COLLOQUIUM MATHEMATICUM

VOL. 110 2008 NO. 2

WHICH BERNOULLI MEASURES ARE GOOD MEASURES?

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Abstract. For measures on a Cantor space, the demand that the measure be "good" is a useful homogeneity condition. We examine the question of when a Bernoulli measure on the sequence space for an alphabet of size n is good. Complete answers are given for the $n = 2$ cases and the rational cases. Partial results are obtained for the general cases.

1. Introduction. Cantor space, unique up to homeomorphism, is a perfect, zero-dimensional, compact metrizable space. On a Cantor space, X, we consider full, nonatomic, Borel probability measures μ. For example, if $p = (p_1, \ldots, p_N)$ is a positive probability vector on a finite set A of cardinality $N > 1$ and I is a countable index set then the *Bernoulli measure associated with* p is such a measure. This Bernoulli measure, denoted $\beta(p)$, is the product measure on $X = A^I$ having independent, identically distributed coordinate projections with distribution given by p on each factor.

Ultimately, one would like to classify these measures, up to homeomorphism. Some of these can be distinguished by the invariant $S(\mu)$, the *clopen values set* for μ, which is the set of values $\mu(U)$ obtained as U varies over the clopen subsets of X. The set $S(\mu)$ is always countable and dense in the unit interval $I = [0, 1]$ with $0, 1 \in S(\mu)$. A subset $D \subset I$ with $1 \in D$ is called *grouplike* when $D = I \cap G$ with G an additive subgroup of the reals, $\mathbb{R}$. In that case, $G = D + \mathbb{Z}$. Furthermore, D is called *ringlike* or *fieldlike* if $D + \mathbb{Z}$ is a subring or a subfield, respectively, of $\mathbb{R}$.

We call μ a *good measure* when for every pair of clopen subsets U, V of X with $\mu(V) \le \mu(U)$ there exists a clopen subset V_1 of U such that $\mu(V) = \mu(V_1)$. The label "good" for this mild-appearing homogeneity property was introduced in Akin (2005). In fact, good measures have a number of strong properties:

- The automorphism group $H(X, \mu)$ of a good measure μ acts transitively on X. That is, given two points $x_1, x_2 \in X$ there exists $h \in$

2000 *Mathematics Subject Classification*: Primary 37B05; Secondary 28D05, 28C10.

Key words and phrases: Bernoulli measure, Cantor space measure, clopen values set, uniquely ergodic.

The third author was supported in part by NSF grant: DMS 0400481.

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$H(X,\mu)$ such that $h(x_1) = x_2$. Furthermore, if U_1, U_2 are clopen subsets of X with $\mu(U_1) = \mu(U_2)$ then there exists $h \in H(X,\mu)$ such that $h(U_1) = U_2$.

- For a good measure μ there exists $h \in H(X,\mu)$ such that the dynamical system on X obtained by iterating h is a uniquely ergodic, minimal system. Conversely, for any uniquely ergodic minimal homeomorphism on a Cantor space the invariant measure is good (Glasner and Weiss, 1995).
- If μ is good then the clopen values set $S(\mu)$ is group-like. Conversely, for every countable, dense group-like subset D of $[0,1]$ there is a good measure μ on a Cantor space such that $S(\mu) = D$. Furthermore, μ is unique up to homeomorphism. In particular, the clopen values set is a complete homeomorphism invariant for good measures.

Thus the good measures are exactly the measures which arise from minimal, uniquely ergodic systems on a Cantor space, the so-called Jewett–Krieger measures. So the natural examples come from adding machine systems, also called odometers, and from lifts of irrational rotations on the circle. For this reason Bernoulli measures, associated with the (far from minimal) shift maps, were not seriously considered in Akin (2005).

Much earlier Oxtoby and his students had been examining the homeomorphism relation among Bernoulli measures: see, for example, Navarro-Bermúdez (1979) and Navarro-Bermúdez and Oxtoby (1988). This line of work was continued by Mauldin (1990), and eventually by Dougherty *et al.* (2007) where the question of our title arose. One crucial observation was that for a Bernoulli measure $\mu = \beta(p)$ the clopen values set is closed under multiplication and so $S(\mu)$ is ringlike if it is grouplike. Furthermore, the ring generated by $S(\mu)$ is exactly $\mathbb{Z}[p_1, \ldots, p_N]$. From this we obtain our major result:

THEOREM 1.1. *Let p be a positive probability vector on a finite set A. For the associated Bernoulli measure $\beta(p)$ on $X = A^{\mathbb{N}}$ the following conditions are equivalent*:

