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THE SIZE OF THE CHAIN RECURRENT SET FOR GENERIC MAPS ON AN n-DIMENSIONAL LOCALLY $(n-1)$-CONNECTED COMPACT SPACE

BY

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Abstract. For $n \geq 1$, given an n-dimensional locally $(n-1)$-connected compact space X and a finite Borel measure μ without atoms at isolated points, we prove that for a generic (in the uniform metric) continuous map $f : X \to X$, the set of points which are chain recurrent under f has μ-measure zero. The same is true for $n = 0$ (skipping the local connectedness assumption).

1. Introduction. Chain recurrent points have been introduced by C. Conley [7]. They play an important role in the theory of attractors and in several other aspects of topological dynamics of a continuous map f on a compact metric space X. The key theorem here is Conley's Decomposition Theorem which says that the space X decomposes into the chain recurrent set $\mathrm{CR}(f)$ (see §2 for definition) and the rest, where the action is *gradient-like* (see [7] for definition; roughly speaking, each orbit in this part heads "one way" from some repeller A^* toward its dual attractor A). Moreover, the set $\mathrm{CR}(f)$ is the intersection over all attractors A of $A \cup A^*$ ([7]). Note that the chain recurrent set contains all nonwandering points, including the "genuine" recurrent points x (i.e., such that x belongs to the closure of its forward orbit), minimal subsets and periodic orbits.

Another motivation for studying chain recurrent sets in this particular context (of n-dimensional locally $(n-1)$-connected spaces) is provided by two other results. The first one is Pugh's Closing Lemma, which allows one to replace chain recurrent points by periodic ones (by slightly perturbing the map):

THEOREM ([13]). *Let (X, d) be an n-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connectedness assumption), and $f : X \to X$ be a map. If $x \in \mathrm{CR}(f)$, then for every $\varepsilon > 0$,*

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there exists a map $g : X \to X$ *such that* $d(f, g) < \varepsilon$ (*where* d *is the uniform distance*) *and* x *is a periodic point of* g.

The second is the result by Block and Franke [4, Theorem H], which characterizes the case where all chain recurrent points are nonwandering, in terms of stability of the nonwandering set under perturbations:

THEOREM ([4]). *Let* (X, d) *be an* n*-dimensional locally* $(n-1)$*-connected compact metric space, where* $n \geq 0$ *(for* $n = 0$*, skip the local connectedness assumption), and* $f : X \to X$ *be a map. Then* $\Omega(f) = \mathrm{CR}(f)$ *if and only if* f *does not permit* Ω*-explosions; that is, for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $g : X \to X$ *with* $d(f, g) < \delta$*, then each point of* $\Omega(g)$ *belongs to the* ε*-neighborhood of* $\Omega(f)$*. Here* $\Omega(h)$ *means the nonwandering set of a map* h.

It is hence quite important to know how large the set $\mathrm{CR}(f)$ is. If it is small, the gradient-like behavior dominates in the system, which makes the dynamics rather nonchaotic. In many systems the chain recurrent set indeed turns out to be small; for example, Franzová [9] proved that if X is the interval then for a generic (in the uniform metric) continuous maps the chain recurrent set has Lebesgue measure zero.

In this paper we will generalize that result to all n-dimensional locally $(n-1)$-connected compact spaces and any finite Borel measures without atoms at isolated points. This includes all measures on compact manifolds and on polyhedra. The same is proved to hold on any compact zero-dimensional space, which can be thought of as the case $n = 0$ in the main result. We also discuss some other (topological) properties of the set $\mathrm{CR}(f)$.

2. Preliminaries. We now give the terminology and notation needed in what follows. A *map on* X is a continuous function $f : X \to X$ from a space X to itself; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. The dimension $\dim X$ of a space X means the covering dimension (see [8] and [12]). By a *graph*, we mean a connected one-dimensional compact polyhedron.

We let $f : X \to X$ be a map from a compact metric space (X, d) to itself. Let $x, y \in X$. An ε*-chain* from x to y is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X such that $x_0 = x$, $x_n = y$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. We say x *can be chained to* y if for every $\varepsilon > 0$ there exists an ε-chain from x to y, and we say x is *chain recurrent* if it can be chained to itself. The set of all chain recurrent points is called the *chain recurrent set* of f and denoted by $\mathrm{CR}(f)$. The following two lemmas are basic properties of the chain recurrent set of a map.

LEMMA 2.1 ([3, p. 114]). *The chain recurrent set* $\mathrm{CR}(f)$ *is closed and* $f(\mathrm{CR}(f)) = \mathrm{CR}(f)$.

