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INVERSE SEQUENCES WITH PROPER BONDING MAPS

BY

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Abstract. Some topological properties of inverse limits of sequences with proper bonding maps are studied. We show that (non-empty) limits of euclidean half-lines are one-ended generalized continua. We also prove the non-existence of a universal object for such limits with respect to closed embeddings. A further result states that limits of end-preserving sequences of euclidean lines are two-ended generalized continua.

1. Introduction. This paper is concerned with spaces obtained as inverse limits of sequences whose bonding maps are proper (1). Among other results, we prove that inverse limits of end-preserving sequences of euclidean lines ($\mathbb{R}$-type spaces) or half-lines ($\mathbb{R}_{\geq 0}$-type spaces) preserve connectedness and Freudenthal ends (Theorems 5.2 and 6.1). In contrast, this is no longer true for trees without terminal vertices with three or more ends. Furthermore, we show that the category of $\mathbb{R}_{\geq 0}$-type spaces and proper maps does not admit a universal space (Theorem 5.4).

As the space of Freudenthal ends of inverse limits of sequences with proper bonding maps may fail to be metrizable (see Example 4.3), we will use the general theory of ends based on ultrafilters as in [H] and [FG]. In Appendix A we collect the elements of that theory needed in this paper. A second appendix contains an explicit proof of the fact that for generalized continua, Freudenthal ends can be equivalently defined by the use of nested sequences of quasicomponents (Theorem B.6).

2. Preliminaries. By a *space* we mean a locally compact σ-compact Hausdorff space. It is clear that local compactness and σ-compactness yield the existence of *exhausting sequences*, that is, increasing sequences of compact sets $K_n \subset X$ such that $X = \bigcup_{n=1}^{\infty} K_n$ and $K_n \subset \operatorname{int} K_{n+1}$. The Freudenthal ends of a space X are defined as follows (see [Fr1], [Fr2] and [H]). Let $\mathcal{A}$ denote the family of all closed sets in X with compact frontier. The *Freuden-*

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(1) The terminology is introduced in Section 2.

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thal compactification of X is the space $\widehat{X}$ of all $\mathcal{A}$-ultrafilters endowed with the compact topology whose basic closed sets are of the form

$$\mathcal{B}(A) = \{\mathcal{U} \in \widehat{X};\ A \in \mathcal{U}\}$$

where A ranges over $\mathcal{A}$; see [H, 2.1]. Moreover, each $x \in X$ is identified with the $\mathcal{A}$-ultrafilter $\mathcal{U}_x$ such that $x \in U$ for all $U \in \mathcal{U}_x$ (termed a *trivial $\mathcal{A}$-ultrafilter*). This way $\widehat{X}$ can be regarded as the union of X with the set of all non-trivial $\mathcal{A}$-ultrafilters in X. The difference $\mathcal{F}(X) = \widehat{X} - X$ turns out to be a zero-dimensional closed subspace whose elements are called the *Freudenthal ends* of X. A space X is said to be *one-ended* (*two-ended*, respectively) if it has exactly one Freudenthal end (two Freudenthal ends, respectively). In Appendix A we give a brief account of the basic properties of the Freudenthal compactification used in this paper.

Most of the results in this paper deal with metrizable spaces (*admissible spaces*, for short) and all maps considered are proper. Recall that a continuous map $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact subset $K \subset Y$. It is well-known that proper maps between admissible spaces are closed ([E, 3.7.18]). Any proper map $f : X \to Y$ between admissible spaces extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ which restricts to a continuous map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Namely, if $\mathcal{U} \in \mathcal{F}(X)$, then $\widehat{f}(\mathcal{U}) = f_*(\mathcal{U})$ is the unique element in $\bigcap_{U \in \mathcal{U}} \overline{f(U)}^{\widehat{Y}}$. See Lemma A.6.

Notice that admissible spaces are second countable. Connected admissible spaces are termed *generalized continua*. The ends of a generalized continuum X can be described in a more geometrically appealing way as nested sequences of quasicomponents. More precisely, there is a homeomorphism

$$\mathcal{F}(X) \cong \varprojlim \mathcal{Q}(X - \operatorname{int} K_n)$$

where $\mathcal{Q}(X - \operatorname{int} K_n)$ is the space of quasicomponents of $X - \operatorname{int} K_n$ and $\{K_n\}_{n \geq 1}$ is an exhausting sequence of X. In particular, $\mathcal{F}(X)$ is homeomorphic to a closed subset of the Cantor set. All this is stated without proof in [Sh]. For the sake of completeness we give explicit proofs of these facts in Appendix B.

In general, the Freudenthal compactification of an admissible space X may fail to be metrizable. In fact, the metrizability of $\widehat{X}$ and $\mathcal{F}(X)$ are equivalent and both are equivalent to the compactness of the space of quasicomponents of X. Explicitly,

THEOREM 2.1 ([I, Thm. VI.42]). *Let X be a separable metric space in which every point has arbitrarily small neighbourhoods with compact frontier. Then $\widehat{X}$ is metrizable and compact if and only if the space of quasicomponents of X is metrizable and compact.*

In particular, the Freudenthal compactification of a generalized continuum is always a metrizable space and hence a continuum. This fact allows us to prove the following

LEMMA 2.2. *Let $\{K_n\}_{n\geq 1}$ be an exhausting sequence of the generalized continuum X and let $\varepsilon = (Q_n)_{n\geq 1}$ be the Freudenthal end defined by the nested sequence of quasicomponents $Q_n \subset X - \operatorname{int} K_n$. Then, for every $n \geq 1$ there is a continuum $L \subset \widehat{X} - \operatorname{int} K_n$ joining ε and $\operatorname{Fr} K_n$. Moreover each Q_n $(n \geq 1)$ contains at least one non-compact component.*

The following well-known result is crucial in the proof of Lemma 2.2.

LEMMA 2.3 ([K, Thm. 2, p. 172]). *If $A \subset X$ is a non-trivial subset of the continuum X and C is a component of $X - A$, then $\overline{C} \cap \operatorname{Fr} A \neq \emptyset$. In particular, if $A = \{p\}$ reduces to one point, then p lies in the closure of each component $C \subset X - \{p\}$.*

Proof of Lemma 2.2. Any sequence $\{x_k\}_{k\geq n}$ with $x_k \in Q_k$ converges to ε in $\widehat{X}$. Let D_k denote the component of x_k in $\widehat{X} - \operatorname{int} K_n$. We claim that $D_k \cap X \subset Q_n$ for all $k \geq n$. Indeed, for any closed-open set H in $X - \operatorname{int} K_n$ with $Q_n \subset H$ we know from the topology of $\widehat{X}$ (see Appendix B) that the set $\widehat{H} = H \cup H^{\mathcal{F}}$ with $H^{\mathcal{F}} = \overline{H}^{\widehat{X}} \cap \mathcal{F}(X)$ is closed-open in $\widehat{X} - \operatorname{int} K_n$ containing x_k. Therefore $D_k \subset \widehat{H}$ by connectedness and so $D_k \cap X \subset H$; that is, $D_k \cap X \subset Q_n$ by definition of a quasicomponent.

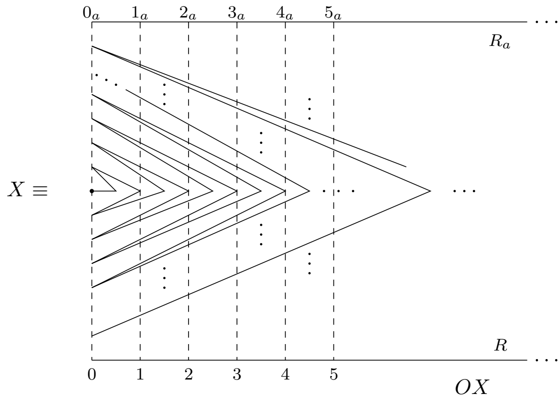
Next we apply Lemma 2.3 to $\operatorname{int} K_n \subset \widehat{X}$ to show that D_k meets $\operatorname{Fr}(\operatorname{int} K_n) \subset \operatorname{Fr} K_n$ for all $k \geq n$. By compactness of $\operatorname{Fr} K_n$, we can assume without loss of generality that there is a sequence $y_k \in D_k \cap \operatorname{Fr} K_n \subset Q_n$ converging to some $y_0 \in Q_n \cap \operatorname{Fr} K_n$. Here we use the fact that Q_n is a closed set.