(a) *$\beta(p)$ is a good measure on X.*

(b) *The clopen values set $S(\beta(p))$ is grouplike and $p_1, \ldots, p_N$ are units of the ring $\mathbb{Z}[p_1, \ldots, p_N]$.*

(c) *The clopen values set $S(\beta(p))$ is grouplike and there exists a polynomial P in N variables with integer coefficients such that*

$$1 = p_1 \cdot \ldots \cdot p_N \cdot P(p_1, \ldots, p_N).$$

(d) *The clopen values set $S(\beta(p))$ is grouplike and the automorphism group $H(X, \beta(p))$ acts minimally on X.*

(e) *There exists a homeomorphism h on X such that $\beta(p)$ is the unique invariant measure for h.*

As this result suggests, the major difficulty in recognizing a good measure is detecting whether or not the clopen values set is grouplike. The units condition of (b) is usually easy to check, but there are many simple looking examples where we cannot tell whether the set is grouplike and so do not know whether the measure is good.

In the two-letter alphabet case the situation is completely understood thanks to Dougherty *et al.* (2007) and Yingst's thesis, Yingst (2008).

By Theorem 1.1 we need only consider the case where $p_1 = r$ is algebraic. For $r \in (0,1)$ algebraic we call $R(t)$ a *minimal polynomial* for r if R is a polynomial of minimum degree among those with integer coefficients having r as a root and if, in addition, the greatest common divisor of the coefficients is 1. Thus, $R(t)$ is uniquely defined up to multiplication by ± 1.

THEOREM 1.2. *For $r \in (0,1)$ algebraic, let $R(t)$ be a minimal polynomial for r. Let $\mu = \beta(r, 1-r)$ be the associated Bernoulli measure on $X = \mathbf{2}^{\mathbb{N}}$.*

(a) *The following conditions are equivalent*:
   (i) *r and $1-r$ are units in the polynomial ring $\mathbb{Z}[r]$.*
   (ii) *$1/(r(1-r))$ is an algebraic integer.*
   (iii) *The automorphism group $H(X,\mu)$ acts minimally on X.*
   (iv) *The automorphism group $H(X,\mu)$ acts transitively on X.*
   (v) *$R(0), R(1) \in \{-1,+1\}$.*

(b) *$\beta(r, 1-r)$ is a good measure iff $R(0), R(1) \in \{-1,+1\}$ and r is the unique root of $R(t)$ in $(0,1)$.*

(c) *$S(\beta(r,1-r))$ is grouplike iff $R(0), R(1) \in \{-2,-1,+1,+2\}$ and r is the unique root of $R(t)$ in $(0,1)$.*

In particular, when r is rational, $\beta(r, 1-r)$ is good only for $r = 1/2$ and $S(\beta(r,1-r))$ is grouplike only for $r = 1/2, 1/3$ and $2/3$. Thus, $S(\beta(1/3,2/3))$ is grouplike but $\beta(1/3,2/3)$ is not good.

There exist examples which satisfy the conditions of (a) but not (b). One is described after Corollary 6.9 below. These provide the first examples that we know about of a minimal group action on a Cantor space with an invariant measure which is not good.

2. Good measures and their clopen values sets. The spaces X which we will consider are compact and metrizable, or equivalently, compact, second countable, Hausdorff spaces. Since every clopen (= closed and open) set is a finite union of basic open sets, such a space contains only countably many distinct clopen sets. The space is *zero-dimensional* when the clopen sets form a basis. A nonempty, perfect zero-dimensional space is called a *Cantor space* and any two such are homeomorphic.

A measure is *full* when every nonempty open subset has positive measure. For a full measure any isolated point is an atom. A measure *has only isolated atoms* if any atom of the measure is an isolated point of the space. In particular, if the space is perfect then such a measure is nonatomic. From now on we assume that all of our measures are full Borel probability measures with only isolated atoms. The two cases we will use most are the full, nonatomic measures on a Cantor space and the measures on a finite set with each point given positive mass.

If $f : X_1 \to X_2$ is a continuous map and μ_1 is a measure on X_1 then $f_*\mu_1$ denotes the induced measure on X_2 given by

$$f_*\mu_1(B) := \mu_1(f^{-1}(B)). \tag{2.1}$$

With $\mu_2 = f_*\mu_1$ we say that f *maps* μ_1 *to* μ_2. We say that μ_1 *maps to* μ_2 if such a continuous map f exists. Since the measures are full, such a map f is necessarily surjective. If f can be chosen to be a homeomorphism then we say that μ_1 is *homeomorphic* to μ_2, written $\mu_1 \approx \mu_2$. Of course, then f^{-1} maps μ_2 to μ_1. In general, we write $\mu_1 \sim \mu_2$ if μ_1 maps to μ_2 and μ_2 maps to μ_1. As we will see, this equivalence need not imply that the two measures are homeomorphic.

