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TOTALLY UMBILICAL SUBMANIFOLDS IN SOME SEMI-RIEMANNIAN MANIFOLDS

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Abstract. We investigate totally umbilical submanifolds in manifolds satisfying some curvature conditions of either recurrent or pseudosymmetry type in the sense of Ryszard Deszcz and derive the respective condition for submanifolds. We also prove some relations involving the mean curvature and the Weyl conformal curvature tensor of submanifolds. Some examples are discussed.

1. Introduction. Let M be a totally umbilical submanifold of a semi-Riemannian manifold N. There are several results of the following type: if a tensor field $\widetilde{T}$ that comes from the metric of the manifold N satisfies on N some relation, then the analogous relation is satisfied on M by the tensor field T arrising in the same way from the induced metric. Moreover, the Weyl conformal curvature tensor C of the submanifold M satisfies the equation

$$LC = 0,$$

where L is some quantity depending on the mean curvature vector field h ([3], [7], [11], [12], [14], [15], [18], [17], [19]).

In this paper we deal with results of the above type.

The paper is organized as follows. Basic definitions, notations and conventions are presented in Section 2. In Section 3 we give in local coordinates all formulas necessary for further computations. In Section 4 we review some known results and give new ones. Section 5 provides some examples. In the next sections we give the proofs of the new results.

All manifolds under consideration are assumed to be connected, smooth, Hausdorff and their metrics need not be definite.

2. Notation and conventions. Suppose that N is a manifold, $n = \dim N \geq 3$, $\widetilde{g}$ is a semi-Riemannian metric on N and $\widetilde{\nabla}$ is its Levi-Civita

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connection. Throughout the paper we adopt the convention that the quantity derived from the metric of the manifold N is marked with a tilde $\widetilde{\ }$. If $\widetilde{T}$ is such a quantity, then $\overline{T}$ denotes the projection of $\widetilde{T}$ on a submanifold while T is the analogue of $\widetilde{T}$ obtained from the induced metric. The Riemann curvature tensor $\widetilde{\mathcal{R}}$ is a trilinear multiplication in the Lie algebra $\mathfrak{X}(N)$ of vector fields on N defined by

$$\widetilde{\mathcal{R}}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

so that in the local coordinate system we have

$$R(\partial_k, \partial_j)\partial_i = R^h{}_{ijk}\partial_h = (\partial_k \Gamma^h_{ij} - \partial_j \Gamma^h_{ik} + \Gamma^h_{ks}\Gamma^s_{ij} - \Gamma^h_{js}\Gamma^s_{ik})\partial_h,$$

where the indices h, i, j, k, r, s run through the range $1, \dots, n$.

With $\widetilde{\mathcal{R}}$ we associate the $(0,4)$ Riemann curvature tensor $\widetilde{R}$ setting

$$\widetilde{R}(X,Y,Z,V) = \widetilde{g}(\widetilde{\mathcal{R}}(X,Y)Z,V)$$

with components $R_{hijk} = g_{hr}R^r{}_{ijk}$.

In terms of n local orthogonal vector fields $X_1, \dots, X_n$,

$$\widetilde{S}(Y,Z) = \sum_{j=1}^{n} \frac{\widetilde{R}(X_j, Y, Z, X_j)}{\widetilde{g}(X_j, X_j)}$$

defines the Ricci tensor $\widetilde{S}$ of type $(0,2)$ with local components $R_{ij} = R^r{}_{ijr}$.

For symmetric $(0,2)$ tensors A and B their Kulkarni–Nomizu product $A \wedge B$ is given by

$$\begin{aligned}(A \wedge B)(U,X,Y,V) &= A(X,Y)B(U,V) - A(X,V)B(U,Y) \\ &\quad + A(U,V)B(X,Y) - A(U,Y)B(X,V).\end{aligned}$$

Then the Weyl conformal curvature tensor $\widetilde{C}$ of type $(0,4)$ is defined as

$$\widetilde{C} = \widetilde{R} - \tfrac{1}{n-2}\widetilde{g} \wedge \widetilde{S} + \tfrac{r}{2(n-1)(n-2)}\widetilde{g} \wedge \widetilde{g},$$

with components C_{hijk}, r being the scalar curvature of N.

We extend the action of $\wedge$ to tensors B of type $(0,4)$ with symmetries

$$B(U,X,Y,V) = B(X,U,Y,V) = -B(U,X,V,Y)$$

setting

$$\begin{aligned}&(A \wedge B)(U,X,Y,V,Z,W) \\ &\qquad = A(X,Y)B(U,V,Z,W) - A(X,V)B(U,Y,Z,W) \\ &\qquad\quad + A(U,V)B(X,Y,Z,W) - A(U,Y)B(X,V,Z,W).\end{aligned}$$

We also put

$$\begin{aligned}(A \vee B)(U,X,Y,V) &= A(X,Y)B(U,V) - A(X,V)B(U,Y) \\ &\quad + A(U,Y)B(X,V) - A(U,V)B(X,Y).\end{aligned}$$

For a trilinear multiplication $\mathcal{P}$ of vector fields, skew-symmetric in the first two arguments, let P be a $(0,4)$ tensor associated with $\mathcal{P}$ by

$$P(X,Y,Z,V) = \widetilde{g}(\mathcal{P}(X,Y)Z,V).$$

One extends the endomorphism $\mathcal{P}(X,Y)$ to a derivation $\mathcal{P}(X,Y)\cdot$ of the Lie algebra of tensor fields on N assuming it commutes with contractions and setting

$$\mathcal{P}(X,Y)\cdot f = 0,$$

f being a function on N, and

$$\begin{aligned}(P{\cdot}T)(X_1,X_2,\dots,X_k;X,Y) &= (\mathcal{P}(X,Y)\cdot T)(X_1,X_2,\dots,X_k)\\ &= -T(\mathcal{P}(X,Y)X_1,X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},\mathcal{P}(X,Y)X_k),\end{aligned}$$

T being a $(0,k)$ tensor, $k \geq 1$. In the case $\mathcal{P} = \widetilde{\mathcal{R}}$ we obtain the well known Ricci identity:

$$\begin{aligned}&(\widetilde{R}{\cdot}T)(X_1,X_2,\dots,X_k;X,Y)\\ &\qquad = \widetilde{\nabla}_Y\widetilde{\nabla}_X T(X_1,X_2,\dots,X_k) - \widetilde{\nabla}_X\widetilde{\nabla}_Y T(X_1,X_2,\dots,X_k).\end{aligned}$$

In the same manner, for a symmetric tensor B of type $(0,2)$ and its associated $\mathcal{B}$,

$$B(X,Y) = \widetilde{g}(\mathcal{B}X,Y),$$

we define $B\cdot T$ by

$$\begin{aligned}&(B{\cdot}T)(X_1,X_2,\dots,X_k)\\ &\qquad = -T(\mathcal{B}X_1,X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},\mathcal{B}X_k).\end{aligned}$$

Thus, in local components, we have for example

$$\begin{aligned}(1)\qquad &(\widetilde{C}{\cdot}\widetilde{R})_{hijklm} = R_{rijk}C^r_{hlm} + R_{hrjk}C^r_{ilm} + R_{hirk}C^r_{jlm} + R_{hijr}C^r_{klm},\\ &(\widetilde{C}{\cdot}\widetilde{S})_{hklm} = R_{rk}C^r_{hlm} + R_{hr}C^r_{klm}.\end{aligned}$$

For a $(0,2)$ tensor A on N we define

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

If, moreover, A is symmetric and T is of type $(0,k)$, $k \geq 1$, we define the tensor $Q(A,T)$ of type $(0,k+2)$ setting

$$\begin{aligned}Q(A,T)(X_1,X_2,\dots,X_k;X,Y) &= ((X\wedge_A Y)\cdot T)(X_1,X_2,\dots,X_k)\\ &= -T((X\wedge_A Y)X_1,X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k).\end{aligned}$$

We easily find that

$$\begin{aligned}
&Q(A,\widetilde{C})(U,V,W,Z;X,Y)\\
&\quad = A(U,X)\widetilde{C}(Y,V,W,Z) + A(V,X)\widetilde{C}(U,Y,W,Z) + A(W,X)\widetilde{C}(U,V,Y,Z)\\
&\quad + A(Z,X)\widetilde{C}(U,V,W,Y) - A(U,Y)\widetilde{C}(X,V,W,Z) - A(V,Y)\widetilde{C}(U,X,W,Z)\\
&\qquad - A(W,Y)\widetilde{C}(U,V,X,Z) - A(Z,Y)\widetilde{C}(U,V,W,X).
\end{aligned}$$

We have

$$\begin{aligned}
(2)\qquad Q(g,\widetilde{C})(\partial_h,\partial_i,\partial_j,\partial_k;\partial_l,\partial_m) &= (Q(g,\widetilde{C}))_{hijklm} = g_{hl}C_{mijk} - g_{hm}C_{lijk}\\
&+ g_{il}C_{hmjk} - g_{im}C_{hljk} + g_{jl}C_{himk} - g_{jm}C_{hilk} + g_{kl}C_{hijm} - g_{km}C_{hijl}.
\end{aligned}$$

More details can be found for example in [1] and [5].

Observe that for a given vector field, say $p^r\partial_r$, and arbitrary coordinate vector fields $\partial_h,\partial_i,\partial_j,\partial_k,\partial_m$ we have

$$\begin{aligned}
(3)\qquad &(((p^r\partial_r)\wedge\partial_m)\cdot\widetilde{C})(\partial_h,\partial_i,\partial_j,\partial_k)\\
&\quad = p_hC_{mijk} + p_iC_{hmjk} + p_jC_{himk} + p_kC_{hijm}\\
&\qquad - g_{hm}p^rC_{rijk} - g_{im}p^rC_{hrjk} - g_{jm}p^rC_{hirk} - g_{km}p^rC_{hijr}.
\end{aligned}$$

Finally, for a symmetric $(0,2)$ tensor field B and its associated $\mathcal{B}$, we define a $(0,k+2)$ tensor field $Q(B,A,T)$ setting

$$Q(B,A,T)(X_1,X_2,\ldots,X_k;X,Y) = (\mathcal{B}(X\wedge_A Y)\cdot T)(X_1,X_2,\ldots,X_k).$$

Thus, for the Ricci tensor $\widetilde{S}$, in local coordinates we have

$$\begin{aligned}
&Q(\widetilde{S},g,\widetilde{C})(\partial_h,\partial_i,\partial_j,\partial_k;\partial_l,\partial_m) = (Q(\widetilde{S},g,\widetilde{C}))_{hijklm}\\
&\quad = g_{hl}R^r_mC_{rijk} - g_{hm}R^r_lC_{rijk} + g_{il}R^r_mC_{hrjk} - g_{im}R^r_lC_{hrjk}\\
&\qquad + g_{jl}R^r_mC_{hirk} - g_{jm}R^r_lC_{hirk} + g_{kl}R^r_mC_{hijr} - g_{km}R^r_lC_{hijr}.
\end{aligned}$$

The next lemma summarizes some of the properties of the operators we have defined.

LEMMA 1. *Let K be a $(0,4)$ tensor with the symmetries*

$$K(X_1,X_2,X_3,X_4) = -K(X_2,X_1,X_3,X_4) = K(X_3,X_4,X_1,X_2),$$

g a metric tensor, A, B, T, $\underline{K}$ $(0,2)$ tensors and $G=\frac{1}{2}g\wedge g$.

Then the following identities hold:

$$\begin{aligned}
(4)\qquad G\cdot K &= Q(g,K),\\
K\cdot G &= 0,
\end{aligned}$$

$$
\begin{aligned}
(5)\qquad & (g \wedge T)\cdot K = Q(T,g,K) + Q(T,K), \quad T \text{ symmetric},\\
& K\cdot (g\wedge T) = g \wedge (K\cdot T),\\
& Q(g,G) = 0,\\
(6)\qquad & Q(g, g\wedge T) = -Q(T,G), \quad T \text{ symmetric},\\
(7)\qquad & g\wedge (g\vee T) = -Q(T,G),\\
(8)\qquad & g\wedge (\underline{K}\vee T) = Q(\underline{K}, g\wedge T) - T\wedge(\underline{K}\vee g), \quad T \text{ symmetric},\\
& g\wedge (T\vee \underline{K}) = Q(T, g\wedge \underline{K}) - \underline{K}\wedge (T\vee g), \quad \underline{K} \text{ symmetric},\\
& 0 = Q(A, B\wedge T) + Q(B, T\wedge A) + Q(T, A\wedge B),\\
& \qquad A, B, T \text{ symmetric},\\
& \underline{K}\vee T = -(T\vee \underline{K}),\\
& \underline{K}\wedge T = T\wedge \underline{K}.
\end{aligned}
$$

Proof. Direct calculations. ∎

LEMMA 2. *Let K be a Riemann curvature tensor, $\underline{K} = \mathrm{Ricc}(K)$, $\underline{\underline{K}} = \mathrm{Tr}(\underline{K})$. Then*

$$
\begin{aligned}
Q(T,g,C) &= Q(T,g,K) + \tfrac{1}{m-2}Q(T, g\wedge \underline{K})\\
&\quad - \tfrac{1}{(m-1)(m-2)}\underline{\underline{K}}Q(T,G) + \tfrac{1}{m-2}Q(\underline{K},T,G) - \tfrac{1}{m-2} g\wedge (\underline{K}\vee T),
\end{aligned}
$$

m being the dimension of the manifold.