As y_0 lies in the lower limit $\operatorname{Li} D_k$, [K, Thm. 6, p. 171] implies that the upper limit $L = \operatorname{Ls} D_k \subset \widehat{X} - \operatorname{int} K_n$ is a continuum with $\varepsilon \in L$ and $y_0 \in \operatorname{Fr} K_n \cap L$.

We apply again Lemma 2.3 to $L_0 = \mathcal{F}(X) \cap L \subset L$ to show that the closure in L of the component of y_0, $C \subset L - L_0$, contains at least one end of L_0. Hence C is a connected non-compact closed set in $X - \operatorname{int} K_n$ containing $y_0 \in Q_n$. Therefore $C \subset Q_n$ by definition of a quasicomponent, and the component of y_0 in Q_n is necessarily non-compact. ∎

The unbounded component given by Lemma 2.2 need not be unique, as shown by the generalized continuum $X \subset \mathbb{R}_{\geq 0} \times [0, 1]$ depicted in the figure below.

It is clear that X is one-ended; however, for all $n \geq 1$ the quasicomponent outside $[0, n) \times [0, 1]$ consists of two unbounded components, namely $[n_a, \infty)$ and $[n, \infty)$.



Uniqueness of such components holds for the so-called Peano generalized continua. Recall that a *Peano continuum* X is a metrizable, compact, connected and locally connected space. If compactness is replaced by local compactness, then the space X is called a *generalized Peano continuum.* Any generalized Peano continuum is separable ([E, 4.4 F.(c)]) and hence second countable; that is, a generalized Peano continuum is a locally connected generalized continuum. For locally connected spaces, quasicomponents coincide with components; in particular, Freudenthal ends are defined by components; see [ShV].

3. Inverse limits with proper bonding maps. We will use the notation $X = \varprojlim_p \{X_n, f_n\}$ to represent the inverse limit of a sequence with proper bonding maps f_n. Notice that X may be the empty space (e.g., the sequence of inclusions $X_1 \supset X_2 \supset \cdots$ where $X_n = [n, \infty)$). For non-empty inverse limits the following lemma can be easily proved; compare ([E, 3.7.12]).

Lemma 3.1. *Any non-empty inverse limit $X = \varprojlim_p \{X_n, f_n\}$ of admissible spaces is an admissible space. Moreover, the natural projections $\pi_n : X \to X_n$ are proper maps. Furthermore, if the f_n's are monotone then so are the π_n's.*

Recall that a map $f : X \to Y$ is said to be *monotone* if it is a continuous surjection such that $f^{-1}(y)$ is connected for each $y \in Y$. It is known (see [E, 6.1.29]) that if f is a monotone closed map then $f^{-1}(C)$ is connected for any connected set $C \subset Y$.

Corollary 3.2. *Any inverse limit of generalized continua $X = \varprojlim_p \{X_n, f_n\}$ with monotone proper bonding maps is a generalized continuum.*

By using the Aleksandrov one-point compactification, $X^+ = X \cup \{\infty\}$, we next show that inverse limits with proper bonding maps can be regarded as ordinary "pointed" inverse limits. For this, if X_n is pointed by $x_n \in X_n$, by writing $X = \varprojlim_{*}\{X_n, g_n\}$ we mean that X is the limit of an inverse sequence whose bonding maps satisfy $g_n^{-1}(x_n) = x_{n+1}$ for all n. Recall that any proper map $f : X \to Y$ extends to a continuous map $f^+ : X^+ \to Y^+$ by setting $f^+(\infty) = \infty$. With this notation, the following proposition is a straightforward consequence of the universal property of inverse limits.

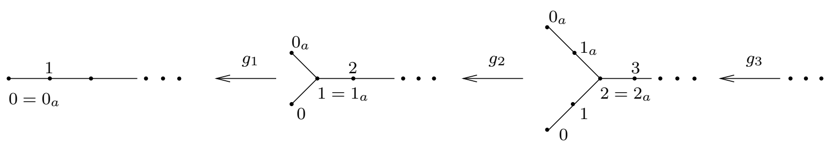
Proposition 3.3. *For any admissible space X the following two statements are equivalent:*

(a) $X = \varprojlim_{p}\{X_n, f_n\}$.
(b) $X^+ = \varprojlim_{*}\{X_n^+, f_n^+\}$ *where X_n^+ is pointed by $\infty \in X_n^+$ for each n.*

As an admissible space X is embedded as a closed subset in $\mathbb{R}^n$ if and only if X^+ embeds in the n-sphere S^n, the following corollary is an immediate consequence of the embedding theorem ([N, 2.36]) due to Isbell.

Corollary 3.4. *If $X = \varprojlim_{p}\{X_n, f_n\}$ where each X_n is homeomorphic to a non-trivial closed subset of $\mathbb{R}^k$, then X can be embedded as a closed set in $\mathbb{R}^{2k}$.*

4. Inverse limits preserving Freudenthal ends. Compactness is crucial, not only for the existence of non-empty inverse limits, but also for the preservation of connectedness. For instance, the inverse limit of one-ended trees X_n sketched in the next figure consists of two copies of the half-line $\mathbb{R}_{\geq 0}$.



Here the (proper) maps g_n are the obvious projections.

We proceed to study the relationship between the connectedness of $X = \varprojlim_{p}\{X_n, f_n\}$ and the behaviour of the bonding maps f_n with respect to ends. We start with the following

Proposition 4.1. *If $X = \varprojlim_{p}\{X_n, f_n\}$ is an inverse limit of generalized continua, then there is a canonical continuous surjection $\varphi : \widehat{X} \to \varprojlim\{\widehat{X_n}, \widehat{f}_n\}$.*

Proof. It is clear that the maps $\widehat{\pi}_n : \widehat{X} \to \widehat{X}_n$ induced by the projections $\pi_n : X \to X_n$ define a canonical map $\varphi : \widehat{X} \to L = \varprojlim\{\widehat{X}_n, \widehat{f}_n\}$. Moreover, the image $\varphi(\widehat{X}) \subset L$ is compact, and hence its complement $D = L - \varphi(\widehat{X})$ is an open set contained in the compact set $F = \varprojlim\{\mathcal{F}(X_n), f_{n*}\}$. If $D \neq \emptyset$ and $\varepsilon \in D$, the 0-dimensionality of F yields an open and closed neighbourhood of ε in F, $\Omega \subset D$. Therefore, Ω is open in L as well as closed in F, and hence compact in L. This contradicts the connectedness of L, proving $D = \emptyset$. Here we use the fact that the $\widehat{X_n}$'s are continua. ∎

COROLLARY 4.2. *Under the above assumptions, there exists a continuous surjection of the Freudenthal ends of* $X = \varprojlim_p \{X_n, f_n\}$ *onto the inverse limit* $\varprojlim\{\mathcal{F}(X_n), f_{n*}\}$.

We say that an inverse sequence $\{X_1 \xleftarrow{g_1} X_2 \xleftarrow{g_2} \cdots\}$ with proper bonding maps is *end-faithful* if the induced maps $g_{n*} : \mathcal{F}(X_{n+1}) \cong \mathcal{F}(X_n)$ are homeomorphisms for all $n \geq 1$. Moreover, the limit $X = \varprojlim_p \{X_n, g_n\}$ is said to be *end-preserving* if the canonical projections $\pi_n : X \to X_n$ induce homeomorphisms $\pi_{n*} : \mathcal{F}(X) \cong \mathcal{F}(X_n)$ for all $n \geq 1$.

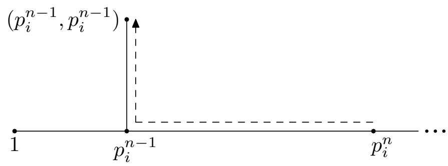
Obviously, sequences of one-ended spaces are end-faithful. The following example shows that end-faithful sequences may have non-metrizable spaces of ends.

