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DECOMPOSITIONS OF CYCLIC ELEMENTS OF LOCALLY CONNECTED CONTINUA

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Abstract. Let X denote a locally connected continuum such that cyclic elements have metrizable G_δ boundary in X. We study the cyclic elements of X by demonstrating that each such continuum gives rise to an upper semicontinuous decomposition G of X into continua such that X/G is the continuous image of an arc and the cyclic elements of X correspond to the cyclic elements of X/G that are Peano continua.

1. Introduction. In Section IV of [15], G. T. Whyburn develops the theory of cyclic elements of a metric continuum. In [6], B. Lehman extends much of Whyburn's results to connected, locally connected, Hausdorff spaces. The reader is referred to the survey by B. McAllister [8] for an outline of the development of cyclic element theory up to 1980. In addition to the utility of cyclic element theory to the study and classification of Peano continua, it has also been very useful in the study of non-metric continua. Among others, Nikiel [9]–[11] and Nikiel, Tuncali and Tymchatyn [12] have utilized cyclic element theory in studies of continuous Hausdorff images of generalized arcs.

Many of the standard techniques and constructions in the theory of upper semicontinuous decompositions are not applicable to the collection of cyclic elements of a locally connected continuum simply because the cyclic elements need not be disjoint. As a result, decompositions of continua often fail to preserve the intrinsic cut point structure and thus the cyclic element structure of the base continua. If, however, X is a locally connected continuum and cyclic elements are G_δ in X and have metrizable boundary in X, we demonstrate that X does give rise to an upper semicontinuous decomposition G of X into continua such that X/G is the continuous image of an arc and such that the cyclic elements of X are in 1-1 correspondence with the cyclic elements of X/G. Furthermore, the cyclic elements of X/G are Peano subcontinua of X/G.

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The boundary conditions on X are quite strong. However, if one wishes to utilize such a decomposition in the setting of continuous images of compact ordered spaces (which was the basis for the author's interest) then the boundary conditions are in fact necessary. For example, it is shown in [3] that an IOK (see Section 2) M is metrizable if and only if M is separable and M may be embedded as a closed G_δ subset of some locally connected continuum Y.

These results (particularly those in Section 3) generalize a similar construction (proof of Lemma 4 of [14]) of L. B. Treybig. It should also be noted that a related unpublished study has been done by B. Pearson [13].

2. Preliminaries. All spaces are assumed to be Hausdorff. A *compactum* is a compact space. A *continuum* is a connected compactum. A space X is said to be an *IOK* if it is the continuous image of some compact ordered space K. If K is also connected, then X is said to be an *IOC*. An *arc* is a continuum X which admits a linear ordering such that the order topology coincides with the given topology.

If U is a cover of a space X and $A \subseteq X$ then the *star* of A is denoted $\operatorname{Star}(A, U)$ and is defined by $\operatorname{Star}(A, U) = \bigcup\{u \in U : A \cap u \neq \emptyset\}$. If U and V are covers of X then U is a *star refinement* of V if and only if for each $u \in U$ there is some $v \in V$ such that $\operatorname{Star}(u, U) \subseteq v$.

A point p of connected space X is said to *separate* X if $X - \{p\}$ is not connected. If S, T, and R are non-empty subsets of X then S is said to *separate* T and R if $X - S$ is the union of two mutually separated sets T' and R' such that $T \subseteq T'$ and $R \subseteq R'$. A *cyclic element* C of a continuum X is a subcontinuum of X that is maximal with respect to the property that no point separates C. If a cyclic element C of X is non-degenerate, C is said to be a *true cyclic element* of X. Otherwise, we say that C is a *degenerate cyclic element* of X.

3. Structure theorems. We begin by establishing some structural theorems concerning locally connected continua. Throughout this section, X is a locally connected continuum and Y is a closed subset of X with metric G_δ boundary in X.

The proof of Lemma 1 below may be found in [2].

