is called *excessive* when $\nu P \leq \nu$, and *invariant* when $\nu P = \nu$.

For two transition kernels P, Q, their product is defined as

$$PQ(x,U) = \int_{\mathsf{X}} Q(y,U)\, P(x,dy).$$

In particular, P^k is the k-fold iterate. The first part of the following is well-known; we outline the proof because it is needed in the second part, regarding $\mathsf{supp}(\nu)$.

(2.6) LEMMA. *If the locally contractive SDS is recurrent, then every excessive measure ν is invariant. Furthermore,* $\mathsf{supp}(\nu) = \mathsf{L}$.

Proof. For any pair of Borel sets $U, V \subset \mathsf{X}$, define the transition kernel $P_{U,V}$ and the measure ν_U by

$$P_{U,V}(x,B) = \mathbf{1}_U(x)\, P(x, B \cap V) \quad \text{and} \quad \nu_U(B) = \nu(U \cap B),$$

where $B \subset \mathsf{X}$ is a Borel set. We abbreviate $P_{U,U} = P_U$. Also, consider the stopping time $\tau_x^U = \inf\{n \geq 1 : X_n^x \in U\}$, and for $x \in U$ let

$$P^U(x,B) = \mathsf{Pr}[\tau_x^U < \infty,\, X_{\tau_x^U}^x \in B]$$

be the probability that the first return of X_n^x to the set U occurs at a point of $B \subset \mathsf{X}$. Then we have

$$\nu_U \geq \nu_U P_U + \nu_{U^c} P_{U^c,U},$$

and by a typical inductive ("balayage") argument,

$$\nu_U \geq \nu_U \Big(P_U + \sum_{k=0}^{n-1} P_{U,U^c} P_{U^c}^k P_{U^c,U} \Big) + \nu_{U^c} P_{U^c}^n P_{U^c,U}.$$

In the limit,

$$\nu_U \geq \nu_U \Big(P_U + \sum_{k=0}^{\infty} P_{U,U^c} P_{U^c}^k P_{U^c,U} \Big) = \nu_U P^U.$$

Now suppose that U is open and relatively compact, and $U \cap \mathsf{L} \neq \emptyset$. Then, by recurrence, for any $x \in U$, we have $\tau_x^U < \infty$ almost surely. This means that P^U is stochastic, that is, $P^U(x, U) = 1$. But then $\nu_U P^U(U) = \nu_U(U) = \nu(U) < \infty$. Therefore $\nu_U = \nu_U P^U$. We now can set $U = \mathsf{B}(r)$ and let $r \to \infty$. Then monotone convergence implies $\nu = \nu P$, and ν is invariant.

Let us next show that $\mathsf{supp}(\nu) \subset \mathsf{L}$.

Take an open, relatively compact set V such that $V \cap \mathsf{L} = \emptyset$. Choose r large enough such that $U = \mathsf{B}(r)$ contains V and intersects L. Let $Q = P^U$. We know from the above that $\nu_U = \nu_U\, Q = \nu_U\, Q^n$. We get

$$\nu(V) = \nu_U(V) = \int_U Q^n(x, V)\, d\nu_U(x).$$

Now $Q^n(x, V)$ is the probability that the SDS starting at x visits V at the instant when it returns to U for the nth time. As

$$\mathsf{Pr}[X_n^x \in V \text{ for infinitely many } n] = 0,$$

it is an easy exercise to show that $Q^n(x, V) \to 0$. Since the measure ν_U has finite total mass, we can use dominated convergence to see that $\int_U Q^n(x, V)\, d\nu_U(x) \to 0$ as $n \to \infty$.

We conclude that $\nu(V) = 0$, and $\mathsf{supp}(\nu) \subset \mathsf{L}$.

Since $\nu P = \nu$, we have $f(\mathsf{supp}(\nu)) \subset \mathsf{supp}(\nu)$ for every $f \in \mathsf{supp}(\widetilde{\mu})$, where (recall) $\widetilde{\mu}$ is the distribution of the random functions F_n in $\mathfrak{G}$. But then almost surely $X_n^x \in \mathsf{supp}(\nu)$ for all $x \in \mathsf{supp}(\nu)$ and all n, that is, $\mathsf{L}^x(\omega) \subset \mathsf{supp}(\nu)$ for Pr-almost every ω. Lemma (2.5) implies that $\mathsf{L} \subset \mathsf{supp}(\nu)$. ∎

The following holds in more generality than just for recurrent locally contractive SDS.

(2.7) PROPOSITION. *If the locally contractive SDS is recurrent, then it possesses an invariant Radon measure ν.*

Proof. Fix $\psi \in \mathcal{C}_c^+(\mathsf{X})$ such that its support intersects L. Recurrence implies that

$$\sum_{k=1}^{\infty} P^k \psi(x) = \infty \quad \text{for every } x \in \mathsf{X}.$$

The statement now follows from a result of Lin [31, Thm. 5.1]. ■

Thus we have an invariant Radon measure ν with $\nu P = \nu$ and $\mathsf{supp}(\nu) = \mathsf{L}$. It is now easy to see that the attractor depends only on $\mathsf{supp}(\widetilde{\mu}) \subset \mathfrak{G}$.

(2.8) Corollary. *In the recurrent case,* L *is the smallest non-empty closed subset of* X *with the property that* $f(\mathsf{L}) \subset \mathsf{L}$ *for every* $f \in \mathsf{supp}(\widetilde{\mu})$.

Proof. The reasoning at the end of the proof of Lemma (2.6) shows that L is indeed a closed set with that property. On the other hand, if $C \subset \mathsf{X}$ is closed, non-empty and such that $f(C) \subset C$ for all $f \in \mathsf{supp}(\widetilde{\mu})$ then $(X_n^x(\omega))$ evolves almost surely within C when the starting point x is in C. But then $\mathsf{L}^x(\omega) \subset C$ almost surely, and on the other hand $\mathsf{L}^x(\omega) = \mathsf{L}$ almost surely. ■

(2.9) Remark. Suppose that the SDS induced by the probability measure $\widetilde{\mu}$ on $\mathfrak{G}$ is not necessarily locally contractive, resp. recurrent, but that there is another probability measure $\widetilde{\mu}'$ on $\mathfrak{G}$ which does induce a locally contractive, recurrent SDS and which satisfies $\mathsf{supp}(\widetilde{\mu}) = \mathsf{supp}(\widetilde{\mu}')$. Let L be the limit set of this second SDS. Since it depends only on $\mathsf{supp}(\widetilde{\mu}')$, the results that we have so far imply that also for the SDS (X_n^x) associated with $\widetilde{\mu}$, the attractor L is the unique "essential class" in the following sense: it is the unique minimal non-empty closed subset of X such that

(i) for every open set $U \subset \mathsf{X}$ that intersects L and every starting point $x \in \mathsf{X}$, the sequence (X_n^x) visits U with positive probability, and
(ii) if $x \in \mathsf{L}$ then $X_n^x \in \mathsf{L}$ for all n. ■

For $\ell \geq 2$, we can lift each $f \in \mathfrak{G}$ to a continuous mapping

$$f^{(\ell)} : \mathsf{X}^\ell \to \mathsf{X}^\ell, \qquad f^{(\ell)}(x_1, \dots, x_\ell) = (x_2, \dots, x_\ell, f(x_\ell)).$$

In this way, the random mappings F_n induce the SDS

$$(F_n^{(\ell)} \circ \cdots \circ F_1^{(\ell)}(x_1, \dots, x_\ell))_{n \geq 0}$$

on X^ℓ. For $n \geq \ell - 1$ this is just $(X_{n-\ell+1}^{x_\ell}, \dots, X_n^{x_\ell})$.