We denote by $H(X, \mu)$ the automorphism group of the measure μ, i.e. $f \in H(X, \mu)$ iff f is a homeomorphism on X with $f_*\mu = \mu$.

For a measure μ on X we define the *clopen values set* $S(\mu)$ to be the set of values on the clopen subsets.

$$S(\mu) := \{\mu(U) : U \subset X \text{ with } U \text{ clopen}\}. \tag{2.2}$$

This is always a countable subset of the unit interval and at least $0, 1 \in S(\mu)$. Of course, if X is connected these are the only clopen values, but for any *Cantor space measure* (that is, full, nonatomic measure) the clopen values set is a countable dense subset of the interval.

Since the continuous preimage of a clopen set is clopen,

$$\mu_1 \text{ maps to } \mu_2 \;\Rightarrow\; S(\mu_2) \subset S(\mu_1). \tag{2.3}$$

In particular, we have three progressively weaker equivalence relations between measures on X:

$$\mu_1 \approx \mu_2 \;\Rightarrow\; \mu_1 \sim \mu_2 \;\Rightarrow\; S(\mu_1) = S(\mu_2). \tag{2.4}$$

In Mauldin (1990) it is observed that for Bernoulli measures on two symbols, the last two statements are equivalent. This is not generally the case, however, even for Bernoulli measures. For example, we will see that $\beta(1/3, 2/3)$ and $\beta(1/3, 1/3, 1/3)$ satisfy the last statement, but not the second (see Proposition 5.8 below).

Let D be a subset of the unit interval which contains 1. We say that D is *symmetric* if $\alpha \in D$ implies $1 - \alpha \in D$. We say that D is *multiplicative* if it is closed under multiplication. A number $\delta \in [0,1]$ is called a *divisor* of D if for all $\alpha \in [0,1]$,

$$\alpha \in D \;\Leftrightarrow\; \alpha \cdot \delta \in D. \tag{2.5}$$

The set of all divisors of D is denoted $\mathrm{Div}(D)$.

D is called *grouplike* if it is the unit interval portion of a subgroup of the reals, i.e. there exists an additive subgroup G of $\mathbb{R}$ such that $D = G \cap [0,1]$. If G is a subring of the reals then D is called *ringlike* and if G is a subfield then D is called *fieldlike*. As usual, we will write

$$D + \mathbb{Z} := \{\alpha + n : \alpha \in D,\, n \in \mathbb{Z}\}. \tag{2.6}$$

PROPOSITION 2.1. *Let D be a countable subset of the unit interval with $0, 1 \in D$.*

(a) $1 \in \mathrm{Div}(D)$, $0 \notin \mathrm{Div}(D)$ *and* $\mathrm{Div}(D)$ *is a multiplicative subset of D. If D itself is multiplicative then for $\delta_1, \delta_2 \in D$,*

$$\delta_1, \delta_2 \in \mathrm{Div}(D) \;\Leftrightarrow\; \delta_1 \cdot \delta_2 \in \mathrm{Div}(D). \tag{2.7}$$

(b) *The following conditions are equivalent*:

  (1) *D is grouplike.*
  (2) *$D + \mathbb{Z}$ is a subgroup of $\mathbb{R}$.*
  (3) *D is symmetric and satisfies*

$$\alpha, \beta \in D \text{ and } \alpha + \beta \le 1 \;\Rightarrow\; \alpha + \beta \in D. \tag{2.8}$$

(c) *The following conditions are equivalent*:

  (1) *D is ringlike.*
  (2) *$D + \mathbb{Z}$ is a subring of $\mathbb{R}$.*
  (3) *D is grouplike and multiplicative.*

(d) *If D is ringlike and $\delta \in [0,1]$ then $\delta \in \mathrm{Div}(D)$ iff δ is a unit of the ring $D + \mathbb{Z}$, i.e. iff the reciprocal $1/\delta \in D + \mathbb{Z}$.*

(e) *The following conditions are equivalent*:

  (1) *D is fieldlike.*
  (2) *$D + \mathbb{Z}$ is a subfield of $\mathbb{R}$.*
  (3) *D is grouplike, $D \cap (0,1) \neq \emptyset$ and $\mathrm{Div}(D) = D \setminus \{0\}$.*

Proof. (a) Since $1 \in D$, $\delta \in \mathrm{Div}(D)$ implies that $\delta = 1 \cdot \delta \in D$. Hence, $\mathrm{Div}(D) \subset D$. Further, $D \neq [0,1]$, so there is $\alpha \in [0,1] \setminus D$, but $0 \cdot \alpha \in D$, so $0 \notin \mathrm{Div}(D)$.