EXAMPLE 4.3. The end space of the inverse limit of one-ended trees needs not be metrizable. Indeed, let $\{p_i\}_{i\geq 1}$ be an increasing sequence of prime numbers where $p_1 = 2$ and consider the inverse sequence formed by the trees ($n \geq 1$)

$$X_n = ([1, \infty) \times \{0\}) \cup \{\{p_i^n\} \times [0, p_i^n]\}_{i\geq 1}$$

and proper maps $f_n : X_{n+1} \to X_n$ defined as follows:

$$f_n(x, 0) = \begin{cases} (x, 0) & \text{if } x \in [1, \infty), \\ (p_i^{n-1} + x, 0) & \text{if } 0 \leq x \leq p_i^{n-1}(p_i - 1), \\ (p_i^{n-1}, x - p_i^{n-1}(p_i - 1)) & \text{if } p_i^{n-1}(p_i - 1) \leq x \leq p_i^n. \end{cases}$$



The dotted line in the figure depicts the image under f_{n-1} of the segment $\{p_i^n\} \times [0, p_i^n] \subset X_n$. It is not hard to check that $X = \varprojlim_p \{X_n, f_n\}$ is homeomorphic to the disjoint union $[1, \infty) \sqcup \bigsqcup_{i\geq 1} [p_i, \infty)$, and so $\mathcal{F}(X)$ is not metrizable by Theorem 2.1.

PROPOSITION 4.4. *Assume that the admissible space* $X = \varprojlim_p \{X_n, g_n\}$ *is the end-preserving limit of an end-faithful sequence of generalized continua. Then X is connected, and hence a generalized continuum.*

In the proof of Proposition 4.4 we will use the following straightforward generalizations of [E, 2.5.7] and [N, 2.19], respectively. We include the proof of Lemma 4.6 for the sake of completeness.

LEMMA 4.5. *Let $A \subset X = \varprojlim_p\{X_i, f_i\}$ be a closed set where the X_i's are admissible spaces. If $\pi_i : X \to X_i$ are the canonical projections, then $A = \varprojlim_p\{\pi_i(A), f_i'\}$ for the obvious restrictions f_i'.*

LEMMA 4.6. *Let $A_1, A_2 \subset X$ be closed subsets of an inverse limit $X = \varprojlim_p\{X_i, f_i\}$ of admissible spaces. If $A_1 \cap A_2 \neq \emptyset$ and either A_1 or A_2 is compact, then $A_1 \cap A_2 = \varprojlim_p\{\pi_i(A_1) \cap \pi_i(A_2), f_i^1\}$ for the obvious restrictions f_i^1.*

Proof. As each intersection $\pi_i(A_1) \cap \pi_i(A_2)$ is compact, the limit $L = \varprojlim_p\{\pi_i(A_1) \cap \pi_i(A_2), f_i^1\}$ is not empty. Furthermore, by Lemma 4.5, $A_1 \cap A_2 = \varprojlim_p\{\pi_i(A_1 \cap A_2), f_i^2\} \subset L$ for the corresponding restrictions f_i^2. Conversely, given $x = (x_i)_{i\geq 1} \in L$ there is an element $y_i^j \in A_j$ such that $\pi_i(y_i^j) = x_i$ for all $i, j = 1, 2$. Then the sequences y_i^1 and y_i^2 converge to x in X and so $x \in A_1 \cap A_2$. Here we use the fact that A_1 and A_2 are closed sets. ■

REMARK 4.7. The inverse limit at the beginning of this section shows that Lemma 4.6 fails to hold if compactness is dropped.

Proof of Proposition 4.4. Suppose that $X = U_1 \cup U_2$ is a disjoint union of two open (and hence closed) sets. Consider the induced maps $\widehat{\pi}_n : \widehat{X} \to \widehat{X}_n$ between Freudenthal compactifications, and set $A_i = \overline{U_i}^{\widehat{X}}$ for $i = 1, 2$. By Lemma 4.5, $A_i = \varprojlim\{\widehat{\pi}_n(A_i), \widehat{g}_n'\}$ for the obvious restrictions. Moreover, the connectedness of $\widehat{X}_n$ leads to $\widehat{\pi}_n(A_1) \cap \widehat{\pi}_n(A_2) \neq \emptyset$ for each n. Applying Lemma 4.6, we get $A_1 \cap A_2 = \varprojlim\{\widehat{\pi}_n(A_1) \cap \widehat{\pi}_n(A_2), \widehat{g}_n''\}$ for the corresponding restrictions, and hence $A_1 \cap A_2 = A_1 \cap A_2 \cap \mathcal{F}(X) \neq \emptyset$. Thus, for any end $\mathcal{U} \in A_1 \cap A_2$, Lemma A.3 yields $U_1, U_2 \in \mathcal{U}$, whence $\emptyset = U_1 \cap U_2 \in \mathcal{U}$, which contradicts that $\mathcal{U}$ is a filter. ■

We also have the following partial converse of Proposition 4.4:

PROPOSITION 4.8. *A path connected inverse limit $X = \varprojlim_p\{D_i, f_i\}$ of an end-faithful sequence of generalized dendrites is end-preserving.*

Recall that a *(generalized) dendrite* is a (generalized) Peano continuum in which any two different points can be separated by the omission of some third point. It is known that the Freudenthal compactification of a generalized dendrite is a dendrite; see [FeQ, Sect. 4] for a proof.

Proof of Proposition 4.8. Suppose that there are two distinct Freudenthal ends (i.e., sequences of quasicomponents) $\varepsilon_1 = (Q_n^1)_{n\geq 1}$ and $\varepsilon_2 = (Q_n^2)_{n\geq 1}$ with $\varepsilon = \pi_{i*}(\varepsilon_1) = \pi_{i*}(\varepsilon_2)$ for each $i \geq 1$. Here $\pi_{i*} : \mathcal{F}(X) \to \mathcal{F}(X_i)$ are the maps induced by the projections $\pi_i : X \to X_i$. As $\varepsilon_1 \neq \varepsilon_2$ the quasicomponents Q_n^1 and Q_n^2 are disjoint for n large enough. Let $L_j \subset \widehat{X} - \operatorname{int} K_n$ $(j = 1, 2)$ be continua in $\widehat{X}$ given by Lemma 2.2 with $\varepsilon_j \in L_j$

and $L_j \cap \operatorname{Fr} K_n \neq \emptyset$. Then, for the induced map $\widehat{\pi}_i : \widehat{X} \to \widehat{X}_i$, the image $\widehat{\pi}_i(L_j)$ is a connected set in the dendrite $\widehat{D}_i$ with $\varepsilon \in \widehat{\pi}_i(L_1) \cap \widehat{\pi}_i(L_2)$. Hence $\Omega_i = \widehat{\pi}_i(L_1) \cup \widehat{\pi}_i(L_2)$ is also a connected set in $\widehat{D}_i$. If $\Omega_0^i = \Omega_i \cap \mathcal{F}(D_i)$, then $\Omega_i - \Omega_0^i = \pi_i(L_1 - L_0^1) \cup \pi_i(L_2 - L_0^2)$ with $L_0^j = L_j \cap \mathcal{F}(X)$. Notice that $L_j - L_0^j$ is closed in X.

On the other hand, we choose a path $\Gamma \subset X$ with $\Gamma \cap L_j \neq \emptyset$ for $j = 1, 2$. Hence $\pi_i(\Gamma)$ is a continuum in D_i and so is $\Sigma_i = \widehat{\pi}_i(\Gamma) \cap \Omega_i = \pi_i(\Gamma) \cap (\Omega_i - \Omega_0^i)$ since dendrites are hereditarily unicoherent. Therefore, for the restrictions $f_i' : \Sigma_{i+1} \to \Sigma_i$ the inverse limit $\Sigma = \varprojlim\{\Sigma_i, f_i'\}$ is a continuum in X. However, Lemma 4.6 yields

$$\Sigma = \Gamma \cap ((L_1 - L_0^1) \cup (L_2 - L_0^2))$$

where the right-hand side space is not connected. This is a contradiction and the proof is finished. ■

REMARK 4.9. As any generalized Peano continuum is path connected ([Shu, 4.2.5]), Proposition 4.8 holds for X being a generalized Peano continuum.

Monotone bonding maps produce end-faithful inverse sequences. More precisely:

THEOREM 4.10. *Any sequence* $\{X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots\}$ *of admissible spaces with monotone proper bonding maps is end-faithful. Moreover, its inverse limit* $X = \varprojlim_p\{X_n, f_n\}$ *is end-preserving.*

The proof is an immediate consequence of the following

LEMMA 4.11. *Any monotone proper map* $f : X \to Y$ *between admissible spaces induces a homeomorphism* $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$.

