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A WIENER TYPE THEOREM FOR $(U(p,q),H_n)$

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Abstract. It is well known that $(U(p,q),H_n)$ is a generalized Gelfand pair. Applying the associated spectral analysis, we prove a theorem of Wiener Tauberian type for the reduced Heisenberg group, which generalizes a known result for the case $p=n$, $q=0$.

1. Introduction. A classical Tauberian theorem due to Wiener (see e.g. [7]) states that a closed ideal J of the convolution algebra $L^1(\mathbb{R}^n)$ is the full algebra if and only if there exists $g\in J$ such that $\widehat{g}(\xi)\neq 0$ for every $\xi\in\mathbb{R}^n$, where $\widehat{g}$ denotes the Fourier transform of g. Equivalently, given $g\in L^1(\mathbb{R}^n)$, the smallest closed, translation-invariant subspace generated by g is $L^1(\mathbb{R}^n)$ if and only if $\widehat{g}(\xi)\neq 0$ for every $\xi\in\mathbb{R}^n$.

Analogues of Wiener's theorem have been proved in the context of Heisenberg groups.

Let us consider the Heisenberg group H_n defined by $H_n=\mathbb{C}^n\times\mathbb{R}$ with group law $(z,t)(z',t')=(z+z',t+t'-\frac{1}{2}\operatorname{Im} z.\overline{z'})$ where $z.\overline{z'}=\sum_{j=1}^n z_j\overline{z'_j}$. Then the unitary group $U(n)$ acts on H_n by automorphisms, in the natural way, $g.(z,t)=(gz,t)$. It is well known that $(U(n),H_n)$ is a Gelfand pair, that is, the convolution algebra $L^1_{U(n)}(H_n)$ of $U(n)$-invariant, integrable functions on H_n is commutative. Its spectrum is identified, via integration, with the set of bounded spherical functions. In [5], it is proved that if J is a closed ideal of $L^1_{U(n)}(H_n)$ and if for each bounded spherical function φ, there exists $f\in J$ such that $\langle f,\varphi\rangle:=\int_{H_n} f(z,t)\varphi(z,t)\,dz\,dt\neq 0$, then $J=L^1_{U(n)}(H_n)$.

The motion group of the Heisenberg group is $G=U(n)\ltimes H_n$ (semidirect product) acting on $L^1(H_n)$ in the canonical way. For $f\in L^1(H_n)$ and $g\in G$, let $f^g(z,t)=f(g.(z,t))$ and let V_f be the smallest closed subspace spanned by $\{f^g : g\in G\}$. In [6] sufficient conditions on f are given in order to get $V_f=L^1(H_n)$. Moreover, a similar problem is considered in [11, Th. 4.2.4] for $f\in L^p(H_n^r)$ (for some range of p), where H_n^r denotes the reduced Heisenberg group H_n/Γ, $\Gamma=\{0\}\times 2\pi\mathbb{Z}$.

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In this article, we study a similar problem but considering $U(p,q)$, $p+q = n$, $p,q \geq 1$, in place of $U(n)$ (which is isomorphic to $U(n,0)$).

For $z,w \in \mathbb{C}^n$, we set $B(z,w) = \sum_{j=1}^{p} z_j\overline{w}_j - \sum_{j=p+1}^{n} z_j\overline{w}_j$ and we write, for $x \in \mathbb{R}^n$, $x = (x', x'')$ with $x' \in \mathbb{R}^p$, $x'' \in \mathbb{R}^q$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ by $\varphi(x', x'', y', y'') = (x' + iy', x'' - iy'')$, $x', y' \in \mathbb{R}^p$, $x'', y'' \in \mathbb{R}^q$. Via φ, the form $-\operatorname{Im} B(z,w)$ agrees with the standard symplectic form on $\mathbb{R}^{2(p+q)}$, and so H_n is isomorphic to $\mathbb{C}^n \times \mathbb{R}$ with the product $(z,t)(w,s) = (z+w, t+s-\frac{1}{2}\operatorname{Im} B(z,w))$. From now on, we will use freely this identification. We note that $U(p,q) = \{g \in Gl(n,\mathbb{C}) : B(gz, gw) = B(z,w)\}$ and so it acts on H_n, by automorphisms, by $g.(z,t) = (gz, t)$. Let

$$X_j = -\frac{1}{2}\, y_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \qquad Y_j = \frac{1}{2}\, x_j \frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}, \qquad j = 1, \ldots, n, \qquad T = \frac{\partial}{\partial t}$$

denote the standard basis of the Heisenberg Lie algebra. Then the algebra of left invariant and $U(p,q)$-invariant differential operators on H_n is generated by T and $L = \sum_{j=1}^{p}(X_j^2 + Y_j^2) - \sum_{j=p+1}^{n}(X_j^2 + Y_j^2)$. The pair $(U(p,q), H_n)$ is a generalized Gelfand pair and the corresponding harmonic analysis has been developed in [1], [2], [4].

