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ON THE NEUMANN PROBLEM FOR SYSTEMS OF ELLIPTIC EQUATIONS INVOLVING HOMOGENEOUS NONLINEARITIES OF A CRITICAL DEGREE

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Abstract. We establish the existence of solutions for the Neumann problem for a system of two equations involving a homogeneous nonlinearity of a critical degree. The existence of a solution is obtained by a constrained minimization with the aid of P.-L. Lions' concentration-compactness principle.

1. Introduction. In this paper we investigate the nonlinear Neumann problem

$$
(1.1)\qquad \begin{cases} -\Delta u + \lambda u = Q(x)H_u(u,v) + P_u(u,v) & \text{in } \Omega,\\ -\Delta v + \mu v = Q(x)H_v(u,v) + P_v(u,v) & \text{in } \Omega,\\ \partial u/\partial \nu = \partial v/\partial \nu = 0 \quad \text{on } \partial\Omega,\ u \geq 0,\ u \not\equiv 0,\ v \geq 0,\ v \not\equiv 0 \text{ on } \Omega, \end{cases}
$$

where λ and μ are positive parameters, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and ν is the unit outward normal to $\partial\Omega$. The coefficient $Q(x)$ is continuous and positive on $\bar{\Omega}$. The nonlinearities H and P are of class $C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, where $\mathbb{R}_+ = [0,\infty)$, and are homogeneous of degree 2^* and 2, respectively. This means that $H(\lambda u, \lambda v) = \lambda^{2^*} H(u,v)$ and $P(\lambda u, \lambda v) = \lambda^2 H(u,v)$ for every $\lambda > 0$ and $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$. Here 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$, $N \geq 3$. Further assumptions on H and P will be formulated later. A special case of problem (1.1) has been considered in [8], namely, $H(u,v) = u^\alpha v^\beta$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$, with $\alpha + \beta = 2^*$, $\alpha, \beta > 1$, and $P(u,v) \equiv 0$. The corresponding problem with the Dirichlet boundary conditions has been considered in [9], [5]. In this paper we use some ideas from paper [9]. The nonlinear Neumann problem involving the critical Sobolev exponent has an extensive literature. We refer to [1]–[7] where further bibliographical references can be found.

Solutions of problem (1.1) are sought in the Sobolev space $W := H^1(\Omega) \times H^1(\Omega)$. We recall that $H^1(\Omega)$ is the usual Sobolev space equipped with the

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norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

The norm in W is given by

$$\|(u,v)\|_W^2 = \int_\Omega (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2)\,dx.$$

In a given Banach space we denote by "$\to$" strong convergence and by "$\rightharpoonup$" weak convergence. The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, are denoted by $\|\cdot\|_p$.

The paper is organized as follows. In Section 2 we state our assumptions and recall some properties of homogeneous functions of two variables that will be used in our approach to problem (1.1). In particular, we recall an extension of the Sobolev inequality from [9] involving homogeneous functions of two variables of a critical degree. The existence of solutions to problem (1.1), through a constrained minimization (3.1), is presented in Section 3 (see Theorem 3.2). The existence of minimizers of problem (3.1) depends on the shape of the graph of the coefficient Q (see condition (3.6)). Section 4 is devoted to the verification of this condition. In the final Section 5 we describe some properties of solutions of problem (1.1).

2. Preliminaries. The nonlinearities H and P satisfy the following assumptions:

- (H_1) $H \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, $H(u,v) \ge 0$, $\not\equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}_+$ and H is homogeneous of degree 2^*, that is, $H(\lambda u, \lambda v) = \lambda^{2^*} H(u,v)$ for every $\lambda > 0$ and all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$,
- (H_2) $G(s^{2^*}, t^{2^*}) = H(s,t)$ is a concave function for $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$.

We extend H to $\mathbb{R}^2$ by setting $H(s,t) = H(s^+, t^+)$, where $s^+ = \max(0,s)$. This extension is of class C^1 provided $H_u(0,1) = H_v(1,0) = 0$. This assumption is needed in the proof of Theorem 4.1 (see (3.4)).

It is assumed that the nonlinearity P satisfies the following condition:

- (P_1) $P \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, and P is homogeneous of degree 2, that is, $P(\lambda u, \lambda v) = \lambda^2 P(u,v)$ for every $\lambda > 0$ and all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$.

In Theorem 4.1 we assume that $P_u(0,1) > 0$ and $P_v(1,0) > 0$. So we extend P to $\mathbb{R}^2$ in the following way:

$$P(s,t) = \begin{cases} P(s,t) & \text{for } s,t \ge 0, \\ P(0,t) + P_s(0,t)s & \text{for } s \le 0 \le t, \\ P(s,0) + P_t(s,0)t & \text{for } t \le 0 \le s, \\ 0 & \text{for } s,t \le 0. \end{cases}$$

This extension is of class C^1. From now on we mean by H and P the extended functions.