Proof. Let $\{L_n\}_{n\geq 1}$ be an exhausting sequence of Y. It is readily checked that $\{K_n\}_{n\geq 1}$ with $K_n = f^{-1}(L_n)$ is an exhausting sequence of X. Given two ends (i.e., $\mathcal{A}$-ultrafilters) $\mathcal{U}_1 \neq \mathcal{U}_2$ in $\mathcal{F}(X)$, there exist two closed sets with compact frontier, $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$, with $U_1 \cap U_2 = \emptyset$ and $\operatorname{Fr} U_1 \cup \operatorname{Fr} U_2 \subset \operatorname{int} K_{n_0}$ for n_0 sufficiently large. By Lemma A.2, $X - \operatorname{int} K_{n_0} \in \mathcal{U}_1 \cap \mathcal{U}_2$ and so $F_1 = U_1 - \operatorname{int} K_{n_0}$ and $F_2 = (X - \operatorname{int} K_{n_0}) - F_1$ form a partition of $X - \operatorname{int} K_{n_0}$ into two closed-open sets with $F_1 \in \mathcal{U}_1$ and $F_2 \in \mathcal{U}_2$ since $U_2 - \operatorname{int} K_{n_0} \subset F_2$.

On the other hand, $f(F_1) \cap f(F_2) = \emptyset$; indeed, if $f(x_1) = f(x_2)$ for $x_i \in F_i$ then the connected set $f^{-1}(f(x_1))$ meets F_1 and F_2, which is a contradiction. Here we use the monotonicity of f. This way $f(F_1)$ and $f(F_2)$ form a partition of $Y - \operatorname{int} L_{n_0} = f(X - \operatorname{int} K_{n_0})$ into two closed (and hence open) sets. Therefore the frontier of $f(F_i)$ in Y is compact for $i = 1, 2$. Moreover $\widehat{f}(\mathcal{U}_i) = f_*(\mathcal{U}_i) \in \overline{f(F_i)}^{\widehat{Y}}$ by definition of the induced map $\widehat{f} : \widehat{X} \to \widehat{Y}$. Hence $f(F_i)$ belongs to the $\mathcal{A}$-ultrafilter $f_*(\mathcal{U}_i)$; see Lemma A.3.

Moreover, $f(F_1) \cap f(F_2) = \emptyset$ yields $f_*(\mathcal{U}_1) \neq f_*(\mathcal{U}_2)$. This shows that f_* is injective.

The surjectivity of f_* follows from the fact that monotone maps are supposed to be onto and hence, given any end $\mathcal{W} \in \mathcal{F}(Y)$, $f^{-1}(\mathcal{W})$ is an $\mathcal{A}$-filter in X by Lemma A.7. It is readily checked from the definition of f_* that $f_*(\mathcal{U}) = \mathcal{W}$ for any $\mathcal{A}$-ultrafilter $\mathcal{U}$ with $f^{-1}(\mathcal{W}) \subset \mathcal{U}$. ■

Proof of Theorem 4.10. It follows from Lemma 4.11 that the induced maps $f_{n*} : \mathcal{F}(X_{n+1}) \to \mathcal{F}(X_n)$ are homeomorphisms. To check that the inverse limit is end-preserving we observe that the projections $\pi_n : X \to X_n$ are monotone by Lemma 3.1, and we apply Lemma 4.11 again. ■

5. Ray-type spaces. Next we consider the proper analogue of the well-known class of arc-like spaces in continuum theory. Namely, we say that a space X is a *ray-type space* if $X = \varprojlim_p \{X_n, f_n\}$ where $X_n = \mathbb{R}_{\geq 0}$ is the euclidean half-line for each $n \geq 1$.

LEMMA 5.1. *Let $X = \varprojlim_p \{\mathbb{R}_{\geq 0}, f_n\}$ be a ray-type space. If F_1 and F_2 are two non-compact closed connected subsets in X, then either $F_1 \subset F_2$ or $F_2 \subset F_1$.*

Proof. By Lemma 3.1 the projections $\pi_n : X \to \mathbb{R}_{\geq 0}$ are proper, and so $\pi_n(F_1)$ and $\pi_n(F_2)$ are non-compact closed connected subsets in $\mathbb{R}_{\geq 0}$. Hence $\pi_n(F_j)$ ($j = 1, 2$) is an unbounded closed interval and hence either $\pi_n(F_1) \subset \pi_n(F_2)$ or $\pi_n(F_2) \subset \pi_n(F_1)$. Notice that if $\pi_m(F_1) \subset \pi_m(F_2)$ for some m, then $\pi_n(F_1) \subset \pi_n(F_2)$ for all $n \leq m$. Therefore, the existence of an infinite subsequence $\{n_j\}_{j\geq 1}$ with $\pi_{n_j}(F_1) \subset \pi_{n_j}(F_2)$ yields $\pi_n(F_1) \subset \pi_n(F_2)$ for all $n \geq 1$. Moreover, by Lemma 4.5, $F_i = \varprojlim_p \{\pi_n(F_i), f'_n\}$ where $f'_n : \pi_{n+1}(F_i) \to \pi_n(F_i)$ are the restrictions ($i = 1, 2$), and so $F_1 \subset F_2$.

If the subsequence $\{n_j\}_{j\geq 1}$ does not exist, then necessarily there is n_0 for which $\pi_n(F_2) \subset \pi_n(F_1)$ for $n \geq n_0$, and so $F_2 \subset F_1$. ■

THEOREM 5.2. *Any ray-type space X is a one-ended generalized continuum.*

Proof. X has at most one non-compact component by Lemma 5.1. On the other hand, Proposition 3.3 implies that the Aleksandrov compactification X^+ is an (arc-like) continuum, and so Lemma 2.3 shows that the closure of any component $C \subset X = X^+ - \{\infty\}$ must contain $\infty \in C$. Hence, X has non-compact components and so it is a generalized continuum.

Suppose that X has two distinct ends $\varepsilon_i = (Q^i_n)_{n\geq 1}$ ($i = 1, 2$) defined by sequences of quasicomponents $Q^i_n \subset X - \operatorname{int} K_n$ for the exhausting sequence $\{K_n\}_{n\geq 1}$. As $\varepsilon_1 \neq \varepsilon_2$, there exists an n_0 such that the quasicomponents $Q^1_{n_0}$ and $Q^2_{n_0}$ are disjoint. Let $C_i \subset Q^i_{n_0}$ be a non-compact component given by Lemma 2.2. As each Q^i_n is a closed set, so is C_i, and Lemma 5.1 yields either

$C_1 \subset C_2 \subset Q^2_{n_0}$, or $C_2 \subset C_1 \subset Q^1_{n_0}$. In both cases, $Q^1_{n_0} \cap Q^2_{n_0} \neq \emptyset$, which is a contradiction. ∎

REMARK 5.3. In continuum theory, arc-like spaces are characterized by the existence for each $\epsilon > 0$ of an ϵ-map $f : X \to [0,1]$ (i.e., for each $x \in X$, $\operatorname{diam}(f^{-1}(f(x))) < \epsilon$).

A crucial step in the proof is the fact that if f is an ϵ-map then there exists $\delta > 0$ such that $\operatorname{diam}(f^{-1}(A)) < \epsilon$ whenever $\operatorname{diam}(A) < \delta$; see [N, 2.33]. This property does not hold for ray-type spaces; indeed, the linear homeomorphism $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $g(n) = n$ and $g(n + 1/2) = n + 1/n$ is an ϵ-map for all $\epsilon > 0$ but for the sets $A_n = [n, n + 1/n]$ we have $\operatorname{diam}(A_n) < 1/n$ and $\operatorname{diam}(g^{-1}(A_n)) = 1/2$ for all n.

In order to obtain a characterization of ray-type spaces in terms of ϵ-maps $f : X \to \mathbb{R}_{\geq 0} \cong [0,1)$ we have to consider metrics on X which are *controlled at infinity*, that is, for each $\eta > 0$ there exists a compact set $K \subset X$ such that $d(x,y) < \eta$ if $x, y \in X - K$ (these metrics are exactly restrictions of metrics on the Aleksandrov compactification X^+).