Proof. Direct calculations. ∎

3. Preliminaries. Let $(N,\widetilde{g})$ be a manifold covered by a system of coordinate neighbourhoods $\{U; x^r\}$. We denote by g_{ij}, Γ^k_{ij}, R_{hijk}, R_{ij}, r, C_{hijk} the components of the metric tensor $\widetilde{g}$, the Christoffel symbols, the curvature tensor $\widetilde{R}$, the Ricci tensor $\widetilde{S}$, the scalar curvature and the Weyl conformal curvature tensor $\widetilde{C}$ of $(N,\widetilde{g})$ respectively. Here and throughout, the indices h, i, j, k, l, m, r, s, t, u, v run over the range $1,\ldots,n$. Let (M,g) be an m-dimensional manifold covered by a system of coordinate neighbourhoods $\{V; y^a\}$ immersed in $(N,\widetilde{g})$ and let $x^r = x^r(y^a)$ be the local expression of the immersion F. Then the local components g_{ab} of the induced metric tensor of (M,g) are related to g_{rs} by $g_{ab} = g_{rs}B^r_aB^s_b$, where $B^r_a = \partial x^r/\partial y^a$. In what follows we shall adopt the convention

$$
B^{rs}_{ab} = B^r_aB^s_b, \quad B^{rst}_{abc} = B^r_aB^s_bB^t_c, \quad B^{rstu}_{abcd} = B^r_aB^s_bB^t_cB^u_d.
$$

We denote by Γ^c_{ab}, K_{abcd}, K_{bc}, K, C_{abcd} the components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor C of (M,g). Here and below, the indices a, b, c, d, e, f, g run over the range $1,\ldots,m$, $m<n$.

The van der Waerden–Bortolotti covariant derivative of B^r_a is given by

$$B^r_{a;b} = \nabla_b B^r_a = \partial_b B^r_a + \Gamma^r_{st} B^{st}_{ba} - B^r_c \Gamma^c_{ba}, \tag{9}$$

where the semicolon denotes covariant differentiation with respect to the metric of the submanifold.

The vector field $h = H^r \partial_r$, where

$$H^r = \tfrac{1}{m} g^{ef} \nabla_e B^r_f, \tag{10}$$

is called the mean curvature vector of (M, g). Using (9) and the equation

$$\Gamma^a_{bc} = (\partial_c B^r_b + \Gamma^r_{st} B^{st}_{cb}) B^u_d g^{da} g_{ru},$$

we obtain on (M, g) the relation

$$g_{rs} H^r B^s_a = 0. \tag{11}$$

Let N^r_x, $x, y, z = m+1, \ldots, n$, be mutually orthogonal unit vectors normal to M. Then

$$g_{rs} N^r_x N^s_x = e_x, \qquad g_{rs} N^r_x N^s_y = 0,\ x \neq y, \qquad g_{rs} N^r_x B^s_a = 0 \tag{12}$$

and

$$g^{rs} = B^{rs}_{ab} g^{ab} + \sum_x e_x N^r_x N^s_x, \tag{13}$$

where e_x is the indicator of the vector N^r_x.

The Schouten curvature tensor H^r_{ab} of M is defined by $H^r_{ab} = \nabla_b B^r_a$. The second fundamental form H_{abx} is related to H^i_{ab} by $H^i_{ab} = \sum_x e_x H_{abx} N^i_x$.

If

$$H^r_{ab} = g_{ab} H^r, \tag{14}$$

then M is called a *totally umbilical submanifold* of N. Then $H_{abx} = g_{ab} H_x$, where $H_y = H^r N^s_y g_{rs}$, and

$$H^r = \sum_x e_x H_x N^r_x.$$

Furthermore, on a totally umbilical submanifold the Gauss, Codazzi and Weingarten equations take the form ([17], [19])

$$K_{abcd} = R_{rstu} B^{rstu}_{abcd} + H(g_{bc} g_{ad} - g_{bd} g_{ac}), \tag{15}$$

$$R_{rstu} N^r_x B^{stu}_{bcd} = g_{bc} A_{dx} - g_{bd} A_{cx} \tag{16}$$

and

$$N^s_{z;a} = -H_z B^s_a + \sum_y e_y L_{azy} N^s_y \tag{17}$$

respectively, where the mean curvature H is given by

$$H = g_{rs} H^r H^s = \sum_x e_x H_x H_x,$$

and

$$A_{ax} = \partial_a H_x + \sum_y e_y L_{ayx} H_y, \quad L_{azy} = g_{rs} N^r_y N^s_{z;a}. \tag{18}$$

Moreover, we have ([17], [19])

$$R_{rstu} H^r B^{stu}_{bcd} = \tfrac{1}{2}(g_{bc} H_d - g_{bd} H_c), \quad H_c = H_{;c}, \tag{19}$$

$$\begin{aligned} K_{abcd;e} = R_{rstu,v} B^{rstuv}_{abcde} &+ H_e(g_{bc} g_{ad} - g_{bd} g_{ac}) \\ &+ \tfrac{1}{2}[H_a(g_{bc} g_{ed} - g_{bd} g_{ec}) + H_b(g_{ec} g_{ad} - g_{ed} g_{ac}) \\ &+ H_c(g_{be} g_{ad} - g_{bd} g_{ae}) + H_d(g_{bc} g_{ae} - g_{be} g_{ac})] \end{aligned} \tag{20}$$

and

$$H^r_{;a} = -H B^r_a + \sum_y e_y A_{ay} N^r_y. \tag{21}$$

Letting

$$E_{bc} = R_{hijk} H^h B^{ij}_{bc} H^k, \qquad A_{bc} = \sum_x e_x A_{bx} A_{cx}, \qquad H_{ae} = H_{;ae},$$

from the results of [19, p. 108] we find

$$\begin{aligned} H K_{abce} = R_{rstu,v} B^{rst}_{abc} H^u B^v_e &+ g_{ae} E_{bc} - g_{be} E_{ac} + A_{ae} g_{bc} - A_{be} g_{ac} \\ &+ H^2(g_{ae} g_{bc} - g_{be} g_{ac}) - \tfrac{1}{2}(H_{ae} g_{bc} - H_{be} g_{ac}). \end{aligned} \tag{22}$$

Formulas for $\widetilde{C}$, corresponding to (19) and (22), are

$$C_{rstu} H^r B^{stu}_{bcd} = g_{bc} M_d - g_{bd} M_c, \tag{23}$$

$$\begin{aligned} C_{rstu,v} H^r B^{stuv}_{bcde} &= H K_{ebcd} - E_{bc} g_{de} + E_{bd} g_{ce} - g_{bc} A_{de} + g_{bd} A_{ce} \\ &- H^2(g_{bc} g_{ed} - g_{bd} g_{ec}) + \tfrac{1}{2}(g_{bc} H_{de} - g_{bd} H_{ce}) - \tfrac{1}{n-2}(g_{bc} P_{de} - g_{bd} P_{ce}), \end{aligned} \tag{24}$$

where

$$M_c = \tfrac{1}{2} H_c - \tfrac{1}{n-2} R_{ru} H^r B^u_c, \qquad P_{ae} = R_{ru,v} B^r_a H^u B^v_e \tag{25}$$

and

$$\begin{aligned} C_{hijk} = R_{hijk} &- \tfrac{1}{n-2}(g_{ij} R_{hk} - g_{ik} R_{hj} + g_{hk} R_{ij} - g_{hj} R_{ik}) \\ &+ \tfrac{r}{(n-1)(n-2)}(g_{ij} g_{hk} - g_{ik} g_{hj}). \end{aligned} \tag{26}$$

LEMMA 3 ([7, Lemma 1]). *Let M ($\dim M \geq 4$) be a totally umbilical submanifold of a manifold N. Then the components C_{abcd} of the Weyl conformal curvature tensor C of M satisfy the relation*

$$\begin{aligned} C_{abcd} = \overline{C}_{abcd} &- \tfrac{1}{m-2}(g_{bc} T_{ad} - g_{bd} T_{ac} + g_{ad} T_{bc} - g_{ac} T_{bd}) \\ &+ \tfrac{P}{(m-1)(m-2)}(g_{bc} g_{ad} - g_{bd} g_{ac}), \end{aligned}$$

where

$$\overline{C}_{abcd} = C_{rstu}B^{rstu}_{abcd}, \qquad T_{ab} = K_{ab} - \tfrac{m-2}{n-2}R_{rs}B^{rs}_{ab},$$
$$P = K + (m-1)(m-2)H + \tfrac{(m-1)(m-2)}{(n-1)(n-2)}r.$$

LEMMA 4. *Let C be a generalized curvature tensor and a, b be 1-forms.*

(a) ([23]) *If*

$$b_e C_{abcd} + a_a C_{ebcd} + a_b C_{aecd} + a_c C_{abed} + a_d C_{abce} = 0$$

then

$$(b_e + 2a_e)C_{abcd} = 0.$$

(b) *If, moreover, C is a trace free generalized curvature tensor and for some $\Psi \in R$, $\Psi \neq n-1$,*

$$\begin{aligned}\Psi a_e C_{abcd} &+ a_a C_{ebcd} + a_b C_{aecd} + a_c C_{abed} + a_d C_{abce} \\ &- g_{ae}a^r C_{rbcd} - g_{be}a^r C_{arcd} - g_{ce}a^r C_{abrd} - g_{de}a^r C_{abcr} = 0,\end{aligned} \tag{27}$$

then

$$(\Psi + 2)a_e C_{abcd} = 0.$$

Proof. Transvecting (27) with g^{de} and using the Bianchi identity, we get $(\Psi - n + 1)a^p C_{abcp} = 0$, which, together with part (a), yields (b). ∎

4. Review of old and new results

4.1. Pseudosymmetry and recurrent type conditions. Let N be a manifold, $\widetilde{g}$ denote a Riemannian or semi-Riemannian metric on N and $\widetilde{\nabla}$ be its Levi-Civita connection. A $(0, k)$ tensor field $\widetilde{T}$ on N is said to be *recurrent* if for all vector fields X, X_j, Y_j, $j = 1, \ldots, k$, on N,

$$\widetilde{\nabla}_X\widetilde{T}(X_1, \ldots, X_k)\widetilde{T}(Y_1, \ldots, Y_k) = \widetilde{T}(X_1, \ldots, X_k)\widetilde{\nabla}_X\widetilde{T}(Y_1, \ldots, Y_k).$$

It follows that at each point of the set $U_{\widetilde{T}} = \{x \in N : \widetilde{T}(x) \neq 0\}$ there exists a unique 1-form a such that

$$\widetilde{\nabla}_X\widetilde{T}(X_1, \ldots, X_k) = a(X)\widetilde{T}(X_1, \ldots, X_k).$$

In [24] Roter proved the existence of manifolds with recurrent covariant derivative of the Riemann curvature tensor $\widetilde{R}$, i.e. such that

$$\widetilde{\nabla}_Y\widetilde{\nabla}_X\widetilde{R} = a(Y)\widetilde{\nabla}_X\widetilde{R} \tag{28}$$

for some 1-form a at each point $x \in N$ at which $\widetilde{\nabla}_X\widetilde{R} \neq 0$. Hence,

$$\widetilde{\nabla}_Y\widetilde{\nabla}_X\widetilde{R} - \widetilde{\nabla}_X\widetilde{\nabla}_Y\widetilde{R} = a(Y)\widetilde{\nabla}_X\widetilde{R} - a(X)\widetilde{\nabla}_Y\widetilde{R}. \tag{29}$$

It is clear that condition (28), resp. (29), yields an analogous condition for the Weyl conformal curvature tensor $\widetilde{C}$:

$$\widetilde{\nabla}_Y\widetilde{\nabla}_X\widetilde{C} = a(Y)\widetilde{\nabla}_X\widetilde{C}, \tag{30}$$

resp.

$$\widetilde{\nabla}_Y \widetilde{\nabla}_X \widetilde{C} - \widetilde{\nabla}_X \widetilde{\nabla}_Y \widetilde{C} = a(Y)\widetilde{\nabla}_X \widetilde{C} - a(X)\widetilde{\nabla}_Y \widetilde{C}. \tag{31}$$

It is also well known that the condition

$$\widetilde{\nabla}_X \widetilde{R} = a(X)\widetilde{R}$$

implies

$$\widetilde{\nabla}_X \widetilde{C} = a(X)\widetilde{C}. \tag{32}$$

Moreover, the class of manifolds satisfying either of the above conditions is contained in the class characterized by

$$\widetilde{\nabla}_Y \widetilde{\nabla}_X \widetilde{C} = b(X,Y)\widetilde{C}$$

and in the class of those satisfying

$$\widetilde{\nabla}_Y \widetilde{\nabla}_X \widetilde{C} - \widetilde{\nabla}_X \widetilde{\nabla}_Y \widetilde{C} = a(X,Y)\widetilde{C} + \widetilde{A}((X \wedge_{\widetilde{g}} Y) \cdot \widetilde{C}) \tag{33}$$

for some function $\widetilde{A}$ on N and $(0,2)$ tensor field a. In this connection we have

THEOREM 5 ([17], [19]). *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If condition* (32) *holds on N, then on M we have:*

(a) $\nabla_{F_*X} C = a(F_*X)C$.
(b) $HC = 0$.

