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ON LIFTING OF IDEMPOTENTS AND SEMIREGULAR ENDOMORPHISM RINGS

BY

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Abstract. Starting with some observations on (strong) lifting of idempotents, we characterize a module whose endomorphism ring is semiregular with respect to the ideal of endomorphisms with small image. This is the dual of Yamagata's work [Colloq. Math. 113 (2008)] on a module whose endomorphism ring is semiregular with respect to the ideal of endomorphisms with large kernel.

1. Introduction. In this paper, rings R are associative with identity and modules M are unitary right modules. Homomorphisms of modules are written on the left of their arguments. For a submodule X of a module M, we write $X \leq_e M$ and $X \ll M$ to indicate that X is a large, respectively small, submodule of M. For an R-module M, S denotes the endomorphism ring of M, and we let

$$\Delta = \{u \in S : \operatorname{Ker} u \leq_e M\} \quad \text{and} \quad \nabla = \{u \in S : uM \ll M\}.$$

Note that Δ and ∇ are proper ideals of S. The Jacobson radical of a ring R is denoted by $J(R)$. A ring R is *semiregular* if $R/J(R)$ is (von Neumann) regular and idempotents lift modulo $J(R)$. It is well-known from Utumi [14] that S is semiregular and $\Delta = J(S)$ for an injective module M. This result was generalized to quasi-injective modules by Faith and Utumi [2], to continuous modules by Utumi [15], and later to direct-injective, kernel-extending modules by Nicholson [9]. Dually, S is semiregular and $\nabla = J(S)$ for a discrete module (also called d-continuous module) M as shown by Mohamed and Singh [8], and more generally for a direct-projective, image-lifting module M by Nicholson [9].

This paper is motivated by recent work of Yamagata [16] who characterized a module M for which S/Δ is regular and idempotents lift modulo Δ. His results are used to obtain characterizations of a module M for which S is semiregular and $J(S) = \Delta$. Dually, we characterize a module M for which S/∇ is regular and idempotents lift modulo ∇, and further a module

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M for which S is semiregular and $J(S) = \nabla$. Because of the role of lifting and strong lifting of idempotents in this paper, Section 2 is devoted to some basic relations between lifting and strong lifting of idempotents. If I is an ideal of R, we write $\overline{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$.

2. Lifting and strong lifting of idempotents. Lifting idempotents is a basic method in determining the structure of a ring. For a left ideal I of a ring R, we say that *idempotents lift modulo I* if, whenever $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $a - e \in I$. Following [12], we say that *idempotents lift strongly modulo I* if $a^2 - a \in I$ implies that $a - e \in I$ for some $e^2 = e \in aR$ (equivalently $e^2 = e \in aRa$, or $e^2 = e \in Ra$). By [12, Proposition 16], for an ideal I of R, idempotents lift strongly modulo I if and only if every direct sum decomposition of $\overline{R}$ into left ideals lifts to a direct sum decomposition of R into left ideals, that is, $\overline{R} = \overline{R}\bar{a}_1 \oplus \cdots \oplus \overline{R}\bar{a}_n$ implies that $R = T_1 \oplus \cdots \oplus T_n$ where $T_i \subseteq Ra_i$ is a left ideal for each i. Lifting and strong lifting of idempotents are the same for several ideals including $I = J(R)$, but they differ in general (see [12]). In this section, we discuss basic relations between the two conditions through a third condition of lifting regular elements.

Following [5], we say that *regular elements lift modulo a left ideal I* of R if, whenever $a - aba \in I$, there exists a regular element r of R such that $a - r \in I$.

LEMMA 2.1. *Let I be an ideal of a ring R. The following are equivalent:*

(1) *Idempotents lift modulo I.*
(2) *Each idempotent of R/I lifts to a regular element of R.*

Proof. Obviously, (1) implies (2). Suppose (2) holds and let $a^2 - a \in I$. By (2), there exist $r, s \in R$ such that $r = rsr$ and $r - a \in I$. Let $e = rs$. Then $er = r$ and $f := e + er(1 - e)$ is an idempotent. Thus, $\bar{r}\bar{e} = \bar{r}^2\bar{s} = \bar{a}^2\bar{s} = \bar{a}\bar{s} = \bar{r}\bar{s} = \bar{e}$ and

$$\bar{f} = \bar{e} + \overline{er}(\bar{1} - \bar{e}) = \bar{e} + \bar{r}(\bar{1} - \bar{e}) = \bar{e} + \bar{a}(\bar{1} - \bar{e}) = \bar{a} + (\bar{1} - \bar{a})\bar{e} = \bar{a} + (\bar{1} - \bar{r})\bar{e} = \bar{a}$$

in R/I. This proves (1). ∎

Hence, regular elements lifting modulo an ideal I implies that idempotents lift modulo I. But the converse is false.

EXAMPLE 2.2. Let $R = \mathbb{Z}$ and let $I = 5\mathbb{Z}$. Then idempotents lift modulo I by [12, Example 2]. Notice that $3 - 3\cdot 2\cdot 3 \in I$. Assume that regular elements lift modulo I. Then there exist $a, b \in R$ such that $a = aba$ and $3 - a \in I$. Since $a \neq 0$, we have $ab = 1$; so $a = 1$ or $a = -1$. But this contradicts that $3 - a \in I$.