This way, a space X is ray-type if and only if, given a metric d on X controlled at infinity, there exists an ϵ-map $f : (X,d) \to \mathbb{R}_{\geq 0}$ for any $\epsilon > 0$. For this we observe that $X = \varprojlim_p \{\mathbb{R}_{\geq 0}, g_n\}$ is ray-type if and only if $X^+ = \varprojlim_\infty \{\mathbb{R}^+_{\geq 0}, g^+_n\}$ (Proposition 3.3). Then, a careful inspection of the arguments in the proof of [N, 12.19] shows that the latter is equivalent to the existence of an ϵ-map $f : X^+ \to \mathbb{R}^+_{\geq 0} \cong [0,1]$ with $f^{-1}(\infty) = \infty$ for any $\epsilon > 0$.

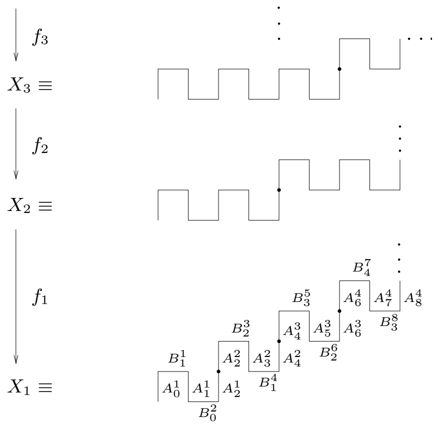
It is known that the class of arc-like spaces contains a universal space (see [S]). In contrast, the class of ray-type spaces admits no universal space. Recall that a space U is said to be *universal* in a topological category $\mathcal{C}$ if every space of $\mathcal{C}$ can be embedded in U.

THEOREM 5.4. *There is no universal space in the category $\mathcal{R}$ of ray-type spaces and proper maps.*

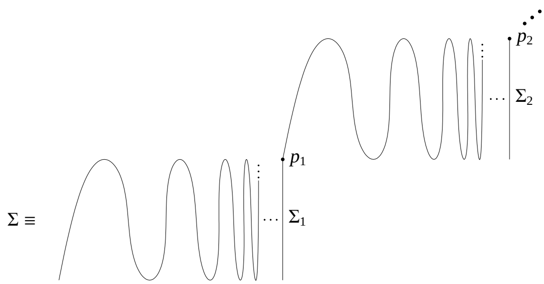
For this we define a *rayless space* to be a space which does not admit a proper embedding of the half-line $\mathbb{R}_{\geq 0}$.

Proof of Theorem 5.4. Let $X = \mathbb{R}_{\geq 0}$ be the half-line and Y be a rayless ray-type space in $\mathcal{R}$ (see Example 5.5 below for an example of such a space). Assume that there is a universal space $U \in \mathcal{R}$. This implies the existence of closed embeddings $X, Y \subset U$. By Lemma 5.1 we have either $X \subset Y$ or $Y \subset X$. The former is ruled out since Y is rayless, and thus Y is a closed connected subset in X and hence an interval. This is a contradiction and the theorem follows. ∎

EXAMPLE 5.5. Next we describe an example of a rayless ray-type space. For this we consider the family of unit segments in the planar grid of unit squares given by $A_i^j = \{(x,y) \in \mathbb{R}^2;\ x = i,\ j-1 \le y \le j\}$ and $B_j^i = \{(x,y) \in \mathbb{R}^2;\ y = j,\ i-1 \le x \le i\}$. Let X_n be the ray in the plane grid obtained by adding to the union $\bigcup\{A_i^j;\ 1 \le j < \infty,\ 2n(j-1) \le i \le 2nj\}$ a minimum set of horizontal segments B_j^i; see the figure below. The proper bonding maps $f_n : X_{n+1} \to X_n$ are given by the obvious maps which carry the segment $A_i^j \subset X_{n+1}$ linearly onto $A_{i-2(j-1)}^j \subset X_n$ if $2(n+1)(j-1) \le i \le 2(n+1)j-2$ and onto A_{2nj}^j if $2(n+1)j-2 \le i \le 2(n+1)j$.



We claim that $X = \varprojlim_p\{X_n, f_n\}$ is homeomorphic to the space $\Sigma = \bigcup_{k=1}^\infty \Sigma_k$ depicted below.



For this we observe that X_n decomposes as a union $X_n = \bigcup_{k=1}^\infty X_n^k$ where X_n^k is the arc in X_n containing $\bigcup_{2(k-1)\le i \le 2k} A_i^k$. Moreover $f_n(X_{n+1}^k) = X_n^k$ for all $n, k \ge 1$. From this, it is readily checked that for the restrictions $f_n^k = f|X_n^k$, the inverse limit $X^k = \varprojlim\{X_n^k, f_n^k\}$ is a closed subset in X and $X = \bigcup_{k=1}^\infty X^k$. Moreover, the definition of the bonding maps f_n yields homeomorphisms $\varphi_k : X^k \cong \Sigma^k$ onto the topologists's sine curve $\Sigma_k \subset \Sigma$, which are compatible at the points $\{p_k\}_{k\ge 1}$. This way we get a homeomorphism $\varphi = \bigcup_{k=1}^\infty \varphi_k : X = \bigcup_{k=1}^\infty X^k \cong \Sigma = \bigcup_{k=1}^\infty \Sigma_k$, and the result follows.

6. Further results and final remarks. As an extension of ray-type spaces, a space X is said to be a *T-type space* if it is the limit $X = \varprojlim_p\{X_n, f_n\}$ of an end-faithful sequence where each $X_n = T$ is the locally finite tree T. For $T = \mathbb{R}$ the euclidean line, Theorem 5.2 extends to $\mathbb{R}$-type spaces. Namely,

THEOREM 6.1. *Any $\mathbb{R}$-type space X is a two-ended generalized continuum.*

Proof. Assume that $X = \varprojlim_p\{\mathbb{R}, f_n\}$ is not connected with a compact component D. As X is $\mathbb{R}$-type, the one-point compactification X^+ is an (S^1-like) continuum. Here we use Proposition 3.3. Then, by Lemma 2.3, $\infty \in D$, which is a contradiction. Therefore, all components of X are non-compact.

Suppose that X has at least three components C_1, C_2 and C_3. As the canonical projections $\pi_n : X \to \mathbb{R}$ are proper maps, it follows that $\pi_n(C_i) \subset \mathbb{R}$ $(i = 1, 2, 3)$ are non-compact closed connected sets, and hence unbounded closed intervals. Thus, at least two of them are related by inclusion, say $\pi_n(C_1) \subset \pi_n(C_2)$. By arguing as in the proof of Lemma 5.1 and by using Lemma 4.5 we get

$$C_1 = \varprojlim_p\{\pi_n(C_1), f_n^1\} \subset \varprojlim_p\{\pi_n(C_2), f_n^2\} = C_2,$$

which is a contradiction. Here $f_n^i : \pi_{n+1}(C_i) \to \pi_n(C_i)$ are the corresponding restrictions of the bonding maps f_n.

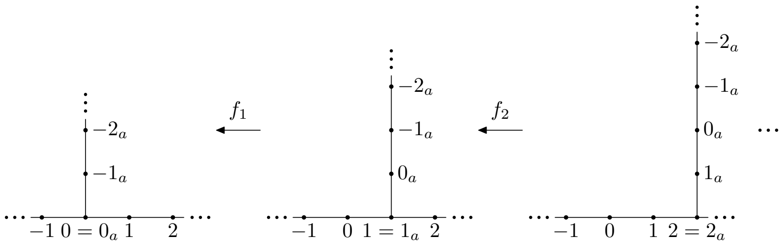
It remains to rule out the case that X has exactly two non-compact components C_1 and C_2. For this we observe that the connectedness of $\mathbb{R}$ implies that for each n the unbounded intervals $\pi_n(C_i)$ $(i = 1,\ 2)$ must have a non-empty intersection $A_n = \pi_n(C_1) \cap \pi_n(C_2) \neq \emptyset$, which can be assumed to be a compact interval for all $n \geq n_0$, since otherwise $\pi_n(C_1) \subset \pi_n(C_2) = \mathbb{R}$ (or vice versa) for each n, and we would proceed as in the previous case.