LEMMA 1. *There is a sequence $\{U_n\}$ of X-open sets and a sequence $\{H_n\}$ of finite covers of Y by connected X-open sets such that*

(i) $Y = \bigcap U_n = \bigcap \operatorname{Cl}(U_n)$ *and*

$$X \supseteq \operatorname{Cl}(U_1) \supseteq U_1 \supseteq \operatorname{Cl}(U_2) \supseteq U_2 \supseteq \operatorname{Cl}(U_3) \supseteq U_3 \supseteq \cdots,$$

(ii) H_{n+1} *is a star-refinement of* H_n *for each* $n \geq 1$,

(iii) $\delta(h \cap \mathrm{Bd}(Y)) \leq (1/3)^n$ *for each* $h \in H_n$,
(iv) *if* $p \in \mathrm{Bd}(Y)$, $\{p\} = \bigcap \mathrm{Star}(p, H_n)$,
(v) *if* $h \in H_n$ *then* h *does not intersect both* U_{n+1} *and* $X - U_n$.

LEMMA 2. *Under the conditions of the previous lemma, it may be assumed that for each* H_n *and* $g \in H_n$ *there exists a* $p \in \mathrm{Bd}(g) - \mathrm{Cl}(\bigcup h)$ *where the union is over all* $h \in H_n$ *such that* $h \neq g$.

Proof. By selecting and relabeling an appropriate subsequence of $\{U_n\}$, we may assume that

$$U_1 \supseteq \mathrm{Cl}\Big(\bigcup H_1\Big) \supseteq \bigcup H_1 \supseteq \mathrm{Cl}(U_2) \supseteq U_2 \supseteq \mathrm{Cl}\Big(\bigcup H_2\Big) \supseteq \bigcup H_2 \supseteq \cdots .$$

Fix $n \geq 1$ and suppose $H_n = \{g_1, \ldots, g_j\}$. Select j distinct points $x_1, \ldots, x_j$ such that $x_i \in g_i - \mathrm{Cl}(U_{n+1})$ for each $i = 1, \ldots, j$. Apply normality to obtain disjoint open sets B_1 and O_1 such that $x_1 \in B_1$ and $O_1 \supseteq \mathrm{Cl}(U_{n+1}) \cup (X - g_1) \cup \{x_2, \ldots, x_j\}$. Apply normality again to obtain disjoint open sets B_2 and O_2 such that $x_2 \in B_2$ and

$$O_2 \supseteq \mathrm{Cl}(U_{n+1}) \cup (X - g_2) \cup \{x_3, \ldots, x_j\} \cup \mathrm{Cl}(B_1).$$

Assume that B_m and O_m are defined for all m such that $2 \leq m < j$. Apply normality to obtain disjoint open sets B_{m+1} and O_{m+1} such that $x_{m+1} \in B_{m+1}$ and

$$O_{m+1} \supseteq \mathrm{Cl}(U_{n+1}) \cup (X - g_{m+1}) \cup \{x_{m+2}, \ldots, x_j\} \cup \mathrm{Cl}\Big(\bigcup_{i=1}^{m} B_i\Big).$$

For each $i = 1, \ldots, j$, select an open neighborhood A_i of x_i so that $\mathrm{Cl}(A_i) \subseteq B_i$. Then for each $i = 1, \ldots, j$ we have $x_i \in A_i \subseteq \mathrm{Cl}(A_i) \subseteq B_i \subseteq \mathrm{Cl}(B_i) \subseteq g_i - \mathrm{Cl}(U_{n+1})$.