THEOREM 6 ([15]). *Let M be a totally umbilical submanifold immersed in a manifold N and suppose that*

$$\widetilde{\nabla}_Y \widetilde{\nabla}_X \widetilde{C} - \widetilde{\nabla}_X \widetilde{\nabla}_Y \widetilde{C} = 0, \quad X, Y \in \mathfrak{X}(N).$$

Then, on M,

$$A_{ey} C_{abcd} = 0. \tag{34}$$

Moreover, on some neighbourhood of each point $x \in M$ such that $C(x) \neq 0$, the vectors $\nabla_a H^r$, $a = 1, \dots, m$, are tangent to M and the mean curvature is constant:

$$\nabla_a H^r = -HB^r_a, \quad H = \text{const}.$$

THEOREM 7 ([7, Lemma 2]). *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If condition* (33) *holds on N, then for any $X, Y \in \mathfrak{X}(M)$ we have*

$$\begin{aligned}\nabla_{F_*Y} \nabla_{F_*X} C &- \nabla_{F_*X} \nabla_{F_*Y} C \\ &= c(F_*X, F_*Y)C + (\widetilde{A} - H)((F_*X \wedge_{\widetilde{g}} F_*Y) \cdot C).\end{aligned}$$

We shall prove

THEOREM 8. *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If the condition* (33) *holds on N, then the Weyl conformal curvature tensor C of the manifold M satisfies*

$$
\begin{aligned}
0 &= P_{ey}C_{abcd} + A_{ay}C_{ebcd} + A_{by}C_{aecd} + A_{cy}C_{abed} + A_{dy}C_{abce} \\
&\quad - g_{ae}A^g_yC_{gbcd} - g_{be}A^g_yC_{agcd} - g_{ce}A^g_yC_{abgd} - g_{de}A^g_yC_{abcg},
\end{aligned}
\tag{35}
$$

where $P_{ey} = a_{rs}B^r_aN^s_y$, $A^g_y = g^{gd}A_{dy}$, and

$$P^g_yC_{gbcd} = (m-1)A^g_yC_{gbcd}. \tag{36}$$

If, moreover, one of the following conditions is satisfied on M:

$$a_{hk} = 0, \quad P_{ey} = 0, \quad P_{ey} = \alpha A_{ey}, \quad \alpha \neq m-1,$$

then, on M,

$$A_{ay}C_{ebcd} = 0. \tag{37}$$

As a consequence, if $C(x) \neq 0$, $x \in M$, then on some neighbourhood of x we obtain

$$\nabla_a H^r = -HB^r_a, \quad H = \text{const.}$$

Another class is formed by manifolds for which the Weyl conformal curvature tensor $\widetilde{C}$ satisfies

$$\widetilde{\nabla}_Y\widetilde{\nabla}_X\widetilde{C} - \widetilde{\nabla}_X\widetilde{\nabla}_Y\widetilde{C} = a(Y)\widetilde{\nabla}_X\widetilde{C} - a(X)\widetilde{\nabla}_Y\widetilde{C} + \widetilde{A}((X \wedge_{\widetilde{g}} Y)\cdot\widetilde{C}). \tag{38}$$

This includes manifolds satisfying (28)–(31). For such manifolds we shall prove

THEOREM 9. *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If condition* (38) *holds on N, then for any $X, Y \in \mathfrak{X}(M)$ we have:*

(a) $\nabla_{F_*Y}\nabla_{F_*X}C - \nabla_{F_*X}\nabla_{F_*Y}C$

$$= a(F_*Y)\nabla_{F_*X}C - a(F_*X)\nabla_{F_*Y}C + (\widetilde{A} - H)((F_*X \wedge_g F_*Y)\cdot C).$$

(b) *Moreover, the Weyl conformal curvature tensor C of the manifold M satisfies*

$$
\begin{aligned}
0 &= 2H_ya_eC_{abcd} - P_y\nabla_eC_{abcd} \\
&\quad + A_{ay}C_{ebcd} + A_{by}C_{aecd} + A_{cy}C_{abed} + A_{dy}C_{abce} \\
&\quad - g_{ae}A^g_yC_{gbcd} - g_{be}A^g_yC_{agcd} - g_{ce}A^g_yC_{abgd} - g_{de}A^g_yC_{abcg},
\end{aligned}
$$

and

$$2H_ya^fC_{fbcd} + P_y\nabla_fC^f_{bcd} = (m-1)A^f_yC_{fbcd}, \quad P_y = a_sN^s_{y.}$$

At any point such that $P_y = a_sN^s_y \neq 0$, there exist a 1-form p and a vector b such that for all vectors $X \in T_xM$,

$$\nabla_XC = p(X)\otimes C + (b\wedge X)\cdot C.$$

A semi-Riemannian manifold $(N,\ g)$, $\dim M \geq 3$, is said to be *pseudosymmetric* ([5, Section 3.1]; [25]; [1]) if at every point of N the tensors $\widetilde{R} \cdot \widetilde{R}$ and $Q(g,\ \widetilde{R})$ are linearly dependent. This is equivalent to

$$\widetilde{R} \cdot \widetilde{R} = L_{\widetilde{R}} Q(\widetilde{g}, \widetilde{R}) \tag{39}$$

on the set $U_{\widetilde{R}} = \{x \in M : \widetilde{R}(x) \neq 0\}$, $L_{\widetilde{R}}$ being a function on $U_{\widetilde{R}}$.

If $\dim N \geq 4$ and the tensors $\widetilde{R} \cdot \widetilde{C}$, $Q(\widetilde{g},\ \widetilde{C})$ are linearly dependent at every point of $\widetilde{N}$, the manifold is said to be *Weyl-pseudosymmetric* ([5, Section 4.1]; [1]). Thus $(N,\ g)$ is Weyl-pseudosymmetric if and only if

$$\widetilde{R} \cdot \widetilde{C} = L_{\widetilde{C}} Q(\widetilde{g}, \widetilde{C}) \tag{40}$$

on the set $U_{\widetilde{C}}$, $L_{\widetilde{C}}$ being a function on $U_{\widetilde{C}}$. The conditions (39) and (40) are equivalent on the subset $U_{\widetilde{C}}$ if either N is a 4-dimensional warped product manifold or $\dim N \geq 5$ ([5, Sections 9.2 and 9.3]). Therefore (39) and (40) can be considered as special cases of either (33) or (38). We have

Theorem 10 ([3, Propositions 2 and 4]). *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If condition* (40) *holds on N, then on M we have:*

(a) $R \cdot C = (L_{\widetilde{C}} - H) Q(g, C)$.
(b) *Relation* (34) *is satisfied.*

A semi-Riemannian manifold $(N,\ g)$, $\dim N \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([5, Section 12.6]) if at every point of N the tensors $\widetilde{C} \cdot \widetilde{C}$ and $Q(g,\ \widetilde{C})$ are linearly dependent. This is equivalent to

$$\widetilde{C} \cdot \widetilde{C} = L_{\widetilde{C}} Q(\widetilde{g}, \widetilde{C}) \tag{41}$$

on the set $U_{\widetilde{C}}$, $L_{\widetilde{C}}$ being a function on $U_{\widetilde{C}}$.

Finally, we consider a manifold N such that the tensors $\widetilde{C} \cdot \widetilde{R}$ and $Q(g, \widetilde{C})$ are linearly dependent at every point of N. This condition is equivalent to

$$\widetilde{C} \cdot \widetilde{R} = L_{\widetilde{C}} Q(\widetilde{g}, \widetilde{C}) \tag{42}$$

on the set $U_{\widetilde{C}}$. We shall prove

Theorem 11. *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If one of the conditions* (41) *or* (42) *holds on N, then on M we have:*

(a)

$$C \cdot C + \tfrac{1}{m-2} Q(T, C) + \tfrac{1}{m-2} Q(T, g, C) = \big(\tfrac{P}{(m-1)(m-2)} + L_{\widetilde{C}}\big) Q(g, C), \tag{43}$$

where g is the metric tensor on M induced from the one on N, $T = S - \frac{m-2}{n-2}\bar{S}$ (cf. Lemma 3).

(b)
$$U_{fy}C_{abcd} = 0,$$
where
$$U_{fy} = A_{fy} - \tfrac{1}{n-2}R_{rs}B^r_f N^s_y.$$
Hence, if $C(x) \neq 0$, $x \in M$, then on some neighbourhood of x we obtain
$$\nabla_a H^k = -HB^k_a + \tfrac{1}{n-2}\sum_y (e_y R_{rs}B^r_f N^s_y)N^k_y,$$
$$\nabla_a H = \tfrac{2}{n-2}R_{rs}B^r_a H^s.$$

The condition $\widetilde{C}\cdot\widetilde{C} = L_{\widetilde{C}}Q(\widetilde{g},\widetilde{C})$ on an ambient space is not preserved on a totally umbilical submanifold. However, we have

Theorem 12. *Let $\widetilde{J}$ be a symmetric $(0,2)$ tensor on a manifold N and*
$$\widetilde{C}\cdot\widetilde{C} + \tfrac{1}{n-2}Q(\widetilde{J},\widetilde{C}) + \tfrac{1}{n-2}Q(\widetilde{J},\widetilde{g},C) = L_{\widetilde{C}}Q(\widetilde{g},\widetilde{C}).$$
Then, on a totally umbilical submanifold M,
$$C\cdot C + \tfrac{1}{m-2}Q(T+\bar{J},C) + \tfrac{1}{m-2}Q(T+\bar{J},g,C) = \Big(\tfrac{P}{(m-1)(m-2)} + L_{\widetilde{C}}\Big)Q(g,C).$$

4.2. Some equivalences. In this section and in Sections 6.3–6.5, the symbols AR, UC, AU, AC, UR, AF and UF represent tensors obtained by applying various bilinear pairings to the factors A, C, U, F and R. To simplify the notation, components of such tensors will be written without parentheses (so that, for instance, AR_{ydef} stands for $(AR)_{ydef}$ etc.).

Theorem 13. *Let M be a totally umbilical submanifold isometrically immersed in a manifold N and suppose that on N one of the conditions* (33), (38), (41), (42) *is satisfied. Let T_{dy} be a tensor of mixed type on M given by*
$$T_{ey} = A_{ex} = \partial_e H_x + \sum_y e_y L_{eyx}H_y \qquad \textit{in case } (33),$$
$$T_{ey} = A_{ey} + a_r B^r_e H_y \qquad \textit{in case } (38),$$
$$T_{ex} = U_{ex} = A_{ex} - \tfrac{1}{n-2}R_{pu}N^p_x B^u_e \qquad \textit{in case } (41) \textit{ and } (42)$$
and put
$$AR_{ydef} = -AR_{ydfe} = \sum_x e_x T_{dx}R_{qrvw}N^q_x N^r_y B^{vw}_{ef}.$$
Then
$$T_{ey}C_{abcd} = 0$$

if and only if

$$AR_{ydef} + \tfrac{1}{m-1}(g_{de}AR_y{}^a{}_{fa} - g_{df}AR_y{}^a{}_{ea}AR_y{}^a{}_{ea}) = 0. \tag{44}$$

A version of the above theorem can be stated as follows. Define

$$AR_{def} = -AR_{dfe} = \sum_x e_x T_{dx} R_{qrvw} N_x^q H^r B_{ef}^{vw}$$

with T_{dx} as above.