The compactness of the A_n's yields $\emptyset \neq \varprojlim\{A_n, f'_n\} \subset C_1 \cap C_2$ for the restrictions $f'_n = f_n|_{A_{n+1}}$, which is a contradiction. Hence X is a generalized continuum.

Moreover, Corollary 4.2 shows that X has at least two ends. Here we use the fact that the sequence defining X is end-faithful. Next we check that the number of ends is at most 2. Indeed, assume on the contrary that $\varepsilon_i = (Q_n^i)_{n\geq 1}$ $(1 \leq i \leq 3)$ are distinct ends where, for each n, $Q_n^i \subset X - \operatorname{int} K_n$ is a quasicomponent for the exhausting sequence $\{K_n\}_{n\geq 1}$. Then there exists m such that the quasicomponents Q_m^1, Q_m^2, Q_m^3 are pairwise disjoint. If, for each $i \leq 3$, $C_i \subset Q_m^i$ is a non-compact component given by Lemma 2.2, then for each $n \geq 1$, at least two of the three non-compact and connected sets $\pi_n(C_i)$ share one of the ends of $\mathbb{R}$. In particular, we find a pair of indices $1 \leq i < j \leq 3$ and a subsequence $\{n_k\}_{k\geq 1}$ such that both $\pi_{n_k}(C_i)$ and $\pi_{n_k}(C_j)$ contain the same end of $\mathbb{R}$. Hence, for each $k \geq 1$, either $\pi_{n_k}(C_i) \subset \pi_{n_k}(C_j)$

or $\pi_{n_k}(C_j) \subset \pi_{n_k}(C_i)$. As in the proof of Lemma 5.1, we can readily infer that either $C_i \subset C_j$ or $C_j \subset C_i$. This contradicts the assumption $\varepsilon_i \neq \varepsilon_j$, and the proof is finished. ∎

Easy examples show that Theorems 5.2 and 6.1 on $\mathbb{R}_{\geq 0}$-type and $\mathbb{R}$-type spaces, respectively, do not hold for other trees. More precisely, a T-type space X, with T a one-ended tree, may fail to be connected and end-preserving, as shown by the example at the beginning of Section 4. Moreover, Example 4.3 shows that the end space of X may even fail to be metrizable. It is also easy to obtain an example showing that a T-type space Y with T a tree without end vertices may fail to be connected and end-preserving; see the figure below representing an inverse sequence of infinite triods whose limit is the union $\mathbb{R} \sqcup \mathbb{R}_a$ of two disjoint copies of the euclidean line.



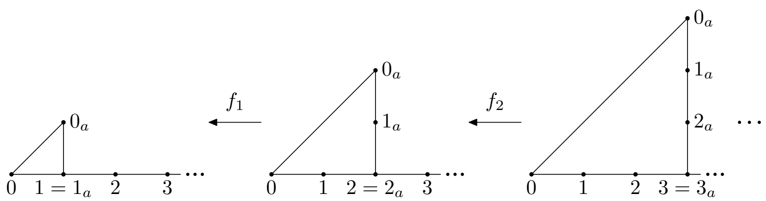
Example 4.3 suggests the following question:

QUESTION 6.2. Assume that $X = \varprojlim_p\{T_n, f_n\}$ is the limit of trees T_n with a finite number of branching points. Is the end space of X metrizable?

As a consequence of Proposition 4.4, if X is an end-preserving T-type space, then X is connected. But we do not have yet a positive answer or a counterexample for the converse:

QUESTION 6.3. Is any connected T-type space end-preserving?

A partial positive answer was given in Proposition 4.8. Also the following simple example shows that for some one-ended locally finite graphs G there exist connected G-type spaces which are not end-preserving. For instance, the euclidean line $\mathbb{R}$ is the inverse limit of the following sequence of one-ended graphs where the bonding maps are the obvious extensions of the ones of the example at the beginning of Section 4.



Appendix A. This appendix collects the basic facts of the theory of ends for (not necessarily metrizable) locally compact σ-compact Hausdorff spaces. We follow [H] and [FG]. Throughout this appendix we will use the notation introduced in Section 2. In particular, $\mathcal{A}$ stands for the family of all closed subsets with compact frontier of a space X. We start with some elementary lemmas whose proofs follow basically from the definitions and so are omitted.

LEMMA A.1. *An $\mathcal{A}$-ultrafilter $\mathcal{U}$ is trivial if and only if $\mathcal{U}$ contains a compact set.*

LEMMA A.2. *If $\{K_n\}_{n\geq 1}$ is an exhausting sequence in X, then all complements $X - \operatorname{int} K_n$ $(n \geq 1)$ belong to any non-trivial $\mathcal{A}$-ultrafilter $\mathcal{U}$.*

For any set $B \subset X$, let $B^{\mathcal{F}}$ denote the intersection $\overline{B}^{\widehat{X}} \cap \mathcal{F}(X)$ in the Freudenthal compactification of X.

LEMMA A.3. *For any $A \in \mathcal{A}$, $\mathcal{B}(A) = \overline{A}^{\widehat{X}} = A \cup A^{\mathcal{F}}$. In particular, $\mathcal{U} \in \overline{A}^{\widehat{X}}$ if and only if $A \in \mathcal{U}$.*

Let $\mathcal{G}$ denote the family of all open subsets with compact frontier in X. Then by using Lemmas A.1 and A.3 one gets

LEMMA A.4. *For each $G \in \mathcal{G}$ the set $G^{\natural} = G \cup G^{\mathcal{F}}$ is open in $\widehat{X}$, and these sets together with the open sets of X form a basis of open sets in $\widehat{X}$.*

The following statement is an immediate consequence of the previous lemma.

LEMMA A.5. *For any compact set $K \subset X$ and any closed set $F \subset X$ with compact frontier $\operatorname{Fr} F \subset K$ the difference $F - K$ is an open set in $\mathcal{G}$ and $(F - K)^{\natural}$ is an open set in $\widehat{X}$.*

Next we prove the main result of this appendix.

LEMMA A.6. *Let $f : X \to Y$ be a proper map between admissible spaces. Then f induces a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ which restricts to a map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$.*

In the proof of Lemma A.6 we will use the following

LEMMA A.7. *Let $f : X \to Y$ a proper map and $A \subset Y$. If $\operatorname{Fr} A$ is compact then so is $\operatorname{Fr}(f^{-1}(A))$.*

Proof. By continuity the closed set $\operatorname{Fr}(f^{-1}(A))$ is contained in the compact set $f^{-1}(\operatorname{Fr} A)$. Here we use the fact that f is proper. Hence, $\operatorname{Fr}(f^{-1}(A))$ is compact. ∎

Proof of Lemma A.6. Let $\mathcal{U}$ be a non-trivial ultrafilter. By compactness of $\widehat{Y}$, the filter $f(\mathcal{U})$ generated by the images of elements of $\mathcal{U}$ has at least one cluster point (see [E, 3.1.24]) and so

$$Z_{\mathcal{U}} = \bigcap_{U\in\mathcal{U}} \overline{f(U)}^{\widehat{Y}} \neq \emptyset.$$

Furthermore, this intersection contains no elements of Y. Indeed, otherwise take $y \in Z_{\mathcal{U}} \cap Y = \bigcap_{U\in\mathcal{U}} f(U)$. Here we use the fact that each $f(U)$ is a closed set in Y. Thus, the compact set $f^{-1}(y)$ meets all $U \in \mathcal{U}$ and so $\bigcap_{U\in\mathcal{U}} U \cap f^{-1}(y) \neq \emptyset$ ([E, 3.1.24]), which is a contradiction since $\mathcal{U}$ is not trivial; see Lemma A.1.

Moreover, $Z_{\mathcal{U}}$ reduces to one element. To prove this, assume that $\mathcal{W}_1$ and $\mathcal{W}_2$ are two distinct $\mathcal{A}$-ultrafilters in $Z_{\mathcal{U}}$. Then we can find two disjoint closed sets $W_i \in \mathcal{W}_i$ $(i = 1,2)$ with compact frontier. Moreover, by Lemma A.5, $(W_i - \mathrm{Fr}\, W_i)^{\natural}$ is an open neighbourhood of $\mathcal{W}_i$ in $\widehat{Y}$ and hence

$$f(U) \cap (W_i - \mathrm{Fr}\, W_i)^{\natural} = f(U) \cap (W_i - \mathrm{Fr}\, W_i) \neq \emptyset.$$

Thus, $W_i \cap f(U) \neq \emptyset$, and so $U \cap f^{-1}(W_i) \neq \emptyset$ for all $U \in \mathcal{U}$ and $i = 1,2$. By Lemma A.7, $f^{-1}(W_i)$ is a closed set with compact frontier. Hence $f^{-1}(W_1)$ and $f^{-1}(W_2)$ are disjoint sets in the $\mathcal{A}$-ultrafilter $\mathcal{U}$, which is a contradiction.