(2.10) Lemma. *Let* $x \in \mathsf{X}$*, and let* $U_0, \dots, U_{\ell-1} \subset \mathsf{X}$ *be Borel sets such that*

$$\mathsf{Pr}[X_n^x \in U_0 \text{ for infinitely many } n] = 1,$$
$$\mathsf{Pr}[X_1^y \in U_j] \geq \alpha > 0 \quad \text{for every } y \in U_{j-1},\ j = 1, \dots, \ell - 1.$$

Then also

$$\mathsf{Pr}[X_n^x \in U_0,\ X_{n+1}^x \in U_1, \dots, X_{n+\ell-1}^x \in U_{\ell-1} \text{ for infinitely many } n] = 1.$$

Proof. This is quite standard and true for general Markov chains and not just SDS. Let $\tau(n)$, $n \ge 1$, be the stopping times of the successive visits of (X_n^x) in U_0. They are all a.s. finite by assumption. We consider the events

$$\Lambda_n = [X^x_{\tau(\ell n)+1} \in U_1, \dots, X^x_{\tau(\ell n)+\ell-1} \in U_{\ell-1}] \quad \text{and} \quad \Lambda_{k,m} = \bigcup_{n=k+1}^{m-1} \Lambda_n,$$

where $k < m$. We need to show that $\mathsf{Pr}(\limsup_n \Lambda_n) = 1$. By the strong Markov property, we have

$$\mathsf{Pr}(\Lambda_n \mid X^x_{\tau(\ell n)} = y) \ge \alpha^\ell \quad \text{for every } y \in U_0.$$

Let $k, m \in \mathbb{N}$ with $k < m$. Just for the purpose of the next lines of the proof, consider the measure on X defined by

$$\sigma(B) = \mathsf{Pr}([X^x_{\tau(\ell m)} \in B] \cap \Lambda^c_{k,m-1}).$$

It is concentrated on U_0, and using the Markov property,

$$\begin{aligned}\mathsf{Pr}(\Lambda^c_{k,m}) &= \int_{U_0} \mathsf{Pr}(\Lambda^c_m \mid X^x_{\tau(\ell m)} = y)\, d\sigma(y) \\ &\le (1-\alpha^\ell)\sigma(U_0) = (1-\alpha^\ell)\,\mathsf{Pr}(\Lambda^c_{k,m-1}) \le \cdots \le (1-\alpha^\ell)^{m-k}.\end{aligned}$$

Letting $m \to \infty$, we see that $\mathsf{Pr}(\bigcap_{n>k} \Lambda^c_n) = 0$ for every k, so that

$$\mathsf{Pr}\Big(\bigcap_k \bigcup_{n>k} \Lambda_n\Big) = 1,$$

as required. ■

(2.11) PROPOSITION. *If the SDS is locally contractive and recurrent on* X*, then so is the lifted process on* X^ℓ*. The limit set of the latter is*

$$\mathsf{L}^{(\ell)} = \big\{(x, f_1(x), f_2 \circ f_1(x), \dots, f_{\ell-1} \circ \cdots \circ f_1(x)) : x \in \mathsf{L},\ f_i \in \mathsf{supp}(\widetilde{\mu})\big\}^-,$$

and if the Radon measure ν *is invariant for the original SDS on* X*, then the measure* $\nu^{(\ell)}$ *is invariant for the lifted SDS on* X^ℓ*, where*

$$\begin{aligned}&\int_{\mathsf{X}^\ell} f\, d\nu^{(\ell)} \\ &\qquad = \int_{\mathsf{X}} \cdots \int_{\mathsf{X}} f(x_1, \dots, x_\ell)\, P(x_{\ell-1}, dx_\ell)\, P(x_{\ell-2}, dx_{\ell-1}) \cdots P(x_1, dx_2)\, d\nu(x_1).\end{aligned}$$

Proof. It is a straightforward exercise to verify that the lifted SDS is locally contractive and has $\nu^{(\ell)}$ as an invariant measure. We have to prove that it is recurrent. For this purpose, we just have to show that there is some relatively compact subset of X^ℓ that is visited infinitely often with positive probability. We can find relatively compact open subsets $U_0, \dots, U_{\ell-1}$ of X that intersect L such that

$$\mathsf{Pr}[F_1(U_{j-1}) \subset U_j] \ge \alpha > 0 \quad \text{for } j = 1, \dots, \ell - 1.$$

We know that for an arbitrary starting point $x \in \mathsf{X}$, with probability 1, the SDS (X_n^x) visits U_0 infinitely often. Lemma (2.10) implies that the lifted SDS on X^ℓ visits $U_0 \times \cdots \times U_{\ell-1}$ infinitely often with probability 1.

By Lemma (2.2), the lifted SDS on X^ℓ is recurrent. Now that we know this, it is clear from Corollary (2.8) that its attractor is the set $\mathsf{L}^{(\ell)}$, as stated. ∎

As outlined in the introduction, we can equip the trajectory space $X^{\mathbb{N}_0}$ of our SDS with the infinite product σ-algebra and the measure Pr_ν, which is in general σ-finite.

(2.12) Lemma. *If the SDS is locally contractive and recurrent, then the shift T is conservative on* $(\mathsf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathsf{X}^{\mathbb{N}_0}), \mathsf{Pr}_\nu)$.

Proof. Let $\varphi = \mathbf{1}_U$, where $U \subset \mathsf{X}$ is open, relatively compact, and intersects L. We can extend it to a positive function in $L^1(\mathsf{X}^{\mathbb{N}_0}, \mathsf{Pr}_\nu)$ by setting $\varphi(\mathbf{x}) = \varphi(x_0)$ for $\mathbf{x} = (x_n)_{n\ge 0}$. We know from recurrence that

$$\sum_n \varphi(X_n^x) = \infty \qquad \mathsf{Pr}\text{-almost surely, for every } x \in \mathsf{X}.$$

This translates into

$$\sum_n \varphi(T^n \mathbf{x}) = \infty \qquad \mathsf{Pr}_\nu\text{-almost surely, for every } \mathbf{x} \in \mathsf{X}^{\mathbb{N}_0}.$$

Conservativity follows; see e.g. [35, Thm. 5.3]. ∎

The uniqueness part of the following theorem is contained in [4] and [5]; see also Brofferio [10, Thm. 3], who considers SDS of affine mappings. We modify and extend the proof in order to be able to conclude that our SDS is ergodic with respect to T. (This, as well as Proposition (2.11), is new with respect to Benda's work.)