Now suppose that k and k' are distinct natural numbers and $k < k'$. Then $O_k \supseteq \mathrm{Cl}(B_{k'})$ and $O_k \cap \mathrm{Cl}(B_k) = \emptyset$, and therefore $\mathrm{Cl}(B_k) \cap \mathrm{Cl}(B_{k'}) = \emptyset$. For each $i = 1, \ldots, j$, define $H_{n_i} = \{h : h$ is a component of $(g_i - \mathrm{Cl}(A_i)) - \mathrm{Cl}(\bigcup_{l \neq i} B_l)$ and $h \cap Y \neq \emptyset\}$. Let $H'_n = \bigcup_{i=1}^{j} H_{n_i}$. Then, by construction, there exists, for each $h \in H_{n_i} \subseteq H'_n$, a point $y \in \mathrm{Bd}(h) \cap \mathrm{Bd}(A_i)$ such that $y \notin \mathrm{Cl}(\bigcup\{h' : h' \in H'_n$ and $h' \neq h\})$. ∎

In the following four results, let $\{U_n\}$ and $\{H_n\}$ denote the sequence of open sets and the sequence of finite covers, respectively, as constructed in the preceding two lemmas. Also, we set $G = \{h : h \subseteq X,\ h = \bigcap_n \mathrm{Star}(x, H_n)$ for some $x \in Y\} \cup \{\{z\} : z \in X - Y\}$.

LEMMA 3. *The elements of* G *partition* X *into continua.*

Proof. Define a relation R on X by xRy if and only if $x \in \bigcap_n \mathrm{Star}(y, H_n)$. Trivially, R is reflexive. Suppose xRy. Then for each n, there is an $h_n \in H_n$ such that $x, y \in h_n$. Therefore, $y \in \bigcap_n \mathrm{Star}(x, H_n)$ and R is symmetric.

Now suppose that xRy and yRz. For each n, there exist $h_{n+1}, g_{n+1} \in H_{n+1}$ such that without loss of generality $x, y \in h_{n+1}$ and $y, z \in g_{n+1}$. Since $h_{n+1} \cup g_{n+1} \subseteq \mathrm{Star}(y, H_{n+1}) \subseteq g_n$ for some $g_n \in H_n$, we have $x, z \in g_n$. Then $x \in \bigcap_n \mathrm{Star}(z, H_n)$ so that xRz and R is transitive. R is therefore an equivalence relation on X, and G is clearly the set of R-equivalence classes. If $x \in X$ and n is a natural number then $\mathrm{Star}(x, H_{n+2}) \subseteq h_{n+1} \subseteq \mathrm{Cl}(h_{n+1}) \subseteq \mathrm{Star}(h_{n+1}, H_{n+1}) \subseteq h_n \subseteq \mathrm{Star}(x, H_n)$ for some $h_n \in H_n$ and some $h_{n+1} \in H_{n+1}$. Therefore, $\bigcap \mathrm{Star}(x, H_n) = \bigcap \mathrm{Cl}(\mathrm{Star}(x, H_n))$ so that each element of G is a continuum. ∎

Lemma 4. *G is an upper semicontinuous decomposition of X into continua.*

Proof. Suppose $x \in g \in G$ and $g \subseteq U$ with U open in X. There exists a natural number n such that $\mathrm{Cl}(\mathrm{Star}(x, H_n)) \subseteq U$. Let $y \in \mathrm{Star}(x, H_{n+2}) - g$. Note that $\mathrm{Star}(y, H_{n+2}) \subseteq h_{n+1}$ for some $h_{n+1} \in H_{n+1}$. But $x \in h_{n+1}$ and $\mathrm{Star}(x, H_{n+1}) \subseteq U$. Therefore, if $y \in k \in G$ then $k \subseteq U$. ∎

Theorem 5. *Let $\phi : X \to X/G$ denote the natural map. Then X/G is a locally connected continuum and $\phi(Y)$ is metrizable.*

Proof. Since the set $\{\mathrm{Star}(x, H_n) : x \in Y\}$ is countable, the set $B = \{K : K = \{h \in G : h \subseteq \mathrm{Star}(x, H_n)$ for some fixed $x \in Y$ and fixed natural number $n\}$ and $\bigcup K$ is open in $Y\}$ is a countable basis for $\phi(Y)$. ∎

4. Application to cyclic elements. We consider applications of the results of the previous section to the collection of cyclic elements of a locally connected continuum. Although several of the results hold in a more general setting (e.g. Lemmas 6 and 7), we assume throughout this section that X is a locally connected continuum such that each cyclic element of X has metrizable G_δ boundary in X.