The previous observations show that f extends to a well-defined map $\widehat{f} : \widehat{X} \to \widehat{Y}$ by setting $\widehat{f}(\mathcal{U}) = f_*(\mathcal{U})$ where $f_*(\mathcal{U})$ is the only element in $Z_{\mathcal{U}}$. In order to prove the continuity of $\widehat{f}$, let $G^{\natural}$ be a basic open set as in Lemma A.4. Then $\widehat{f}^{-1}(G^{\natural}) = f^{-1}(G) \cup f_*^{-1}(G^{\mathcal{F}})$. Moreover $\mathrm{Fr}(f^{-1}(G))$ is compact by Lemma A.7, and the continuity of $\widehat{f}$ will follow if we check the equality

$$f_*^{-1}(G^{\mathcal{F}}) = (f^{-1}(G))^{\mathcal{F}}$$

yielding $\widehat{f}^{-1}(G^{\natural}) = (f^{-1}(G))^{\natural}$. For this, given $\mathcal{U} \in f_*^{-1}(G^{\mathcal{F}})$, it follows that $G^{\natural}$ is an open neighbourhood of $f_*(\mathcal{U})$ in $\widehat{Y}$ and hence $G \cap f(U) = G^{\natural} \cap f(U) \neq \emptyset$ for all $U \in \mathcal{U}$. Therefore, $(f^{-1}(G))^{\natural} \cap U = f^{-1}(G) \cap U \neq \emptyset$ for all $U \in \mathcal{U}$; that is, $\mathcal{U} \in f^{-1}(G)^{\mathcal{F}}$. The converse is similar. ∎

Appendix B. Freudenthal ends of generalized continua can be defined alternatively by using nested sequences of quasicomponents. This appendix contains a proof of the equivalence of both approaches (Theorem B.7). A third description of the Freudenthal compactification by using sequences is given in [B].

Recall that, given a space X, the *quasicomponent* of $x \in X$, denoted by $Q = Q(x)$, is defined to be the intersection of all closed-open sets of

X containing x. The partition into quasicomponents of X refines the partition into components (i.e., each component is contained in a quasicomponent); moreover, the continuous image of a quasicomponent is contained in a quasicomponent. For compact metric spaces, quasicomponents coincide with components; see [K] for details.

LEMMA B.1. *Let X be an admissible space. Given a compact set $K \subset X$ and a disjoint quasicomponent $Q \subset X$, there exists a closed-open set U with $Q \subset U$ and $U \cap K = \emptyset$.*

Proof. As K and Q are disjoint, for each $x \in K$ there is a closed-open set H_x with $x \in H_x$ and $Q \subset X - H_x$. By compactness, $K \subset H = \bigcup_{i=1}^{n} H_{x_i}$ for some $n \geq 1$, and we are done by setting $U = X - H$. ∎

The *space of quasicomponents of* X is the set $\mathcal{Q}(X)$ of quasicomponents of X endowed with the topology generated by the basis of open sets consisting of all the sets $A^{\Diamond} = \{Q; Q \in \mathcal{Q}(X) \text{ and } Q \subset A\}$ where $A \subset X$ ranges over all closed-open subsets in X. Any continuous map $f : X \to Y$ between admissible spaces induces a continuous map $f_{\#} : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ which carries a quasicomponent $Q \subset X$ to the unique quasicomponent $Q' \subset Y$ with $f(Q) \subset Q'$.

LEMMA B.2. *Let X be a generalized continuum. For any compact set $K \subset X$ the space of quasicomponents $\mathcal{Q}(X - \operatorname{int} K)$ is compact.*

Proof. Consider any cover $\mathcal{Q}(X - \operatorname{int} K) = \bigcup_{\alpha\in\Lambda} A_{\alpha}^{\Diamond}$ where each A_α is a closed-open set in $X - \operatorname{int} K$. The connectedness of X guarantees that $A_\alpha \cap \operatorname{Fr} K \neq \emptyset$ for all α, and the compactness of $\operatorname{Fr} K$ yields $\operatorname{Fr} K \subset \bigcup_{i=1}^{s} A_{\alpha_i}$ for some $s \geq 1$. We claim that $\mathcal{Q}(X - \operatorname{int} K) \subset \bigcup_{j=1}^{s} A_{\alpha_j}^{\Diamond}$. Indeed, by Lemma B.1 and connectedness of X, $Q \cap \operatorname{Fr} K \neq \emptyset$ for all $Q \in \mathcal{Q}(X - \operatorname{int} K)$. Hence, given $x \in Q \cap \operatorname{Fr} K$ there is $i \leq s$ with $x \in A_{\alpha_i}$, thus $Q \subset A_{\alpha_i}$, or equivalently, $Q \in A_{\alpha_i}^{\Diamond}$. ∎

PROPOSITION B.3. *Let X be a generalized continuum and $K \subset X$ be a compact subset. Then $\mathcal{Q}(X - \operatorname{int} K)$ is homeomorphic to a closed subspace of the Cantor set.*

Proof. By Lemma B.2, $\mathcal{Q}(X - \operatorname{int} K)$ is compact, and by [K, Thm. 3 p. 148 and Thm. 5 p. 151] there exists an embedding $\mathcal{Q}(X - \operatorname{int} K) \hookrightarrow \prod_{i=1}^{\infty}\{0,1\}$. Here we use the fact that $X - \operatorname{int} K$ is second countable. ∎

Given an exhausting sequence $\{K_n\}_{n\geq 1}$ of X, a $\mathfrak{q}$-*end* of X is a sequence $(Q_n)_{n\geq 1}$ of quasicomponents $Q_n \subset X - \operatorname{int} K_n$ with $Q_{n+1} \subset Q_n$. Let $\mathcal{E}(X)$ denote the set of all $\mathfrak{q}$-ends of X. The set ${}^{\mathfrak{q}}X = X \cup \mathcal{E}(X)$ admits a compact topology whose basis consists of all open sets of X together with the sets

$$ {}^{\mathfrak{q}}\Omega = \Omega \cup \{(Q_n)_{n\geq 1}; \text{ there is } n_0 \text{ with } Q_n \subset \Omega \text{ for } n \geq n_0\} $$

where $\Omega \subset X$ is any open set with compact frontier. We call ${}^{\mathfrak{q}}X$ the $\mathfrak{q}$-*compactification* of X. Moreover, the subspace $\mathcal{E}(X) \subset {}^{\mathfrak{q}}X$ turns out to be homeomorphic to $\varprojlim \mathcal{Q}(X - \operatorname{int} K_n)$, and hence, by Proposition B.3, to a closed subset of the Cantor set.