(2.13) Theorem. *For a recurrent locally contractive SDS, let ν be the measure of Proposition* (2.7)*. Then the shift T on* $\mathsf{X}^{\mathbb{N}_0}$ *is ergodic with respect to* Pr_ν*. In particular, ν is the unique invariant Radon measure for the SDS up to multiplication with constants.*

Proof. Let $\mathfrak{I}$ be the σ-algebra of T-invariant sets in $\mathfrak{B}(\mathsf{X}^{\mathbb{N}_0})$. For $\varphi \in L^1(\mathsf{X}^{\mathbb{N}_0}, \mathsf{Pr}_\nu)$, we write $\mathsf{E}_\nu(\varphi) = \int \varphi \, d\mathsf{Pr}_\nu$ and $\mathsf{E}_\nu(\varphi \,|\, \mathfrak{I})$ for the conditional "expectation" of φ with respect to $\mathfrak{I}$. The quotation marks refer to the fact that it does not have the meaning of an expectation when ν is not a probability measure. As a matter of fact, what is well defined in the latter case are the quotients $\mathsf{E}_\nu(\varphi \,|\, \mathfrak{I})/\mathsf{E}_\nu(\psi \,|\, \mathfrak{I})$ for suitable $\psi \ge 0$; compare with the explanations in Revuz [35, pp. 133–134].

In view of Lemma (2.12), we can apply the ergodic theorem of Chacon and Ornstein [13] (see also [35, Thm. 3.3]). Choosing an arbitrary function

$\psi \in L^1(\mathsf{X}^{\mathbb{N}_0}, \mathsf{Pr}_\nu)$ with

$$\mathsf{Pr}_\nu\Big(\Big\{\mathbf{x} \in \mathsf{X}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} \psi(T^n\mathbf{x}) < \infty\Big\}\Big) = 0, \tag{2.14}$$

one has, for every $\varphi \in L^1(\mathsf{X}^{\mathbb{N}_0}, \mathsf{Pr}_\nu)$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \varphi(T^k\mathbf{x})}{\sum_{k=0}^{n} \psi(T^k\mathbf{x})} = \frac{\mathsf{E}_\nu(\varphi \,|\, \mathfrak{I})}{\mathsf{E}_\nu(\psi \,|\, \mathfrak{I})} \quad \text{for } \mathsf{Pr}_\nu\text{-almost every } \mathbf{x} \in \mathsf{X}^{\mathbb{N}_0}. \tag{2.15}$$

In order to show ergodicity of T, we need to show that the right hand side is just

$$\frac{\mathsf{E}_\nu(\varphi)}{\mathsf{E}_\nu(\psi)}.$$

It is sufficient to show this for non-negative functions that depend only on finitely many coordinates. For a function φ on X^ℓ, we also write φ for its extension to $\mathsf{X}^{\mathbb{N}_0}$, given by $\varphi(\mathbf{x}) = \varphi(x_0, \dots, x_{\ell-1})$.

That is, we need to show that for every $\ell \geq 1$ and non-negative Borel functions φ, ψ on X^ℓ, with ψ satisfying (2.14),

(2.16)
$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \varphi(X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega))}{\sum_{k=0}^{n} \psi((X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega)))} = \frac{\int_{\mathsf{L}} \mathsf{E}(\varphi(X_0^y, \dots, X_{\ell-1}^y))\, d\nu(y)}{\int_{\mathsf{L}} \mathsf{E}(\psi(X_0^y, \dots, X_{\ell-1}^y))\, d\nu(y)}$$

for ν-almost every $x \in \mathsf{X}$ and Pr-almost every $\omega \in \Omega$, when the integrals on the right hand side are finite.

At this point, we observe that we need to prove (2.16) only for $\ell = 1$. Indeed, once we have the proof for this case, we can reconsider our SDS on X^ℓ, and using Proposition (2.11), our proof for $\ell = 1$ applies to the new SDS as well.

So now let $\ell = 1$. By regularity of ν, we may assume that φ and ψ are non-negative, compactly supported, continuous functions on L that both are non-zero.

We consider the random variables $S_n^x\varphi(\omega) = \sum_{k=0}^{n} \varphi(X_k^x(\omega))$ and $S_n^x\psi(\omega)$. Since the SDS is recurrent, both functions satisfy (2.14), i.e., almost surely we have $S_n^x\varphi$, $S_n^x\psi > 0$ for all but finitely many n and all x. We shall show that

$$\lim_{n\to\infty} \frac{S_n^x\varphi}{S_n^x\psi} = \frac{\int_{\mathsf{L}} \varphi\, d\nu}{\int_{\mathsf{L}} \psi\, d\nu} \quad \mathsf{Pr}\text{-almost surely and for } \textit{every } x \in \mathsf{L}, \tag{2.17}$$

which is more than what we need (namely that it just holds for ν-almost every x). We know from (2.15) that the limit exists in terms of conditional expectations for ν-almost every x, so that we only have to show that it is $\mathsf{Pr} \otimes \nu$-almost everywhere constant.

STEP 1: *Independence of* x. Let $K_0 \subset \mathsf{L}$ be compact such that the support of φ is contained in K_0. Define $K = \{x \in \mathsf{L} : d(x, K_0) \le 1\}$. Given $\varepsilon > 0$, let $0 < \delta \le 1$ be such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $d(x,y) < \delta$.

By (2.15), there is x such that the limits $\lim_n S_n^x \mathbf{1}_K \big/ S_n^x \varphi$ and $Z_{\varphi,\psi} = \lim_n S_n^x\varphi \big/ S_n^x \psi$ exist and are finite Pr-almost surely.

Local contractivity implies that for this specific x and each $y \in \mathsf{X}$, we have the following. Pr-almost surely, there is a random $N \in \mathbb{N}$ such that

$$|\varphi(X_k^x) - \varphi(X_k^y)| \le \varepsilon \cdot \mathbf{1}_K(X_k^x) \quad \text{for all } k \ge N.$$

Therefore, for every $\varepsilon > 0$ and $y \in \mathsf{X}$,

$$\limsup_{n\to\infty} \frac{|S_n^x\varphi - S_n^y\varphi|}{S_n^x\varphi} \le \varepsilon \cdot \lim_{n\to\infty} \frac{S_n^x \mathbf{1}_K}{S_n^x \varphi} \quad \mathsf{Pr}\text{-almost surely.}$$

This implies that for every $y \in \mathsf{L}$,

$$\lim_{n\to\infty} \frac{S_n^x\varphi - S_n^y\varphi}{S_n^x\varphi} = 0, \quad \text{that is,} \quad \lim_{n\to\infty} \frac{S_n^y\varphi}{S_n^x\varphi} = 1 \quad \mathsf{Pr}\text{-almost surely.}$$