THEOREM 14. *Let M be a totally umbilical submanifold isometrically immersed in a manifold N and suppose that on N one of the conditions* (33), (38), (41), (42) *is satisfied. Then on M the equality*

$$AR_{cef} + \tfrac{1}{m-1}(g_{ce}AR^a{}_{fa} - g_{cf}AR^a{}_{ea}) = 0 \tag{45}$$

is equivalent to

$$\nabla H \otimes C = 0 \quad \textit{in case } (33), \tag{46}$$

$$(\nabla_{F_*X}H + 2Ha(F_*X)) \otimes C = 0 \quad \textit{in case } (38), \tag{47}$$

and

$$M \otimes C = 0 \quad \textit{in cases } (41) \textit{ and } (42), \tag{48}$$

where $M = \frac{1}{2}\nabla H - \frac{1}{n-2}\nabla F \lrcorner (h \lrcorner S)$, $\lrcorner$ *denotes the interior product and S is the Ricci tensor of the submanifold* (*cf.* (25)).

In the next section we shall give an application of the last theorem.

From the definitions of AR_{ycef} and AR_{cef} we get immediately

REMARK 15. (44) and (45) hold on an arbitrary hypersurface of N.

A manifold $\widetilde{N}$ is said to be *of quasi-constant curvature* if $\widetilde{R} = \frac{a}{2}\widetilde{g} \wedge \widetilde{g} + b\widetilde{g} \wedge (v \otimes v)$ for a 1-form v, a, b being functions on $\widetilde{N}$. Then $\widetilde{C} = 0$ and $\widetilde{R} \cdot \widetilde{R} = (a + bv \lrcorner v)Q(\widetilde{g}, \widetilde{R})$.

REMARK 16. $AR_{ycef} = 0$ and, consequently, $AR_{def} = 0$ on every submanifold in a manifold of quasi-constant curvature.

4.3. Quasi-recurrent type conditions. For tensors a of type $(0,1)$ and R of type $(0,4)$ we put

$$\begin{aligned} D(a,R)(W)(X,Y,U,V) &= 2a(W)R(X,Y,U,V) + a(X)R(W,Y,U,V) \\ &+ a(Y)R(X,W,U,V) + a(U)R(X,Y,W,V) + a(V)R(X,Y,U,W). \end{aligned}$$

Let $\widetilde{G}$ be the $(0,4)$ tensor field given by

$$\widetilde{G} = \tfrac{1}{2}\widetilde{g} \wedge \widetilde{g}.$$

THEOREM 17 ([13]). *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If the condition*

$$\widetilde{\nabla}_W \widetilde{R} = D(a, \widetilde{R})(W) + D(b, \widetilde{G})(W) \tag{49}$$

holds on N for any $W \in \mathfrak{X}(N)$, then on M we have:

(a) $\nabla_{F_*W} K = D(a,K)(F_*W) + D(c,G)(F_*W)$, *where* $c(F_*W) = b(F_*W) - Ha(F_*W) + \frac{1}{2}\nabla_{F_*W} H$, $W \in \mathfrak{X}(M)$.

(b) $(H - a(h))C = 0$.

A manifold whose curvature tensor satisfies (49) is called *extended quasi-recurrent* ([22], [20], [21]).

For a $(1,3)$ tensor field $\mathcal{T}$, its associated T and $(0,1)$ tensor field A let

$$\begin{aligned} M(A,T)(W)(X,Y,U,V) &\\ = \widetilde{g}(W,X)(T\cdot A)(Y,U,V) &+ \widetilde{g}(W,Y)(T\cdot A)(X,V,U)\\ + \widetilde{g}(W,U)(T\cdot A)(V,X,Y) &+ \widetilde{g}(W,V)(T\cdot A)(U,Y,X),\end{aligned}$$

where

$$(T\cdot A)(X,Y,Z) = -A(\mathcal{T}(X,Y)Z).$$

Observe that

$$D(a,T)(\partial_m) - M(\widetilde{g},T)(\partial_m) = 2a(\partial_m)\otimes T + (\alpha\wedge\partial_m)\cdot T,$$

where $a(X) = \widetilde{g}(\alpha, X)$ for all vector fields X.

Theorem 18 ([12, Theorems 5 and 6]). *Let M be a totally umbilical submanifold immersed in a manifold N and let F be the immersion. If the condition*

$$\widetilde{\nabla}_X \widetilde{C} = v(X)\widetilde{C} + D(a,\widetilde{C})(X) \tag{50}$$

holds on N for any $X \in \mathfrak{X}(N)$, then on M we have:

(a) $\nabla_{F_*X} C = v(F_*X)C + D(a,C)(F_*X) - M(a,C)(F_*X)$, $X \in \mathfrak{X}(M)$.

(b) $(H - a(h))C = 0$.

We shall prove

Theorem 19. *The statements of Theorem 18 remain true if (50) is replaced with*

$$\widetilde{\nabla}_X \widetilde{C} = v(X)\widetilde{C} + D(a,\widetilde{C})(X) - M(a,\widetilde{C})(X).$$

5. Examples

5.1. A manifold satisfying (30). Let a manifold $N = \mathbb{R}^n$, $n \geq 4$, be endowed with the metric $\widetilde{g}$ given by

$$g_{ij}dx^i dx^j = Q(dx^1)^2 + k_{ab}dx^a dx^b + 2dx^1 dx^n, \tag{51}$$

where

$$Q = [Ak_{ab} + Bc_{ab} + Dd_{ab}]x^a x^b,$$

A, B, D being functions depending on x^1 only, and $k_{ab} = k^{ab} = e_a\delta_{ab}$, $|e_a| = 1$, $c_{ab} = e_a\delta_{ab}$ for $a, b \neq n-1$, $c_{n-1,n-1} = e_{n-1}(3-n)$, $d_{ab} = e_a\delta_{ab}$ for $a, b \neq 2$, $d_{22} = e_2(3-n)$.

The only components of the Riemann curvature tensor $\widetilde{R}$ and the Weyl conformal curvature tensor $\widetilde{C}$ which may not vanish are

$$R_{1221} = \tfrac{1}{2}e_2[A+B+(3-n)D], \qquad R_{1ff1} = \tfrac{1}{2}e_f[A+B+D],$$
$$R_{1,n-1,n-1,1} = \tfrac{1}{2}e_{n-1}[A+(3-n)B+D],$$
$$C_{1ff1} = \tfrac{1}{2}e_f[B+D], \qquad f = 3,\dots,n-2, \text{ if } n \neq 4, \tag{a}$$
$$C_{1221} = \tfrac{1}{2}e_2[B+(3-n)D], \tag{b}$$
$$C_{1,n-1,n-1,1} = \tfrac{1}{2}e_{n-1}[(3-n)B+D]. \tag{c}$$

Moreover, the only components of their covariant derivatives of an arbitrary order k which may not vanish are those related to $\nabla_1^k\widetilde{R} = \partial_1^k\widetilde{R}$, $\nabla_1^k\widetilde{C} = \partial_1^k\widetilde{C}$. For $n = 4$ the tensor field $\widetilde{C}$ is recurrent. For more detailed computations see [10].

We shall say that the Weyl conformal curvature tensor $\widetilde{C}$ of a manifold N satisfies (30) *in an essential* way if it satisfies (30) and neither $\widetilde{C}$ is recurrent nor $\widetilde{R}$ satisfies (28).

THEOREM 20. *For each $n > 4$ there exists a non-conformally flat manifold N, $\dim N = n$, with the following properties:*

(i) *The Weyl conformal curvature tensor $\widetilde{C}$ satisfies* (30) *in an essential way.*
(ii) *The scalar curvature of N vanishes.*
(iii) *The Ricci tensor is recurrent.*
(iv) *N is semi-symmetric, hence conformally semi-symmetric.*

Moreover, for each $m = 4, 5, \dots, n-1$, there exists a manifold M, $\dim M = m$, isometrically immersed in N as a totally umbilical submanifold having the above described properties of the ambient space.

Proof. Consider the manifold $(\mathbb{R}^n, \widetilde{g})$, with metric $\widetilde{g}$ given by (51) and suppose $\widetilde{C} \neq 0$. Applying (30) to the pairs of components (a)–(b) , (b)–(c), (c)–(a) respectively, we get

$$[B+D]' = M_1[B+(3-n)D]',$$
$$[B+(3-n)D]' = M_2[(3-n)B+D]',$$
$$[B+D]' = M_3[(3-n)B+D]',$$

M_j being constants. For $n > 4$ straightforward computations show that $\widetilde{C}$

satisfies (30) if and only if either

$$D' = \frac{M_1 - 1}{(n-3)M_1 + 1} B', \qquad M_2 = \frac{-1}{(n-4)M_1 + 1}, \qquad M_3 = M_1 M_2,$$
$$(n-4)M_1 \neq -1, \qquad (n-3)M_1 \neq -1, \qquad M_1 = \text{const},$$

or

$$B' = 0, \quad (n-3)M_1 = -1, \quad M_2 = 3 - n, \quad M_3 = 1, \quad D \text{ arbitrary},$$

or

$$D' = 0, \quad M_1 = 1, \quad (n-3)M_2 = (n-3)M_3 = -1, \quad B \text{ arbitrary}.$$

For the case $n = 4$, the condition holds if and only if

$$D' = \frac{M_1 - 1}{M_1 + 1} B', \quad M_1 \neq -1, \quad M_2 = -1, \quad M_1 + M_3 = 0, \quad M_1 = \text{const}.$$

$\widetilde{R}$ does not satisfy (28) if and only if one of the following inequalities hold:

$$A''B' \neq A'B'', \qquad A''D' \neq A'D'',$$
$$[A''(B' - D') - A'(B'' - D'')] - (n-4)(B'D'' - B''D') \neq 0.$$

$\widetilde{C}$ is not recurrent if and only if $n > 4$ and $D'B - DB' \neq 0$. Letting for example $B = e^{x^1}$, $D = 1$, $A = e^{2x^1}$ we get a manifold with nowhere vanishing tensor $\widetilde{C}$ satisfying (30) in an essential way.

Finally, (ii)–(iv) are satisfied by [10, Lemma 1].

Let U be an open subset of $\mathbb{R}^m$, $m < n$, covered by the coordinate system $(y^1, \ldots, y^m)$, and consider the immersion of U in $N = \mathbb{R}^n$, $n > 4$, given $x^p = y^p$, $p,\ q = 1,\ 2,\ l+1, \ldots, n,\ l < n-1,\ x^a = C_a$, $a = 3, \ldots, l$, C_a being constant, $l = n - m + 2$. For the metric h on U induced from $\widetilde{g}$ on N, with components h_{pq}, we have $h_{pq} = g_{pq|U}$ and U is totally geodesic. This completes the proof. ∎

5.2. Manifolds satisfying (40)–(42). Let $N = V_{n-m} \times_F V_m$ be a warped product manifold with warping function F. Then on a neighbourhood of each point there exists a coordinate system in which the metric g has the form

$$g_{ij}dx^i dx^j = g_{\alpha\beta}dx^\alpha dx^\beta + F g_{ab} dx^a dx^b, \tag{52}$$
$$\partial_a g_{\alpha\beta} = \partial_a F = 0, \qquad \partial_\alpha g_{ab} = 0,$$

$i, j = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, n-m$, $a, b = n-m+1, \ldots, n$.

According to [16, Theorem 1], the immersion $x^\alpha = u^\alpha$, $x_a = C_a$, defines a totally geodesic submanifold in N, while that with $x^\alpha = C_\alpha$, $x^a = y^a$ defines a totally umbilical submanifold in N, C_j being constants. In particular, if $F = \text{const}$, both V_{n-m} and V_m are totally geodesic.

In [2] examples of pseudosymmetric ($m = n-1$) and non-pseudosymmetric ($m = n-2$) warped product manifolds with pseudosymmetric Weyl tensor are given. An example of a compact pseudosymmetric but non-semisymmetric ($R \cdot R \neq 0$) warped product $S^{n-m} \times_F S^m$, $m \leq n-2$, with pseudosymmetric Weyl tensor is presented in [6]. Further examples of warped product manifolds realizing pseudosymmetry type conditions are given in [9, Theorem 4.1] and [8, Example 5.1]. We shall prove

LEMMA 21. *Each warped product manifold M admits a totally umbilical submanifold such that the mean curvature vector $h = H^r \partial_r$ satisfies the condition*

$$R_{hijk} H^i B^j_c B^k_d = 0, \tag{53}$$

where $B^j_c = \partial x^j / \partial y^c$ and R_{hijk} are the components of the curvature tensor $\widetilde{R}$.