LEMMA 2.3. *The following are equivalent for a left ideal I of R:*

(1) *If $a - aba \in I$, there exists a regular element $r \in aR$ such that $a - r \in I$.*
(2) *If $a - aba \in I$, there exists a regular element $r \in aRa$ such that $a - r \in I$.*

Proof. Suppose that (1) holds and let $a - aba \in I$. Then

$$aba - (aba)b(aba) = (1 + ab)(ab)(a - aba) \in I.$$

By (1), there exists a regular element $r \in (aba)R$ such that $r - aba \in I$, i.e., $r - a \in I$. Write $r = rsr$ with $s \in R$ and $r = (aba)c$ with $c \in R$. Define $d = rsa \in aRa$. Then $d - a = rsa - a = (rs - 1)(a - r) \in I$ and moreover

$$d(bacs)d = rsa \cdot bacs \cdot rsa = rs \cdot abac \cdot srsa = rsr \cdot srsa = rsa = d.$$

This proves (2). ∎

We say that *regular elements lift strongly modulo a left ideal I* if the conditions in Lemma 2.3 are satisfied. Clearly, regular elements lift strongly modulo an ideal I if and only if, whenever $a - aba \in I$, there exists a regular element $r \in Ra$ such that $a - r \in I$. An ideal I of R is called an *enabling ideal* if whenever $a - e \in I$ with $e^2 = e \in R$ there exists $f^2 = f \in aR$ (equivalently $f^2 = f \in aRa$ or $f^2 = f \in Ra$) such that $a - f \in I$ ([1]).

The next theorem shows that, for an ideal I of R, regular elements lift strongly modulo I if and only if idempotents lift strongly modulo I.

THEOREM 2.4. *The following are equivalent for an ideal I of R:*

(1) *Idempotents lift strongly modulo I.*
(2) *Regular elements lift strongly modulo I.*
(3) *Idempotents lift modulo I and I is an enabling ideal.*
(4) *Regular elements lift modulo I and I is an enabling ideal.*

Proof. (1)⇔(3). This is [1, Theorem 2].

(1)⇒(2). Suppose that $a - aba \in I$. Then $(ab)^2 - ab \in I$. By hypothesis, there exists $e^2 = e \in (ab)R$ such that $e - ab \in I$. Write $e = (ab)c$ with $c \in R$ and let $d = ea \in aR$. Then $a - d = a - ea = (a - aba) + (ab - e)a \in I$ and moreover $d(bc)d = e(abc)d = e^2d = d$.

(2)⇒(4). Let $a - e \in I$ with $e^2 = e$. Then $\bar{a}^3 = \bar{a}^2 = \bar{a}$. Thus, by hypothesis, there exists a regular element $r \in aR$ such that $a - r \in I$. Write $r = rsr$ where $s \in R$. Then $f := rs + r(1 - rs) \in aR$ is an idempotent and moreover

$$\bar{f} = \bar{r}\bar{s} + \bar{r} - \bar{r}\bar{r}\bar{s} = \bar{r}\bar{s} + \bar{a} - \bar{a}^2\bar{s} = \bar{r}\bar{s} + \bar{a} - \bar{a}\bar{s} = \bar{r}\bar{s} + \bar{a} - \bar{r}\bar{s} = \bar{a};$$

so $a - f \in I$. This shows that I is an enabling ideal.

(4)⇒(3). Apply Lemma 2.1. ∎

There exists a ring R with an ideal I such that regular elements lift modulo I, but not strongly.

EXAMPLE 2.5. *Let $R = \mathbb{Z}$ and $I = 2\mathbb{Z}$. Then regular elements lift modulo I, but not strongly.*

Proof. The ring R has only three regular elements: 0, 1 and -1. If $a \in R$ is even, $a - 0 \in I$; if $a \in R$ is odd, $a - 1 \in I$. So regular elements lift modulo I. We see that $3 - 3 \cdot 1 \cdot 3 \in I$. The only regular element in $3R$ is 0, but $3 - 0 \notin I$. Thus there does not exist a regular element $r \in 3R$ such that $3 - r \in I$. So regular elements do not lift strongly modulo I. ∎

COROLLARY 2.6. *Let I be an enabling ideal of a ring R. Then idempotents lift modulo I if and only if regular elements lift modulo I if and only if regular elements lift strongly modulo I.*

Various examples of enabling ideals of a ring are given in [1]. In particular, every ideal contained in $J(R)$ is an enabling ideal of R by [1, Proposition 5]. So Corollary 2.6 has the following consequence.

COROLLARY 2.7 ([5, Corollary 9.4], [17, Lemma 2.4]). *Let $I \subseteq J(R)$ be an ideal of R. Then idempotents lift modulo I if and only if regular elements lift modulo I.*

Khurana and Lam [5, Theorem 9.3] proved that, for an ideal I of R, if idempotents lift modulo every left ideal contained in I then regular elements lift modulo every left ideal contained in I. The equivalence (1)⇔(3) of our next theorem proves the converse, and extends the result to the case when I is a left ideal.

THEOREM 2.8. *Let I be a left ideal of R. The following are equivalent:*

(1) *Idempotents lift modulo every left ideal contained in I.*
(2) *Idempotents lift strongly modulo every left ideal contained in I.*
(3) *Regular elements lift modulo every left ideal contained in I.*
(4) *Regular elements lift strongly modulo every left ideal contained in I.*

Proof. (1)⇒(2). Let $K \subseteq I$ be a left ideal of R and suppose that $a^2 - a \in K$. Then $R(a^2 - a) \subseteq I$. By hypothesis, there exists $e^2 = e \in R$ such that $e - a \in R(a^2 - a)$. Thus, $e - a \in K$ and $e \in Ra$. This shows that idempotents lift strongly modulo K.