The same applies to ψ in place of φ. We deduce that for all y,

$$\frac{S_n^x\varphi}{S_n^x\psi} - \frac{S_n^y\varphi}{S_n^y\psi} = \frac{S_n^y\varphi}{S_n^y\psi}\left(\frac{S_n^x\varphi}{S_n^y\varphi}\frac{S_n^y\psi}{S_n^x\psi} - 1\right) \to 0 \quad \mathsf{Pr}\text{-almost surely.}$$

In other terms, for the positive random variable $Z_{\varphi,\psi}$ given above in terms of our x,

$$\lim_{n\to\infty} \frac{S_n^y\varphi}{S_n^y\psi} = Z_{\varphi,\psi} \quad \mathsf{Pr}\text{-almost surely, for every } y \in \mathsf{L}.$$

STEP 2: $Z_{\varphi,\psi}$ *is a.s. constant.* Recall the random variables $X_{m,n}^x$ of (2.3) and set $S_{m,n}^x\varphi(\omega) = \sum_{k=m}^n \varphi(X_{m,k}^x(\omega))$, $n > m$. Then Step 1 also shows that for our given x and each m,

$$\lim_{n\to\infty} \frac{S_{m,n}^y\varphi}{S_{m,n}^y\psi} = \lim_{n\to\infty} \frac{S_{m,n}^x\varphi}{S_{m,n}^x\psi} \quad \mathsf{Pr}\text{-almost surely, for every } x \in \mathsf{L}. \tag{2.18}$$

Let $\Omega_0 \subset \Omega$ be the set on which the convergence in (2.18) holds for all m, and both $S_n^x\varphi, S_n^x\psi \to \infty$ on Ω_0. We have $\mathsf{Pr}(\Omega_0) = 1$. For fixed $\omega \in \Omega_0$ and $m \in \mathbb{N}$, let $y = X_m^x(\omega)$. Then (because in the ratio limit we can omit the first m terms of the sums)

$$Z_{\varphi,\psi}(\omega) = \lim_{n\to\infty} \frac{S_n^x\varphi(\omega)}{S_n^x\psi(\omega)} = \lim_{n\to\infty} \frac{S_{m,n}^y\varphi(\omega)}{S_{m,n}^y\psi(\omega)} = \lim_{n\to\infty} \frac{S_{m,n}^x\varphi(\omega)}{S_{m,n}^x\psi(\omega)}.$$

Thus, $Z_{\varphi,\psi}$ is independent of $F_1, \dots, F_m$, whence it is constant by Kolmogorov's 0-1 law. This completes the proof of ergodicity. It is immediate from (2.17) that ν is unique up to multiplication by constants. ∎

(2.19) COROLLARY. *Let the locally contractive SDS* (X_n^x) *be recurrent with invariant Radon measure* ν. *For relatively compact, open* $U \subset \mathsf{X}$ *which intersects* L, *consider the probability measure* m_U *on* X *defined by* $\mathsf{m}_U(B) = \nu(B \cap U)/\nu(U)$. *Consider the SDS with initial distribution* m_U, *and let* τ^U *be its return time to* U.

(a) *If* $\nu(\mathsf{L}) < \infty$ *then the SDS is* positive recurrent, *that is,*

$$\mathsf{E}(\tau^U) = \nu(\mathsf{L})/\nu(U) < \infty.$$

(b) *If* $\nu(\mathsf{L}) = \infty$ *then the SDS is* null recurrent, *that is,*

$$\mathsf{E}(\tau^U) = \infty.$$

This follows from the well known formula of Kac (see e.g. Aaronson [1, 1.5.5, p. 44]).

(2.20) LEMMA. *In the positive recurrent case, let the invariant measure be normalized so that* $\nu(\mathsf{L}) = 1$. *Then, for every starting point* $x \in X$, *the sequence* (X_n^x) *converges in law to* ν.

Proof. Let $\varphi : \mathsf{X} \to \mathbb{R}$ be continuous and compactly supported. Since φ is uniformly continuous, local contractivity implies for all $x, y \in X$ that $\varphi(X_n^x) - \varphi(X_n^y) \to 0$ almost surely. By dominated convergence, $\mathsf{E}(\varphi(X_n^x) - \varphi(X_n^y)) \to 0$. Thus,

$$\begin{aligned} P^n\varphi(x) - \int \varphi\, d\nu &= \int (P^n\varphi(x) - P^n\varphi(y))\, d\nu(y) \\ &= \int \mathsf{E}(\varphi(X_n^x) - \varphi(X_n^y))\, d\nu(y) \to 0. \ \blacksquare \end{aligned}$$

3. Basic example: affine stochastic recursion. Here we briefly review the main known results regarding the SDS on $\mathsf{X} = \mathbb{R}$ given by

$$Y_0^x = x, \quad Y_{n+1}^x = A_n Y_n^x + B_{n+1}, \tag{3.1}$$

where $(A_n, B_n)_{n \ge 0}$ is a sequence of i.i.d. random variables in $\mathbb{R}_*^+ \times \mathbb{R}$. The following results are known.

(3.2) PROPOSITION. *If* $\mathsf{E}(\log^+ A_n) < \infty$ *and*

$$-\infty \le \mathsf{E}(\log A_n) < 0$$

then (Y_n^x) *is strongly contractive on* $\mathbb{R}$. *If in addition* $\mathsf{E}(\log^+ |B_n|) < \infty$ *then the affine SDS has a unique invariant probability measure* ν, *and is* (*positive*) *recurrent on* $\mathsf{L} = \mathsf{supp}(\nu)$. *Furthermore, the shift on the trajectory space is ergodic with respect to the probability measure* $\Pr_\nu$.

Proof (outline). This is *the* classical application of Furstenberg's contraction principle. One verifies that for the associated right process,

$$R_n^x \to Z = \sum_{n=1}^{\infty} A_1 \cdots A_{n-1} B_n$$

almost surely for every $x \in \mathbb{R}$. The series that defines Z is almost surely abolutely convergent by the assumptions on the two expectations. Recurrence is easily deduced via Lemma (2.2). Indeed, we cannot have $|Y_n^x| \to \infty$ almost surely, because then by dominated convergence $\nu(U) = \nu\, P^n(U) \to 0$ for every relatively compact set U. Ergodicity now follows from strong contractivity. ∎

(3.3) PROPOSITION. *Suppose that* $\mathsf{Pr}[A_n = 1] < 1$ *and* $\mathsf{Pr}[A_n x + B_n = x] < 1$ *for all* $x \in \mathbb{R}$ *(non-degeneracy). If* $\mathsf{E}(|\log A_n|) < \infty$ *and* $\mathsf{E}(\log^+ B_n) < \infty$*, and if*

$$\mathsf{E}(\log A_n) = 0,$$

then (Y_n^x) *is locally contractive on* $\mathbb{R}$*. If in addition* $\mathsf{E}(|\log A_n|^2) < \infty$ *and* $\mathsf{E}((\log^+ |B_n|)^{2+\varepsilon}) < \infty$ *for some* $\varepsilon > 0$ *then the affine SDS has a unique invariant Radon measure* ν *with infinite mass, and it is (null) recurrent on* $\mathsf{L} = \mathsf{supp}(\nu)$*.*

This goes back to [3], with a small gap that was later filled in [5]. With the moment conditions as stated here, a nice and complete "geometric" proof is given in [10]: it is shown that under the stated hypotheses,

$$A_1 \cdots A_n \cdot \mathbf{1}_K(Y_n) \to 0 \quad \text{almost surely}$$

for every compact set K. Recurrence was shown earlier in [18, Lemma 5.49].