If $\delta_1, \delta_2 \in \mathrm{Div}(D)$ then for $\alpha \in [0,1]$,

$$\alpha \in D \;\Leftrightarrow\; \alpha\delta_1 \in D \;\Leftrightarrow\; \alpha\delta_1\delta_2 \in D, \tag{2.9}$$

and so $\delta_1\delta_2 \in \mathrm{Div}(D)$.

Now assume that D is multiplicative and that $\delta_1, \delta_2 \in D$ with $\delta_1\delta_2 \in \mathrm{Div}(D)$. If $\alpha \in D$ then $\alpha\delta_1 \in D$ because D is multiplicative. If $\alpha\delta_1 \in D$ then because D is multiplicative, $\alpha\delta_1\delta_2 \in D$, which implies $\alpha \in D$ since $\delta_1\delta_2$ is a divisor. Thus, δ_1 is in $\mathrm{Div}(D)$ and similarly δ_2 is.

In each of (b), (c) and (e), (2)⇒(1) and (1)⇒(3) are obvious.

Now assume (b3). If $\beta > \alpha$ in D then

$$\beta - \alpha = 1 - ((1-\beta) + \alpha) \tag{2.10}$$

lies in D by (b3). For $n_1, n_2 \in \mathbb{Z}$ the equations

$$\begin{aligned}(n_1 + \alpha_1) - (n_2 + \alpha_2) &= (n_1 - n_2) + (\alpha_1 - \alpha_2) \\ &= (n_1 - n_2 - 1) + (\alpha_1 + (1 - \alpha_2))\end{aligned} \tag{2.11}$$

imply that $D + \mathbb{Z}$ is closed under subtraction and so it is a group. Thus, (b3) implies (b2). It is easy to check that (c3) implies $D+\mathbb{Z}$ is closed under multiplication, i.e. (c2).

(d) If $\delta \in [0,1]$ is a unit of the ring, and $\alpha\delta \in D$ for some $\alpha \in [0,1]$, then $\alpha = (\alpha\delta)\cdot(1/\delta)$ is in the intersection of the ring with $[0,1]$ and so $\alpha \in D$. Thus, δ is a divisor of D.

Conversely, if $\delta \in \mathrm{Div}(D)$, then write $1/\delta = n + \alpha$ with $\alpha \in [0,1)$. To show that δ is a unit it suffices to show that $\alpha \in D$. But $\alpha\delta = 1 - n\delta$. Since D is grouplike, $1 - n\delta \in D$. Since δ is a divisor, $\alpha \in D$.

(e) Assume (e3). Since $\mathrm{Div}(D) = D \setminus \{0\}$ is multiplicative, D is multiplicative and so $D+\mathbb{Z}$ is at least a ring. It suffices to show that any positive element g of the ring is a unit. Choose $\delta \in D \cap (0,1)$. For some power n the product $g \cdot \delta^n$ lies in $(0,1)$ and so it is in D. Since it and δ are units by assumption it follows that g is a unit. ∎

If U is a clopen, nonempty subset of X then from a measure μ on X we define the relative measure μ_U on U by

$$\mu_U(A) := \mu(A)/\mu(U) \quad \text{for } A \subset U. \tag{2.12}$$

Since a clopen subset of U is a clopen subset of X we clearly have

$$\mu(U)\cdot S(\mu_U) \subset S(\mu) \cap [0, \mu(U)]. \tag{2.13}$$

The question of equality here turns out to be an important issue. Following Akin (2005) we define a measure on a Cantor space X to be *good* when equality holds for every clopen, nonempty subset of X. That is,

Definition 2.2. A full, nonatomic measure μ on a Cantor space X is called *good* if for clopen subsets U, V of X, $\mu(V) \le \mu(U)$ implies there exists a clopen subset V_1 of U such that $\mu(V) = \mu(V_1)$.

This appears to be a rather mild homogeneity condition, but in fact it has very strong consequences. On the other hand, there are many distinct examples. The following collects the key results from Akin (2005) concerning such measures.