Proof. Let f be an immersion into the warped product manifold (52) given by $x^\alpha = C_\alpha$, $x^a = y^a$, C_α being constants. By the use of (9) and (10) we find $H^\alpha = -\frac{1}{2} g^{\alpha\beta} \partial_\beta F$, $H^a = 0$. On the other hand, the only components of the Riemann curvature tensor $\widetilde{R}$ which may not vanish are $R_{\alpha\beta\gamma\delta}$, $R_{\alpha bc\delta}$, R_{abcd} ([4]). ∎

By the above results and Theorem (14) we get immediately

PROPOSITION 22. *There exist totally umbilical submanifolds which are not totally geodesic realizing* (53)*. Consequently, there exist manifolds realizing the pseudosymmetry type condition* (40) *or* (41) *or* (42) *that admit totally umbilical submanifolds satisfying* (46) *or* (48) *or, respectively,* (48)*.*

6. Proofs

6.1. Proof of Theorem 8. From (16) and (26) we readily get

$$C_{rijk} N^r_x B^{ijk}_{bcd} = g_{bc} U_{dx} - g_{bd} U_{cx}, \tag{54}$$

where

$$U_{fy} = A_{fy} - \tfrac{1}{n-2} R_{rs} B^r_f N^s_y. \tag{55}$$

Transvecting (2) with $B^{hijkl}_{abcde} N^m_y$, we get

$$Q(g, \widetilde{C})_{hijklm} B^{hijkl}_{abcde} N^m_y = U_{ay} G_{ebcd} + U_{by} G_{aecd} + U_{cy} G_{abed} + U_{dy} G_{abce}. \tag{56}$$

Put

$$P_{ay} = a_{rs} B^r_a N^s_y,$$

$$F_{adey} = F_{daey} = \sum_x e_x (U_{ax} R_{xdey} + U_{dx} R_{xaey}), \quad R_{xdey} = R_{hijk} N^h_x B^{ij}_{de} N^k_y. \tag{57}$$

Transvecting (33) with $B^{hijkl}_{abcde}N^m_y$, we obtain

$$
\begin{aligned}
(58)\qquad & g_{ae}A^g_y\overline{C}_{gbcd} + g_{be}A^g_y\overline{C}_{agcd} + g_{ce}A^g_y\overline{C}_{abgd} + g_{de}A^g_y\overline{C}_{abcg} \\
&\quad - A_{ay}\overline{C}_{ebcd} - A_{by}\overline{C}_{aecd} - A_{cy}\overline{C}_{abed} - A_{dy}\overline{C}_{abce} \\
&\quad + g_{bc}F_{adey} - g_{bd}F_{acey} + g_{ad}F_{bcey} - g_{ac}F_{bdey} - P_{ey}\overline{C}_{abcd} \\
&\quad + \widetilde{A}(U_{ay}G_{ebcd} + U_{by}G_{aecd} + U_{cy}G_{abed} + U_{dy}G_{abce}) = 0.
\end{aligned}
$$

Making use of Lemma 3 and contracting the resulting equation with g^{bc} we get

$$
\begin{aligned}
(59)\qquad & (m-2)F_{adey} + \big(g^{bc}F_{bcey} + \tfrac{P-T^g_g}{m-2}P_{ey}\big)g_{ad} + A^g_yT_{ga}g_{de} + A^g_yT_{gd}g_{ae} \\
&\quad - P_{ey}T_{ad} - A_{ay}T_{de} - A_{dy}T_{ae} \\
&\quad - \widetilde{A}[(m-2)(g_{ae}U_{dy} + g_{de}U_{ay}) + 2g_{ad}U_{ey})] = 0,
\end{aligned}
$$

whence, by contraction with g^{ad}, we find

$$
(60)\qquad g^{bc}F_{bcey} = 2\widetilde{A}U_{ey} + \tfrac{1}{m-2}\big(T^g_g - \tfrac{mP}{2(m-1)}\big)P_{ey}.
$$

Now, from (59) and (60), we get

$$
\begin{aligned}
F_{adey} &= \widetilde{A}(g_{ae}U_{dy} + g_{de}U_{ay}) \\
&\quad + \tfrac{1}{m-2}\big[\big(T_{ad} - \tfrac{P}{2(m-1)}g_{ad}\big)P_{ey} - A^g_yT_{ga}g_{de} - A^g_yT_{gd}g_{ae} + A_{ay}T_{de} + A_{dy}T_{ae}\big].
\end{aligned}
$$

Applying the last equation to (58), in virtue of Lemma 3, we readily find (35) and (36). This completes the proof of the first part of the theorem. The final statements are consequences of (37), (21) and (18). ∎

6.2. Proof of Theorem 9. To prove Theorem 9 we shall need the following

LEMMA 23. *Let M, $\dim M > 2$, be a totally umbilical submanifold immersed isometrically in a manifold N. Then at any point $x \in M$ there exist tensors $S_{ade} = S_{dae}$ and $V_{xad} = V_{xda}$ such that the following decompositions hold:*

$$
(61)\qquad
\begin{aligned}
C_{hijk,l}B^{hijkl}_{abcde} &= K_{abcd;e} - g_{bc}S_{ade} + g_{bd}S_{ace} - g_{ad}S_{bce} + g_{ac}S_{bde}, \\
C_{hijk,l}B^{hijk}_{abcd}N^l_x &= 2H_xK_{abcd} - g_{bc}V_{xad} + g_{bd}V_{xac} - g_{ad}V_{xbc} + g_{ac}V_{xbd}.
\end{aligned}
$$

Proof. If we put

$$
S_{ade} = \tfrac{1}{2}(g_{ad}H_e + g_{ed}H_a + g_{ae}H_d) + \tfrac{1}{n-2}\big(R_{hk,l}B^{hkl}_{ade} - \tfrac{1}{2(n-1)}r_{,l}B^l_eg_{ad}\big),
$$

then the first formula results from (20) and (26). To prove the second one differentiate covariantly (16) with respect to ∂_e. By the use of (17), (16),

(14) and (15) we obtain

$$R_{hijk,l}N^h_x B^{ijkl}_{bcde} = H_x K_{ebcd} - HH_x G_{ebcd} - g_{bc}L_{xed} + g_{bd}L_{xec} + g_{be}(T_{xcd} - T_{xdc}) + g_{ce}T_{xbd} - g_{de}T_{xbc} + g_{bc}A_{dx;e} - g_{bd}A_{cx;e},$$

where

$$L_{xed} = \sum_y e_y L_{exy} A_{dy}, \quad T_{xed} = R_{hijk} N^h_x B^{ij}_{ed} H^k.$$

Transvecting $R_{hijk,l} = R_{lkhi,j} + R_{ljih,k}$ with $B^{hijk}_{abcd}N^l_x$, making use of the last identity and (26), we get (61), where

$$V_{xad} = L_{xda} + T_{xda} - A_{ax;d} + S_{adx} + HH_x g_{ad}$$

and

$$S_{adx} = \tfrac{1}{n-2}\big(R_{ij,k} - \tfrac{1}{2(n-1)}g_{ij}r_{,k}\big)B^{ij}_{ad}N^k_x.$$

Summing (61) cyclically over (b,c,d), adding the resulting equations and contracting the sum with g^{ad}, we readily get $V_{xad} = V_{xda}$. ∎

Proof of Theorem 9. In local coordinates, (38) takes the form

$$C_{rijk}R^r_{hlm} + C_{hrjk}R^r_{ilm} + C_{hirk}R^r_{jlm} + C_{hijr}R^r_{klm} = a_m C_{hijk,l} - a_l C_{hijk,m} + \widetilde{A}[g_{hl}C_{mijk} - g_{hm}C_{lijk} + g_{il}C_{hmjk} - g_{im}C_{hljk} + g_{jl}C_{himk} - g_{jm}C_{hilk} + g_{kl}C_{hijm} - g_{km}C_{hijl}].$$

By transvecting with $B^{hijkl}_{abcde}N^m_y$ and the use of (13), (16), (54)–(57), we obtain

$$\begin{aligned} &g_{ae}A^g_y\overline{C}_{gbcd} + g_{be}A^g_y\overline{C}_{agcd} + g_{ce}A^g_y\overline{C}_{abgd} + g_{de}A^g_y\overline{C}_{abcg} \\ &\quad - A_{ay}\overline{C}_{ebcd} - A_{by}\overline{C}_{aecd} - A_{cy}\overline{C}_{abed} - A_{dy}\overline{C}_{abce} + g_{bc}F_{adey} \\ &\quad - g_{bd}F_{acey} + g_{ad}F_{bcey} - g_{ac}F_{bdey} - a_y C_{hijk,l}B^{hijkl}_{abcde} + a_e C_{hijk,m}B^{hijk}_{abcd}N^m_y \\ &\quad - \widetilde{A}(U_{ay}G_{ebcd} + U_{by}G_{aecd} + U_{cy}G_{abed} + U_{dy}G_{abce}) = 0, \end{aligned}$$

where $a_e = a_r B^r_e$, $a_y = a_r N^r_y$. Hence, making use of Lemmas 3 and 23, we can follow step by step the proof of Theorem 7 ([7, Lemma 2]) to obtain Theorem 9(b). Similarly, by transvecting with B^{hijklm}_{abcdef}, following the proof of Theorem 8, we get Theorem 9(a). ∎

6.3. Proof of Theorem 11. First we sketch the proof of (a) in the case (41). Let U be the $(0,2)$ tensor with the components

$$U_{df} = \sum_x e_x U_{dx} U_{fx},$$

where U_{dx} are defined by (55). By the definition of the operator "$\cdot$" we have

(62) $$(\widetilde{C}\cdot\widetilde{C})_{hijklm} = (C_{rijk}C_{shlm} + C_{hrjk}C_{silm} + C_{hirm}C_{sjlm} + C_{hijr}C_{sklm})g^{rs}.$$

Transvecting (41) with B^{hijklm}_{abcdef}, by the use of (62), (2), (13), (54) and (55), we get

$$\overline{C}\cdot\overline{C}+Q(U,G)=L_{\widetilde{C}}Q(g,\overline{C})$$

on M, whence, from Lemma 3, we obtain

$$\begin{aligned}(63)\qquad & C\cdot C+\tfrac{1}{m-2}Q(Z,G)+\tfrac{1}{m-2}Q(T,C)+\tfrac{1}{m-2}Q(T,g,C)\\ &\quad+\tfrac{1}{m-2}g\wedge(C\cdot T)=\big(L_{\widetilde{C}}+\tfrac{P}{(m-1)(m-2)}\big)Q(g,C),\end{aligned}$$

where Z is the tensor of type $(0,2)$ with the components

$$Z_{de}=(m-2)U_{ed}-\tfrac{1}{m-2}T_{ad}T^a_e+\big(L_{\widetilde{C}}+\tfrac{P}{(m-1)(m-2)}\big)T_{de}.$$

Contraction of (63) with g^{bc} yields $C\cdot T=g\vee Z$, which, applied to (63), in virtue of (7), completes the proof of (a).

To prove (b), by transvection of (41) with $B^{hijkl}_{abcde}N^m_y$ and the use of (62), (13), (16), (55) and (56) we get

$$\begin{aligned}& g_{ae}U^g_y\overline{C}_{gbcd}+g_{be}U^g_y\overline{C}_{agcd}+g_{ce}U^g_y\overline{C}_{abgd}+g_{de}U^g_y\overline{C}_{abcg}\\ &\quad-U_{ay}\overline{C}_{ebcd}-U_{by}\overline{C}_{aecd}-U_{cy}\overline{C}_{abed}-U_{dy}\overline{C}_{abce}\\ &\quad+g_{bc}UC_{adey}-g_{bd}UC_{acey}+g_{ad}UC_{bcey}-g_{ac}UC_{bdey}\\ &\quad-L_{\widetilde{C}}(U_{ay}G_{ebcd}+U_{by}G_{aecd}+U_{cy}G_{abed}+U_{dy}G_{abce})=0,\end{aligned}$$

where

$$UC_{adey}=UC_{daey}=\sum_x e_x(U_{ax}\overline{C}_{xdey}+U_{dx}\overline{C}_{xaey}),\qquad \overline{C}_{xdey}=C_{hijk}N^h_xB^{ij}_{de}N^k_y.$$

Now, the proof follows that of Theorem 8.