(3.4) PROPOSITION. *If* $\mathsf{E}(|\log A_n|) < \infty)$ *and* $\mathsf{E}(\log^+ B_n) < \infty$*, and if*

$$\mathsf{E}(\log A_n) > 0,$$

then (Y_n^x) *is transient, that is,* $|Y_n^x| \to \infty$ *almost surely for every starting point* $x \in \mathbb{R}$*.*

A proof is given, e.g., by Elie [19].

4. Iteration of random contractions. Let us now consider a more specific class of SDS: within $\mathfrak{G}$, we consider the closed submonoid $\mathfrak{L}_1$ of all *contractions* of X, i.e., mappings $f : \mathsf{X} \to \mathsf{X}$ with Lipschitz constant $\mathfrak{l}(f) \le 1$. We suppose that the probability measure $\widetilde{\mu}$ that governs the SDS is supported by $\mathfrak{L}_1$, that is, each random function F_n of (1.1) satisfies $\mathfrak{l}(F_n) \le 1$. In this case, one does not need local contractivity in order to obtain Lemma (2.2); this follows directly from properness of X and the inequality

$$D_n(x,y) \le d(x,y), \quad \text{where} \quad D_n(x,y) = d(X_n^x, X_n^y).$$

When $\mathsf{Pr}[d(X_n^x, x) \to \infty] = 0$ for every x, we can in general only speak of conservativity, since we do not yet have an attractor on which the SDS is topologically recurrent. Let $\mathfrak{S}(\widetilde{\mu})$ be the closed subsemigroup of $\mathfrak{L}_1$ generated by $\mathsf{supp}(\widetilde{\mu})$.

(4.1) REMARK. For strong contractivity it is sufficient that $\mathsf{Pr}[D_n(x,y) \to 0] = 1$ pointwise for all $x, y \in \mathsf{X}$.

Indeed, by properness, X has a dense, countable subset Y. If $K \subset \mathsf{X}$ is compact and $\varepsilon > 0$ then there is a finite $W \subset Y$ such that $d(y, W) < \varepsilon$ for every $y \in K$. Therefore

$$\sup_{y\in K} D_n(x,y) \le \underbrace{\max_{w\in W} D_n(x,w)}_{\to 0 \text{ a.s.}} + \varepsilon,$$

since $D_n(x,y) \le D_n(x,w) + D_n(w,y) \le D_n(x,w) + d(w,y)$.

The following key result of [4] (whose statement and proof is slightly strengthened here) is inspired by [28, Thm. 2.2], where reflected random walk is studied; see also [29].

(4.2) THEOREM. *If the SDS of contractions is conservative, then it is strongly contractive if and only if $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function.*

Proof. Keeping Remark (4.1) in mind, first assume that $D_n(x,y) \to 0$ almost surely for all x, y. We can apply all previous results on (local) contractivity, and the SDS has the non-empty attractor L. If $x_0 \in \mathsf{L}$, then with probability 1 there is a random subsequence (n_k) such that $X^x_{n_k} \to x_0$ for every $x \in \mathsf{X}$, and by the above, this convergence is uniform on compact sets. Thus, the constant mapping $x \mapsto x_0$ is in $\mathfrak{S}(\widetilde{\mu})$.

Conversely, assume that $\mathfrak{S}(\widetilde{\mu})$ contains a constant function. Since $D_{n+1}(x,y) \le D_n(x,y)$, the limit $D_\infty(x,y) = \lim_n D_n(x,y)$ exists and is between 0 and $d(x,y)$. We set $w(x,y) = \mathsf{E}(D_\infty(x,y))$. First of all, we claim that

$$\lim_{m\to\infty} w(X^x_m, X^y_m) = D_\infty(x,y) \quad \text{almost surely.} \tag{4.3}$$

To see this, consider $X^x_{m,n}$ as in (2.3). Then $D_{m,\infty}(x,y) = \lim_n d(X^x_{m,n}, X^y_{m,n})$ has the same distribution as $D_\infty(x,y)$, whence $\mathsf{E}(D_{m,\infty}(x,y)) = w(x,y)$. Therefore, we also have

$$\mathsf{E}(D_{m,\infty}(X^x_m, X^y_m) \mid F_1, \dots, F_m) = w(X^x_m, X^y_m).$$

On the other hand, $D_{m,\infty}(X^x_m, X^y_m) = D_\infty(x,y)$, and the bounded martingale

$$(\mathsf{E}(D_\infty(x,y) \mid F_1, \dots, F_m))_{m\ge 1}$$

converges almost surely to $D_\infty(x,y)$. Statement (4.3) follows.

Now let $\varepsilon > 0$ be arbitrary, and fix $x, y \in X$. We have to show that the event $\Lambda = [D_\infty(x,y) \ge \varepsilon]$ has probability 0.

(i) By conservativity,

$$\Pr\Big(\bigcup_{r\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{n\ge m} [X^x_n, X^y_n \in \mathsf{B}(r)]\Big) = 1.$$

On $\varLambda$, we have $D_n(x,y) \ge \varepsilon$ for all n. Therefore we need to show that $\Pr(\varLambda_r) = 0$ for each $r \in \mathbb{N}$, where

$$\varLambda_r = \bigcap_{m\in\mathbb{N}} \bigcup_{n\ge m} [X_n^x, X_n^y \in \mathsf{B}(r),\, D_n(x,y) \ge \varepsilon].$$

(ii) By assumption, there is $x_0 \in X$ which can be approximated uniformly on compact sets by functions of the form $f_k \circ \cdots \circ f_1$, where $f_j \in \mathsf{supp}(\widetilde{\mu})$. Therefore, given r there is $k \in \mathbb{N}$ such that

$$\Pr(\varGamma_{k,r}) > 0, \quad \text{where} \quad \varGamma_{k,r} = \Big[\sup_{u\in\mathsf{B}(r)} d(X_k^u, x_0) \le \varepsilon/4\Big].$$

On $\varGamma_{k,r}$ we have $D_\infty(u,v) \le D_k(u,v) \le \varepsilon/2$ for all $u,v \in \mathsf{B}(r)$. Therefore, setting $\delta = \Pr(\varGamma_{k,r}) \cdot (\varepsilon/2)$, we find for all $u,v \in \mathsf{B}(r)$ with $d(u,v) \ge \varepsilon$ that

$$\begin{aligned} w(u,v) &= \mathsf{E}(\mathbf{1}_{\varGamma_{k,r}}\, D_\infty(u,v)) + \mathsf{E}(\mathbf{1}_{\Omega\setminus\varGamma_{k,r}}\, D_\infty(u,v)) \\ &\le \Pr(\varGamma_{k,r}) \cdot (\varepsilon/2) + (1 - \Pr(\varGamma_{k,r})) \cdot d(u,v) \le d(u,v) - \delta. \end{aligned}$$