We now give examples of homogeneous functions satisfying the above conditions:

(1) $H(s,t) = s^{2^*} + t^{2^*} + \sum_{j=1}^{k} a_j s^{\alpha_j} t^{\beta_j}$ for $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, where $\alpha_j, \beta_j > 1$, $a_j > 0$ and $\alpha_j + \beta_j = 2^*$.
(2) Let $H(s,t)$ be as in (1) and set $\tilde{H}(s,t) = H(s,t)^q/(s^{2^*} + t^{2^*})^{q-1}$ for $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, where $q > 1$.

Both functions H and $\tilde{H}$ satisfy (H_1). As noted in [9], H is so far the only example of a homogeneous function of degree 2^* satisfying (H_2). To obtain the existence of a solution (u,v), with both components nonzero, of problem (1.1) we need to assume that (see Theorem 3.2 below) $P_s(0,1) > 0$ and $P_t(1,0) > 0$. As examples of homogeneous functions of degree 2 satisfying this requirement we can give $P(u,v) = uv$ and $P_1(u,v) = (u+v)\sqrt{u^2+v^2}$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$.

We associate with a homogeneous function H satisfying (H_1) and (H_2) the best Sobolev constant. First, we recall that the usual best Sobolev constant is defined by

$$S = \inf_{u \in H_0^1(\Omega) - \{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{(\int_\Omega |u|^{2^*}\, dx)^{2/2^*}},$$

where $H_0^1(\Omega)$ is the subspace of functions of $H^1(\Omega)$ having zero trace on the boundary $\partial\Omega$. It is known that S is independent of Ω and it is only attained when $\Omega = \mathbb{R}^N$ (see [11, Chapter 3, Section 1] or [12]). In this case, as the corresponding Sobolev space we can take $D^{1,2}(\mathbb{R}^N)$, the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

It is well-known that the best Sobolev constant is attained on $\mathbb{R}^N$ by a family of functions

$$U_{\epsilon,y}(x) = \epsilon^{-(N-2)/2} U\left(\frac{x-y}{\epsilon}\right), \qquad y \in \mathbb{R}^N,\ \epsilon > 0, \tag{2.1}$$

where

$$U(x) = \left(\frac{N(N-2)}{N(N-2) + |x|^2}\right)^{(N-2)/2}.$$

The function U, called an *instanton*, satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla U_{\epsilon,y}|^2\,dx = \int_{\mathbb{R}^N} U_{\epsilon,y}^{2^*}\,dx = S^{N/2}.$$

If $y = 0$, we write $u_\epsilon = U_{\epsilon,0}$.

Let H be a homogeneous function satisfying (H_1) and (H_2). We define

$$S_H = \inf\Big\{\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\,dx : (u,v) \in W,\ \int_\Omega H(u^+, v^+)\,dx = 1\Big\}.$$

Due to the 2^*-homogeneity of H, there is a relation between S and S_H. Namely, let $F(s,t) = H(s,t)^{2/2^*}$ and set $M_F = \max\{F(s,t) : (s,t) \in \mathbb{R}\times\mathbb{R},\ s^2 + t^2 = 1\}$. Letting $m = M_F^{-1}$ we have $mH(s,t)^{2/2^*} \le s^2 + t^2$ for every $(s,t) \in \mathbb{R}\times\mathbb{R}$ and there exists a point (s_0,t_0) such that

$$mH(s_0,t_0)^{2/2^*} = s_0^2 + t_0^2.$$

It follows from Lemma 3 in [9] that

$$S_H = mS. \tag{2.2}$$

We point out here that condition (H_2) implies the following form of the Hölder inequality:

$$\int_\Omega H(u,v)\,dx \le H(\|u\|_{2^*}, \|v\|_{2^*})$$

for all $(u,v) \in W$. This inequality is needed to establish (2.2) (for further details we refer to [9]).

3. Constrained minimization. A solution to problem (1.1) will be found as a minimizer of the constrained minimization

(3.1)
$$S_{\lambda\mu} = \inf\Big\{\int_\Omega [I(u,v) - P(u,v)]\,dx : (u,v) \in W,\ \int_\Omega Q(x)H(u,v)\,dx = 1\Big\},$$

where

$$I(u,v) = \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2).$$

To find a minimizer for $S_{\lambda\mu}$, we use the following version of P.-L. Lions' concentration-compactness principle [10].