There, the associated spherical distributions were computed: for $\lambda \in \mathbb{R} - \{0\}$ and $k \in \mathbb{Z}$, there exists a tempered $U(p,q)$, invariant distribution (on H_n) $S_{\lambda,k}$ satisfying

$$LS_{\lambda,k} = -|\lambda|(2k + p - q)S_{\lambda,k}, \quad iTS_{\lambda,k} = \lambda S_{\lambda,k} \tag{1.1}$$

and, for $\nu \in \mathbb{R}$, there exists a tempered $U(p,q)$-invariant distribution S_ν such that

$$LS_\nu = \nu S_\nu, \qquad iTS_\nu = 0. \tag{1.2}$$

Up to a constant, $S_{\lambda,k}$ is the unique tempered solution of (1.1) (see [2]). In contrast, the solution space of (1.2) in $\mathcal{S}'(H_n)$ is one-dimensional for $\nu \neq 0$, and for $\nu = 0$ it is two-dimensional, generated by S_0 and the trivial solution 1 (see [4]).

For f in the Schwartz space $\mathcal{S}(H_n)$, the Plancherel inversion formula provides the spectral decomposition

$$f = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k} |\lambda|^n \, d\lambda.$$

For the reduced Heisenberg group we have a similar spectral analysis. Indeed, we write $H_n^r = \mathbb{R}^n \times \mathbb{R}^n \times S^1$, where $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$. The algebra of left invariant and $U(p,q)$-invariant differential operators on H_n^r is generated by L and $\overline{T} := \partial/\partial\theta$, and the associated eigendistributions are given by $S_{l,k} := S_{\lambda,k}$ for $\lambda = l \in \mathbb{Z}$ and S_ν, $\nu \in \mathbb{R}$.

THEOREM (cf. [11, Theorem 4.2.4]). *Let K be a $U(p,q)$-invariant function in $L^\infty(H_n^r)$ and let $J = \{g \in L^1(H_n^r) : g * K = 0\}$. Assume that*

(i) *for all $k, l \in \mathbb{Z}$, $l \neq 0$, there exists $g \in J$ such that $g * S_{l,k} \neq 0$,*
(ii) *for each real ν, there exists $g \in J$ such that $g * S_\nu \neq 0$.*

Then $K = 0$ (and $J = L^1(H_n)$).

For the proof of the Theorem, we cannot argue as in Th. 4.2.4 of [11], since there is no nontrivial $U(p,q)$-invariant, integrable function on H_n.

Instead, we use some results obtained in [3], pertaining to a Plancherel inversion formula.

2. Preliminaries. Let us introduce some notation and recall some known facts. For $p, q \geq 1$, $p + q = n$, the space $\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ of $U(p,q)$-invariant, tempered distributions on $\mathbb{C}^n$ was described in [10]. There, the space $\mathcal{H}$ of functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that $\varphi(\tau) = \varphi_1(\tau) + \tau^{n-1} H(\tau)\varphi_2(\tau)$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$, was introduced (where H denotes the Heaviside function, i.e. $H(\tau) = \chi_{(0,\infty)}(\tau)$). $\mathcal{H}$ is endowed with a suitable Fréchet topology, and there is a linear, continuous and surjective map $N : \mathcal{S}(\mathbb{C}^n) \to \mathcal{H}$ whose adjoint $N' : \mathcal{H}' \to \mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ is a homeomorphism.

For $f \in \mathcal{S}(H_n)$, we will write $Nf(\tau, t)$ for $N(f(\cdot, t))(\tau)$. We have

$$Nf(\tau, t) = \int_{B(z,z)=\tau} f(z,t)\, d\sigma_\tau(z) \tag{2.1}$$

where $d\sigma_\tau(z)$ denotes a surface measure on the hyperboloid $B(z,z) = \tau$ such that

$$\int_{H_n} f(z,t)\, dz\, dt = \frac{1}{2^n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Nf(\tau, t)\, d\tau\, dt \tag{2.2}$$

(cf. (2.12) in [2]).