We conclude that on $\varLambda_r$, there is a (random) sequence (n_ℓ) such that

$$w(X_{n_\ell}^x, X_{n_\ell}^y) \le D_{n_\ell}(x,y) - \delta.$$

Passing to the limit on both sides, we see that (4.3) is violated on $\varLambda_r$, since $\delta > 0$. Therefore $\Pr(\varLambda_r) = 0$ for each r. ∎

(4.4) Corollary. *If the semigroup $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function, then the SDS is locally contractive.*

Proof. In the transient case, X_n^x can visit any compact K only finitely often, whence $d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^x) = 0$ for all but finitely many n. In the conservative case, we even have strong contractivity by Proposition (4.2). ∎

5. Some remarks on reflected random walk. As outlined in the introduction, the reflected random walk on $\mathbb{R}^+$ induced by a sequence $(B_n)_{n\ge0}$ of i.i.d. real valued random variables is given by

$$X_0^x = x \ge 0, \quad X_{n+1}^x = |X_n^x - B_{n+1}|. \tag{5.1}$$

Let μ be the distribution of the B_n, a probability measure on $\mathbb{R}$. The transition probabilities of the reflected random walk are

$$P(x,U) = \mu(\{y : |x-y| \in U\}),$$

where $U \subset \mathbb{R}^+$ is a Borel set. If $B_n \le 0$ almost surely, then (X_n^x) is an ordinary random walk (resulting from a sum of i.i.d. random variables). We shall exclude this, and we shall always assume to be in the *non-lattice* situation. That is,

$$\mathsf{supp}(\mu) \cap (0,\infty) \ne \emptyset, \quad \text{and} \quad \mathsf{supp}(\mu) \subset \kappa\cdot\mathbb{Z} \quad \text{for no } \kappa > 0. \tag{5.2}$$

For the lattice case, see [33], and for higher-dimensional variants, see [32].

For $b \in \mathbb{R}$, consider $g_b \in \mathfrak{L}_1(\mathbb{R}^+)$ given by $g_b(x) = |x - b|$. Then our reflected random walk is the SDS on $\mathbb{R}^+$ induced by the random continuous contractions $F_n = g_{B_n}$, $n \geq 1$. The law $\widetilde{\mu}$ of the F_n is the image of μ under the mapping $b \mapsto g_b$.

In [29, Prop. 3.2], it is shown that $\mathfrak{S}(\widetilde{\mu})$ contains the constant function $x \mapsto 0$. Note that this statement and its proof in [29] are completely deterministic, regarding topological properties of the set $\mathsf{supp}(\mu)$. In view of Theorem (4.2) and Corollary (4.4), we get the following.

(5.3) Proposition. *Under the assumptions* (5.2)*, the reflected random walk on* $\mathbb{R}^+$ *is locally contractive, and strongly contractive if it is recurrent.*

A. Non-negative B_n. We first consider the case when $\mathsf{Pr}[B_n \geq 0] = 1$. Let

$$N = \sup \mathsf{supp}(\mu) \quad \text{and} \quad \mathsf{L} = \begin{cases} [0, N] & \text{if } N < \infty, \\ \mathbb{R}^+ & \text{if } N = \infty. \end{cases}$$

The distribution function of μ is

$$F_\mu(x) = \mathsf{Pr}[B_n \leq x] = \mu([0, x]), \qquad x \geq 0.$$

We next summarize basic properties that are due to [20], [28] and [29]; they do not depend on recurrence.

(5.4) Lemma. *Suppose that* (5.2) *is satisfied and that* $\mathsf{supp}(\mu) \subset \mathbb{R}^+$*. Then the following holds.*

(a) *The reflected random walk with any starting point is absorbed after finitely many steps by the interval* L*.*
(b) *The reflected random walk is topologically irreducible on* L*, that is, for every* $x \in \mathsf{L}$ *and open set* $U \subset \mathsf{L}$*, there is* n *such that* $P^n(x, U) = \mathsf{Pr}[X_n^x \in U] > 0$.
(c) *The measure* ν *on* L *given by*

$$\nu(dx) = (1 - F_\mu(x))\, dx,$$

where dx *is Lebesgue measure, is an invariant measure for the transition kernel* P*.*

At this point Lemma (2.6) implies that in the recurrent case, the above set is indeed the attractor, and ν is the unique invariant measure up to multiplication with constants. We now want to understand when we have recurrence.

(5.5) Theorem. *Suppose that* (5.2) *is satisfied and* $\mathsf{supp}(\mu) \subset \mathbb{R}^+$*. Then each of the following conditions implies the next one and is sufficient for* *recurrence of the reflected random walk on* L:

(i) $$\mathsf{E}(B_1) < \infty,$$

(ii) $$\mathsf{E}\big(\sqrt{B_1}\big) < \infty,$$

(iii) $$\int_{\mathbb{R}^+} (1 - F_\mu(x))^2\,dx < \infty,$$

(iv) $$\lim_{y\to\infty} (1 - F_\mu(y)) \int_0^y (F_\mu(y) - F_\mu(x))\,dx = 0.$$

In particular, one has positive recurrence precisely when $\mathsf{E}(B_1) < \infty$.

The proof of (i)⇒(ii)⇒(iii)⇒(iv) is a basic exercise. For condition (i), see [28]. The implication (ii) ⇒ recurrence is due to [37], while the recurrence condition (iii) was proved by ourselves in [33]. However, we had not been aware of [37], as well as of [34], where it is proved that already (iv) implies recurrence on L. Since ν has finite total mass precisely when $\mathsf{E}(B_1) < \infty$, the statement on positive recurrence follows from Corollary (2.19). In this case, also Lemma (2.20) applies and shows that X_n^x converges in law to $(1/\nu(\mathsf{L}))\nu$. This was already obtained by [28].

Note that the "margin" between conditions (ii), (iii) and (iv) is quite narrow.

B. General reflected random walk. We now drop the restriction that the random variables B_n are non-negative. Thus, the "ordinary" random walk $S_n = B_1 + \cdots + B_n$ on $\mathbb{R}$ may visit the positive as well as the negative half-axis. Since we assume that μ is non-lattice, the closed group generated by $\mathsf{supp}(\mu)$ is $\mathbb{R}$.

We start with a simple observation ([6] has a more complicated proof).

(5.6) LEMMA. *If μ is symmetric, then the reflected random walk is* (*topologically*) *recurrent if and only if the random walk* (S_n) *is recurrent.*

Proof. If μ is symmetric, then also $|S_n|$ is a Markov chain. Indeed, for a Borel set $U \subset \mathbb{R}^+$,

$$\begin{aligned}\mathsf{Pr}[\,|S_{n+1}| \in U \mid S_n = x] &= \mu(-x+U) + \mu(-x-U) - \mu(-x)\delta_0(U)\\ &= \mathsf{Pr}[\,|S_{n+1}| \in U \mid S_n = -x],\end{aligned}$$

and we see that $|S_n|$ has the same transition probabilities as the reflected random walk governed by μ. ∎

Recall the classical result that when $\mathsf{E}(|B_1|) < \infty$ and $\mathsf{E}(B_1) = 0$ then (S_n) is recurrent; see Chung and Fuchs [15].