(2)⇒(4). Suppose that $a - aba \in K$, where $K \subseteq I$ is a left ideal of R. Then $ba - (ba)^2 \in K$. By hypothesis and by [12, Lemma 1], there exists $e^2 = e \in R(ba)$ such that $e - ba \in K$. Write $e = c(ba)$ with $c \in R$ and let $d = ae \in aRa$. Then $a - d = (a - aba) + a(ba - e) \in K$ and moreover $d(cb)d = d(cba)e = de^2 = d$. So (4) holds.

(4)⇒(3). This is clear.

(3)⇒(1). Suppose that $a^2-a \in K$ where $K \subseteq I$ is a left ideal of R. Then $a^3-a=(a+1)(a^2-a) \in R(a^3-a) \subseteq K$. By hypothesis, there exist $r, s \in R$ such that $r=rsr$ and $a-r \in R(a^3-a)$. Let $e=sr$ and $f=e+(1-e)re$. Then f is an idempotent of R. It suffices to show that $f-a \in K$. Since $a-r \in R(a^3-a)$, write $a-r=b(a^3-a)$ with $b \in R$. Then

$$(a-r)a=b(a^3-a)a=ba(a^3-a) \in R(a^3-a) \subseteq K,$$

so $a^2-r^2=(a-r)a+r(a-r) \in K$. It follows that

$$f-a=(sr+r-sr^2)-a=(1+s)(r-a)+s(a-a^2)+s(a^2-r^2) \in K.$$

This proves (1). ∎

By Nicholson [10, Theorem 2.1], a ring R is an exchange ring if and only if idempotents lift modulo every left ideal. Thus, letting $I=R$ in Theorem 2.8 yields the following

Corollary 2.9 ([3, Corollary 5]). *A ring R is an exchange ring if and only if regular elements lift modulo every left ideal of R.*

3. Yamagata's theorem and consequences. For an ideal I of a ring R, [12, Theorem 28] gives equivalent conditions on R such that R/I is regular and idempotents lift strongly modulo I. In this section, we review Yamagata's theorem which gives characterizations of a module M for which S/Δ is regular and idempotents lift modulo Δ, and show that S/Δ is regular and idempotents lift strongly modulo Δ if and only if S is semiregular with $J(S)=\Delta$. As a consequence of Yamagata's theorem, characterizations are obtained for a module M with the latter condition.

Lemma 3.1. *If $u, v \in S$, then* $\operatorname{Ker}(u-uvu)=\operatorname{Ker} u \oplus \operatorname{Ker}(1-vu)$. *In particular,* $\operatorname{Ker}(u-u^2)=\operatorname{Ker} u \oplus \operatorname{Ker}(1-u)$.

Proof. It is clear that $\operatorname{Ker}(u-uvu) \supseteq \operatorname{Ker} u+\operatorname{Ker}(1-vu)$ and that $\operatorname{Ker} u \cap \operatorname{Ker}(1-vu)=0$. For $x \in \operatorname{Ker}(u-uvu)$, $x=vux+(1-vu)x$ with $vux \in \operatorname{Ker}(1-vu)$ and $(1-vu)x \in \operatorname{Ker} u$; so $\operatorname{Ker}(u-uvu)=\operatorname{Ker} u+\operatorname{Ker}(1-vu)$. ∎

Let X, Y be submodules of a module M. Following Yamagata [16], X is called a *semicomplement* of Y in M if $X \cap Y=0$ and $X+Y \leq_e M$. A submodule N of a module M is said to *lie under a direct summand* of M if N is large in a direct summand of M.

Lemma 3.2 ([16]). *The following are equivalent for an idempotent $\bar{u} \in S/\Delta$:*

(1) *$\bar{u}$ lifts to an idempotent of S.*
(2) *There is a semicomplement N of* $\operatorname{Ker} u$ *in M such that uN lies under a direct summand of M.*

If N is a submodule of M, we write $N \hookrightarrow M$ for the inclusion. Let $u \in S$. If N is a semicomplement of $\operatorname{Ker} u$ in M, then $u|_N : N \to uN$ is an isomorphism, so $(u|_N)^{-1} : uN \to N$ is well defined.

LEMMA 3.3 ([16]). *The following are equivalent for $u \in S$:*

(1) *$\bar{u}$ is regular in S/Δ.*
(2) *There exist $v \in S$ and a semicomplement N of* $\operatorname{Ker} u$ *in M such that the following diagram is commutative:*

$$\begin{array}{ccc} uN & \hookrightarrow & M \\ {\scriptstyle (u|_N)^{-1}}\downarrow & & \downarrow{\scriptstyle v} \\ N & \hookrightarrow & M \end{array}$$

(3) *There exists a semicomplement N of* $\operatorname{Ker} u$ *in M such that $N \subseteq$* $\operatorname{Ker}(1 - vu)$ *for some $v \in S$.*

For $u \in S$, $\operatorname{Ker}(1 - u) \leq_e M$ implies that u is a monomorphism because $\operatorname{Ker} u \cap \operatorname{Ker}(1 - u) = 0$.