In order to give explicit expressions for the distributions $S_{\lambda,k}$ we recall the definition of the Laguerre polynomials. For nonnegative integers m, α let $L_m^\alpha(\tau)$ (see e.g. [9, pp. 99–101]) be given by

$$L_m^0(\tau) = \sum_{j=0}^m \binom{m}{j} (-1)^j \frac{\tau^j}{j!}, \qquad L_{m-1}^{\alpha+1}(\tau) = -\frac{d}{d\tau} L_m^\alpha(\tau).$$

It is well known that the family $\{e^{-\tau/2} L_m^0(\tau)\}_{m \geq 0}$ is an orthonormal basis of $L^2(0,\infty)$.

The distributions $S_{\lambda,k}$ can be written as

$$S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t}$$

where $F_{\lambda,k} \in \mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$. Thus, there exists $T_{\lambda,k} \in \mathcal{H}'$ such that

$$\langle F_{\lambda,k}, g \rangle = \langle T_{\lambda,k}, Ng \rangle.$$

Explicitly,

$$\langle F_{\lambda,k}, g \rangle = \left\langle (L^0_{k-q+n-1} H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|}\, e^{-\tau/2} Ng\Big(\frac{2}{|\lambda|}\,\tau\Big)\right\rangle$$

for $k \geq 0$, $\lambda \neq 0$, and for $k < 0$, $\lambda \neq 0$,

$$\langle F_{\lambda,k}, g \rangle = \left\langle (L^0_{-k-p+n-1} H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|}\, e^{-\tau/2} Ng\Big(-\frac{2}{|\lambda|}\,\tau\Big)\right\rangle.$$

Associated to the bilinear form B, we introduce the "Fourier transform" $\mathcal{F}_B$ defined on $L^1(\mathbb{C}^n)$ by

$$(\mathcal{F}_B f)(u) = \int_{H_n} e^{i\,\mathrm{Re}\,B(u,z)} f(z)\, dz. \tag{2.3}$$

For $\nu \in \mathbb{R}$, let

$$\langle S^0_\nu, f \rangle = \int_{B(u,u)=v} \int_{\mathbb{C}^n} e^{i\,\mathrm{Re}\,B(u,z)} f(z)\, dz\, d\sigma_\nu(u)$$

where $d\sigma_v(u)$ is the surface measure on $B(u,u) = v$, involved in (2.1). In other words, $S^0_\nu = \mathcal{F}_B(d\sigma_\nu)$.

For $g \in \mathcal{S}(H_n)$, the distributions S_ν are given by (see [4])

$$\langle S_\nu, g \rangle = \int_{B(u,u)=\nu} \int_{H_n} e^{i\,\mathrm{Re}\,B(u,z)} g(z,t)\, dz\, dt\, d\sigma_\nu(u). \tag{2.4}$$

In [3] a generalized spherical transform on the Schwartz space on H_n is defined: for $g \in \mathcal{S}(H_n)$, $\mathcal{F}g : (\mathbb{R}-\{0\}) \times \mathbb{Z} \to \mathbb{C}$ is given by

$$\mathcal{F}(g)(\lambda, k) = \langle S_{\lambda,k}, g \rangle.$$

It is proved that $\mathrm{Ker}\, N = \mathrm{Ker}\, \mathcal{F}$, the image of $\mathcal{F}$ is described and an inversion Plancherel formula is obtained, recovering Ng in terms of $\mathcal{F}(g)(\lambda, k)$. In order to state the formula, we introduce more notation.

For $m : (\mathbb{R}-\{0\}) \times \mathbb{Z} \to \mathbb{C}$, we set

$$m^*(\lambda, k) = \begin{cases} m(\lambda, k) & \text{if } k \geq 0, \\ (-1)^n m(\lambda, k) & \text{if } k < 0, \end{cases}$$

$$m^{**}(\lambda, k) = \begin{cases} m(\lambda, k) & \text{if } k < 0, \\ (-1)^n m(\lambda, k) & \text{if } k \geq 0. \end{cases}$$

Also we define

$$E(m)(\lambda,k)=\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}m(\lambda,k-l),$$
$$\widetilde{E}(m)(\lambda,k)=\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}m(\lambda,k+l).$$

In the following proposition we state the main results proved in [3, Section 4], concerning the Plancherel inversion formula.