PROPOSITION 3.1. *Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\Omega)$. Suppose that*

$$|\nabla u_n|^2 \rightharpoonup \mu, \qquad |\nabla v|^2 \rightharpoonup \sigma$$

and $H(u_n, v_n) \rightharpoonup \nu$ in the sense of measures. Then there exist an at most countable index set J and sequences $\{x_j\} \subset \bar{\Omega}$, $\{\mu_j\}, \{\sigma_j\}, \{\nu_j\} \subset (0,\infty)$, $j \in J$, such that

(i) $\nu = H(u,v) + \sum_{j\in J} \nu_j \delta_{x_j}$,
(ii) $\mu \ge |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$ *and* $\sigma \ge |\nabla v|^2 + \sum_{j\in J} \sigma_j \delta_{x_j}$,

(iii) $\mu_j + \sigma_j \geq S_H(\nu_j)^{2/2^*}$ *if* $x_j \in \Omega$,

(iv) $\mu_j + \sigma_j \geq 2^{-2/N} S_H(\nu_j)^{2/2^*}$ *if* $x_j \in \partial\Omega$,

where δ_{x_j} *denotes the Dirac measure assigned to* x_j.

In the case of the space $H_0^1(\Omega)$ this modification of P.-L. Lions' concentration-compactness principle can be found in [9]. The proof in our situation is the same as in [9]. We only need to add the proof of inequality (iv). This follows from the following modification of the result due to X. J. Wang [13].

Let $\tilde{B} = B(0,1) \cap \{x_N > h(x')\}$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$, $h(x')$ is a C^1 function defined on $\{x' \in \mathbb{R}^{N-1} : |x'| < 1\}$ with h and ∇h vanishing at 0. Then for every $(u,v) \in H^1(B(0,1))$ with $\operatorname{supp} u, \operatorname{supp} v \subset B(0,1)$ we have

(a) if $h \equiv 0$, then

$$\int_{\tilde{B}} (|\nabla u|^2 + |\nabla v|^2)\,dx \geq 2^{-2/N} S_H \Big(\int_{\tilde{B}} H(u,v)\,dx\Big)^{2/2^*},$$

(b) for every $\epsilon > 0$ there exists a $\delta > 0$ depending only on ϵ such that if $|\nabla h| < \delta$, then

$$\int_{\tilde{B}} (|\nabla u|^2 + |\nabla v|^2)\,dx \geq \Big(\frac{S_H}{2^{2/N}} - \epsilon\Big)\Big(\int_{\tilde{B}} H(u,v)\,dx\Big)^{2/2^*}.$$

Using this result we deduce (iv) (for details see [9]).

To formulate the first existence result for problem (1.1) we introduce the following assumption:

(P_2) $\max_{s^2+t^2=1} P(s,t) =: b > 0$.

This yields $P(s,t) \leq b(s^2+t^2)$ for $(s,t) \in \mathbb{R}^2$. Hence for all $(u,v) \in W$ and $\lambda > 2b$, $\mu > 2b$ we have

$$\begin{aligned}
(3.2)\quad J(u,v) &= \frac{1}{2}\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2)\,dx - \int_{\Omega} P(u,v)\,dx \\
&\geq \int_{\Omega} \Big[\frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) + \Big(\frac{\lambda}{2} - b\Big)u^2 + \Big(\frac{\mu}{2} - b\Big)v^2\Big]\,dx \\
&\geq aS_1\Big[\Big(\int_{\Omega} |u|^{2^*}\,dx\Big)^{2/2^*} + \Big(\int_{\Omega} |u|^{2^*}\,dx\Big)^{2/2^*}\Big].
\end{aligned}$$

where $a = \min(1/2, \lambda/2 - b, \mu/2 - b)$ and S_1 is the best Sobolev constant for the embedding of $H^1(\Omega)$ into $L^{2^*}(\Omega)$. We now observe that the inequality $mH(s,t)^{2/2^*} \leq s^2 + t^2$ for all $(s,t) \in \mathbb{R}^2$ yields $H(s,t) \leq A(|s|^{2^*} + |t|^{2^*})$ for all $(s,t) \in \mathbb{R}^2$, where $A > 0$ is a constant independent of s and t. Applying this to inequality (3.2) we derive

$$
\begin{aligned}
(3.3)\qquad J(u,v) &\geq aS_1\Big[\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*} + \Big(\int_\Omega |v|^{2^*}\,dx\Big)^{2/2^*}\Big] \\
&\geq B\Big(\int_\Omega (|u|^{2^*} + |v|^{2^*})\,dx\Big)^{2/2^*} \\
&\geq \frac{B}{A^{2/2^*}Q_{\mathrm{M}}^{2/2^*}}\Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*},
\end{aligned}
$$

where $B > 0$ is a constant independent of u and v and $Q_{\mathrm{M}} = \max_{x\in\bar\Omega} Q(x)$. Inequality (3.3) implies that $S_{\lambda\mu} > 0$ provided $\lambda > 2b$ and $\mu > 2b$.