(5.7) COROLLARY. *If μ is symmetric and has finite first moment then the reflected random walk is recurrent.*

Let $B_n^+ = \max\{B_n, 0\}$ and $B_n^- = \max\{-B_n, 0\}$, so that $B_n = B_n^+ - B_n^-$. The following is well-known.

(5.8) LEMMA. *If* (a) $\mathsf{E}(B_1^-) < \mathsf{E}(B_1^+) \le \infty$, *or if* (b) $0 < \mathsf{E}(B_1^-) = \mathsf{E}(B_1^+) < \infty$, *then* $\limsup S_n = \infty$ *almost surely, so that there are infinitely many reflections.*

In general, we should exclude that $S_n \to -\infty$, since in that case there are only finitely many reflections, and the reflected random walk tends to $+\infty$ almost surely. In what follows, we assume that $\limsup S_n = \infty$ almost surely. Then the (non-strictly) ascending *ladder epochs*

$$\mathbf{s}(0) = 0, \quad \mathbf{s}(k+1) = \inf\{n > \mathbf{s}(k) : S_n \ge S_{\mathbf{s}(k)}\}$$

are all almost surely finite, and the random variables $\mathbf{s}(k+1) - \mathbf{s}(k)$ are i.i.d. We can consider the *embedded random walk* $S_{\mathbf{s}(k)}$, $k \ge 0$, which tends to ∞ almost surely. Its increments $\overline{B}_k = S_{\mathbf{s}(k)} - S_{\mathbf{s}(k-1)}$, $k \ge 1$, are i.i.d. non-negative random variables with distribution denoted $\overline{\mu}$. Furthermore, if $\overline{X}_k^x$ denotes the reflected random walk associated with the sequence $(\overline{B}_k)$, while X_n^x is our original reflected random walk associated with (B_n), then

$$\overline{X}_k^x = X_{\mathbf{s}(k)}^x,$$

since no reflection can occur between times $\mathbf{s}(k)$ and $\mathbf{s}(k+1)$. When $\mathsf{Pr}[B_n < 0] > 0$, one clearly has $\sup \mathsf{supp}(\overline{\mu}) = +\infty$. Lemma (5.4) implies the following.

(5.9) COROLLARY. *Suppose that* (5.2) *is satisfied,* $\mathsf{Pr}[B_n < 0] > 0$ *and* $\limsup S_n = \infty$. *Then*

(a) *the reflected random walk is topologically irreducible on* $\mathsf{L} = \mathbb{R}^+$,
(b) *the embedded reflected random walk* $\overline{X}_k^x$ *is recurrent if and only the original reflected random walk is recurrent.*

Proof. Statement (a) is clear.

Since both processes are locally contractive, each of them is transient if and only if it tends to $+\infty$ almost surely: If $\lim_n X_n^x = \infty$ then clearly also $\lim_k X_{\mathbf{s}(k)}^x = \infty$ a.s. Conversely, suppose that $\lim_k \overline{X}_k^x \to \infty$ a.s. If $\mathbf{s}(k) \le n < \mathbf{s}(k+1)$ then $X_n^x \ge X_{\mathbf{s}(k)}^x$. (Here, k is random, depending on n and $\omega \in \Omega$, and when $n \to \infty$ then $k \to \infty$ a.s.) Therefore, also $\lim_n X_n^x = \infty$ a.s., so that (b) is also true. ∎

We can now deduce the following.

(5.10) THEOREM. *Suppose that* (5.2) *is satisfied and* $\mathsf{Pr}[B_1 < 0] > 0$. *Then the reflected random walk* (X_n^x) *is (topologically) recurrent on* $\mathsf{L} = \mathbb{R}^+$ *if*

(a) $\mathsf{E}(B_1^-) < \mathsf{E}(B_1^+)$ *and* $\mathsf{E}\Big(\sqrt{B_1^+}\Big) < \infty$, *or*

(b) $0 < \mathsf{E}(B_1^-) = \mathsf{E}(B_1^+)$ *and* $\mathsf{E}\Big(\sqrt{B_1^+}^{\,3}\Big) < \infty$.

Proof. We show that in each case the assumptions imply that $\mathsf{E}(\sqrt{\overline{B}_1}) < \infty$. Then we can apply Theorem (5.5) to deduce recurrence of $(\overline{X}_k^x)$. This in turn yields recurrence of (X_n^x) by Corollary (5.9).

(a) Under the first set of assumptions,

$$\begin{aligned}\mathsf{E}\Big(\sqrt{\overline{B}_1}\Big) &= \mathsf{E}\Big(\sqrt{B_1 + \cdots + B_{\mathbf{s}(1)}}\Big) \le \mathsf{E}\Big(\sqrt{B_1^+ + \cdots + B_{\mathbf{s}(1)}^+}\Big)\\ &\le \mathsf{E}\Big(\sqrt{B_1^+} + \cdots + \sqrt{B_{\mathbf{s}(1)}^+}\Big) = \mathsf{E}\Big(\sqrt{B_1^+}\Big)\cdot \mathsf{E}(\mathbf{s}(1))\end{aligned}$$

by Wald's identity. Thus, we now are left with proving $\mathsf{E}(\mathbf{s}(1)) < \infty$. If $\mathsf{E}(B_1^+) < \infty$, then $\mathsf{E}(|B_1|) < \infty$ and $\mathsf{E}(B_1) > 0$ by assumption, and in this case it is well known that $\mathsf{E}(\mathbf{s}(1)) < \infty$; see e.g. [20, Thm. 2 in §XII.2, pp. 396–397]. If $\mathsf{E}(B_1^+) = \infty$ then there is $M > 0$ such that $B_n^{(M)} = \min\{B_n, M\}$ (which has finite first moment) satisfies $\mathsf{E}(B_n^{(M)}) = \mathsf{E}(B_1^{(M)}) > 0$. The first increasing ladder epoch $\mathbf{s}^{(M)}(1)$ associated with $S_n^{(M)} = B_1^{(M)} + \cdots + B_n^{(M)}$ has finite expectation by what we just said, and $\mathbf{s}(1) \le \mathbf{s}^{(M)}(1)$. Thus, $\mathbf{s}(1)$ is integrable.

(b) If the B_n are centered, non-zero and $\mathsf{E}((B_1^+)^{1+a}) < \infty$, where $a > 0$, then $\mathsf{E}((\overline{B}_1)^a) < \infty$, as was shown by Chow and Lai [14]. In our case, $a = 1/2$. ∎

We conclude our remarks on the reflected random walk by discussing sharpness of the sufficient recurrence conditions $\mathsf{E}\Big(\sqrt{B_1^+}^{\,3}\Big) < \infty$ in the centered case, resp. $\mathsf{E}(\sqrt{B_1}) < \infty$ in the case when $B_1 \ge 0$.