PROPOSITION 2.1. *Let $g\in\mathcal{S}(H_n)$ and set $m(\lambda,k)=\mathcal{F}(g)(\lambda,k)$.*

(i) *For $\tau\ge 0$,*

$$Ng(\tau,\widehat{\lambda})=(-1)^{n-1}\frac{|\lambda|}{2}\sum_{k\ge 0}E(m^*)(\lambda,k+q)L_k^0(|\lambda\tau|/2)e^{-|\lambda|\tau/4},$$

and for $\tau<0$,

$$Ng(\tau,\widehat{\lambda})=(-1)^{n-1}\frac{|\lambda|}{2}\sum_{k\ge 0}\widetilde{E}(m^{**})(\lambda,-k-p)L_k^0(-|\lambda|\tau/2)e^{|\lambda|\tau/4}.$$

(ii) *Let D be the operator defined on the space of polynomial functions by $DL_k^0=L_k^0-L_{k-1}^0$ for $k\ge 1$ and $D1=1$. Then for $k\ge 0$ and $m\ge 0$,*

$$D^m(L_k^0)=\sum_{j=0}^{\min(m,k)}(-1)^j\binom{m}{j}L_{k-j}^0$$

(see [4, Lemma 4.3]). Moreover, for $k\ge n-1$,

$$D^{n-1}(L_k^0)(\tau)=(-1)^{n-1}\,\frac{1}{(n-1)!}\,\tau^{n-1}\mathcal{L}_{k-(n-1)}^{n-1}(\tau)$$

(see [4, Lemma 4.4]).

(iii) *For $\tau\ge 0$,*

$$Ng(\tau,\widehat{\lambda})=\xi_1(\tau,\widehat{\lambda})+\eta_1(\tau,\widehat{\lambda})\quad\text{where}$$
$$\xi_1(\tau,\widehat{\lambda})=\frac{(-1)^{n-1}}{(n-1)!}\frac{|\lambda|}{2}\sum_{k\ge q}m(\lambda,k)\left(\frac{|\lambda|\tau}{2}\right)^{n-1}\mathcal{L}_{k-q}^{n-1}(|\lambda|\tau/2)e^{-|\lambda|\tau/4},$$
$$\eta_1(\tau,\widehat{\lambda})=\frac{|\lambda|}{2}\sum_{-p+1\le k\le q-1}m^*(\lambda,k)(D^{n-1}L_{k-q+n-1}^0)(|\lambda|\tau/2)e^{-|\lambda|\tau/4}.$$

Similarly, for $\tau\le 0$,

$$Ng(\tau,\widehat{\lambda})=\xi_2(\tau,\widehat{\lambda})+\eta_2(\tau,\widehat{\lambda})\quad\text{where}$$

$$\xi_2(\tau,\widehat{\lambda}) = \frac{(-1)^{n-1}}{(n-1)!}\frac{|\lambda|}{2}\sum_{k\le -p} m(\lambda,k)(|\lambda|\tau/2)^{n-1}\mathcal{L}^{n-1}_{-k-p}(-|\lambda|\tau/2)e^{|\lambda|\tau/4},$$

$$\eta_2(\tau,\widehat{\lambda}) = \frac{|\lambda|}{2}\sum_{-p+1\le k\le q-1} m^{**}(\lambda,k)(D^{n-1}L^0_{-k-p+n-1})(-|\lambda|\tau/2)e^{|\lambda|\tau/4}.$$

Moreover, by taking the Fourier transform in the variable t, we can write $\xi_1(\tau,t) = \tau^{n-1}\widetilde{\xi_1}(\tau,t)$ (resp. $\xi_2(\tau,t) = \tau^{n-1}\widetilde{\xi_2}(\tau,t)$) and the series defining $\widetilde{\xi_1}$ converges absolutely and uniformly on $[0,\infty]\times\mathbb{R}$ (resp. on $[-\infty,0]\times\mathbb{R}$) [4, Th. 4.5]). ∎

We now consider $g\in\mathcal{S}(H_n^r)$. We recall that, for $l\in\mathbb{Z}$, the Fourier coefficients of $t\mapsto Ng(\tau,t)$ are given by

$$Ng(\tau,\widehat{l}) = \frac{1}{2\pi}\int_0^{2\pi} Ng(\tau,t)e^{-ilt}\,dt,$$

and

$$Ng(\tau,t) = \sum_{l\ne 0} Ng(\tau,\widehat{l})e^{ilt} + Ng(\tau,\widehat{0})$$

is the corresponding Fourier expansion.