The quantities Q_{M} and $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$ play an important role in establishing the existence of a solution for problem (1.1).

THEOREM 3.2. *Suppose that H and P satisfy (H_1), (H_2), (P_1) and (P_2). Further, assume that*

$$(3.4)\qquad H_u(0,1) = 0, \qquad H_v(1,0) = 0$$

and

$$(3.5)\qquad P_u(0,1) > 0, \qquad P_v(1,0) > 0.$$

If for $\lambda > 2b$ and $\mu > 2b$,

$$(3.6)\qquad S_{\lambda\mu} < S_\infty := \min\Big(\frac{S_H}{Q_{\mathrm{M}}^{(N-2)/N}}, \frac{S_H}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}}\Big),$$

then there exists a minimizer $(u,v) \in W$ for $S_{\lambda\mu}$, which, up to a multiplicative constant, is a solution of problem (1.1).

Proof. Let $\{(u_n, v_n)\} \subset W$ be a minimizing sequence for $S_{\lambda\mu}$. Since $\{u_n\}$ and $\{v_n\}$ are bounded in $H^1(\Omega)$ we may assume that $\{u_n\}$ and $\{v_n\}$ satisfy conditions (i)–(iv) of Proposition 3.1. By the Sobolev compact embeddings we may also assume that $u_n \to u$ and $v_n \to v$ in $L^2(\Omega)$ and so $\int_\Omega P(u_n, v_n)\,dx \to \int_\Omega P(u,v)\,dx$. It follows from Proposition 3.1 that

$$(3.7)\qquad 1 = \int_\Omega Q(x)H(u,v)\,dx + \sum_{j\in J}\nu_j Q(x_j).$$

Thus

$$
\begin{aligned}
S_{\lambda\mu} &\geq J(u,v) + \sum_{j\in J}(\mu_j + \sigma_j) \\
&\geq S_{\lambda\mu}\Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*} + \sum_{x_j\in\Omega} S_H\nu_j^{2/2^*} + \sum_{x_j\in\partial\Omega}\frac{1}{2^{2/N}}\nu_j^{2/2^*}S_H
\end{aligned}
$$

$$
\begin{aligned}
&= S_{\lambda\mu}\Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*} \\
&\quad + \sum_{x_j\in\Omega} S_H \frac{(Q(x_j)\nu_j)^{2/2^*}}{Q(x_j)^{(N-2)/N}} + \sum_{x_j\in\partial\Omega} \frac{S_H}{2^{2/N}} \frac{(Q(x_j)\nu_j)^{2/2^*}}{Q(x_j)^{(N-2)/N}} \\
&\ge S_{\lambda\mu}\Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*} \\
&\quad + \sum_{x_j\in\Omega} \frac{S_H (Q(x_j)\nu_j)^{2/2^*}}{Q_{\mathrm{M}}^{(N-2)/N}} + \sum_{x_j\in\partial\Omega} \frac{S_H}{2^{2/N}} \frac{(Q(x_j)\nu_j)^{2/2^*}}{Q_{\mathrm{m}}^{(N-2)/N}}.
\end{aligned}
$$

Since (3.6) holds we derive from this inequality that

$$S_{\lambda\mu} \ge S_{\lambda\mu}\Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*} + S_\infty \sum_{j\in J}(Q(x_j)\nu_j)^{2/2^*}.$$

If at least one of the constants $\nu_j \neq 0$, then, since $S_\infty > S_{\lambda\mu}$, we get

$$1 > \Big(\int_\Omega Q(x)H(u,v)\,dx\Big)^{2/2^*} + \sum_{j\in J}(Q(x_j)\nu_j)^{2/2^*}.$$

This obviously contradicts (3.7). Hence $\nu_j = 0$ for all $j \in J$. This shows that the pair $(u,v) \in W$ is a minimizer for $S_{\lambda\mu}$. Since $(|u|,|v|)$ is also a minimizer we may assume that $u \ge 0$ and $v \ge 0$ on Ω. Assumptions (3.4) and (3.5) imply that both functions are nonzero. Using the Lagrange multiplier technique we obtain

$$
\begin{aligned}
\text{(3.8)}\qquad &\int_\Omega (\nabla u\nabla\phi + \nabla v\nabla\zeta + \lambda u\phi + \mu v\zeta - P_u(u,v)\phi - P_v(u,v)\zeta)\,dx \\
&\qquad = \kappa\int_\Omega Q(x)(H_u(u,v)\phi + H_v(u,v)\zeta)\,dx
\end{aligned}
$$