LEMMA 3.4 ([16]). *For a module M, $\Delta \subseteq J(S)$ iff every $u \in S$ with* $\operatorname{Ker}(1 - u) \leq_e M$ *is an isomorphism.*

THEOREM 3.5 ([16]). *The following are equivalent for a module M:*

(1) *S/Δ is regular, and idempotents lift modulo Δ.*
(2) *For any $u \in S$, there exist semicomplements N_1, N_2 of* $\operatorname{Ker} u$ *in M such that*
   (a) *$(u|_{N_1})^{-1} : uN_1 \to N_1$ extends to an endomorphism of M,*
   (b) *uN_2 lies under a direct summand of M if $u^2 - u \in \Delta$.*
(3) *For any $u \in S$, there exists a semicomplement N of* $\operatorname{Ker} u$ *in M such that*
   (a) *$(u|_N)^{-1} : uN \to N$ extends to an endomorphism of M,*
   (b) *uN lies under a direct summand of M if $u^2 - u \in \Delta$.*

Next we discuss some consequences of Theorem 3.5. In the literature, various sufficient conditions on a module M are obtained so that S is semiregular and $\Delta = J(S)$; for example, see [2], [7], [9], [11], [14], [15] and [16]. Here we characterize a module M for which S is semiregular and $\Delta = J(S)$. A submodule X of M is called a *kernel submodule* if $X = \operatorname{Ker} u$ for some $u \in S$. The module M is called *kernel-extending* if every kernel submodule of M lies under a direct summand. For $M_R = R_R$, the equivalence (1)$\Leftrightarrow$(2) in Corollary 3.6 below is obtained in [12, Corollary 35].

COROLLARY 3.6. *Let M be a module. The following are equivalent:*

(1) *S/Δ is regular, and idempotents lift strongly modulo Δ.*

(2) *S is semiregular and $J(S) = \Delta$.*
(3) *The following hold:*
   (a) *M is kernel-extending.*
   (b) *Every monomorphism in S with essential image is onto.*
   (c) *For any $u \in S$, there exists a semicomplement N of* $\operatorname{Ker} u$ *in M such that $(u|_N)^{-1} : uN \to N$ extends to an endomorphism of M.*

Proof. (2)⇒(1). Apply [12, Lemma 5].

(1) ⇒ (2). Since S/Δ is regular, $J(S) \subseteq \Delta$. So to show (2), it suffices to show that $J(S) \supseteq \Delta$. Assume that $u \in S$ with $\operatorname{Ker}(1-u) \leq_e M$. We only need to show that u is an isomorphism by Lemma 3.4. Since $u - 1 \in \Delta$, by (1) there exists $e^2 = e \in uSu$ with $u - e \in \Delta$. So $N := \operatorname{Ker}(u-e) \cap \operatorname{Ker}(1-u) \leq_e M$. Thus $N = uN = eN \leq_e eM$. This implies that $eM \leq_e M$. So $M = eM \subseteq uM$ (as $e \in uSu$). Hence $uM = M$. But $\operatorname{Ker} u = 0$ by Lemma 3.1. Hence $u \in S$ is an automorphism.

(3)⇒(2). By Lemma 3.3, (c) means that S/Δ is regular, so it follows that $J(S) \subseteq \Delta$. Moreover, (b) implies that $\Delta \subseteq J(S)$. In fact, for $u \in S$ with $\operatorname{Ker}(1-u) \leq_e M$, we have $\operatorname{Ker} u = 0$ and $uM \leq_e M$, because $\operatorname{Ker} u \cap \operatorname{Ker}(1-u) = 0$ and $\operatorname{Ker}(1-u) \subseteq uM$. So u is an isomorphism by (b). Thus, by Lemma 3.4, $\Delta \subseteq J(S)$. Lastly, (a) implies Lemma 3.2(2). To see this, let $u^2 - u \in \Delta$. Then $\operatorname{Ker}(1-u)$ is a semicomplement of $\operatorname{Ker} u$ in M by Lemma 3.1. Moreover $u \operatorname{Ker}(1-u) = \operatorname{Ker}(1-u)$ is a kernel submodule of M and it lies under a direct summand of M by (a). Hence Lemma 3.2(2) holds.

(2)⇒(3). Suppose that S is semiregular and $J(S) = \Delta$. Then (c) holds by Theorem 3.5. To verify (a), let $u \in S$. Since S is semiregular, there exists $v \in S$ such that $v = vuv$ and $u - uvu \in J(S)$ by [9, Theorem 2.9]. So $M = vuM \oplus (1-vu)M$. It is clear that $\operatorname{Ker} u \subseteq (1-vu)M$. Since $\operatorname{Ker}(u-uvu) \leq_e M$, $\operatorname{Ker}(u-uvu) \cap (1-vu)M \leq_e (1-vu)M$. But

$$\begin{aligned}\operatorname{Ker}(u-uvu) \cap (1-vu)M &= [\operatorname{Ker} u \oplus \operatorname{Ker}(1-vu)] \cap (1-vu)M \\ &= \operatorname{Ker} u \oplus (\operatorname{Ker}(1-vu) \cap (1-vu)M) \\ &= \operatorname{Ker} u \oplus 0 = \operatorname{Ker} u,\end{aligned}$$

so $\operatorname{Ker} u \leq_e (1-vu)M$. This proves that M is kernel-extending. Assume further that $\operatorname{Ker} u = 0$ and $uM \leq_e M$ and let $N := \operatorname{Ker}(u-uvu)$. Since u is monic, $uN \leq_e uM$ and hence $uN \leq_e M$ because $uM \leq_e M$. Since $(1-uv)uN = 0$, $1 - uv \in \Delta = J(S)$ and this shows that uv is a unit of S. Hence $uM = M$ and (b) holds. ∎

A module M is called *extending* if every submodule of M lies under a direct summand. It is worth noting that there exists a module M such that S is semiregular and $J(S) = \Delta$, but M is not extending (see [11, Examples (4), p. 186]). A module M is called *direct-injective* if every submodule that is isomorphic to a direct summand of M is itself a direct summand (see [9]). Such modules are also called C_2-modules in [7].