THEOREM 2.3. *Let μ be a good measure on a Cantor space X. Denote by $H(X,\mu)$ the group of μ-invariant homeomorphisms, i.e. those which map μ to μ.*

(a) *The group $H(X,\mu)$ acts transitively on X. That is, given two points $x_1, x_2 \in X$ there exists $h \in H(X,\mu)$ such that $h(x_1) = x_2$. Furthermore, if U_1, U_2 are clopen subsets of X with $\mu(U_1) = \mu(U_2)$ then there exists $h \in H(X,\mu)$ such that $h(U_1) = U_2$.*

(b) *If U is a nonempty clopen subset of X then μ_U is a good measure on U with*

$$S(\mu_U) = (1/\mu(U)) \cdot (S(\mu) \cap [0, \mu(U)]). \tag{2.14}$$

(c) *There exists $h \in H(X,\mu)$ such that the dynamical system on X obtained by iterating h is a uniquely ergodic, minimal system. Conversely, for any uniquely ergodic minimal homeomorphism on a Cantor space the invariant measure is good.*

(d) *The clopen values set $S(\mu)$ is grouplike. Conversely, for every countable, dense grouplike subset D of $[0,1]$ which contains 0 and 1 there is a good measure μ on a Cantor space such that $S(\mu) = D$ and μ is unique up to homeomorphism. That is, the clopen values set is a complete homeomorphism invariant for good measures.*

Proof. (a) These results are contained in Proposition 2.11(a) and Corollary 2.12 of Akin (2005).

(b) Use Proposition 2.4 of Akin (2005).

(c) See Theorem 4.6 of Akin (2005). In Section 4 of that paper the uniquely ergodic automorphisms are explicitly constructed. For the converse, Lemma 2.5 of Glasner and Weiss (1995) implies that the invariant measure of a uniquely ergodic minimal system on a Cantor space is good.

(d) By Proposition 2.4 of Akin (2005) the set $S(\mu)$ is grouplike when μ is good and Theorem 2.9(a) there says that the clopen values set is a complete invariant for good measures. That every grouplike subset occurs as the clopen values subset for some good measure follows from Theorems 2.6 and 1.7(c) of Akin (2005). ∎

We provide one immediate example.

PROPOSITION 2.4. *If μ is Haar measure for some topological group structure on a Cantor space X then μ is a good measure on X.*

Proof. Recall the quick proof that the identity element of X is the intersection of a decreasing sequence of clopen subgroups. Since X is a Cantor space we can choose an ultrametric d on X, that is, a metric such that $d(x,z) \le \max(d(x,y), d(y,z))$ for all $x, y, z \in X$.

Because X is compact, multiplication is uniformly continuous. Hence, $\sup_{z\in X} d(zx, zy)$ defines an ultrametric on X which is equivalent to d. Thus, we can assume that d is invariant under left translation. Then the ε-balls centered at the identity are the required clopen subgroups.

For each clopen subgroup G the projection of X to the finite space X/G maps μ to the uniform counting measure, normalized to unity, on the cosets. Since X is the inverse limit of these coset spaces, we can choose for any pair U, V of clopen sets a single clopen subgroup G so that each is a union of cosets of G. To say that $\mu(V) \leq \mu(U)$ means that the number of cosets in U (namely $\mu(U)/\mu(G)$) is at least as large as the number of cosets in V. Choose V_1 to be a union of $\mu(V)/\mu(G)$ cosets among those in U. ■

Remark. The measure of any coset of G is the reciprocal of the index of G. It easily follows that for Haar measure the clopen values set is contained in the field of rationals $\mathbb{Q}$. Conversely, Theorem 2.16 of Akin (2005) says that any good measure with clopen values set contained in $\mathbb{Q}$ is homeomorphic to Haar measure for some monothetic—and hence abelian—topological group structure on Cantor space.

For the next result of this section we change the focus of the notion of goodness from the measure to individual clopen sets.

Definition 2.5. Let μ be a full, nonatomic measure on a Cantor space X. A clopen subset U of X is called *good for* μ (or just *good* when the measure is understood) if for every clopen subset V of X with $\mu(V) \leq \mu(U)$ there exists a clopen subset V_1 of U such that $\mu(V) = \mu(V_1)$. Equivalently, U is good iff it satisfies the equation

$$\mu(U) \cdot S(\mu_U) = S(\mu) \cap [0, \mu(U)]. \tag{2.15}$$

The empty set and the entire space are always good. It can happen that these are the only good subsets. (See the examples in Akin (1999).)

The measure μ is good exactly when every clopen subset is good for μ. However, it suffices to find a sufficiently rich collection of clopen sets which are good for μ.

Recall that a *partition* of a clopen set U is a (necessarily finite) pairwise disjoint collection of clopen sets with union U.

Definition 2.6. A *partition basis* $\mathcal{B}$ for a zero-dimensional space X is a collection of clopen subsets of X such that every nonempty clopen subset of X can be partitioned by elements of $\mathcal{B}$.