By adapting the results of Proposition 2.1 to Schwartz functions on the reduced Heisenberg group, we see that for $\tau\ge 0$,

$$Ng(\tau,t) = \xi_1(\tau,t) + \eta_1(\tau,t) + Ng(\tau,\widehat{0})$$

where

$$\xi_1(\tau,t) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{l\ne 0}\frac{|l|}{2}\sum_{k\ge q} m(l,k)(|l|\tau/2)^{n-1}\mathcal{L}^{n-1}_{k-q}(|l|\tau/2)e^{-|l|\tau/4}e^{ilt},$$

(2.5)

$$\eta_1(\tau,t) = \sum_{l\ne 0}\frac{|l|}{2}\sum_{-p+1\le k\le q-1} m^{*}(l,k)(D^{n-1}L^0_{k-q+n-1})(|l|\tau/2)e^{-|l|\tau/4}e^{ilt},$$

and for $\tau\le 0$,

$$Ng(\tau,t) = \xi_2(\tau,t) + \eta_2(\tau,t) + Ng(\tau,\widehat{0})$$

with

$$\xi_2(\tau,t) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{l\ne 0}\frac{|l|}{2}\sum_{k\le -p} m(l,k)(|l|\tau/2)^{n-1}\mathcal{L}^{n-1}_{-k-p}(-|l|\tau/2)e^{|l|\tau/4}e^{ilt},$$

(2.6)

$$\eta_2(\tau,t) = \sum_{l\ne 0}\frac{|l|}{2}\sum_{-p+1\le k\le q-1} m^{**}(l,k)(D^{n-1}L^0_{-k-p+n-1})(-|l|\tau/2)e^{|l|\tau/4}e^{ilt}.$$

Furthermore, the series involved in (2.5) and (2.6) converge absolutely and uniformly.

Let $\Psi^{n-1}_{l,k}(\tau,t) := \mathcal{L}^{n-1}_{k-q}(|l|\tau/2)e^{-|l|\tau/4}e^{ilt}$. Let K be a $U(p,q)$-invariant function in $L^\infty(H^r_n)$. We set $\widetilde{K}(\tau,t) := K(z,t)$ if $B(z,z)=\tau$. We also set $K_0(z) = K(z,\widehat{0})$ and let K_{00} be the distribution given by

$$\langle K_{00}, g\rangle = \int\limits_{H^r_n} K(z,\widehat{0})g(z,t)\,dz\,dt, \tag{2.7}$$

that is, $\langle K_{00}, g\rangle = \langle K_0\otimes 1_t, g\rangle$.

We also define, for $l\neq 0$, $-p+1\le k\le q-1$,

$$a_{l,k}(K) = \frac{|l|}{2}\int\limits_0^{2\pi}\Big(\int\limits_0^{\infty}\widetilde{K}(\tau,t)(D^{n-1}L^0_{k-q+n-1})(|l|\tau/2)e^{-|l|\tau/4}e^{ilt}\,d\tau\Big)\,dt,$$

$$b_{l,k}(K) = \frac{|l|}{2}\int\limits_0^{2\pi}\Big(\int\limits_{-\infty}^{0}\widetilde{K}(\tau,t)(D^{n-1}L^0_{-k-p+n-1})(-|l|\tau/2)e^{|l|\tau/4}e^{ilt}\,d\tau\Big)dt.$$

Proposition 2.2. *Assume that K is a $U(p,q)$-invariant function in $L^\infty(H^r_n)$. Then, in the distribution sense,*

$$K = \sum_{k\in\mathbb{Z}}\sum_{l\neq 0} c_{l,k}(K)S_{l,k} + K_{00}$$

where, for fixed $l\neq 0$: if $k\ge q$,

$$c_{l,k}(K) = \frac{(-1)^{n-1}}{(n-1)!}\Big(\frac{|l|}{2}\Big)^n\int\limits_0^{2\pi}\Big(\int\limits_0^{\infty}\tau^{n-1}\Psi^{n-1}_{l,k-q}(|l|\tau/2,t)\widetilde{K}(\tau,t)\,d\tau\Big)\,dt; \tag{2.8}$$

if $k\le -p$,

$$\begin{aligned} &c_{l,k}(K) \\ &= \frac{(-1)^{n-1}}{(n-1)!}\Big(\frac{|l|}{2}\Big)^n\int\limits_0^{2\pi}\Big(\int\limits_{-\infty}^{0}\tau^{n-1}\Psi^{n-1}_{l,-k-p}(-|l|\tau/2,t)\widetilde{K}(\tau,t)d\tau\Big)\,dt; \end{aligned} \tag{2.9}$$

if $0\le k\le q-1$,

$$c_{l,k}(K) = a_{l,k}(K) + (-1)^n b_{l,k}(K);$$

and if $-p+1\le k\le -1$,

$$c_{l,k}(K) = (-1)^n a_{l,k}(K) + b_{l,k}(K).$$

Proof. By (2.2), for $g \in \mathcal{S}(H_n^r)$ we have

$$\langle K, g\rangle := \int_{H_n^r} K(z,t)g(z,t)\,dz\,dt = \int_0^{2\pi}\Big(\int_{-\infty}^{+\infty} N(Kg)(\tau,t)\,d\tau\Big)\,dt$$
$$= \int_0^{2\pi}\Big(\int_{-\infty}^{+\infty} \widetilde{K}(\tau,t)Ng(\tau,t)\,d\tau\Big)dt.$$