COROLLARY 3.7 ([9]). *If a module M is direct-injective and kernel-extending, then S is semiregular and $J(S) = \Delta$.*

Proof. By Corollary 3.6, it suffices to show that (3)(c) of Corollary 3.6 holds. Let $u \in S$. Since M is kernel-extending, there exists a decomposition $M = X \oplus Y$ such that $\operatorname{Ker} u \leq_e X$. Thus Y is a semicomplement of $\operatorname{Ker} u$ in M. Since M is direct-injective, uY is a direct summand of M, and hence $(u|_Y)^{-1} : uY \to Y$ extends to an endomorphism of M. ∎

A module M is called *mono-injective* if, for any submodule N of M, every monomorphism $N \to M$ can be extended to M ([4]). Mono-injective modules are also called pseudo-injective by Jain and Singh [13].

COROLLARY 3.8. *If M is a mono-injective module, then $S/J(S)$ is regular and $J(S) = \Delta$.*

Proof. For any $u \in S$, there exists $N \leq M$ such that $N \oplus \operatorname{Ker} u \leq_e M$. Since M is mono-injective, $(u|_N)^{-1} : uN \to N$ extends to an endomorphism of M; so S/Δ is regular by Lemma 3.3. It follows that $J(S) \subseteq \Delta$. To finish the proof, it suffices to show that $\Delta \subseteq J(S)$. Let $u \in S$ with $\operatorname{Ker}(1-u) \leq_e M$. We only need to show that u is onto by Lemma 3.4. Since u is monic, $u^{-1} : uM \to M$ is a monomorphism, so there exists $v \in S$ such that $vx = u^{-1}(x)$ for all $x \in uM$. That is, $vu = 1_M$. Hence $M = \operatorname{Ker} v \oplus uM$. If $y \in \operatorname{Ker} v \cap \operatorname{Ker}(1-u)$, then $vy = 0$ and $y = uy$; so $0 = vy = vuy = y$. Hence $\operatorname{Ker} v \cap \operatorname{Ker}(1-u) = 0$. It follows that $\operatorname{Ker} v = 0$ since $\operatorname{Ker}(1-u) \leq_e M$. So v is a unit of S and hence $u = v^{-1}$ is certainly onto. ∎

By Yamagata [16], for a module M which is a direct sum of indecomposable injective modules, S/Δ is regular and idempotents lift modulo Δ, but idempotents do not lift strongly modulo Δ in general. We include two easy examples of the same kind. Recall that the *trivial extension* of a ring R by an R-bimodule M is the ring $R \propto M = \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. For a subset I of R and a subset X of M, we write $I \propto X = \{(a, x) : a \in I, x \in X\}$ for convenience. The right singular ideal of R is denoted by $Z_r(R)$.

EXAMPLE 3.9. *Let $R = \mathbb{Z} \propto \mathbb{Z}_{5^\infty}$, where $\mathbb{Z}_{5^\infty}$ is the Prüfer group. Then $R/Z_r(R)$ is regular and idempotents lift modulo $Z_r(R)$, but regular elements do not lift modulo $Z_r(R)$.*

Proof. It is easily seen that $J(R) = 0 \propto \mathbb{Z}_{5^\infty}$ and $Z_r(R) = 5\mathbb{Z} \propto \mathbb{Z}_{5^\infty}$; so $R/Z_r(R) \cong \mathbb{Z}_5$. Moreover, $R/Z_r(R)$ has only two trivial idempotents which are the images of the two trivial idempotents of R. Hence every idempotent of $R/Z_r(R)$ can be lifted to an idempotent of R. For $a = (3,0) \in R$ and $b = (2,0)$, $a - aba = (-15,0) \in Z_r(R)$. But, for any regular element $c = (n,m) \in R$, either $n = 0$ or $n = 1$ or $n = -1$, so $a - d \notin Z_r(R)$. ■

Example 3.10. *Let $R = \mathbb{Z} \propto \mathbb{Z}_{2^\infty}$, where $\mathbb{Z}_{2^\infty}$ is the Prüfer group. Then $R/Z_r(R)$ is regular and regular elements lift modulo $Z_r(R)$, but idempotents do not lift strongly modulo $Z_r(R)$.*

Proof. As above, $J(R) = 0 \propto \mathbb{Z}_{2^\infty}$, $Z_r(R) = 2\mathbb{Z} \propto \mathbb{Z}_{2^\infty}$, and $R/Z_r(R) \cong \mathbb{Z}_2$. Moreover, every element of $R/Z_r(R)$ can be lifted to an idempotent of R. For $a = (3,0) \in R$, we have $a - 1 \in Z_r(R)$. But, for any idempotent $e = (n,m) \in aR$, $n = 0$, so $a - e \notin Z_r(R)$. ■

4. The dual of Yamagata's theorem and consequences. As the dual of Yamagata's theorem, we characterize a module M for which S/∇ is regular and idempotents lift modulo ∇. We further characterize a module M for which S is semiregular and $J(S) = \nabla$.