In the case (42) the proof of part (a) is quite similar. By transvecting (42) with B^{hijklm}_{abcdef} and the use of (1), (2), (13), (15) and (16) we get

$$\overline{C}\cdot K-Q(AU,G)=L_{\widetilde{C}}Q(g,\overline{C}),$$

where

$$AU_{ef}=\sum_x A_{ax}U_{ex},$$

with U_{dx} defined by (55), K being the Riemann curvature tensor of M. Since

$$\overline{C}=C+\tfrac{1}{m-2}g\wedge T-\tfrac{1}{(m-1)(m-2)}G$$

(cf. Lemma 3), using (5), (4), (7) and (6), we obtain

$$\begin{aligned}(64)\qquad & C\cdot K+\tfrac{1}{m-2}Q(T,G)+\tfrac{1}{m-2}Q(T,K)+\tfrac{1}{m-2}Q(T,g,K)+\tfrac{1}{m-2}Q(T,G)\\ &\quad=L_{\widetilde{C}}Q(g,C)+\tfrac{P}{(m-1)(m-2)}Q(g,K)+Q(AU,G).\end{aligned}$$

Contracting the last equation with g^{bc} we find

$$(65)\quad (C\cdot\underline{K})_{adef}=-2(AU_{ef}-AU_{fe})g_{ad}-(g\vee S)_{adef}+\tfrac{1}{m-2}(\underline{K}\vee T)_{adef},$$

where we have put

$$S = (m-2)AU - \tfrac{P}{(m-1)(m-2)}\underline{K} - L_{\widetilde{C}}T + \tfrac{1}{m-2}\underline{K}\lrcorner T.$$

Contracting (65) with g^{ad} we obtain

$$AU_{ef} = AU_{fe}. \tag{66}$$

Substituting in (64)

$$K = C + \tfrac{1}{m-2}g\wedge\underline{K} - \tfrac{1}{(m-1)(m-2)}\underline{\underline{K}}G$$

and applying (65), (66), (8), (6) and Lemma 2, we obtain (43).

On the other hand, by transvecting (42) with $B^{hijkl}_{abcde}N^m_y$ and the use of (1), (56), (13), (15) and (16) we get

$$\begin{aligned}
&g_{ae}U^g_yK_{gbcd} + g_{be}U^g_yK_{agcd} + g_{ce}U^g_yK_{abgd} + g_{de}U^g_yK_{abcg}\\
&\quad - U_{ay}K_{ebcd} - U_{by}K_{aecd} - U_{cy}K_{abed} - U_{dy}K_{abce}\\
&\quad + g_{bc}AC_{adey} - g_{bd}AC_{acey} + g_{ad}AC_{bcey} - g_{ac}AC_{bdey}\\
&\quad - L_{\widetilde{C}}(U_{ay}G_{ebcd} + U_{by}G_{aecd} + U_{cy}G_{abed} + U_{dy}G_{abce}) = 0,
\end{aligned} \tag{67}$$

where

$$AC_{adey} = AC_{daey} = \sum_x e_x(A_{ax}\bar{C}_{xdey} + A_{dx}\bar{C}_{xaey}), \qquad \bar{C}_{xdey} = C_{hijk}N^h_xB^{ij}_{de}N^k_y.$$

Contracting (67) with g^{bc} and $g^{bc}g^{ad}$ we get, respectively,

$$\begin{gathered}
(m-2)AC_{adey} + g_{ad}AC_{bcey}g^{bc} + (g_{ae}K^g_d + g_{de}K^g_a)U_{gy} - 2L_{\widetilde{C}}g_{ad}U_{ey}\\
-[(m-2)L_{\widetilde{C}}g_{ae} + K_{ae}]U_{dy} - [(m-2)L_{\widetilde{C}}g_{de} + K_{de}]U_{ay} = 0,\\
AC_{bcey}g^{bc} = 2L_{\widetilde{C}}U_{ey},
\end{gathered}$$

whence

$$\begin{aligned}
AC_{adey} = \tfrac{1}{m-2}[&(m-2)L_{\widetilde{C}}(g_{ae}U_{dy} + g_{de}U_{ay})\\
&+ K_{de}U_{ay} + K_{ae}U_{dy} - (g_{ae}K^g_d + g_{de}K^g_a)U_{gy}].
\end{aligned}$$

Applying the last identity to (67) we easily find

$$\begin{aligned}
&U_{ay}C_{ebcd} + U_{by}C_{aecd} + U_{cy}C_{abed} + U_{dy}C_{abce}\\
&\quad - g_{ae}U^g_yC_{gbcd} - g_{be}U^g_yC_{agcd} - g_{ce}U^g_yC_{abgd} - g_{de}U^g_yC_{abcg} = 0,
\end{aligned}$$

which, together with Lemma 4, completes the proof. ∎

6.4. Proof of Theorem 14. Except for condition (42), the proofs of equivalences in Theorem 14 are technically less complicated than the ones in Theorem 13 and are quite similar. Therefore, we prove Theorem 14 first and omit the proofs in the case of Theorem 13, except for (42).

6.4.1. *Equivalence of (45) and (46).* We begin with two identities useful throughout this section. Applying the Ricci identity to $C_{rstu,[vw]}$ and transvecting with $H^r B^{stuvw}_{bcdef}$, by the use of (13), (16), (19), (23), (54) and (55) we get

$$\begin{aligned}(68)\quad & C_{rstu,[vw]}H^r B^{stuvw}_{bcdef} \\ &= \tfrac{1}{2}H_e\overline{C}_{fbcd} - \tfrac{1}{2}H_f\overline{C}_{ebcd} + g_{bc}M_aK^a_{def} - g_{bd}M_aK^a_{cef} \\ &\quad + g_{bc}UR_{def} - g_{bd}UR_{cef} - g_{be}AR_{fcd} + g_{bf}AR_{ecd} \\ &\quad - g_{ce}AF_{fbd} + g_{cf}AF_{ebd} + g_{de}AF_{fbc} - g_{df}AF_{ebc} \\ &\quad - H[M_e(g_{bd}g_{fc} - g_{bc}g_{fd}) - M_f(g_{bd}g_{ec} - g_{bc}g_{ed})],\end{aligned}$$

where

$$\begin{aligned} UR_{def} &= -UR_{dfe} = \sum_x e_xU_{dx}R_{qrvw}N^q_xH^rB^{vw}_{ef}, \\ (69)\qquad AR_{def} &= -AR_{dfe} = \sum_x e_xA_{dx}R_{qrvw}N^q_xH^rB^{vw}_{ef}, \\ AF_{fbd} &= \sum_x e_xA_{fx}F_{bdx}, \quad F_{bdx} = C_{rstp}H^rB^{st}_{bd}N^p_x.\end{aligned}$$

By the use of the first Bianchi identity we readily obtain

$$(70)\qquad AR_{edf} = AF_{edf} - AF_{efd}.$$

Transvecting $Q(g,\widetilde{C})_{rstuvw}$ with $H^rB^{stuvw}_{bcdef}$, in virtue of (11) and (23), we find

$$(71)\quad Q(g,\widetilde{C})_{rstuvw}H^rB^{stuvw}_{bcdef} = M_e(g_{bd}g_{fc} - g_{bc}g_{fd}) - M_f(g_{bd}g_{ec} - g_{bc}g_{ed}).$$

LEMMA 24. *Let*

$$Z_{cbf} = \tfrac{1}{m-2}\big[AR_{cbf} + AR_{fbc} + g_{cb}AR_f + g_{bf}AR_c - \tfrac{2}{m-1}g_{cf}AR_b\big],$$
$$AR_f = AR^a{}_{fa} = AR_{afb}g^{ab}.$$

Under the assumptions of Theorem 8, the relation

$$\begin{aligned}(72)\qquad \tfrac{1}{2}H_bC_{cdef} = &- g_{bc}AR_{def} + g_{bd}AR_{cef} - g_{be}AR_{fcd} + g_{bf}AR_{ecd} \\ &+ g_{ce}Z_{dbf} - g_{cf}Z_{dbe} + g_{df}Z_{cbe} - g_{de}Z_{cbf}\end{aligned}$$

holds on M.

Proof. Transvecting (33) with $H^r B^{stuvw}_{bcdef}$, by the use of (68) and (71), we get

$$\begin{aligned}(73)\qquad & \tfrac{1}{2}H_e\overline{C}_{fbcd} - \tfrac{1}{2}H_f\overline{C}_{ebcd} + g_{bc}UR_{def} - g_{bd}UR_{cef} - g_{be}AR_{fcd}\\ &\quad + g_{bf}AR_{ecd} + g_{bc}M_aK^a_{def} - g_{bd}M_aK^a_{cef}\\ &\quad - g_{ce}AF_{fbd} + g_{cf}AF_{ebd} + g_{de}AF_{fbc} - g_{df}AF_{ebc}\\ &= (H+\widetilde{A})[M_e(g_{bd}g_{fc} - g_{bc}g_{fd}) - M_f(g_{bd}g_{ec} - g_{bc}g_{ed})]\\ &\quad + a_{ef}(g_{bc}M_d - g_{bd}M_c).\end{aligned}$$

Contracting (73) with g^{bc}, in virtue of Lemma 3, we get

$$\begin{aligned}&(m-1)M_aK^a_{dfe} = AF_{efd} - AF_{fed} - AR_{edf} + AR_{fde}\\ &\quad - (m-1)a_{ef}M_d + (m-1)UR_{def} - g_{df}Q_e + g_{de}Q_f - \tfrac{1}{2}H_fT_{de} + \tfrac{1}{2}H_eT_{df},\end{aligned}$$

where $Q_e = AF_e + \frac{P-T}{2(m-2)}H_e - (m-1)(H+\widetilde{A})M_e$, $AF_e = AF_{eab}g^{ab}$, $T = T_{ab}g^{ab}$. Substituting into (73) we have

$$\begin{aligned}(74)\qquad & \tfrac{1}{2}H_fC_{becd} - \tfrac{1}{2}H_eC_{bfcd} - g_{be}AR_{fcd} + g_{bf}AR_{ecd}\\ & - g_{df}\big(AF_{ebc} - \tfrac{1}{2(m-2)}H_eT_{bc}\big) + g_{de}\big(AF_{fbc} - \tfrac{1}{2(m-2)}H_fT_{bc}\big)\\ & + g_{cf}\big(AF_{ebd} - \tfrac{1}{2(m-2)}H_eT_{bd}\big) - g_{ce}\big(AF_{fbd} - \tfrac{1}{2(m-2)}H_fT_{bd}\big)\\ & + \tfrac{1}{m-1}g_{bd}\big[AF_{efc} - AF_{fec} - AR_{ecf} + AR_{fce} + \tfrac{1}{2(m-2)}(H_fT_{ce} - H_eT_{cf})\\ & - g_{cf}\big(AF_e - \tfrac{1}{2(m-2)}H_eT\big) + g_{ce}\big(AF_f - \tfrac{1}{2(m-2)}H_fT\big)\big]\\ & - \tfrac{1}{m-1}g_{bc}\big[AF_{efd} - AF_{fed} - AR_{edf} + AR_{fde} + \tfrac{1}{2(m-2)}(H_fT_{de} - H_eT_{df})\\ & - g_{df}\big(AF_e - \tfrac{1}{2(m-2)}H_eT\big) + g_{de}\big(AF_f - \tfrac{1}{2(m-2)}H_fT\big)\big] = 0.\end{aligned}$$

Notice that the term containing a_{ef} vanished.

Contracting the last equation with g^{cf} and g^{bf} we get two different equations involving H^aC_{adbe} and H^aC_{aecd} respectively. Making the necessary changes of indices and eliminating H^aC_{adbe} we obtain

$$\begin{aligned}(75)\qquad & -AF_{cde} + AF_{ced} - (m^2-3m+1)AF_{dce} - AF_{dec}\\ &\quad + (m-2)(AF_{ecd} - AF_{edc}) - 2AR_{cde} - (m-3)AR_{dce}\\ &\quad + (m^2-2m-1)AR_{ecd} + g_{ce}((m-2)AF_{da}{}^{a} - AF^{a}{}_{da} - AR^{a}{}_{ad})\\ &\quad + (m-1)(g_{cd}AR^{a}{}_{ae} - g_{ce}AF^{a}{}_{ad} + g_{de}AF^{a}{}_{ac} - g_{de}AF^{a}{}_{ca})\\ &\quad + \tfrac{1}{2}H^aT_{ad}g_{ce} - \tfrac{1}{2}TH_dg_{ce} + \tfrac{m-1}{2}H_dT_{ce} = 0,\end{aligned}$$

whence, alternating in (c,e), we find

$$\begin{aligned}&\tfrac{m-3}{m-1}[AF_{cde}-AF_{ced}+AF_{ecd}-AF_{edc}+m(AF_{dec}-AF_{dce})\\&\quad+(m+1)(AR_{cde}+AR_{ecd})-2AR_{dce}]\\&\quad+g_{de}(AF^a{}_{ac}-AF^a{}_{ca}-AR^a{}_{ac})-g_{dc}(AF^a{}_{ae}-AF^a{}_{ea}-AR^a{}_{ae})=0.\end{aligned}$$