A straightforward modification of the proofs of Theorems 6 and 7 (pages 313–315) of [5] yields the following lemma.

Lemma 6. *Each cyclic element E of X is locally connected.*

Lemma 7. *Let E be a cyclic element of X. If $a \in E$ and B and C are disjoint connected subsets of E such that $a \notin \mathrm{Cl}(B) \cup \mathrm{Cl}(C)$ then there exists a connected neighborhood N of a such that N is open in X and N does not separate B from C in E.*

Proof. By normality, there exist disjoint connected X-open sets U and V such that $a \notin U \cup V$, $\mathrm{Cl}(B) \subseteq U$ and $\mathrm{Cl}(C) \subseteq V$. For each $x \in E - (U \cup V \cup \{a\})$, select an open connected neighborhood $N(x)$ of x such that $a \notin \mathrm{Cl}(N(x))$ and $\mathrm{Cl}(N(x)) \cap (\mathrm{Cl}(B) \cup \mathrm{Cl}(C)) = \emptyset$. Then $M = U \cup V \cup \{N(x) : x \in E - (U \cup V \cup \{a\})\}$ is a cover of $E - \{a\}$ by X-open sets. Select $b \in B$ and

$c \in C$. Then there is a chain $M' = \{M_1, \ldots, M_k\} \subseteq M$ from b to c. Clearly, both U and V must be elements of M'. Then $\mathrm{Cl}(\bigcup M')$ is a continuum in X that does not contain a. By normality, we may then select an X-open connected neighborhood N of a such that $N \cap \mathrm{Cl}(\bigcup M') = \emptyset$. ∎

It is well-known that a point being G_δ in a compact space is sufficient for first countability of the space at the point. This condition clearly holds for each point of the boundary of each cyclic element of X. The following lemma is then immediate.

LEMMA 8. *Let E be a cyclic element of X and $p \in \mathrm{Bd}(E)$. Then X is first countable at p.*

LEMMA 9. *Let E be a cyclic element of X. There exists a sequence $\{G_n\}$ of finite covers of E by connected X-open sets such that G_{n+1} is a star refinement of G_n for each $n \geq 1$, $\delta(g \cap \mathrm{Bd}(E)) \leq (1/3)^n$ for each $g \in G_n$, and if h and k are elements of G_n such that $\mathrm{Cl}(h) \cap \mathrm{Cl}(k) = \emptyset$ then no element of G_{n+1} separates h from k in E.*

Proof. For each point $p \in \mathrm{Bd}(E)$, let $N_1(p)$ denote an open connected neighborhood of p such that $\delta(N_1(p) \cap \mathrm{Bd}(E)) \leq 1/3$. For $q \in \mathrm{Int}(E)$, select an open connected neighborhood $M_1(q)$ such that $\mathrm{Cl}(M_1(q)) \cap \mathrm{Bd}(E) = \emptyset$. Select a finite subcover G_1 of E by elements of $\{N_1(p) : p \in \mathrm{Bd}(E)\} \cup \{M_1(p) : p \notin \mathrm{Bd}(E)\}$. Now suppose that G_n has been constructed and if $1 \leq m < n$ then G_m and G_{m+1} satisfy the lemma. Let $g \in G_n$ and define $S_n(g) = \{\{u, w\}$: u and w are elements of G_n such that $\mathrm{Cl}(u) \cap \mathrm{Cl}(w) = \emptyset$ and $g - (u \cup w)$ separates u from w in $E\}$. If $S_n(g) = \emptyset$ then g of course does not separate any pair of elements of G_n. If $S_n(g) \neq \emptyset$ then apply Lemma 7 to obtain for each $p \in g$ an open connected neighborhood $L(p)$ of p so that $L(p)$ does not separate u and w for all $\{u, w\} \in S_n(g)$, $L(p) \subseteq g$, and if $p \in \mathrm{Bd}(E)$ then $\delta(L(p) \cap \mathrm{Bd}(E)) \leq (1/3)^{n+1}$. Let R denote a star refinement of $\{g \in G_n : S_n(g) = \emptyset\}$ with the additional property that if $p \in \mathrm{Bd}(E)$ then each element r of R containing p satisfies $\delta(r \cap \mathrm{Bd}(E)) \leq (1/3)^{n+1}$. Set $G''_{n+1} = R \cup \{L(p) : p \in g$ for some $g \in G_n$ and $S_n(g) \neq \emptyset\}$. Then let G'_{n+1} be a star refinement of G''_{n+1} and let G_{n+1} be a finite cover of E by elements of G'_{n+1}. ∎