LEMMA 2.2 ([3, p. 117]). $\mathrm{CR}(f) = \mathrm{CR}(f|_{\mathrm{CR}(f)})$*; that is, every chain recurrent point remains chain recurrent for the restriction of* f *to* $\mathrm{CR}(f)$.

We need the next lemma by Block and Franke which gives a characteristic property of chain recurrent points.

LEMMA 2.3 ([4, Theorem A]). *Let* $f : X \to X$ *be a map on a compact metric space* X *and* $x \in X$. *Then* $x \notin \mathrm{CR}(f)$ *if and only if there exists an open set* U *of* X *such that* $x \notin U$, $f(x) \in U$ *and* $f(\mathrm{Cl}\, U) \subseteq U$.

We state fundamental facts from geometric topology. A space X is said to be *locally* $(n-1)$*-connected* if for every $x \in X$ and every neighborhood U of x in X, there exists a neighborhood $V \subseteq U$ of x in X such that every map $f : S^k \to V$ extends to a map $\tilde{f} : B^{k+1} \to U$ for every $0 \le k \le n-1$, where S^k and B^{k+1} stand for the unit k-dimensional sphere and the unit $(k+1)$-dimensional ball of the $(k+1)$-dimensional Euclidean space, respectively. We recall a characteristic property of locally $(n-1)$-connected spaces; it is slightly rephrased here.

LEMMA 2.4 ([5, p. 80], [10, p. 150]). *Let* X *be a compact metric space and* $n \in \mathbb{N}$. *Then* X *is locally* $(n-1)$*-connected if and only if for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for every map* $\varphi : A \to X$ *from a closed set* A *of a compact metric space* Z *with* $\dim Z \setminus A \le n$ *and* $\mathrm{diam}[\mathrm{Im}\, \varphi] < \delta$, *there exists an extension* $\tilde{\varphi} : Z \to X$ *of* φ *satisfying* $\mathrm{diam}[\mathrm{Im}\, \tilde{\varphi}] < \varepsilon$.

3. Nullity of the chain recurrent set for a generic map and a given measure. Here is our main result.

THEOREM 3.1. *Let* (X, d) *be an* n*-dimensional locally* $(n-1)$*-connected compact metric space, where* $n \ge 0$ *(for* $n = 0$ *we simply skip the local connectedness assumption), and* μ *be a finite Borel measure on* X *without atoms at the isolated points of* X. *Then the set of maps on* X *with the chain recurrent set of* μ*-measure zero is residual in the space of all maps on* X.

REMARK 1. The interval case modulo Lebesgue measure of the theorem above was proved by Franzová [9].

Proof. Initially we only assume that X is compact metric. Fix any finite Borel measure μ, $\varepsilon > 0$, and a continuous map $f : X \to X$. By regularity of μ, the set $\mathrm{CR}(f)$ can be approximated up to ε in measure from above by an open set, say C. If $x \in X \setminus C$ then x is not chain recurrent, so Lemma 2.3 applies. By making the set U in that lemma slightly smaller we obtain an open set U_x such that $x \notin \mathrm{Cl}\, U_x$, $f(x) \in U_x$ and $f(\mathrm{Cl}\, U_x) \subseteq U_x$. Thus the graph of $f|_{X \setminus C}$ is covered by the open sets $(X \setminus \mathrm{Cl}\, U_x) \times U_x$. By compactness of the graph, a finite collection (say, consisting of $U_i = U_{x_i}$, $i = 1, \ldots, k$) of such sets covers this graph. It is now clear that for any perturbation g of f sufficiently close to f in the uniform distance, the graph of g is covered by the same collection, and the property $g(\mathrm{Cl}\, U_i) \subseteq U_i$ is maintained. Using the other direction of Lemma 2.3 this implies that the chain recurrent set for g is contained in C, hence its measure is not larger than $\mu(\mathrm{CR}(f)) + \varepsilon$. The above implies that $\mu(\mathrm{CR}(f)) < \varepsilon$ is an open property ([1]) of f, and the property $\mu(\mathrm{CR}(f)) = 0$ is of type G_δ.