Also

$$\int_{-\infty}^{+\infty} \widetilde{K}(\tau,t)Ng(\tau,t)\,d\tau = \int_0^{+\infty} \widetilde{K}(\tau,t)\xi_1(\tau,t)\,d\tau + \int_0^{+\infty} \widetilde{K}(\tau,t)\eta_1(\tau,t)\,d\tau$$
$$+ \int_{-\infty}^{0} \widetilde{K}(\tau,t)\xi_2(\tau,t)\,d\tau + \int_{-\infty}^{0} \widetilde{K}(\tau,t)\eta_2(\tau,t)\,d\tau$$
$$+ \int_{-\infty}^{+\infty} \widetilde{K}(\tau,t)Ng(\tau,\widehat{0})\,d\tau.$$

Since the series (2.5) and (2.6) converge uniformly, and since $\tau^{n-1}\Psi_{l,k}^{n-1}(\tau,t) \in \mathcal{S}(H_n^r)$, the dominated convergence theorem implies that

$$\langle \widetilde{K}, \xi_1\rangle = \sum_{l\neq 0,\, k\geq q} c_{l,k}(K)\langle S_{l,k}, g\rangle, \qquad \langle \widetilde{K}, \xi_2\rangle = \sum_{l\neq 0,\, k\leq -p} c_{l,k}(K)\langle S_{l,k}, g\rangle,$$

where $c_{l,k}(K)$ are given by (2.8) and (2.9), respectively.

Also, taking account of the definition of m^* and m^{**}, we can see that

$$\langle \widetilde{K}, \eta_1\rangle + \langle \widetilde{K}, \eta_2\rangle = \sum_{l\neq 0,\, -p+1\leq k\leq q-1} c_{l,k}(K)\langle S_{l,k}, g\rangle,$$

with the desired expression for $c_{l,k}(K)$, $-p+1 \leq k \leq q-1$.

Finally,

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} \widetilde{K}(\tau,t)Ng(\tau,\widehat{0})\,d\tau\,dt = \int_{-\infty}^{+\infty} \widetilde{K}(\tau,\widehat{0})Ng(\tau,\widehat{0})\,d\tau$$
$$= \int_{-\infty}^{+\infty} N(Kg)(\tau,\widehat{0})\,d\tau = \langle K_{00}, g\rangle,$$

where the last equality again follows from (2.2). ■

PROPOSITION 2.3. *Let K and $c_{l,k}(K)$ be as in Proposition 2.2. Assume that for $l \neq 0$, $c_{l,k}(K) = 0$ for all $k \in \mathbb{Z}$. Then $K(z,\widehat{l}) = 0$ for a.e. z.*

Proof. Indeed, for $k \geq q$,

$$0 = c_{l,k}(K) = \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{|l|}{2}\right)^n \int_0^{2\pi}\Big(\int_0^\infty \tau^{n-1}\Psi^{n-1}_{l,k-q}(|l|\tau/2,t)\widetilde{K}(\tau,t)\,d\tau\Big)\,dt$$
$$= \frac{(-1)^{n-1}}{(n-1)!}\int_0^\infty s^{n-1}\mathcal{L}^{n-1}_{k-q}(s)e^{-s/2}\widetilde{K}(2s/|l|,\widehat{l}\,)\,ds.$$

Since, with a suitable normalization, the Laguerre polynomials $\{L^{n-1}_m(s)\}_{m\geq 0}$ form an orthonormal basis of $L^2((0,\infty), s^{n-1}e^{-s/2}\,ds)$ (see [9, p. 100]), and since $\widetilde{K}(s,\widehat{l}\,) \in L^2((0,\infty), s^{n-1}e^{-s/2}\,ds)$, we have $\widetilde{K}(s,\widehat{l}\,) = 0$ for a.e. $s \geq 0$.