for some $\kappa \in \mathbb{R}$ and all $(\phi,\zeta) \in W$. Since

$$uH_u(u,v) + vH_v(u,v) = 2^*H(u,v) \quad\text{and}\quad uP_u(u,v) + vP_v(u,v) = 2P(u,v),$$

we derive from (3.8) that $\kappa = 2S_{\lambda\mu}/2^*$. It is easy to check that the pair

$$\bigg(\Big(\frac{1}{2^*}S_{\lambda\mu}\Big)^{\frac{1}{2^*-1}}u, \Big(\frac{1}{2^*}S_{\lambda\mu}\Big)^{\frac{1}{2^*-1}}v\bigg)$$

is a solution of problem (1.1). ■

4. Validity of condition (3.6). We now formulate conditions guaranteeing that (3.6) holds for $\lambda > 2b$ and $\mu > 2b$ (or at least for some $\lambda > 2b$ and $\mu > 2b$). It is easy to establish this in the case

$$Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}. \tag{4.1}$$

Then we require that

$$S_{\lambda\mu} < \frac{S_H}{2^{2/N} Q_{\mathrm{m}}^{(N-2)/N}}.$$

Indeed, let $M(y)$ denote the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$. It is well-known that (see [1], [2], [13])

$$\frac{\int_\Omega |\nabla U_{\epsilon,y}|^2\,dx}{(\int_\Omega U_{\epsilon,y}^{2^*}\,dx)^{2/2^*}} = \begin{cases} 2^{-2/N}S - A_N M(y)\epsilon\log(1/\epsilon) + O(\epsilon), & N = 3,\\ 2^{-2/N}S - A_N M(y)\epsilon + O(\epsilon^2)\log(1/\epsilon), & N = 4,\\ 2^{-2/N}S - A_N M(y)\epsilon + O(\epsilon^2), & N \geq 5, \end{cases} \tag{4.2}$$

where A_N is a positive constant depending only on N.

THEOREM 4.1. *Let $Q_{\mathrm{M}} \leq 2^{2/(N-2)}Q_{\mathrm{m}}$. Suppose that P and Q satisfy (H_1), (H_2), (P_1), (P_2), (3.4) and (3.5). Moreover, assume that $Q(x)$ satisfies the following condition: there exists $y \in \partial\Omega$ such that $Q(y) = Q_{\mathrm{m}}$, $M(y) > 0$ and*

$$|Q(x) - Q(y)| = o(|x-y|) \quad \text{for } x \text{ near } y. \tag{4.3}$$

Then for every $\lambda > 2b$ and $\mu > 2b$ problem (1.1) *admits a solution.*

Proof. First we observe that condition (4.3) yields the expansion

$$\int_\Omega Q(x)U_{\epsilon,y}(x)^{2^*}\,dx = Q_{\mathrm{m}}\int_\Omega U_{\epsilon,y}(x)^{2^*}\,dx + o(\epsilon).$$

We now test $S_{\lambda\mu}$ with $(s_0U_{\epsilon,y}, t_0U_{\epsilon,y})$ to obtain with the aid of asymptotic estimates (4.2) the following estimate for $S_{\lambda\mu}$:

$$S_{\lambda\mu} \leq \frac{(s_0^2+t_0^2)\int_\Omega |\nabla U_{\epsilon,y}|^2\,dx + O(\epsilon^2)}{H(s_0,t_0)^{2/2^*}(Q_{\mathrm{m}}\int_\Omega U_{\epsilon,y}^{2^*}\,dx + o(\epsilon))^{2/2^*}} < \frac{S_H}{2^{2/N}Q_{\mathrm{m}}^{2/2^*}}$$

for $\epsilon > 0$ sufficiently small. Here we have used the fact that

$$\int_\Omega U_{\epsilon,y}(x)^2\,dx = O(\epsilon^2) \quad\text{and}\quad \int_\Omega P(s_0U_{\epsilon,y}, t_0U_{\epsilon,y})\,dx = O(\epsilon^2). \ \blacksquare$$

In the case $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$, we first observe that the constant functions s,t given by $s = t = \frac{1}{(H(1,1)\int_\Omega Q(x)\,dx)^{1/2^*}}$ satisfy $\int_\Omega Q(x)H(s,s)\,dx = 1$. So for $\lambda > 2b$ and $\mu > 2b$ we have

$$\begin{aligned} J(\lambda,\mu) &= |\Omega|\left[\frac{\lambda+\mu}{2(H(1,1)\int_\Omega Q(x)\,dx)^{2/2^*}} - \frac{P(1,1)}{(H(1,1)\int_\Omega Q(x)\,dx)^{2/2^*}}\right]\\ &= \frac{|\Omega|}{(H(1,1)\int_\Omega Q(x)\,dx)^{2/2^*}}\left(\frac{\lambda+\mu}{2} - P(1,1)\right). \end{aligned}$$