Applying (70) in the last equality we easily obtain

$$AR_{def}+AR_{fde}+AR_{efd}=0. \tag{76}$$

Moreover, applying (70) and (76) to (75) we find

$$\begin{aligned}AF_{dce}=\tfrac{-1}{m-2}\big[&AR_{cde}-(m-1)AR_{ecd}+g_{cd}AR_e+\tfrac{m-2}{m-1}(AF^a{}_{ad}-AF_{da}{}^a)g_{ce}\\&-\tfrac{2}{m-1}g_{ce}AR_d+g_{de}AR_c-\tfrac{1}{2(m-1)}(H^aT_{ad}-TH_d)g_{ce}-\tfrac12 H_dT_{ce}\big],\end{aligned}$$

which, together with (74), yields

$$\begin{aligned}&\tfrac{m-2}{2}(H_eC_{bfcd}-H_fC_{becd})\\&\quad=g_{bc}\big[2AR_{def}+AR_{edf}+AR_{fed}+\tfrac{m+3}{m-1}(g_{de}AR_f-g_{df}AR_e)\big]\\&\qquad-g_{bd}\big[2AR_{cef}+AR_{ecf}+AR_{fec}+\tfrac{m+3}{m-1}(g_{ce}AR_f-g_{cf}AR_e)\big]\\&\qquad+g_{de}[AR_{bcf}+(m-1)AR_{cbf}-g_{cf}AR_b]\\&\qquad-g_{ce}[AR_{bdf}+(m-1)AR_{dbf}-g_{df}AR_b]\\&\qquad+g_{cf}[AR_{bde}+(m-1)AR_{dbe}-g_{de}AR_b]\\&\qquad-g_{df}[AR_{bce}+(m-1)AR_{cbe}-g_{ce}AR_b]\\&\qquad+g_{bf}[(m-2)AR_{ecd}+g_{ec}AR_d-g_{ed}AR_c]\\&\qquad-g_{be}[(m-2)AR_{fcd}+g_{fc}AR_d-g_{fd}AR_c],\end{aligned} \tag{77}$$

where we have put $AR_e=AR^a{}_{ea}$. Summing (77) cyclically in (b,e,f), adding the resulting equations and subtracting (77) we get (72). ∎

We are going back to the proof of Theorem 14.

Proof. Suppose $\nabla H\otimes C=0$ on M. Contracting (72) with g^{bd} and using (76), we easily get $AR_{cef}+\frac{1}{m-1}(g_{ce}AR_f-g_{cf}AR_e)=0$. Conversely, the last equality applied to (72) gives $\nabla H\otimes C=0$. ∎

6.4.2. *Equivalence of (45) and (47).* The proof is quite similar to the last one.

6.4.3. *Equivalence of (45) and (48).* To prove it we shall need the following

LEMMA 25. *Let*

$$\begin{aligned}&Z_{cbf}=\tfrac{1}{m-2}\big[UC_{cbf}+UC_{fbc}+g_{cb}UC_f+g_{bf}UC_c-\tfrac{2}{m-1}g_{cf}UC_b\big],\\&UC_f=UC^a{}_{fa}=UC_{afb}g^{ab}.\end{aligned}$$

Under the assumptions of Theorem 11, *the relation*

$$M_b C_{cdef} = -g_{bc}UC_{def} + g_{bd}UC_{cef} - g_{be}UC_{fcd} + g_{bf}UC_{ecd} + g_{ce}Z_{dbf} - g_{cf}Z_{dbe} + g_{df}Z_{cbe} - g_{de}Z_{cbf} \tag{78}$$

holds on M.

Proof. Put

$$UC_{def} = -UC_{dfe} = \sum_x e_x U_{dx} C_{qrvw} N_x^q H^r B_{ef}^{vw},$$
$$UF_{def} = \sum_x e_x U_{dx} F_{efx} = \sum_x e_x U_{dx} C_{qrvw} H^q B_{ef}^{rv} N_x^w.$$

Transvecting (41) with $H^r B_{bcdef}^{ijklm}$, by the use of (13), (69), (71) and (25), we obtain

$$\begin{aligned} &M_e \overline{C}_{fbcd} - M_f \overline{C}_{ebcd} \\ &\quad + g_{bc}UC_{def} - g_{bd}UC_{cef} - g_{be}UC_{fcd} + g_{bf}UC_{ecd} + g_{bc}M_a\overline{C}^a_{def} \\ &\quad - g_{bd}M_a\overline{C}^a_{cef} - g_{ce}UF_{fbd} + g_{cf}UF_{ebd} + g_{de}UF_{fbc} - g_{df}UF_{ebc} \\ &\qquad = L_{\widetilde{C}}[M_e(g_{bd}g_{fc} - g_{bc}g_{fd}) - M_f(g_{bd}g_{ec} - g_{bc}g_{ed})]. \end{aligned}$$

Following, step by step, the proof of Lemma 24 we complete the proof of (78).

Consequently, by the last lemma, we get the equivalence of (45) and (48). ∎

6.5. Proof of Theorem 13 in the case (42). Let

$$\begin{aligned} UR_{ydef} &= -UR_{ydfe} = \sum_x e_x U_{dx} R_{qrvw} N_x^q N_y^r B_{ef}^{vw}, \\ AR_{ydef} &= -AR_{ydfe} = \sum_x e_x A_{dx} R_{qrvw} N_x^q N_y^r B_{ef}^{vw}, \\ AF_{yfbd} &= \sum_x e_x A_{fx} F_{ybdx}, \quad F_{ybdx} = C_{rstp} N_y^r B_{bd}^{st} N_x^p. \end{aligned} \tag{79}$$

By the Bianchi identity we have

$$UR_{ydef} = AF_{ydef} - AF_{ydfe}. \tag{80}$$

Transvecting (42) with $N_y^h B_{bcdef}^{ijklm}$ and using (79) we obtain

$$\begin{aligned} &-U_{fy}K_{ebcd} + U_{ey}K_{fbcd} + (H - L_{\widetilde{C}})(U_{fy}G_{ebcd} - U_{ey}G_{fbcd}) \\ &- g_{bd}A_y^g\overline{C}_{gcef} + g_{bc}A_y^g\overline{C}_{gdef} + g_{bc}AR_{ydef} - g_{bd}AR_{ycef} - g_{be}UR_{yfcd} \\ &\quad + g_{bf}UR_{yecd} - g_{ce}AF_{yfbd} + g_{cf}AF_{yebd} + g_{de}AF_{yfbc} - g_{df}AF_{yebc} = 0. \end{aligned} \tag{81}$$

Contracting (81) with g^{bc}, by Lemma 3, we obtain

$$
\begin{aligned}
(82)\qquad & AF_{yefd} - AF_{yfed} + (m-1)AR_{ydef} + g_{de}AF_{yf} - g_{df}AF_{ye} \\
&+ UR_{yfde} - UR_{yedf} + (m-1)(H - L_{\widetilde{C}})(g_{de}U_{fy} - g_{df}U_{ey}) \\
&+ K_{df}U_{ey} - K_{de}U_{fy} - (m-1)A^g_yC_{dgef} + \tfrac{m-1}{m-2}(T_{de}A_{fy} - T_{df}A_{ey}) \\
&+ g_{de}\left[\tfrac{m-1}{m-2}A^g_yT_{fg} - \tfrac{P}{m-2}A_{fy}\right] - g_{df}\left[\tfrac{m-1}{m-2}A^g_yT_{eg} - \tfrac{P}{m-2}A_{ey}\right] = 0,
\end{aligned}
$$

where $AF_{yf} = AF_{yfbc}g^{bc}$. Solving (82) for $A^g_yC_{dgef}$ and substituting into (81) we find

$$
\begin{aligned}
(83)\qquad & \tfrac{1}{m-1}[g_{bc}(AF_{yfed} - AF_{yefd}) - g_{bd}(AF_{yfec} - AF_{yefc})] \\
&- g_{ce}\big(AF_{yfbd} - \tfrac{1}{m-1}g_{bd}AF_{yf}\big) + g_{cf}\big(AF_{yebd} - \tfrac{1}{m-1}g_{bd}AF_{ye}\big) \\
&+ g_{de}\big(AF_{yfbc} - \tfrac{1}{m-1}g_{bc}AF_{yf}\big) - g_{df}\big(AF_{yebc} - \tfrac{1}{m-1}g_{bc}AF_{ye}\big) \\
&+ \tfrac{1}{m-2}[g_{bc}(UR_{yedf} - UR_{yfde}) - g_{bd}(UR_{yecf} - UR_{yfce})] \\
&+ \tfrac{1}{m-1}[g_{bc}(U_{fy}K_{ed} - U_{ey}K_{fd}) - g_{bd}(U_{fy}K_{ec} - U_{ey}K_{fc})] \\
&+ g_{bf}UR_{yecd} - g_{be}UR_{yfcd} + U_{ey}K_{fbcd} - U_{fy}K_{ebcd} = 0.
\end{aligned}
$$

In the next step we contract (83) with g^{cf} and g^{bf}. Changing in the second equation the indices (c, d, e) to (b, e, d), subtracting the resulting equation from the first one and applying the identity (80) we get

$$
\begin{aligned}
(84)\qquad & \tfrac{3}{m-1}(AF_{ybed} - AF_{ybde}) + \tfrac{m^2-m-3}{m-1}(AF_{ydeb} - AF_{ydbe}) + \tfrac{m^2-2m-2}{m-1}AF_{yebd} \\
&- \tfrac{m-4}{m-1}AF_{yedb} + g_{bd}\left[\tfrac{m}{m-1}AF_{yage} - \tfrac{2}{m-1}AF_{yaeg} - \tfrac{m-2}{m-1}AF_{yeag}\right]g^{ag} \\
&+ g_{be}(AF_{yadg} - AF_{yagd})g^{ag} + g_{de}(AF_{yabg} - AF_{yagb})g^{ag} \\
&+ \tfrac{K}{m-1}g_{bd}U_{ey} - K_{bd}U_{ey} - \tfrac{1}{m-1}g_{bd}K^g_eU_{gy} = 0.
\end{aligned}
$$

Alternating (84) in (b, d) we readily find

$$(AF_{ybde} - AF_{ybed}) + (AF_{ydeb} - AF_{ydbe}) + (AF_{yebd} - AF_{yedb}) = 0,$$

whence, by (80),

$$UR_{ybde} + UR_{ydeb} + UR_{yebd} = 0. \tag{85}$$

Contracting (84) with g^{bd} we find

$$mK^g_eU_{gy} = (m-2)(m+1)AF_{yaeg}g^{ag} - (m-2)(AF_{yage}g^{ag} + AF_{yeag}g^{ag}) + KU_{ey},$$

which, applied to (84), yields

$$
\begin{aligned}
K_{bd}U_{ey} &= \tfrac{3}{m-1}(AF_{ybed} - AF_{ybde}) + \tfrac{m^2-m-3}{m-1}(AF_{ydeb} - AF_{ydbe}) \\
&+ \tfrac{m^2-2m-2}{m-1}AF_{yebd} - \tfrac{m-4}{m-1}AF_{yedb} + \tfrac{m-2}{m}g_{bd}(AF_{yage} - AF_{yaeg} - AF_{yeag})g^{ag} \\
&+ g_{be}(AF_{yadg} - AF_{yagd})g^{ag} + g_{de}(AF_{yabg} - AF_{yagb})g^{ag} + \tfrac{K}{m}g_{bd}U_{ey}.
\end{aligned}
$$

Substituting it in (83) and making use of (80) and (85) gives us

$$\begin{aligned}(86)\qquad & U_{fy}C_{ebcd} - U_{ey}C_{fbcd}\\ &= g_{bf}UR_{yecd} - g_{be}UR_{yfcd} + \tfrac{3}{m-2}(g_{bc}UR_{ydef} - g_{bd}UR_{ycef})\\ &\quad - g_{ce}B_{ybdf} + g_{de}B_{ybcf} + g_{cf}B_{ybde} - g_{df}B_{ybce} - \tfrac{2}{m-2}G_{cdef}V_{yb}\\ &\quad + \tfrac{1}{m-1}(G_{bcef}V_{yd} - G_{bdef}V_{yc}) + \tfrac{m+3}{(m-1)(m-2)}(G_{ebcd}V_{yf} - G_{fbcd}V_{ye}),\end{aligned}$$

where we have put

$$V_{ye} = UR_{yaeg}g^{ag}, \quad B_{ybdf} = \tfrac{1}{m-2}[UR_{ybdf} + (m-1)UR_{ydbf}].$$

Finally, contracting (86) with g^{bf}, by the use of (85), we easily come to the required conclusion. This completes the proof. ■

6.6. Proof of Theorem 19

LEMMA 26. *Suppose that*

$$(87)\qquad \widetilde{\nabla}_X\widetilde{C} = v(X)\widetilde{C} + D(a,\widetilde{C})(X) + M(b,\widetilde{C})(X)$$

on N. *Then*

$$T\cdot b = -T\cdot a.$$

Proof. By the assumption, in a local coordinate system, (87) takes the form

$$\begin{aligned}(88)\qquad C_{hijk,l} &= p_lC_{hijk} + a_hC_{lijk} + a_iC_{hljk} + a_jC_{hilk} + a_kC_{hijl}\\ &\quad + g_{hl}b^rC_{rijk} + g_{il}b^rC_{hrjk} + g_{jl}b^rC_{hirk} + g_{kl}b^rC_{hijr},\end{aligned}$$

where $p_l = v_l + 2a_l$. By contracting with g^{hl}, we get $C^r_{ijk,r} = (p^r + a^r + mb^r)C_{rijk}$. On the other hand, summing (88) cyclically in (j,k,l) and contracting with g^{hl} we obtain $C^r_{ijk,r} = (p^r - 2a^r + (m-3)b^r)C_{rijk}$. Hence $a^rC_{rijk} + b^rC_{rijk} = 0$. ■

Proof of Theorem 19. To prove the first part of the theorem we modify the proof of [12, Theorem 6].