With the same argument, $c_{l,k}(K) = 0$ for $k \leq -p$ implies that $\widetilde{K}(s,\widehat{l}\,) = 0$ for a.e. $s \leq 0$. Thus $K(z,\widehat{l}\,) = 0$ for a.e. z. ∎

REMARK 2.4. We observe that

$$(g * K_{00})(z,e^{it}) = \int_{\mathbb{C}^n}\int_0^{2\pi} g(z-w, e^{i(t-s-\operatorname{Im}B(z,w))})K(w,\widehat{0})\,ds\,dw$$
$$= \int_{\mathbb{C}^n} g(z-w,\widehat{0})K(w,\widehat{0})\,dw = (g(\cdot,\widehat{0}) \star K_0)(z),$$

where $\star$ denotes the convolution product on $\mathbb{C}^n$.

3. The proof of the Theorem. For $l \neq 0$, we denote by π_l the Schrödinger representation of H^r_n on $L^2(\mathbb{R}^n)$, determined by $\pi_l(0,0,t) = e^{ilt}$, and we set $E_l(\varphi,\psi)(x,y,t) := \langle \pi_l(x,y,t)\varphi,\psi\rangle$, the matrix entry of π_l associated to the vectors φ,ψ. For $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \in \mathbb{N}\cup\{0\}$, let $\|\alpha\| = \sum_{i=1}^p \alpha_i - \sum_{i=p+1}^n \alpha_i$. We pick the orthonormal basis of $L^2(\mathbb{R}^n)$ given by $h_\alpha(x) = h_{\alpha_1}(x_1)\ldots h_{\alpha_n}(x_n)$ where h_j denotes the jth Hermite function.

The Plancherel formula for $g \in \mathcal{S}(H^r_n)$ asserts that (see [11, Ch. 1])

$$g(x,y,t) = \frac{1}{(2\pi)^{n+1}}\sum_{l\neq 0}\operatorname{tr}(\pi_l(g)\pi_l(x,y,t)^{-1})|l|^n + g(x,y,\widehat{0})$$
$$= \frac{1}{(2\pi)^{n+1}}\sum_{l\neq 0}|l|^n\sum_{\alpha,\beta}\langle g, E_l(h_\alpha,h_\beta)\rangle_{L^2(H^r_n)}E_l(h_\alpha,h_\beta)(x,y,t)$$
$$+ g(x,y,\widehat{0})$$

where $\langle\,,\,\rangle_{L^2(H^r_n)}$ denotes the inner product on $L^2(H^r_n)$ and the series converge uniformly. Furthermore, for $l \neq 0$,

$$E_l(h_\alpha,h_\beta)(x,y,t) = e^{ilt}\varphi_{\alpha,\beta}(x,y)$$

where $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$ is an orthonormal basis of $\mathbb{R}^{2n}$ (see [8, Lemma 6.1]).

Also,

$$\int_{H_n^r} E_l(h_\alpha, h_\beta)(z,t) g(z,\widehat{0})\, dz\, dt = \Big(\int_{\mathbb{C}^n} \varphi_{\alpha,\beta}(z) g(z,\widehat{0})\, dz\Big)\Big(\int_0^{2\pi} e^{ilt}\, dt\Big) = 0.$$

Since

$$(g * E_l(h_\alpha, h_\alpha))(x,y,t) = \sum_\beta \langle g, E_l(h_\alpha, h_\beta)\rangle_{L^2(H_n^r)} E_l(h_\alpha, h_\beta)(x,y,t)$$

and since $S_{l,k} = 2^{-n-1}\sum_{\|\alpha\|=k} E_l(h_\alpha, h_\alpha)$ (see [2]), we have the orthogonal decomposition

$$g(x,y,t) = \sum_{l\neq 0,\, k\in\mathbb{Z}} |l|^n (g * S_{l,k})(x,y,t) + g(z,\widehat{0}).$$

Assume that $g \in J$. By hypothesis, we have $g * K = 0$. Thus Proposition 2.2 and Remark 2.4 imply that

$$0 = g * K = \sum_{k\in Z}\sum_{l\neq 0} c_{l,k}(K)(g * S_{l,k}) + (g(\cdot,\widehat{0}) \star K_0).$$

By the formula above, this is the Fourier series expansion of $g * K$, so

$$c_{l,k}(K)(g * S_{l,k}) = 0 \quad \text{for } l \neq 0,\ k \in \mathbb{Z}, \tag{3.1}$$

and

$$g(\cdot,\widehat{0}) \star K_0 = 0. \tag{3.2}$$

Since for $l \neq 0$, $k \in \mathbb{Z}$, hypothesis (i) of the Theorem asserts that there exists $g \in J$ such that $g * S_{l,k} \neq 0$, we conclude that $c_{l,k}(K) = 0$.