It remains to prove that the property $\mu(\mathrm{CR}(f)) < \varepsilon$ is (for every $\varepsilon > 0$) dense in the space of maps. Here we will need the assumptions made on X. Let $\delta < \varepsilon$ be as in Lemma 2.4 (or $\delta = \varepsilon$ in the case $n = 0$) and let $\xi < \delta/2$ be such that $d(x, y) < \xi$ implies $d(f(x), f(y)) < \delta/2$ (by uniform continuity of f). Let $\{z_i \mid i = 1, \ldots, p\}$ be a $\xi/2$-net in X avoiding any atoms of μ (such exists, since possible atoms of μ are not isolated in X). Using the standard fact that there exists a basis of the topology consisting of sets whose boundaries have μ-measure zero, we can easily choose disjoint open neighborhoods V_i of z_i, each of diameter not larger than ξ, whose union has full measure. Using regularity again, we can find open sets $W_i \ni z_i$ with $\mathrm{Cl}\, W_i \subseteq V_i$ which cover all of X but a set of measure ε. In case X is zero-dimensional we choose the sets V_i without boundary (i.e., V_i both closed and open), and we simply let $W_i = V_i$.

We can now create a perturbation g within 2ε-distance from f and whose chain recurrent set has small measure. For each $i \in \{1, \ldots, p\}$, let $m(i) \in \{1, \ldots, p\}$ be such that $d(f(z_i), z_{m(i)}) < \delta/2$. On each $\mathrm{Cl}\, W_i$ we define g by $g(x) = z_{m(i)}$, and we let $g = f$ on the complement of the union of the V_i's. Note that the distance between f and g (where defined) is at most δ, and the image $g(\mathrm{Cl}\, V_i)$ (where defined) has diameter at most δ. Thus, using Lemma 2.4 (separately on each $\mathrm{Cl}\, V_i$), we can extend the map to a continuous map g on the whole of $\mathrm{Cl}\, V_i$ so that the range has diameter at most ε. (In case X has dimension zero there is nothing to do in this step.) The definition is consistent where the sets $\mathrm{Cl}\, V_i$ overlap, since they overlap only on the boundaries, and the map coincides with f there. For the same reason g is continuous everywhere. Its uniform distance from f is now not larger than $\delta + \varepsilon < 2\varepsilon$. It remains to notice that within the open sets W_i only the points z_i may be chain recurrent; any other point is sent together with a neighborhood to this finite set. Thus $\mathrm{CR}(g)$ is contained in the union of $\{z_1, \ldots, z_p\}$ (whose measure is zero) and the complement of the union of the W_i's, which has measure at most ε. ■

[1] This short proof was kindly suggested by the referee.

We note that a manifold and a polyhedron are locally contractible. The *n-dimensional universal Menger compactum M_n^{2n+1}* is obtained by a process of successively deleting cubes from the $(2n+1)$-cube (see [8, p. 96], [2], [11]). When $n = 0$, we obtain the Cantor set, and when $n = 1$, the Menger curve (which is referred to as the Menger sponge in the fractal literature). A compact *n-dimensional Menger manifold* is a compact metric space locally homeomorphic to the n-dimensional universal Menger compactum M_n^{2n+1}. A topological characterization of a compact n-dimensional Menger manifold obtained by Bestvina [2] (cf. Anderson [1] for $n = 1$) is: a compact metric space X is an n-dimensional Menger manifold if and only if it is n-dimensional, locally $(n-1)$-connected, and satisfies the disjoint n-cells property. Kato et al. [11] studied measure-theoretic properties of the dynamics of Menger manifolds.

COROLLARY 3.2. *Let X be either a manifold, Menger manifold or polyhedron with no isolated points, compact and n-dimensional, where $n \in \mathbb{N}$, and μ be a finite Borel measure on X. Then the set of maps on X with the chain recurrent set of μ-measure zero is residual in the space of all maps on X.*

We give an application of the main theorem to dynamical systems of graph maps.

THEOREM 3.3. *Let G be a graph. Then the set of maps on G with the chain recurrent set being totally disconnected is residual in the space of all maps on G.*

Proof. Take a finite Borel measure μ on G which is locally positive (i.e., each nonempty open set has a positive μ-measure). We note that any measurable set in G with a nondegenerate connected component has a positive μ-measure, because it has nonempty interior. Therefore, it follows from the main theorem that a generic map has the chain recurrent set totally disconnected. ∎

Motivated by the result above, we discuss the relation between the chain recurrent set and its connectivity. We need some definitions. A map $f : X \to X$ is said to be *chain transitive* if for every $x, y \in X$, x can be chained to y. *A complete negative trajectory* of a point $x \in X$ with respect to a surjective map $f : X \to X$ is an infinite sequence $\{x_n\}$ with $x_0 = x$ and $f(x_n) = x_{n-1}$ for $n \in \mathbb{N}$ ([3, p. 101]). We note that a complete negative trajectory has a limit point by compactness, and the limit point belongs to the chain recurrent set.

The next is a slight extension of Theorem 2.8 in [6] to the case of the chain recurrent sets of arbitrary surjective maps.