A straightforward (although lengthy and tedious) adaptation of the proofs of several previous results yields the following.

COROLLARY 10. *For each cyclic element E of X, there is a sequence $\{U_n\}$ of X-open sets and a sequence $\{G_n\}$ of finite covers of E by connected X-open sets satisfying the conclusions of each of Lemmas* 1, 2, *and* 9.

THEOREM 11. *For each cyclic element E of X, there exists an upper semicontinuous decomposition $G(E)$ of E into continua such that $E/G(E)$*

is a locally connected metric continuum such that no point separates $E/G(E)$ *and if* $p \in \mathrm{Bd}(E)$ *then* $\{p\} \in G(E)$.

Proof. Select and fix a cyclic element E of X. Suppose that $\{G_n\}$ is the sequence of finite covers of E as in the previous corollary. Define $G(E) = \{S \subseteq E : S = \bigcap \mathrm{Star}(x, G_n) \text{ for some } x \in E\}$. That $G(E)$ is an upper semicontinuous decomposition of E into continua follows from Lemma 4, and that $E/G(E)$ is a locally connected metric continuum follows from Theorem 5. By construction, each boundary point of E is a singleton element of $G(E)$.

We now show that no point separates $E/G(E)$. Assume not. Then there exists an $x \in E$ and $Y = \bigcap \mathrm{Star}(x, G_n) \in G(E)$ such that Y separates $E/G(E)$. Then $E/G(E) - \{Y\} = S_1 \cup S_2$ where S_1 and S_2 are mutually separated and each is non-empty. Suppose that $y_1 \in S_1$ and $y_2 \in S_2$. If $t \in Y$ then there exists a natural number k and $y' \in G_k$ such that $\mathrm{Star}(t, G_{k+1}) \subseteq y' \in G_k$ and so that some component K of $E - y'$ contains y_1 and y_2. Let $\phi_E : E \to E/G(E)$ denote the natural mapping. Then $\phi_E(K)$ is a connected subset of $E/G(E) - \{Y\}$ containing y_1 and y_2, a contradiction. ∎

For each cyclic element E of X, let $G(E)$ denote the upper semicontinuous decomposition of E as in the previous theorem. Let $\phi_E : E \to E/G(E)$ denote the natural mapping. Define $G = \{g : g \in G(E) \text{ for some true cyclic element } E \text{ of } X\} \cup \{\{x\} \text{ for each degenerate cyclic element } \{x\} \text{ of } X\}$.

Theorem 12. *E' is a cyclic element of X/G if and only if $E' = E/G(E)$ for some cyclic element E of X.*

Proof. ($\Leftarrow$) E' is connected and has no cut point by Theorem 11. Suppose F' is a cyclic element of X/G such that F' properly contains E'. Then there exists some cyclic element F of X such that $\phi_F^{-1}(F')$ is connected and has no cut point. Then F properly contains E, a contradiction to the maximality of E.