So $S_{\lambda\mu} < S_H/Q_{\mathrm{M}}^{(N-2)/N}$ provided

$$\frac{|\Omega|}{(H(1,1)\int_\Omega Q(x)\,dx)^{2/2^*}}\left(\frac{\lambda+\mu}{2} - P(1,1)\right) < \frac{S_H}{Q_{\mathrm{M}}^{(N-2)/N}}. \tag{4.4}$$

Let $Q_* = \min_{x\in\bar{\Omega}} Q(x)$. If

$$\frac{|\Omega|^{1-2/2^*}}{H(1,1)^{2/2^*}}\left(\frac{\lambda+\mu}{2} - P(1,1)\right) < S_H\left(\frac{Q_*}{Q_{\mathrm{M}}}\right)^{(N-2)/N} \tag{4.5}$$

then (4.4) holds. For a given $\lambda > 2b$ and $\mu > 2b$, condition (4.5) is satisfied if $|\Omega|$ is small.

Proposition 4.2. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. Suppose that* (H_1), (H_2), (P_1), (P_2), (3.4) *and* (3.5) *hold. If for $\lambda > 2b$ and $\mu > 2b$, inequality* (4.5) *is satisfied, then problem* (1.1) *has a solution.*

Remark 4.3. Condition (4.5) is satisfied in the following two cases:

(i) Let $P(u,v) = uv$. Then $b = 1/2$ and $P(1,1) = 1$ and there exists $\delta_0 > 0$ such that for $\lambda,\mu \in (1, 1+\delta_0)$ condition (4.5) holds.
(ii) Let $P_1(u,v) = (u+v)\sqrt{u^2+v^2}$. Then $b = \sqrt{2}$, $P_1(1,1) = 2\sqrt{2}$ and there exists $\delta_1 > 0$ such that for $\lambda,\mu \in (2\sqrt{2}, 2\sqrt{2}+\delta_1)$ condition (4.5) holds.

We now present a more general result under an additional restriction on P. First, we recall the following result from [7]. Let us consider the following constrained minimization problem:

$$S_\lambda = \inf\left\{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx : u \in H^1(\Omega),\ \int_\Omega Q(x)|u|^{2^*}\,dx = 1\right\},$$

where $\lambda > 0$ is a parameter. Obviously, a minimizer for S_λ is, up to a multiplicative constant, a solution of the Neumann problem

$$\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega, & u > 0 \quad \text{on } \Omega. \end{cases}$$

Proposition 4.4. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. Then there exists a constant $\Lambda > 0$ such that for every $0 < \lambda \le \Lambda$ there exists a minimizer $u_\lambda > 0$ for S_λ and there are no minimizers for $\lambda > \Lambda$. Moreover, $S_\lambda < S/Q_{\mathrm{M}}^{(N-2)/N}$ for $0 < \lambda < \Lambda$ and $S_\lambda = S/Q_{\mathrm{M}}^{(N-2)/N}$ for $\lambda \ge \Lambda$.*

Proposition 4.5. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. Suppose that P and Q satisfy* (Q_1), (Q_2), (P_1), (P_2), (3.4), (3.5).

(i) *If $P(s_0,t_0) = 0$, then there exists a minimizer for $S_{\lambda\mu}$ for $2b < \lambda < \Lambda$ and $2b < \mu < \Lambda$.*
(ii) *If $P(s_0,t_0) > 0$, then there exists a minimizer for $S_{\lambda\mu}$ for $2b < \lambda$ and $2b < \mu$.*

Proof. (i) We may assume that $\mu \le \lambda$. Let u_λ be a minimizer for S_λ with $0 < \lambda < \Lambda$. Testing $S_{\lambda\mu}$ with $(s_0u_\lambda, t_0u_\lambda)$ we obtain

$$S_{\lambda\mu} \le \frac{(s_0^2+t_0^2)\int_\Omega |\nabla u_\lambda|^2\,dx + \lambda s_0^2\int_\Omega u_\lambda^2\,dx + \mu t_0^2\int_\Omega u_\lambda^2\,dx}{H(s_0,t_0)(\int_\Omega Q(x)|u_\lambda|^{2^*}\,dx)^{2/2^*}}$$
$$\le \frac{(s_0^2+t_0^2)\int_\Omega(|\nabla u_\lambda|^2\,dx + \lambda u_\lambda^2)\,dx}{H(s_0,t_0)(\int_\Omega Q(x)|u_\lambda|^{2^*}\,dx)^{2/2^*}} = mS_\lambda < \frac{S_H}{Q_{\mathrm{M}}^{(N-2)/N}}$$

and the result follows from Theorem 3.2. Part (ii) is now obvious. ∎

Proposition 4.5 continues to hold if $2b < \mu < \lambda < \Lambda$ is replaced by $2b < \lambda < \mu < \Lambda$.