Proposition 3.4. *Let* $f : X \to X$ *be a surjective map on a compact metric space* (X, d)*. If the restriction* $f|_{\mathrm{CR}(f)} : \mathrm{CR}(f) \to \mathrm{CR}(f)$ *is chain transitive, then* $\mathrm{CR}(f) = X$*.*

Proof. The strategy of our proof comes from [6, Theorem 2.8].

Let $x \in X$ and $\varepsilon > 0$. We shall construct an ε-chain from x to itself. Since any limit points of a complete negative/positive trajectory of x with respect to f belong to the chain recurrent set, we have $k, \ell \in \mathbb{N}$, $\{x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 = x\} \subseteq X$ and $y, z \in \mathrm{CR}(f)$ such that $f(x_i) = x_{i+1}$ for $-k \le i \le -1$, $d(x_{-k}, y) < \varepsilon$ and $d(f^\ell(x), z) < \varepsilon$. Since $f|_{\mathrm{CR}(f)} : \mathrm{CR}(f) \to \mathrm{CR}(f)$ is chain transitive, there exists an ε-chain $\{z_0 = z, z_1, \ldots, z_m = y'\} \subseteq \mathrm{CR}(f)$ from z to y', where $y' \in \mathrm{CR}(f)$ with $f(y') = y$ (note that $f(\mathrm{CR}(f)) = \mathrm{CR}(f)$ by Lemma 2.1). Then we obtain an ε-chain from x to itself by considering

$$\{x, f(x), \ldots, f^{\ell-1}(x), z, z_1, \ldots, z_m = y', x_{-k}, \ldots, x_{-1}, x\}.$$

Therefore, we conclude $\mathrm{CR}(f) = X$. ∎

Proposition 3.5. *Let* $f : X \to X$ *be a surjective map on a compact metric space* (X, d)*. If the chain recurrent set* $\mathrm{CR}(f)$ *of* f *is connected, then* $\mathrm{CR}(f) = X$*.*

Proof. By Proposition 3.4, it suffices to show that $g \equiv f|_{\mathrm{CR}(f)} : \mathrm{CR}(f) \to \mathrm{CR}(f)$ is chain transitive.

We recall an equivalence relation on $\mathrm{CR}(f)$. For $\varepsilon > 0$, we define a relation on $\mathrm{CR}(f)$ by $x \overset{\varepsilon}{\sim} y$ if there exist two ε-chains from x to y and from y to x with respect to g. Since $\mathrm{CR}(f) = \mathrm{CR}(g)$ by Lemma 2.2, we see that it is an equivalence relation. For $x \in \mathrm{CR}(f)$ and $\varepsilon > 0$, we put $C_\varepsilon(x) = \{y \in \mathrm{CR}(f) \mid x \overset{\varepsilon}{\sim} y\}$. Then the collection $\{C_\varepsilon(x) \mid x \in \mathrm{CR}(f)\}$ gives a partition of $\mathrm{CR}(f)$. We note that $C_\varepsilon(x)$ is open in $\mathrm{CR}(f)$. Therefore, by connectedness of $\mathrm{CR}(f)$, we have $\mathrm{CR}(f) = C_\varepsilon(x)$ for all $x \in \mathrm{CR}(f)$ and $\varepsilon > 0$. This shows that $f|_{\mathrm{CR}(f)} : \mathrm{CR}(f) \to \mathrm{CR}(f)$ is chain transitive. ∎

Remark 2. If $f : X \to X$ is *surjective* and $\mathrm{CR}(f) \neq X$, then $\mathrm{CR}(f)$ must be disconnected by Proposition 3.5. Using a similar argument to that in the proof of Theorem 3.1 (without measure considerations), we find that the property $\mathrm{CR}(f) \neq X$ is generic if X is an n-dimensional locally $(n-1)$-connected compact metric space, where $n \ge 0$ (for $n = 0$, skip the local connectedness condition, but on a further condition: "with an accumulation point").

Question. Is the disconnectedness (or total disconnectedness) property of the chain recurrent set generic?

4. Remarks

Remark 3. Analogous results to Theorem 3.1, Corollary 3.2 and Theorem 3.3 hold for the nonwandering set of a map, because the chain recurrent set contains the nonwandering set.

Remark 4. The main theorem is false if μ has an atom at an isolated point of X, because that point is an element of $\mathrm{CR}(g)$ for any map g which is sufficiently close to the identity map.

Remark 5. It is well known that any f-invariant finite measure μ is supported by the set of recurrent points. In particular $\mu(\mathrm{CR}(f)) > 0$. This implies that with all the assumptions of Theorem 3.1, a generic map f does not preserve a given finite measure μ.

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