(5.11) Example. Define a symmetric probability measure μ on $\mathbb{R}$ by

$$\mu(dx) = \frac{dx}{(1+|x|)^{1+a}},$$

where $a > 0$ and c is the proper normalizing constant (and dx is Lebesgue measure). Then it is well known and quite easy to prove via Fourier analysis that the associated symmetric random walk S_n on $\mathbb{R}$ is recurrent if and only if $a \ge 1$. By Lemma (5.6), the associated reflected random walk is also recurrent, but if $1 \le a \le 3/2$ then condition (b) of Theorem (5.10) does not hold.

Nevertheless, we can also show that in general, the sufficient condition $\mathsf{E}\big(\sqrt{\overline{B}_1}\big) < \infty$ for recurrence of the reflected random walk with non-negative increments $\overline{B}_n$ is very close to being sharp. (We write $\overline{B}_n$ because we shall represent this as an embedded random walk in the next example.)

(5.12) Proposition. *Let μ_0 be a probability measure on $\mathbb{R}^+$ which has a density $\phi_0(x)$ with respect to Lebesgue measure that is decreasing and satis-*

fies

$$\phi(x) \sim c(\log x)^b / x^{3/2} \quad \text{as } x \to \infty,$$

where $b > 1/2$ and $c > 0$. Then the associated reflected random walk on $\mathbb{R}^+$ is transient.

Note that μ_0 has finite moment of order $1/2 - \varepsilon$ for every $\varepsilon > 0$, while the moment of order $1/2$ is infinite.

The proof needs some preparation. Let (B_n) be i.i.d. random variables with values in $\mathbb{R}$ that have finite first moment and are non-constant and centered, and let μ be their common distribution.

The first *strictly ascending* and *strictly descending ladder epochs* of the random walk $S_n = B_1 + \cdots + B_n$ are

$$\mathbf{t}_+(1) = \inf\{n > 0 : S_n > 0\} \quad \text{and} \quad \mathbf{t}_-(1) = \inf\{n > 0 : S_n < 0\},$$

respectively. They are almost surely finite. Let μ_+ be the distribution of $S_{\mathbf{t}_+(1)}$ and μ_- the distribution of $S_{\mathbf{t}_-(1)}$, and—as above—$\overline{\mu}$ the distribution of $\overline{B}_1 = S_{\mathbf{s}(1)}$. We denote the characteristic function associated with any probability measure σ on $\mathbb{R}$ by $\widehat{\sigma}(t)$, $t \in \mathbb{R}$. Then, following Feller [20, (3.11) in §XII.3], *Wiener–Hopf factorization* tells us that

$$\mu = \overline{\mu} + \mu_- - \overline{\mu} * \mu_- \quad \text{and} \quad \overline{\mu} = u \cdot \delta_0 + (1-u) \cdot \mu_+,$$

where

$$u = \overline{\mu}(0) = \sum_{n=1}^{\infty} \mathsf{Pr}[S_1 < 0, \ldots, S_{n-1} < 0,\ S_n = 0] < 1.$$

Here $*$ is convolution. Note that when μ is absolutely continuous (with respect to Lebesgue measure) then $u = 0$, so that

$$\overline{\mu} = \mu_+ \quad \text{and} \quad \mu = \mu_+ + \mu_- - \mu_+ * \mu_-. \tag{5.13}$$

(5.14) Lemma. *Let μ_0 be a probability measure on $\mathbb{R}^+$ which has a decreasing density $\phi_0(x)$ with respect to Lebesgue measure. Then there is an absolutely continuous symmetric probability measure μ on $\mathbb{R}$ such that the associated first* (*non-strictly*) *ascending ladder random variable has distribution μ_0.*

Proof. If μ_0 is the law of the first strictly ascending ladder random variable associated with some absolutely continuous, symmetric measure μ, then by (5.13) we must have $\mu_+ = \mu_0$ and $\mu_- = \check{\mu}_0$, the reflection of μ_0 at 0, and

$$\mu = \mu_0 + \check{\mu}_0 - \mu_0 * \check{\mu}_0. \tag{5.15}$$

We *define* μ in this way. The monotonicity assumption on μ_0 implies that μ is a probability measure: indeed, by the monotonicity assumption it is straightforward to check that the function $\phi = \phi_0 + \check{\phi}_0 - \phi_0 * \check{\phi}_0$ is non-negative; this is the density of μ.

The measure μ of (5.15) is non-degenerate and symmetric. If it induces a recurrent random walk (S_n), then the ascending and descending ladder epochs are a.s. finite. If (S_n) is transient, then $|S_n| \to \infty$ almost surely, but it cannot be that $\mathsf{Pr}[S_n \to \infty] > 0$ since in that case this probability would have to be 1 by Kolmogorov's 0-1-law, while symmetry would yield $\mathsf{Pr}[S_n \to -\infty] = \mathsf{Pr}[S_n \to \infty] \le 1/2$. Therefore $\liminf S_n = -\infty$ and $\limsup S_n = +\infty$ almost surely, a well-known fact (see e.g. [20, Thm. 1 in §XII.2, p. 395]). Consequently, the ascending and descending ladder epochs are again a.s. finite. Therefore the probability measures μ_+ and $\mu_- = \check{\mu}_+$ (the laws of $S_{\mathbf{t}_\pm(1)}$) are well defined. By the uniqueness theorem for Wiener–Hopf factorization [20, Thm. 1 in §XII.3, p. 401], it follows that $\mu_- = \check{\mu}_0$ and that the distribution of the first (non-strictly) ascending ladder random variable is $\overline{\mu} = \mu_0$. ∎

Proof of Proposition (5.12). Let μ be the symmetric measure associated with μ_0 according to (5.15) in Lemma (5.14). Then its characteristic function $\widehat{\mu}(t)$ is non-negative real. A well-known criterion says that the random walk S_n associated with μ is transient if and only if (the real part of) $1/(1-\widehat{\mu}(t))$ is integrable in a neighbourhood of 0. Returning to $\mu_0 = \mu_+$, it is a standard exercise (see [20, Ex. 12 in Ch. XVII, Section 12]) to show that there is $A \in \mathbb{C}$, $A \neq 0$, such that the characteristic function satisfies

$$\widehat{\mu_0}(t) = 1 + A\sqrt{t}\,(\log t)^b(1 + o(t)) \quad \text{as } t \to 0.$$

By (5.13),

$$1 - \widehat{\mu}(t) = (1 - \widehat{\mu_+}(t))(1 - \widehat{\mu_-}(t)).$$

We deduce

$$\widehat{\mu}(t) = 1 - |A|^2|t|(\log|t|)^{2b}(1 + o(t)) \quad \text{as } t \to 0.$$

The function $1/(1-\widehat{\mu}(t))$ is integrable near 0. By Lemma (5.6), the associated reflected random walk is transient. But then also the embedded reflected random walk associated with $S_{\mathbf{s}(n)}$ is transient by Corollary (5.9). This is the reflected random walk governed by μ_0. ∎

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