Lemma 4.1. *Let $u, v \in S$. Then*

$$(u - uvu)M = uM \cap (1 - uv)M \quad \text{and} \quad M = uM + (1 - uv)M.$$

In particular,

$$(u - u^2)M = uM \cap (1 - u)M \quad \text{and} \quad M = uM + (1 - u)M.$$

Proof. It is clear that $(u - uvu)M \subseteq uM \cap (1 - uv)M$ and $uM + (1 - uv)M = M$. For $x \in uM \cap (1 - uv)M$, write $x = uy = (1 - uv)z$ with $y, z \in M$. Then $z = u(y + vz)$ and hence

$$x = (1 - uv)z = (u - uvu)(y + vz) \in (u - uvu)M. \ ■$$

Let X, Y be submodules of a module M. We call Y a *semisupplement* of X in M if $M = X + Y$ and $X \cap Y \ll M$. A submodule N of a module M is said to *lie over a direct summand* of M if there exists a decomposition $M = P \oplus Q$ such that $P \subseteq N$ and $N \cap Q \ll M$.

Lemma 4.2. *The following are equivalent for an idempotent $\bar{u} \in S/\nabla$:*

(1) *$\bar{u}$ lifts to an idempotent of S.*
(2) *There is a semisupplement N of uM in M such that $uN \ll M$ and N lies over a direct summand of M.*

Proof. We may assume $\bar{u} \neq 0$, because the conditions hold trivially for $\bar{u} = 0$.

(1)⇒(2). Let $e^2 = e \in S$ be such that $\bar{e} = \bar{u}$. Let

$$L_1 = (u^2 - u)M, \quad L_2 = (e - u)M, \quad X = (1 - u)M,$$
$$N = X + L_1 + L_2 = X + L_2.$$

We first show that N is a semisupplement of uM in M. Clearly, $N+uM = X+L_2+uM = M$. Let $m \in N \cap uM$ and write $m = uz = (1-u)x+(u-e)y$ with $x, y, z \in M$. Then $x = u(x+z) - (u-e)y$, and so

$$(1-u)x = (u-u^2)(x+z) - (1-u)(u-e)y.$$

Hence $N \cap uM \subseteq L_1 + (1-u)L_2 + L_2 \ll M$, because L_1 and L_2 are small in M. So N is a semisupplement of uM in M. Moreover, $uN \subseteq uL_2+(u^2-u)M \ll M$. Next we show that N lies over $(1-e)M$. For $x \in M$, $(1-e)x = (1-u)x-(e-u)x \in N$, so $(1-e)M \subseteq N$. For $m \in N \cap eM$, write $m = ez = (1-u)x+(e-u)y$ where $x, y, z \in M$. Then $x = ez+ux-(e-u)y$ and so

$$\begin{aligned}(1-u)x &= (1-u)ez + (1-u)ux - (1-u)(e-u)y\\ &= (e-u)ez - (u^2-u)x - (1-u)(e-u)y\\ &\in L_2 + L_1 + (1-u)L_2.\end{aligned}$$

Hence $m = (1-u)x + (e-u)y \in L_2 + L_1 + (1-u)L_2 \ll M$, which gives $N \cap eM \ll M$. So N lies over $(1-e)M$.

(2)⇒(1). By hypothesis, there exist $e^2 = e \in S$ and a submodule N of M such that

$$\begin{aligned}&N + uM = M, \quad N \cap uM \ll M,\\ &(1-e)M \subseteq N, \quad N \cap eM \ll M, \quad \text{and} \quad uN \ll M.\end{aligned}$$

Then $N = (1-e)M + (N \cap eM)$ and $M = uM + N = uM + (1-e)M + (N \cap eM)$. Since $N \cap eM \ll M$, we have

$$M = uM + (1-e)M.$$

Since $(u-ue)M = u(1-e)M \subseteq uN \ll M$, $\bar{u} = \overline{ue} = \bar{u}\bar{e}$. Since $\bar{u}^2 = \bar{u}$, we obtain $(\bar{e}\bar{u})^2 = \bar{e}\bar{u}\bar{e}\bar{u} = \bar{e}\bar{u}^2 = \bar{e}\bar{u}$. Let $f = e + (1-e)ue$. Then $f^2 = f \in S$, and

$$\begin{aligned}(ue-f)M &= (ue-e-(1-e)ue)(uM+(1-e)M)\\ &= (-e+eue)(uM)\\ &= (-eu+eueu)M \ll M \quad (\text{as } (\bar{e}\bar{u})^2 = \bar{e}\bar{u}).\end{aligned}$$

Hence $\bar{u} = \bar{u}\bar{e} = \bar{f}$. ∎

If N is a submodule of M, we write $\pi_N : M \to M/N$ for the natural epimorphism.

LEMMA 4.3. *The following are equivalent for $u \in S$:*

(1) *$\bar{u}$ is regular in S/∇.*
(2) *There exist $v \in S$ and a semisupplement N of uM in M such that*

the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{u} & M \\ {\scriptstyle v}\uparrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\pi_N} & M/N \end{array}$$

(3) *There exists a semisupplement* N *of* uM *in* M *such that* $(1-uv)M \subseteq N$ *for some* $v \in S$.

Proof. (1)⇒(2). Assume that $u - uvu \in \nabla$ where $v \in S$. Then $L := (u-uvu)M \ll M$. So, by Lemma 4.1, $N := (1-uv)M$ is a semisupplement of uM in M. Since $L \subseteq N$, $uvux + N = ux + N$ in M/N for all $x \in M$, i.e., $(\pi_N uv)(ux) = \pi_N(ux)$. Since $M = uM + N$, it follows that $\pi_N uv = \pi_N$.