It is easy to see that a partition basis is a basis for the topology but not every basis is a partition basis. For example, suppose that μ is a good measure such that $S(\mu)$ contains some transcendental α, and all positive powers of α. Then the collection of clopens with measure a positive power of α is a basis for the topology but cannot be used to partition the entire space.

THEOREM 2.7. *In order that a full, nonatomic measure μ on a Cantor space X be good, it suffices that there exist a partition basis $\mathcal{B}$ consisting of clopen sets which are good for μ.*

Proof. Given two clopen sets U, V with $\mu(V) \leq \mu(U)$, we have to construct a clopen subset W of U with $\mu(W) = \mu(V)$. If $\mu(V) = \mu(U)$ (or $= 0$) then W can be, in fact, must be, U itself (resp. the empty set). Now assume that $0 < \mu(V) < \mu(U)$.

Let $\{B_1, \dots, B_k\}$ be a partition of U into good clopen (nonempty) sets. Note that if $k = 1$, then U is good and we are done, so let us assume that $k > 1$.

Our strategy will be to find a partition $\{V_1, \dots, V_k\}$ of V so that $\mu(V_i) < \mu(B_i)$.

Let $m_i = \mu(B_i)\mu(V)/\mu(U)$ for $1 \leq i \leq k$. Note that $0 < m_i < \mu(B_i)$ for each i, and $\sum_{i=1}^{k} m_i = \mu(V)$. Choose $\varepsilon > 0$ so that

$$k\varepsilon < \mu(B_k) - m_k. \tag{2.16}$$

Since $S(\mu_V)$ is dense in the unit interval, we can find a clopen set $V_1 \subset V$ such that $\mu(V_1) = p_1$ satisfies $m_1 - \varepsilon < p_1 < m_1$. Note that $m_2 \leq \sum_{i=2}^{k} m_i = \mu(V) - m_1 < \mu(V) - p_1 = \mu(V \setminus V_1)$. As above, we can find a clopen set $V_2 \subset V \setminus V_1$ whose measure p_2 satisfies $m_2 - \varepsilon < p_2 < m_2$. We continue in this way, choosing V_i for $i = 1, \dots, k-1$.

Conclude by letting $V_k = V \setminus \bigcup_{i=1}^{k-1} V_i$, and $p_k = \mu(V_k) = \mu(V) - \sum_{i=1}^{k-1} p_i$ so that

$$\begin{aligned} m_k = \mu(V) - \sum_{i=1}^{k-1} m_i &< p_k \\ &< \mu(V) - \sum_{i=1}^{k-1} m_i + (k-1)\varepsilon < m_k + k\varepsilon < \mu(B_k). \end{aligned} \tag{2.17}$$

Thus, we have $\mu(V_i) = p_i < \mu(B_i)$ for $i = 1, \dots, k$. Since each B_i is good, we can choose a clopen set $W_i \subset B_i$ with $\mu(W_i) = \mu(V_i)$ for $i = 1, \dots, k$. Their union $W = \bigcup_{i=1}^{k} W_i$ is the required clopen subset of U with $\mu(W) = \mu(V)$.

Hence, the measure is good. ∎

REMARK. The above proof actually shows that if a clopen set can be partitioned by good clopens then it is itself good.

Theorem 2.8. *If μ is a good measure on X and ν is a good measure on Y, then the product $\mu \times \nu$ is a good measure on $X \times Y$ and*

$$S(\mu \times \nu) = \Big\{\sum_i \alpha_i \cdot \beta_i : \alpha_i \in S(\mu),\ \beta_i \in S(\nu) \text{ with the sum at most } 1\Big\}. \tag{2.18}$$

Proof. If $U \subset X$ and $V \subset Y$ are clopens then $U \times V \subset X \times Y$ is a clopen set, which we will call a *rectangular* clopen subset of $X \times Y$. Notice that

$$(U_1 \times V_1) \setminus (U_2 \times V_2) = ((U_1 \setminus U_2) \times V_1) \cup ((U_1 \cap U_2) \times (V_1 \setminus V_2)). \tag{2.19}$$

It easily follows that any union of rectangular clopens can be expressed as a disjoint union of rectangular clopens. Hence, the rectangular clopens form a partition basis for $X \times Y$. By Theorem 2.7, it suffices to show that an arbitrary nonempty rectangular clopen $U \times V$ is good for $\mu \times \nu$.