5. Final remarks. The quantity $S_{\lambda\mu}$ is nondecreasing, that is, if $\lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$, then $S_{\lambda_1\mu_1} \le S_{\lambda_2\mu_2}$.

LEMMA 5.1. *Suppose that the assumptions of Theorem 3.2 hold. Then for $\lambda > 2b$ and $\mu > 2b$ we have*

$$S_{\lambda\mu} \le S_\infty = \min\left(\frac{S_H}{Q_{\mathrm{M}}^{(N-2)/N}}, \frac{S_H}{2^{2/N}Q_{\mathrm{m}}^{(N-2)/N}}\right) \tag{5.1}$$

and

$$\lim_{\lambda\to\infty,\,\nu\to\infty} S_{\lambda\mu} = S_\infty. \tag{5.2}$$

Proof. Testing $S_{\lambda\mu}$ with $(s_0U_{\epsilon,y}, t_0U_{\epsilon,y})$, where $Q(y) = Q_{\mathrm{m}}$ or $Q_{\mathrm{M}} = Q(y)$, we get (5.1). To show (5.2) we argue by contradiction. Since $S_{\lambda\mu}$ is nondecreasing we may assume that $S_{\lambda\mu} < S_\infty$ for all $\lambda > 2b$ and $\mu > 2b$. Let $\lambda_n \to \infty$ and $\mu_n \to \infty$. By Theorem 3.2 for each (λ_n,μ_n) there exists a minimizer (u_n, v_n) for $S_{\lambda_n\nu_n}$. Since

$$\int_\Omega \left[\left(\frac{\lambda_n}{2} - b\right)u_n^2 + \left(\frac{\mu_n}{2} - b\right)v_n^2\right]dx \le S_{\lambda_n\mu_n} \le S_\infty,$$

we have $u_n \to 0$, $v_n \to 0$ in $L^2(\Omega)$ and $\int_\Omega P(u_n,v_n)\,dx \to 0$. It follows from the concentration-compactness principle (see Proposition 3.1) that

$$|\nabla u_n|^2 \rightharpoonup \mu \ge \sum_{j\in J}\mu_j\delta_{x_j}, \quad |\nabla v_n|^2 \rightharpoonup \sigma \ge \sum_{j\in J}\sigma_j\delta_{x_j}$$

and

$$H(u_n,v_n) \rightharpoonup \nu = \sum_{j\in J}\nu_j\delta_{x_j}$$

in the sense of measures. The coefficients μ_j, σ_j and ν_j satisfy conditions (iii) and (iv) of Proposition 3.1. We then have

$$1 = \sum_{j\in J}\nu_j Q(x_j)$$

and

$$S_\infty \geq \lim_{\lambda_n\to\infty,\,\mu_n\to\infty} S_{\lambda_n\mu_n} \geq \sum_{j\in J} S_\infty(\nu_j Q(x_j))^{2/2^*}.$$

Since $2/2^* < 1$, J consists of one point, say x_j. Hence $1 = \nu_j Q(x_j)$ and so $\lim_{\lambda_n\to\infty,\,\mu_n\to\infty} S_{\lambda\mu} \geq S_\infty$. ∎

If $Q(x) \equiv 1$ on Ω, then the system (1.1), in general, may have constant solutions. However, if λ and μ are large then minimizers of $S_{\lambda\mu}$ are not constant.

Proposition 5.2. *Suppose that the assumptions of Theorem* 3.2 *hold and let* $Q(x) \equiv 1$ *on* Ω. *Then for* $\lambda > 2b$ *and* $\mu > 2b$ *large,* $S_{\lambda\mu}$ *does not have constant minimizers.*

Proof. Suppose that for every $\lambda > 2b$ and $\mu > 2b$, $S_{\lambda\mu}$ admits a constant minimizer (s,t). A minimizer (s,t), with $s > 0$ and $t > 0$, depends obviously on λ and μ. We then have

$$S_{\lambda\mu} = |\Omega|\biggl(\frac{\lambda}{2}s^2 + \frac{\mu}{2}t^2\biggr) - |\Omega|P(s,t) \quad \text{and} \quad H(s,t) = \frac{1}{|\Omega|}.$$