($\Rightarrow$) Let p be any point of E'. There exists a cyclic element E of X such that $\phi_E^{-1}(p) \subseteq E$. If $q \in E/G(E)$ then p and q are contained in a connected subset of X/G having no cut point. Therefore, $q \in E'$ so that $E/G(E) \subseteq E'$. Now suppose that $r \in E' - E/G(E)$. Then there exists a cyclic element F of X such that $F \neq E$ and $\phi_F^{-1}(r)$ is defined. There exists a point t such that t separates p and r in X. Then $\phi_F(t)$ is a cut point of E', a contradiction. Therefore, $E' \subseteq E/G(E)$. ∎

Lemma 13. *Let E be a true cyclic element of X and $x \in \mathrm{Bd}(E)$. Suppose that $x \in U$ where U is open in X. Then there exist at most a finite number of true cyclic elements of X that both contain x and intersect $X - U$.*

Proof. Each true cyclic element of X is contained in $\mathrm{Cl}(K)$ for exactly one component K of $X - \{x\}$. Also, each such component K intersects at most one cyclic element containing x. By the local connectivity of X, at most finitely many such components K may intersect $X - U$ for any open set U containing x. ■

THEOREM 14. *Let E be a cyclic element of X. Then the set $\{x \in E :$ there exists a cyclic element D distinct from E such that $x \in D\}$ is countable.*

Proof. Let $\{U_n : n = 1, 2, \dots\}$ be a collection of X-open sets such that $\bigcap_n U_n = E$. By an argument similar to that of Lemma 13, there are only countably many components K of $X - E$ that meet $X - U_n$ for some n. The set $X - E$ therefore has countably many components. If D is a cyclic element that meets E but is distinct from E then D is contained in $\mathrm{Cl}(K)$ for exactly one component K of $X - E$. Any such K may contain at most one such cyclic element D. ■

LEMMA 15. *Let $p \in \mathrm{Bd}(D)$ for some cyclic element D of X and suppose $p \in U$ with U open in X. Suppose also that $\{B_n\}$ is local basis at p of connected open sets such that $U \supseteq \mathrm{Cl}(B_1) \supseteq B_1 \supseteq \mathrm{Cl}(B_2) \supseteq B_2 \supseteq \cdots$. Then there is no sequence $\{h_n\}$ of elements of G such that*

(i) *for each i, $h_i \cap B_i \neq \emptyset$ and $h_i \cap (X - U) \neq \emptyset$,*
(ii) *$\bigcup h_i \subseteq \bigcup \mathcal{F}$, where $\mathcal{F} = \{F : F$ is a true cyclic element of X, $p \notin F\}$.*

Proof. Assume such a sequence exists. We consider two cases.

CASE 1. Suppose infinitely many elements of $\{h_n\}$ are contained in a single true cyclic element E of X; suppose $\bigcup_k h_{n_k} \subseteq E$. Then it follows that E contains a non-degenerate subcontinuum H such that H is the sequential limiting set of $\{h_{n_k}\}$ and such that $p \in H$. Then $p \in \mathrm{Bd}(E)$, a contradiction.

CASE 2. Suppose no single true cyclic element of X contains infinitely many elements of $\{h_n\}$. Then there exist infinitely many distinct true cyclic elements $F_1, F_2, \dots$ such that $F_i \cap B_i \neq \emptyset$ and $F_i \cap (X - U) \neq \emptyset$ for all i. We then obtain a contradiction by employing an argument similar to that used in the proof of Lemma 13. ■

LEMMA 16. *Let $p \in \mathrm{Bd}(E)$ for some true cyclic element E of X and suppose $p \in U$ with U open in X. Then there exists a collection Q of elements of G and an open set O such that $p \in O \subseteq \bigcup Q \subseteq U$.*