Given $\alpha_1, \ldots, \alpha_k \in S(\mu)$, $\beta_1, \ldots, \beta_k \in S(\nu)$ with $\sum_i \alpha_i \cdot \beta_i < \mu(U) \cdot \nu(V)$, we can use density of $S(\mu)$ in the unit interval to choose $\gamma_i \in S(\mu)$, $i = 1, \ldots, k$, so that $\alpha_i \cdot \beta_i < \gamma_i \cdot \nu(V)$ for all i and, in addition, $\sum_i \gamma_i < \mu(U)$. Because the measure μ is good, we can inductively choose clopens $U_1 \subset U$, $U_2 \subset U \setminus U_1, \ \ldots, \ U_k \subset U \setminus (U_1 \cup \cdots \cup U_{k-1})$ so that $\mu(U_i) = \gamma_i$. It then suffices to find a clopen subset of $U_i \times V$ with product measure equal to $\alpha_i \cdot \beta_i$.

Thus, we are reduced to considering a pair of nonempty rectangular clopens $U \times V$ and $R \times S$ with $\mu(R) \cdot \nu(S) < \mu(U) \cdot \nu(V)$. We have to find a clopen subset of $U \times V$ with measure $\mu(R) \cdot \nu(S)$. We will call the latter value the *donor measure*. Let ε be the difference $\mu(U)\nu(V) - \mu(R)\nu(S)$. We will call ε the *gap* for this problem. If $\varepsilon > \mu(R)\nu(S)$ then we will say that the problem has a *big gap*.

Case I: $\mu(R) \leq \mu(U)$ and $\nu(S) \leq \nu(V)$. This is the easy case. Since the measures are both good, we can choose clopens $U_1 \subset U$, $V_1 \subset V$ with $\mu(R) = \mu(U_1)$, $\nu(S) = \nu(V_1)$ and $U_1 \times V_1$ is the required clopen subset of $U \times V$.

Case II: $\mu(R) > \mu(U)$. Divide to obtain the integer $k \geq 1$ and remainder $0 \leq r_1 < \mu(U)$ so that $\mu(R) = k \cdot \mu(U) + r_1$. Notice that

$$r_1 < \frac{1}{2}\,\mu(R). \tag{2.20}$$

Next observe that, if $\nu(V) \leq k \cdot \nu(S)$, then $k\mu(U) \leq \mu(R)$ would imply $k\mu(U)\nu(V) \leq k\mu(R)\nu(S)$. Dividing by k we obtain a contradiction. Hence,

$$k \cdot \nu(S) < \nu(V). \tag{2.21}$$

Furthermore, if $\nu(V) \leq (k+1) \cdot \nu(S)$ then we obtain

$$k\mu(U)\nu(V) \leq (k+1)\mu(R)\nu(S) \quad \text{and so} \quad k \cdot \varepsilon \leq \mu(R)\nu(S).$$

Hence,

$$\text{big gap} \;\Rightarrow\; (k+1)\cdot\nu(S) < \nu(V). \tag{2.22}$$

Because the measures are good, we can choose disjoint clopens $U_1, \dots, U_k \subset R$ with $\mu(U_1) = \cdots = \mu(U_k) = \mu(U)$ and $S_1, \dots, S_k \subset V$ with $\nu(S_1) = \cdots = \nu(S_k) = \nu(S)$. Define

$$R_1 := R \setminus (U_1 \cup \cdots \cup U_k), \qquad V_1 := V \setminus (S_1 \cup \cdots \cup S_k). \tag{2.23}$$

Observe that

$$\mu(R_1) = r_1 < \min\Big(\mu(U), \frac{1}{2}\,\mu(R)\Big). \tag{2.24}$$

$R\times S$ is partitioned by $R_1\times S, U_1\times S, \dots, U_k\times S$ while $U\times V$ is partitioned by $U \times V_1, U \times S_1, \dots, U \times S_k$. Furthermore, $\mu(U_i)\nu(S) = \mu(U)\nu(S_i)$ for $i = 1, \dots, k$. Thus, we have replaced the original problem by the transformation $(R, S, U, V) \mapsto (R_1, S, U, V_1)$. It is important to note that the gap ε remains unchanged and in the transformed problem the measure of the donor is less than half of what it was in the original problem. Notice that $\mu(R_1) < \mu(U)$. If $\nu(S) \le \nu(V_1)$ then the transformed problem is in Case I. In particular, this is true if the gap is big by (2.22), since $\nu(V_1) = \nu(V) - k\cdot\nu(S)$. Otherwise, we are in Case II with R and S exchanged.

We continue performing the Case II transformations. At each stage the donor measure is reduced to less than half its previous value while the gap remains unchanged. Eventually the gap is big and we move into Case I to complete the proof.