Given a set $M \subset X$, let $M^{\mathcal{E}}$ denote the intersection $\overline{M}^{\,{}^{\mathfrak{q}}X} \cap \mathcal{E}(X)$. If $M = \Omega$ is an open set with compact frontier, then it is readily checked that ${}^{\mathfrak{q}}\Omega = \Omega \cup \Omega^{\mathcal{E}}$. Moreover:

LEMMA B.4. *The family of sets of the form $A^{\flat} = A \cup A^{\mathcal{E}}$ where A ranges over all closed subsets with compact frontier in X forms a basis of closed sets in ${}^{\mathfrak{q}}X$.*

Proof. The difference $\Omega = X - A$ is an open set with compact frontier $\operatorname{Fr} \Omega = \operatorname{Fr} A$ contained in the interior of some K_n. Hence $A \cap (X - \operatorname{int} K_n)$ and $\Omega \cap (X - \operatorname{int} K_n)$ form a partition of $X - \operatorname{int} K_n$ into two open sets and so $A^{\mathcal{E}} = \mathcal{E}(X) - \Omega^{\mathcal{E}}$. Thus, $A^{\flat} = {}^{\mathfrak{q}}X - {}^{\mathfrak{q}}\Omega$, and the result follows. ∎

Let $\{K_n\}$ be an exhausting sequence of the generalized continuum X. Given an $\mathcal{A}$-ultrafilter $\mathcal{U} \in \mathcal{F}(X)$, we consider, for each $i \geq 1$, the filter

$$\mathcal{U}_i = \{U \in \mathcal{U};\ \operatorname{Fr} U \subset \operatorname{int} K_i\}. \tag{B.1}$$

Notice that $\mathcal{U}_i \neq \emptyset$ for each $i \geq 1$ since $X - \operatorname{int} K_{i-1} \in \mathcal{U}_i$. Notice also that for any $U \in \mathcal{U}$ there exists n_0 such that $U \in \mathcal{U}_n$ for all $n \geq n_0$. Moreover, the connectedness of X yields $U \cap \operatorname{Fr} K_i \neq \emptyset$ for all $U \in \mathcal{U}_i$. Therefore, by [E, 3.1.24], the compactness of $\operatorname{Fr} K_i$ guarantees that for each $i \geq 1$ the intersection of closed sets

$$L_i = \Big(\bigcap_{U \in \mathcal{U}_i} U\Big) \cap \operatorname{Fr} K_i$$

is a non-empty compact subset of $\operatorname{Fr} K_i$. Moreover, the family $\{\operatorname{Fr} K_i\}_{i \geq 1}$ is locally finite and so the union $L = \bigcup_{i=1}^{\infty} L_i$ is a closed set in X. In addition we have

LEMMA B.5. *The set $L^{\mathcal{E}}$ consists of exactly one $\mathfrak{q}$-end $\varepsilon_{\mathcal{U}}$.*

Proof. Let $\{x_i\}_{i \geq 1}$ be any sequence with $x_i \in L_i$. By compactness of ${}^{\mathfrak{q}}X$, there is a subsequence converging to some end $\varepsilon \in L^{\mathcal{E}}$, and so $L^{\mathcal{E}} \neq \emptyset$.

Next we show that $L^{\mathcal{E}}$ consists of exactly one $\mathfrak{q}$-end. For this, assume on the contrary that $\varepsilon = (Q_n)_{n \geq 1}$ and $\varepsilon' = (Q'_n)_{n \geq 1}$ are two $\mathfrak{q}$-ends in $L^{\mathcal{E}}$. Then one finds $i \geq 1$ such that there is a closed-open set H in $X - \operatorname{int} K_i$ with $Q_n \subset H$ and $Q'_n \cap H = \emptyset$ for all $n \geq i$. If we set $H' = (X - \operatorname{int} K_i) - H$, then ${}^{\mathfrak{q}}H$ and ${}^{\mathfrak{q}}H'$ are basic open neighbourhoods of ε and ε', respectively. As $\varepsilon, \varepsilon' \in \overline{L}^{\,{}^{\mathfrak{q}}X}$, there are subsequences $\{x_{n_s}\}_{s \geq 1}$ and $\{x_{n_t}\}_{t \geq 1}$ of elements $x_n \in L_n$ with $x_{n_s} \in H$ and $x_{n_t} \in H'$ for all $s, t \geq 1$. Thus, $x_{n_s} \in \bigcap_{U \in \mathcal{U}_{n_s}} U \cap H \neq \emptyset$, and so $H \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ since each $U \in \mathcal{U}$ belongs to some $\mathcal{U}_{n_s}$. Since

$\mathcal{U}$ is an ultrafilter, we get $H \in \mathcal{U}$. Similarly, $H' \in \mathcal{U}$ and so $\emptyset = H \cap H' \in \mathcal{U}$, which is a contradiction. Thus, $L^{\mathcal{E}}$ reduces necessarily to a single element. ∎

Lemma B.5 yields a well-defined map

$$\tilde{\Psi} : \mathcal{F}(X) \to \mathcal{E}(X)$$

by setting $\tilde{\Psi}(\mathcal{U}) = \varepsilon_{\mathcal{U}}$.

THEOREM B.6. *The map $\tilde{\Psi}$ is a homeomorphism.*

Proof. First we show that $\tilde{\Psi}$ is bijective. For this, given two distinct $\mathcal{A}$-ultrafilters $\mathcal{U}, \mathcal{W} \in \mathcal{F}(X)$ we find for any $U \in \mathcal{U}$ a set $W \in \mathcal{W}$ with $U \cap W = \emptyset$. Here we use the fact that $\mathcal{W} \neq \mathcal{U}$ is an ultrafilter. With the notation of (B.1) above, we can assume without loss of generality that $U \in \mathcal{U}_{n_0}$ and $W \in \mathcal{W}_{n_0}$ for some n_0. As the $\mathfrak{q}$-ends $\varepsilon_{\mathcal{U}} = (Q_n)_{n\geq 1}$ and $\varepsilon_{\mathcal{W}} = (Q'_n)_{n\geq 1}$ are in $U^{\mathcal{E}}$ and $W^{\mathcal{E}}$, respectively, we have $Q_{n_0} \subset U$ and $Q'_{n_0} \subset W$. Hence $Q_{n_0} \cap Q'_{n_0} = \emptyset$, and $\varepsilon_{\mathcal{U}} \neq \varepsilon_{\mathcal{W}}$. This shows that $\tilde{\Psi}$ is injective.

Furthermore, $\tilde{\Psi}$ is onto. In fact, given $\varepsilon = (Q_n)_{n\geq 1} \in \mathcal{E}(X)$, the union $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n$ is the family of all closed-open sets in $X - \operatorname{int} K_n$ with $\bigcap\{H; H \in \mathcal{H}_n\} = Q_n$, forms a basis for an $\mathcal{A}$-filter. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{H}$. Then, for any $n \geq 1$,

$$L_n = \bigcap_{U\in\mathcal{U}_n} U \cap \operatorname{Fr} K_n \subset \bigcap_{H\in\mathcal{H}_n} H \cap \operatorname{Fr} K_n \subset Q_{n-1}.$$

Hence $\varepsilon \in L^{\mathcal{E}}$; that is, $\varepsilon = \varepsilon_{\mathcal{U}} = \tilde{\Psi}(\mathcal{U})$.

Finally, as both $\mathcal{F}(X)$ and $\mathcal{E}(X)$ are Hausdorff compact spaces, it will be enough to check that the bijection $\tilde{\Psi}$ is continuous. By using Lemma B.4 it suffices to show

$$\tilde{\Psi}^{-1}(A^{\mathcal{E}}) = \mathcal{B}(A) \cap \mathcal{F}(X)$$

for any closed set A with compact frontier, say $\operatorname{Fr} A \subset \operatorname{int} K_n$. To check this, let $\mathcal{U} \in \mathcal{B}(A)$. By definition $A \in \mathcal{U}$, and so $A \in \mathcal{U}_m$ for all $m \geq n+1$. Hence $L_m \subset A$ and so $\tilde{\Psi}(\mathcal{U}) = \varepsilon_{\mathcal{U}} \in A^{\mathcal{E}}$. Conversely, if $\varepsilon_{\mathcal{U}} \in A^{\mathcal{E}}$ then $A \cap L_{n_j} \neq \emptyset$ for a subsequence $\{n_j\}_{j\geq 1}$, and so $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}_{n_j}$. As any set in $\mathcal{U}$ belongs to $\mathcal{U}_{n_j}$ for some n_j, it follows that $A \in \mathcal{U}$; that is, $\mathcal{U} \in \mathcal{B}(A)$. Here we use the fact that $\mathcal{U}$ is an ultrafilter. ∎

We extend $\tilde{\Psi}$ to a map $\Psi : \widehat{X} \to {}^{\mathfrak{q}}X$ by setting $\Psi(\mathcal{U}_x) = x$ if $x \in X$.

THEOREM B.7. *Ψ is a homeomorphism.*

Proof. Clearly Ψ is a bijection. Moreover, the proof of Theorem B.6 shows that $\Psi^{-1}(A^{\flat}) = \mathcal{B}(A)$ for any closed set with compact frontier A, and Lemma B.4 implies the continuity of Ψ. Thus Ψ is a homeomorphism between Hausdorff compact spaces. ∎

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