Hence for $\lambda > 2b$ and $\mu > 2b$ we have

$$S_{\lambda\mu} \geq \biggl(\frac{\lambda}{2} - b\biggr)s^2|\Omega| + \biggl(\frac{\mu}{2} - b\biggr)t^2|\Omega| \tag{5.3}$$

and

$$H(s,t) = \frac{1}{|\Omega|} \leq a(s^{2^*} + t^{2^*}) \tag{5.4}$$

for some constant $a > 0$ independent of s and t. Then for large λ and μ inequalities (5.3) and (5.4) contradict the estimate $S_{\lambda\mu} \leq S_\infty$. ∎

Proposition 5.3. *Let* $\lambda = \mu$. *Suppose that* P *and* H *satisfy the conditions*

$$P_u(u,v) - P_v(u,v) = P_1(u,v)(u-v) \tag{5.5}$$

for all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$, *where* P_1 *is a bounded and continuous function on* $\mathbb{R}_+ \times \mathbb{R}_+$, *and there exists a nonnegative function* H_1 *on* $\mathbb{R}_+ \times \mathbb{R}_+$ *such that*

$$H_v(u,v) - H_u(u,v) = H_1(u,v)(u-v) \tag{5.6}$$

for all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$. *Then for* $\lambda > 0$ *sufficiently large any solution* (u,v) *of problem* (1.1) *satisfies* $u = v$ *on* Ω.

Proof. We observe

$$-\Delta(u-v) + (\lambda + Q(x)H_1(u,v) - P_1(u,v))(u-v) = 0$$

in Ω and for $\lambda > 0$ sufficiently large $\lambda + Q(x)H_1(u,v) - P_1(u,v) > 0$ on Ω. Multiplying the above equation by $u - v$ and integrating over Ω, we get

$$\int_\Omega |\nabla(u-v)|^2\,dx + \int_\Omega (\lambda + Q(x)H_1(u,v) - P_1(u,v))(u-v)^2\,dx = 0.$$

This obviously yields $u = v$ on Ω. ■

Both functions $P(u,v) = uv$ and $P_1(u,v) = (u+v)\sqrt{u^2+v^2}$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfy (5.5). An example of a function satisfying (5.6) is $H(u,v) = u^\alpha v^\alpha$ with $2\alpha = 2^*$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$.

In the case where $H(u,v) = u^\alpha v^\beta$ with $1 < \alpha < \beta$ and $\alpha + \beta = 2^*$, for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$ we have the following result:

PROPOSITION 5.4. *Let $0 < \lambda < \mu$, $H(u,v) = u^\alpha v^\beta$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$, with $1 < \alpha < \beta$ and $\alpha + \beta = 2^*$. Moreover, assume that $P(u,v) = uv$ for $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$. If (u,v) is a solution of problem* (1.1), *then $tu > sv$ on Ω, where $s > 0$ and $t > 0$ are constants satisfying*

$$\alpha s^{\alpha-2} t^\beta = \beta t^{\beta-2} s^\alpha = 1. \tag{5.7}$$

Proof. First, we perform a change of unknown functions $u = su_1$ and $v = tv_1$. Then (u_1, v_1) satisfies the following system of equations:

$$\begin{cases} -\Delta u_1 + \lambda u_1 - \frac{1}{s} P_u(su_1, tv_1) - \alpha Q(x) s^{\alpha-2} t^\beta u_1^{\alpha-1} v_1^\beta = 0, \\ -\Delta v_1 + \mu v_1 - \frac{1}{t} P_v(su_1, tv_1) - \beta Q(x) s^\alpha t^{\beta-2} u_1^\alpha v^{\beta-1} = 0. \end{cases}$$

Selecting s and t so that (5.7) holds we obtain

$$-\Delta(u_1 - v_1) + \lambda(u_1 - v_1) + (\lambda - \mu)v_1 + P_v\left(\frac{s}{t}u_1, v_1\right) - P_u\left(u_1, \frac{t}{s}v_1\right) \\ + Q(x) u_1^{\alpha-1} v_1^{\beta-1}(u_1 - v_1) = 0.$$

We now observe that

$$P_v\left(\frac{s}{t}u_1, v_1\right) - P_u\left(u_1, \frac{t}{s}v_1\right) = \frac{s}{t}(u_1 - v_1) + \frac{s^2 - t^2}{ts} v_1.$$

Since $\alpha < \beta$, we see that $s < t$. Hence

$$-\Delta(u_1 - v_1) + \left(\lambda + \frac{t}{s} + Q(x) u_1^{\alpha-1} v_1^{\beta-1}\right)(u_1 - v_1) \geq 0 \quad \text{in } \Omega.$$

By the maximum principle $u_1 > v_1$ on Ω and the result readily follows. ■

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