(2)⇔(3). It is clear.

(3)⇒(1). By (3), there exists a semisupplement N of uM in M such that $(1-uv)M \subseteq N$ where $v \in S$. Then $uM \cap N \ll M$ and

$$(u-uvu)M = uM \cap (1-uv)M \subseteq uM \cap N \ll M,$$

so $\bar{u} = \bar{u}\bar{v}\bar{u}$. ∎

For $u \in S$, $(1-u)M \ll M$ implies that u is an epimorphism because $uM + (1-u)M = M$.

LEMMA 4.4 ([6]). *For a module* M, $\nabla \subseteq J(S)$ *iff every* $u \in S$ *with* $(1-u)M \ll M$ *is an isomorphism.*

The following theorem is the dual of Theorem 3.5.

THEOREM 4.5. *The following are equivalent for a module* M*:*

(1) S/∇ *is regular, and idempotents lift modulo* ∇.
(2) *For any* $u \in S$, *there exist semisupplements* N_1, N_2 *of* uM *in* M *such that*
   (a) $(1-uv)M \subseteq N_1$ *for some* $v \in S$,
   (b) $uN_2 \ll M$ *and* N_2 *lies over a direct summand of* M *if* $u^2 - u \in \nabla$.
(3) *For any* $u \in S$, *there exists a semisupplement* N *of* uM *in* M *such that*
   (a) $(1-uv)M \subseteq N$ *for some* $v \in S$,
   (b) $uN \ll M$ *and* N *lies over a direct summand of* M *if* $u^2 - u \in \nabla$.

Proof. The implications (1)⇒(2) and (3)⇒(1) follow from Lemmas 4.2 and 4.3.

(2)⇒(3). Let $u \in S$. If $u^2 - u \notin \nabla$, then we simply take $N = N_1$. So we can assume that $u^2 - u \in \nabla$, and let N_1, N_2 be given as in (2). It is enough to show that we can choose a common submodule N as N_1 and N_2. Let

$N = N_2 + uN_2 + (u - u^2)M$. To see that N satisfies (3)(a), let $v = 1_M$. Then

$$\begin{aligned}(1-uv)M &= (1-uv)(N_2+uM) = (1-u)(N_2+uM)\\ &= (1-u)N_2 + (u-u^2)M \le N.\end{aligned}$$

Next we show that N satisfies (3)(b). One sees that $N \cap uM = N_2 \cap uM + [uN_2 + (u-u^2)M]$ is small in M, because $N_2 \cap uM, uN_2, (u-u^2)M$ are all small in M. Since $N + uM = M$, N is a semisupplement of uM in M. Moreover, $uN = uN_2 + u[uN_2 + (u-u^2)M] \ll M$. By our assumption on N_2, there exists $e^2 = e \in S$ such that $(1-e)M \subseteq N_2$ and $N_2 \cap eM \ll M$. Then $(1-e)M \subseteq N$, and

$$\begin{aligned}eM \cap N &= eM \cap [N_2 + uN_2 + (u-u^2)M]\\ &= eM \cap [N_2 \cap eM + (1-e)M + uN_2 + (u-u^2)M]\\ &= N_2 \cap eM + eM \cap [(1-e)M + uN_2 + (u-u^2)M]\\ &\le N_2 \cap eM + e(uN_2 + (u-u^2)M) \ll M.\end{aligned}$$

So N lies over $(1-e)M$ and hence N satisfies (3)(b). The proof is complete. ∎

Next we show that S/∇ is regular and idempotents lift strongly modulo ∇ if and only if S is semiregular with $J(S) = \nabla$, and characterize modules M with the latter condition. We refer to [7], [8] and [9] for some sufficient conditions on a module M for which S is semiregular and $J(S) = \nabla$. A submodule X of M is called an *image submodule* if $X = uM$ for some $u \in S$. The module M is called *image-lifting* if every image submodule of M lies over a direct summand.

Corollary 4.6. *Let M be a module. The following are equivalent:*

(1) *S/∇ is regular, and idempotents lift strongly modulo ∇.*
(2) *S is semiregular and $J(S) = \nabla$.*
(3) *The following hold:*
    (a) *M is image-lifting.*
    (b) *Every epimorphism in S with small kernel is one-to-one.*
    (c) *For any $u \in S$, there exists a semisupplement N of uM in M such that $(1-uv)M \subseteq N$ for some $v \in S$.*

Proof. (2)⇒(1). Apply [12, Lemma 5].

(1)⇒(2). Since S/∇ is regular, $J(S) \subseteq \nabla$. So to show (2), it suffices to show that $J(S) \supseteq \nabla$. Assume that $u \in S$ with $(1-u)M \ll M$. We only need to show that u is an isomorphism by Lemma 4.4. Since $u - 1 \in \nabla$, by (1) there exists $e^2 = e \in uSu$ such that $u - e \in \nabla$. So $N := (u-e)M + (1-u)M \ll M$. For $x \in M$, $(1-e)x = (u-e)x + (1-u)x$, so $(1-e)M \subseteq (u-e)M + (1-u)M$. Thus $(1-e)M \ll M$. It follows that $eM = M$. So $1 = e \in uSu$. This shows that $u \in S$ is an automorphism.