Proof. By the previous lemmas, there is an open connected neighborhood B at p such that only finitely many cyclic elements of X both contain p and meet $X - U$, and such that if F is a cyclic element of X that does not contain p then no element $g \in G(F)$ may intersect both B and $X - U$. Define $Q = \{g \in G : g \subseteq U\}$ and suppose $\{C_n\}$ is a countable local basis at p of connected open sets such that $B \supseteq \mathrm{Cl}(C_1) \supseteq C_1 \supseteq \mathrm{Cl}(C_2) \supseteq C_2 \supseteq \cdots$. Now assume that no such open set O exists. Then there exists a sequence

$\{p_n\}$ such that $\{p_n\}$ converges to p and $p_n \in X - \bigcup Q$ for each n. We may clearly assume that $p_n \in C_n$ for each n. Each p_n must be contained in some $g_n \in G$. Each g_n belongs to $G(E_n)$ for some cyclic element E_n of X. Since each g_n contains $p_n \in C_n \subseteq B$ and $g_n \cap (X - U) \neq \emptyset$, we may assume without loss of generality that $E_n = F$ for each n. Since $G(F)$ is upper semicontinuous, there is a relatively open set W in F that contains p such that $W \subseteq U$, and if $g \in G(F)$ and $g \cap W \neq \emptyset$ then $g \subseteq B \cap F$. This implies that $g_n \subseteq B \cap F \subseteq U$ for some n, a contradiction. ∎

Theorem 17. *G is an upper semicontinuous decomposition of X into continua, each cyclic element of X/G is a Peano continuum, and X/G is an IOC.*

Proof. Suppose $g \in G$ and $g \subseteq U$ with U open in X. We consider three cases.

Case 1. Suppose that $g = \{p\}$, where $p \in \mathrm{Bd}(E)$ for some true cyclic element E of X. By the previous lemma, there is a collection Q of elements of G and an X-open set O such that $p \in O \subseteq \bigcup Q \subset U$.

Case 2. Suppose $g \subseteq \mathrm{Int}(E)$ for some true cyclic element E of X. Since $G(E)$ is upper semicontinuous, there is an E-open set O' such that $O' \subseteq U$, and if $h \in G$ and $h \cap O' \neq \emptyset$ then $h \subseteq U$.

Case 3. Suppose that $g = \{p\}$ and $p \notin E$ for any true cyclic element E of X. Then there exists an open set W such that $p \in W$ and no true cyclic element of X intersects both W and $U - W$. Then no element of G that intersects W also intersects $X - U$.

Local connectivity of cyclic elements of X/G follows from Lemma 6 and metrizability thereof follows from Theorem 5. X/G is an IOC by a result of Cornette [1]. ∎

Finally we demonstrate an application of the results of this section in light of recent results on liftings of quotient mappings. We continue to assume that X is a locally connected continuum such that each cyclic element of X has metrizable G_δ boundary in X. Also, let X/G be the decomposition of Theorem 17 with $\phi : X \to X/G$ the natural mapping.

Theorem (Theorem 1 of Daniel, Nikiel, Treybig, Tuncali, and Tymchatyn [4]). *Let Z be a locally connected continuum and H an upper semicontinuous decomposition of Z such that*

(i) *each $g \in H$ is connected and has zero-dimensional boundary,*
(ii) *each $g \in H$ is a continuous image of an ordered compactum.*

If the quotient space Z/H is the continuous image of an ordered continuum then so too is Z.

COROLLARY 18. *If each $g \in G$ is an IOK and has zero-dimensional boundary in X then X is an IOC.*

Proof. For each true cyclic element C of X, $\phi(C)$ is a Peano continuum and is therefore perfectly normal. Hence $\phi^{-1}(g)$ has closed, zero-dimensional, G_δ boundary in X for each $g \in G$. It follows from Mardešić [7] that $\mathrm{Bd}(g)$ is in fact metrizable for each $g \in G$. Then each cyclic element is an IOC by the Theorem above. It then follows from a result of J. Cornette [1] that X is an IOC. ∎

We chose to include the proof above because it reveals slightly more of the structure of the elements of G, but an alternate proof is given simply by recalling the result of Cornette [1] and then directly applying the result above from [4].

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