On M define tensors

$$p_a = p_rB^r_a, \quad a_a = a_rB^r_a, \quad r_e = (r_{,r} - p_rr)B^r_e, \quad S_{abe} = (R_{rs,t} - p_tR_{rs})B^{rst}_{abc}.$$

We apply (26) to the components C_{hijk} and $C_{hijk,l}$ in (88) to obtain

$$\begin{aligned}(89)\qquad & R_{hijk,l} - p_lR_{hijk} - \tfrac{1}{n-2}[g_{ij}(R_{hk,l} - p_lR_{hk}) - g_{ik}(R_{hj,l} - p_lR_{hj})\\ &\quad + g_{hk}(R_{ij,l} - p_lR_{ij}) - g_{hj}(R_{ik,l} - p_lR_{ik})] + \tfrac{r_{,l}-p_lr}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})\\ &= a_hC_{lijk} + a_iC_{hljk} + a_jC_{hilk} + a_kC_{hijl}\\ &\quad - g_{hl}a^rC_{rijk} - g_{il}a^rC_{hrjk} - g_{jl}a^rC_{hirk} - g_{kl}a^rC_{hijr}.\end{aligned}$$

In virtue of (13) and Lemma 3 we have

$$a_r C_{sijk} g^{rs} B^{ijk}_{bcd} = a_e \overline{C}_{fbcd} g^{ef} + \sum_x e_x a_r N^r_x C_{sijk} N^s_x B^{ijk}_{bcd} \tag{90}$$
$$= a_f C^f_{bcd} + \tfrac{1}{m-2}[g_{bc}t_d - g_{bd}t_c + a_d T_{bc} - a_c T_{bd}]$$
$$- \tfrac{P}{(m-1)(m-2)}(g_{bc}a_d - g_{bd}a_c) + g_{bc}T_d - g_{bd}T_c$$

for some $(0,1)$ tensor t_d.

Transvecting (89) with $B^{hijkl}_{abcde,}$ then applying (15), (20) and (90), in virtue of Lemma 3, we get

$$\begin{aligned} &K_{abcd;e} - p_e K_{abcd} + (Hp_e - H_e) g_{abcd} \\ &- \tfrac{1}{2}[H_a G_{ebcd} + H_b G_{aecd} + H_c G_{abed} + H_d G_{abce}] \\ &- \tfrac{1}{n-2}[g_{bc}S_{ade} - g_{bd}S_{ace} + g_{ad}S_{bce} - g_{ac}S_{bde}] + \tfrac{r_e}{(n-1)(n-2)} G_{abcd} \\ = &\tfrac{1}{m-2}[a_a(g_{bc}T_{ed} - g_{bd}T_{ec}) + a_b(g_{ad}T_{ec} - g_{ac}T_{ed}) + a_c(g_{ad}T_{be} - g_{bd}T_{ae}) \\ &+ a_d(g_{bc}T_{ae} - g_{ac}T_{be})] + a_a C_{ebcd} + a_b C_{aecd} + a_c C_{abed} + a_d C_{abce} \\ &- g_{ae}a^f C_{fbcd} - g_{be}a^f C_{afcd} - g_{ce}a^f C_{abfd} - g_{de}a^f C_{abcf} \\ &+ \big(T_a - \tfrac{1}{m-2}t_a\big) G_{ebcd} + \big(T_b - \tfrac{1}{m-2}t_b\big) G_{aecd} \\ &+ \big(T_c - \tfrac{1}{m-2}t_c\big) G_{abed} + \big(T_d - \tfrac{1}{m-2}t_d\big) G_{abce}. \end{aligned} \tag{91}$$

On the other hand, from the Gauss equation (15) and (20) we have

$$K_{abcd;e} - p_e K_{abcd} = (R_{hijk,l} - p_l R_{hijk}) B^{hijkl}_{abcde}$$
$$+ (H_e - p_e H)(g_{bc}g_{ad} - g_{bd}g_{ac}) + \tfrac{1}{2}[H_a G_{ebcd} + H_b G_{aecd} + H_c G_{abed} + H_d G_{abce}],$$

whence, by transvecting with g^{bc} and the use of (13) we obtain

$$K_{ad;e} - p_e K_{ad} = S_{ade} - \sum_x e_x (R_{hijk,l} - p_l R_{hijk}) B^{hkl}_{ade} N^i_x N^j_x \tag{92}$$
$$+ (m-1)(H_e - p_e H) g_{ad} + \tfrac{1}{2}[(m-2)(H_a g_{ed} + H_d g_{ae}) + 2H_e g_{ad}].$$

Let

$$E_{ed} = \sum_x e_x R_{hijk} B^h_e N^i_x N^j_x B^k_d, \quad F_e = \sum_x e_x (R_{ij,l} - p_l R_{ij}) N^i_x N^j_x B^l_e,$$
$$\alpha = \sum_x e_x R_{ij} N^i_x N^j_x, \quad \Sigma_e = \sum_x e_x a_i N^i_x \big(A_{ax} - \tfrac{1}{n-2} R_{hj} B^h_a N^j_x\big).$$

Notice that the Gauss equation yields

$$E_{ad} = \overline{R}_{ad} - K_{ad} + (m-1) H g_{ad}. \tag{93}$$

Then, in virtue of (26) and (12), we have

$$\begin{aligned} M_{ed} &= \sum_x e_x C_{lijk} B^l_e N^i_x N^j_x B^k_d \\ &= E_{ed} - \tfrac{1}{n-2}[(n-m)\bar{R}_{ed} + \alpha g_{ed}] + \tfrac{(n-m)r}{(n-1)(n-2)} g_{ed} \\ &= \tfrac{m-2}{n-2}\bar{R}_{ed} - K_{ed} + (m-1)H g_{ed} - \tfrac{\alpha}{n-2} g_{ed} + \tfrac{(n-m)r}{(n-1)(n-2)} g_{ed}, \end{aligned}$$

$$\sum_x e_x a_i N^i_x C_{hljk} B^{hl}_{ae} N^j_x B^k_d = g_{ad}\Sigma_e - g_{ed}\Sigma_a.$$

Transvecting (89) with $B^{hkl}_{ade} N^i_x N^j_x$, by the use of (12) and (13), we obtain

$$\begin{aligned} &\sum_x e_x (R_{hijk,l} - p_l R_{hijk}) B^{hkl}_{ade} N^i_x N^j_x = 2g_{ad}\Sigma_e - g_{ed}\Sigma_a - g_{ea}\Sigma_d \\ &+ \tfrac{n-m}{n-2} S_{ade} + \tfrac{1}{n-2} g_{ad} F_e - \tfrac{n-m}{(n-1)(n-2)} r_e g_{ad} + a_a M_{ed} + a_d M_{ea} + g_{ea} Q_d + g_{ed} Q_a \end{aligned}$$

for some $(0,1)$ tensor Q. Substituting the last relation into (92) we obtain

$$\begin{aligned} K_{ad;e} - p_e K_{ad} = {}& \tfrac{m-2}{n-2} S_{ade} - \tfrac{1}{n-2} g_{ad} F_e + \tfrac{n-m}{(n-1)(n-2)} r_e g_{ad} \\ & - g_{ea} Q_d - g_{ed} Q_a - a_a M_{ed} - a_d M_{ea} - 2g_{ad}\Sigma_e + g_{ed}\Sigma_a + g_{ea}\Sigma_d \\ & + (m-1)(H_e - p_e H) g_{ad} + \tfrac{1}{2}[(m-2)(H_a g_{ed} + H_d g_{ae}) + 2H_e g_{ad}]. \end{aligned} \tag{94}$$

Observe that $S_{ade} g^{ad} = r_e - F_e$.

Then, transvecting (94) with g^{ad}, we find that for some $(0,1)$ tensor Z,

$$\begin{aligned} F_e = {}& -\tfrac{n-2}{2(m-1)}(K_{;e} - p_e K) + \tfrac{1}{2}\big[\tfrac{m-2}{m-1} + \tfrac{m(n-m)}{(n-1)(m-1)}\big] r_e + Z_e \\ & + (n-2)[\tfrac{m}{2}(H_e - p_e H) + H_e - \Sigma_e], \end{aligned}$$

which, by substituting into (94), yields

$$\begin{aligned} \tfrac{1}{n-2} S_{ade} = {}& \tfrac{1}{m-2}(K_{ad,e} - p_e K_{ad}) - \tfrac{1}{2(m-1)(m-2)}(K_e - p_e K) g_{ad} \\ & + \tfrac{1}{m-2}[g_{ad}\Sigma_e - g_{ed}\Sigma_a - g_{ea}\Sigma_d] - \tfrac{1}{2}(H_e - p_e H) g_{ad} - \tfrac{1}{2}(H_a g_{ed} + H_d g_{ae}) \\ & + \tfrac{1}{2(n-1)(n-2)} r_e g_{ad} + \tfrac{1}{m-2}(g_{ea} Q_d + g_{ed} Q_a + a_a M_{ed} + a_d M_{ea}) + \tfrac{1}{m-2} Z_e g_{ad}. \end{aligned} \tag{95}$$

Substituting (95) into (91) and making use of (93) and the definition of T_{ad} we obtain

$$\begin{aligned} C_{abcd;e} = {}& p_e C_{abcd} + a_a C_{ebcd} + a_b C_{aecd} + a_c C_{abed} + a_d C_{abce} \\ & + v_e G_{abcd} + (z_a G_{ebcd} + z_b G_{aecd} + z_c G_{abed} + z_d G_{abce}) \\ & + a_a(g_{ed} U_{bc} - g_{ec} U_{bd}) + a_b(g_{ec} U_{ad} - g_{ed} U_{bc}) \\ & + a_c(g_{be} U_{ad} - g_{ae} U_{bd}) + a_d(g_{ae} U_{bc} - g_{be} U_{ac}) \end{aligned} \tag{96}$$

for some 1-forms v_e and z_e on M, while $U_{ad} = \frac{1}{m-2} T_{ad}$.

Contracting (96) with g^{ae} and $g^{ae}g^{bc}$ we get

$$C^e_{bcd;e} = (p_e + a_e)C^e_{bcd} + g_{bc}v_d - g_{bd}v_c + m(g_{bc}z_d - g_{bd}z_c) + m(U_{bc}a_d - U_{bd}a_c) \tag{97}$$

and

$$(m-1)v_d + m(m-1)z_d + mUa_d - mt_d = 0, \tag{98}$$

respectively, while summing (96) cyclically in (c, d, e) and contracting the resulting equality with g^{ae} and $g^{ae}g^{bc}$ we obtain

$$\begin{aligned} C^e_{bcd;e} &= (p_e - 2a_e)C^e_{bcd} - (m-2)(g_{bc}v_d - g_{bd}v_c) + 2(m-2)(g_{bc}z_d - g_{bd}z_c) \\ &\quad + (m-3)(U_{bc}a_d - U_{bd}a_c) + U(g_{bc}a_d - g_{bd}a_c) - g_{bc}t_d + g_{bd}t_c \end{aligned} \tag{99}$$

and

$$2(m-1)z_d + 2Ua_d - 2t_d - (m-1)v_d = 0, \tag{100}$$

where $U = U_{ad}g^{ad}$ and $t_d = a^eU_{ed}$. Solving (98) and (100) we find $t_d = (m-1)z_d + Ua_d$, $v_d = 0$. Finally, subtracting (97) from (99), in virtue of the last relations, we obtain

$$a_eC^e_{bcd} = -(g_{bc}z_d - g_{bd}z_c) - (U_{bc}a_d - U_{bd}a_c),$$

which, by substituting into (96), ends the proof of Theorem 19(a). The proof of (b) is exactly the same as the proof of [12, Theorem 5]. ∎

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