Thus, by Proposition 2.3, $K(z,\widehat{l}\,) = 0$ for a.e. $z \in \mathbb{C}^n$ and all $l \neq 0$.

In order to finish the proof of the Theorem, we must prove that $K(z,\widehat{0}) = 0$ for a.e. z.

For $\mu \in \mathcal{S}'(\mathbb{C}^n)$, we denote by $\operatorname{supp}\mu$ the support of μ. Let $\mathcal{F}_B$ be defined by (2.3). We recall that $S_\nu^0 = \mathcal{F}_B(d\sigma_\nu)$, where $d\sigma_\nu$ denotes the surface measure on $B_\nu := \{z : B(z,z) = \nu\}$.

Proposition 3.1. *Let K_0 be a $U(p,q)$-invariant function in $L^\infty(\mathbb{C}^n)$ and let $J_0 = \{f \in L^1(\mathbb{C}^n) : f \star K_0 = 0\}$. Assume that for each real ν, there exists $f \in J_0$ such that $f \star S_\nu^0 \neq 0$. Then $K_0(z) = 0$ for a.e. $z \in \mathbb{C}^n$.*

Proof. Since $\mathcal{F}_B(f \star h) = \mathcal{F}_B(f)\mathcal{F}_B(h)$ we follow the lines of the proof of the classical Wiener Theorem (see [7]) to obtain

$$\operatorname{supp}\mathcal{F}_B(K_0) \subset \bigcap_{f\in J_0} \{\xi \in \mathbb{C}^n : \mathcal{F}_B(f)(\xi) = 0\}. \tag{3.3}$$

Since K_0 is a $U(p,q)$-invariant function, $\mathcal{F}_B(K_0)$ is a $U(p,q)$-invariant distribution. Assume, by contradiction, that $\xi_0 \in \operatorname{supp}\mathcal{F}_B(K_0)$ and $B(\xi_0)$

$= \nu$. Then $B_\nu \subset \operatorname{supp} \mathcal{F}_B(K_0)$. Thus, by (3.3), $\mathcal{F}_B(f)(\xi) = 0$ for all $\xi \in B_\nu$ and all $f \in J_0$.

On the other hand, we have assumed that for each $\nu \in \mathbb{R}$, there exists $f \in J_0$ such that $f \star S_\nu^0 \neq 0$. For this f we have

$$\langle \mathcal{F}_B(f \star S_\nu^0), \varphi \rangle = \langle \mathcal{F}_B(f)\mathcal{F}_B(S_\nu^0), \varphi \rangle = \int_{B_\nu} \mathcal{F}_B(f)(\xi)\varphi(\xi)\, d\sigma_\nu(\xi) = 0.$$

This is a contradiction and we conclude that $\operatorname{supp} \mathcal{F}_B(K_0)$ is empty and $K_0 = 0$. ■

Let $K_0(z) = K(z, \widehat{0})$ and let J_0 be as in Proposition 3.1. We observe that by (3.2), $g \in J$ implies that $g(\cdot, \widehat{0}) \in J_0$. Also, by hypothesis (ii) of the Theorem, for $\nu \in \mathbb{R}$, there exists $g \in \mathcal{S}(H_n^r)$ such that $g * S_\nu \neq 0$. By (2.4), $(g * S_\nu)(z,t) = (g(, \widehat{0}) \star S_\nu^0)(z)$, so Proposition 3.1 implies that $K_0(z) = 0$ and the Theorem is proved. ■

The converse of the Theorem is stated in the following

Remark 3.2. For $\lambda \neq 0$, $k \in \mathbb{Z}$, there exists $f \in \mathcal{S}(H_n)$ such that $\langle S_{\lambda,k}, f \rangle \neq 0$.

Indeed, let

$$f_\lambda(z) = e^{-|\lambda|\,|z|^2/4} L_{|k|}^{n-1}(|\lambda|\,|z|^2/2) \quad \text{and} \quad f(z,t) = \int_{-\infty}^{\infty} f_\lambda(z) e^{-i\lambda t}\, d\lambda$$

(see [2, p. 343]). For $\nu \in \mathbb{R}$, let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be a nonnegative function such that $\varphi(\rho, \tau) = 1$ on a neighborhood of $(|\nu|, \nu)$ and let $f(z) = \varphi(|z|^2, B(z))$. Then $S_\nu^0(\mathcal{F}_B f) = N f(\nu) \neq 0$ by (2.1).

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