(3)⇒(2). By Lemma 4.3, (c) means that S/∇ is regular, so it follows that $J(S) \subseteq \nabla$. Moreover, (b) implies that $\nabla \subseteq J(S)$. In fact, for $u \in S$ with $(1-u)M \ll M$, we have $uM = M$ and $\operatorname{Ker} u \ll M$, because $M = uM + (1-u)M$ and $\operatorname{Ker} u \subseteq (1-u)M$. So u is an isomorphism by (b). Thus, by Lemma 4.4, $\nabla \subseteq J(S)$. Lastly, (a) implies Lemma 4.2(2). To see this, let $u^2 - u \in \nabla$. Then $(1-u)M$ is a semisupplement of uM in M by Lemma 4.1, $u(1-u)M \ll M$, and $(1-u)M$ lies over a direct summand of M by (a). Hence Lemma 4.2(2) holds.

(2)⇒(3). Suppose that S is semiregular and $J(S) = \nabla$. Then (c) holds by Theorem 4.5. To verify (a), let $u \in S$. Since S is semiregular, there exists $v \in S$ such that $v = vuv$ and $u - uvu \in J(S)$ by [9, Theorem 2.9]. So $M = uvM \oplus (1-uv)M$. Since $uvM \subseteq uM$ and $uM \cap (1-uv)M = (u-uvu)M \ll M$, uM lies over uvM. This proves that M is image-lifting. To verify (b), we assume further that $uM = M$ and $\operatorname{Ker} u \ll M$, and prove $\operatorname{Ker} u = 0$. Since $J(S) = \nabla$, it suffices to show $(1-vu)M \ll M$. Let $M = (1-vu)M + N$ for some submodule N. Then $M = uM = u(1-vu)M + uN$, and this implies that $uM = uN$ since $u(1-vu)M \ll M$. Hence $M = N + \operatorname{Ker} u$, and this shows that $M = N$ since $\operatorname{Ker} u \ll M$. So $(1-vu)M \ll M$. ■

A module M is called *lifting* if every submodule of M lies over a direct summand. There exists a module M such that S is semiregular with $J(S) = \nabla$, but M is not lifting. Indeed, if R is a semiregular ring that is not semiperfect, then $M := R_R$ is such a module by [7, 4.38, p. 69; 4.42, p. 71]. A module M is called *direct-projective* if, whenever a factor module M/K is isomorphic to a direct summand of M, K is a direct summand of M ([9]). These modules are also called D_2-modules in [7].

Corollary 4.7 ([9]). *If a module M is direct-projective and image-lifting, then S is semiregular and $J(S) = \nabla$.*

Proof. By Corollary 4.6, it suffices to show that (3)(c) of Corollary 4.6 holds. Let $u \in S$. Since M is image-lifting, there exists $e^2 = e \in S$ such that $eM \subseteq uM$ and $uM \cap (1-e)M \ll M$. Thus $(1-e)M$ is a semisupplement of uM in M. Since $eu : M \to eM$ is onto and since M is direct-projective, $\operatorname{Ker}(eu)$ is a direct summand of M. Write $M = \operatorname{Ker}(eu) \oplus Z$. Then $eu|_Z : Z \to eM$ is an isomorphism. Define $v \in S$ by $v(x+y) = (eu|_Z)^{-1}(x)$ for $x \in eM, y \in (1-e)M$. Then, for $x \in eM$, $euv(x) = x$ so $(1-uv)x = euvx - uvx = -(1-e)uvx \in (1-e)M$. Moreover, $(1-uv)y = y$ for all $y \in (1-e)M$. So $(1-uv)M \subseteq (1-e)M$. This shows (3)(c) of Corollary 4.6. Hence S is semiregular and $J(S) = \nabla$. ■

A module M is said to be *epi-projective* if, for any submodule N of M, every epimorphism $f : M \to M/N$ can be lifted to M, that is, there exists $g \in S$ such that $\pi_N = fg$ (see [4]).

COROLLARY 4.8. *Suppose that every submodule of M has a semisupplement in M. If M is an epi-projective module, then $S/J(S)$ is regular and $J(S) = \nabla$.*

Proof. For any $u \in S$, there exists $N \leq M$ such that $M = N + uM$ and $N \cap uM \ll M$. Thus $\pi_N u : M \to M/N$ is an epimorphism. Since M is epi-projective, $\pi_N = \pi_N uv$ for some $v \in S$; so S/∇ is regular by Lemma 4.3. It follows that $J(S) \subseteq \nabla$. To finish the proof, it suffices to show that $\nabla \subseteq J(S)$. Let $u \in S$ with $(1-u)M \ll M$. We only need to show that u is one-to-one by Lemma 4.4. Since u is onto and since M is epi-projective, there exists $v \in S$ such that $uv = 1_M$. Hence $M = \operatorname{Ker} u \oplus vM$. But $\operatorname{Ker} u \subseteq (1-u)M$, so $\operatorname{Ker} u \ll M$. It follows that $M = vM$, and hence $\operatorname{Ker} u = 0$. ∎

In contrast to Corollary 3.8, the assumption in Corollary 4.8 that every submodule of M has a semisupplement in M is not superfluous. In fact, it is easy to check that the module $\mathbb{Z}_{\mathbb{Z}}$ is epi-projective, $\mathrm{End}(\mathbb{Z}_{\mathbb{Z}}) \cong \mathbb{Z}$ is semiprimitive, but $\mathbb{Z}$ is not regular. As seen in Section 3, there exist modules M for which S/Δ is regular, idempotents lift modulo Δ, but $\Delta \neq J(S)$. We do not know an example of a module M such that S/∇ is regular, idempotents lift modulo ∇, but $\nabla \neq J(S)$.

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