Finally, $S(\mu)$ is contained in the set of finite sums given in (2.18) because the rectangular clopens form a partition basis. Because the product measure is good, the clopen values set is grouplike and so the reverse inclusion holds as well. ∎

COROLLARY 2.9. *The product of a finite or infinite sequence of good measures is a good measure.*

Proof. The result for finite products follows from Theorem 2.8 by induction on the number of factors. The infinite product result is reduced to the finite case by using the observation that for any clopen subset in the product only finitely many coordinates are restricted. ∎

A useful weakening of the notion of a good measure was introduced by Dougherty *et al.* (2007).

DEFINITION 2.10. Let μ be a full, nonatomic measure on a Cantor space X. A clopen $U \subset X$ is called *refinable for* μ (or just *refinable* when the measure is understood) if, whenever

$$\alpha_1, \dots, \alpha_k \in S(\mu) \quad \text{with} \quad \alpha_1 + \cdots + \alpha_k = \mu(U), \tag{2.25}$$

there exists a clopen partition $\{U_1, \ldots, U_k\}$ of U with $\mu(U_i) = \alpha_i$ for $i = 1, \ldots, k$. The measure μ is called *refinable* if every clopen subset is refinable, and μ is called *weakly refinable* if there exists a partition basis $\mathcal{B}$ for X with $X \in \mathcal{B}$ consisting of refinable clopen subsets.

Although the weakening from good to refinable is very mild, we will eventually see that refinability does not imply goodness. (For example, see Theorem 6.4.)

Observe that if $\mathcal{B}$ is a partition basis for X then to show that the elements of $\mathcal{B}$ are refinable it suffices to check that there exists a partition for those lists of numbers which satisfy (2.25) where all the α_i's are measures of members of $\mathcal{B}$, since any list of numbers not having this additional property can be refined into one which does.

PROPOSITION 2.11. *Let μ be a measure on a Cantor space X.*

(a) *If $S(\mu)$ is grouplike and a clopen $U \subset X$ is refinable then U is good.*
(b) *If the measure μ is good then every clopen subset of X is refinable.*
(c) *Let $U, V \subset X$ be nonempty clopens with $V \subset U$. If V is refinable* (*or good*) *for μ then V is refinable* (*resp. good*) *for μ_U.*

Proof. (a) Let V be a clopen subset of X with $\mu(V) < \mu(U)$. Because $S(\mu)$ is grouplike, $\alpha_2 = \mu(U) - \mu(V)$ as well as $\alpha_1 = \mu(V)$ are elements of $S(\mu)$. Because U is refinable, there exists a partition $\{U_1, U_2\}$ of U with $\mu(U_1) = \mu(V)$. Hence, the clopen U is good.

(b) We check the refinability condition by induction on the number k of terms in (2.25). The condition holds trivially for $k = 1$. For any clopen U let $\alpha_1, \ldots, \alpha_k$ satisfy (2.25). Because the measure is good, $S(\mu)$ is grouplike and so $\alpha_{k-1} + \alpha_k \in S(\mu)$. By inductive hypothesis, there is a partition $\{U_1, \ldots, U_{k-2}, \widetilde{U}\}$ of U with $\mu(U_i) = \alpha_i$ for $i = 1, \ldots, k-2$ and $\mu(\widetilde{U}) = \alpha_{k-1} + \alpha_k$. Because the measure is good, there exists a clopen $U_{k-1} \subset \widetilde{U}$ with $\mu(U_{k-1}) = \alpha_{k-1}$. Let $U_k = \widetilde{U} \setminus U_{k-1}$ to define the required partition of U.

(c) Let $\alpha = \mu(U)$. If $\beta_1, \ldots, \beta_k \in S(\mu_U)$ with sum $\mu_U(V)$ then $\beta_1 \cdot \alpha, \ldots, \beta_k \cdot \alpha \in S(\mu)$ with sum $\mu(V)$. Since V is refinable for μ there exists a partition $\{V_1, \ldots, V_k\}$ of V such that $\mu(V_i) = \beta_i \cdot \alpha$ and so $\mu_U(V_i) = \beta_i$ for $i = 1, \ldots, k$.

Similarly, if $\beta \in S(\mu_U)$ with $\beta < \mu_U(V)$ then $\beta \cdot \alpha \in S(\mu)$ is less than $\mu(V)$. If V is good for μ then there exists a clopen $V_1 \subset V$ with $\mu(V_1) = \beta \cdot \alpha$ and so $\mu_U(V_1) = \beta$. ∎

COROLLARY 2.12. *For a measure μ on a Cantor space X the following are equivalent*:

(a) *μ is a good measure.*

(b) *μ is refinable and $S(\mu)$ is grouplike.*
(c) *μ is weakly refinable and $S(\mu)$ is grouplike.*

Proof. (a)⇒(b). If μ is good then $S(\mu)$ is grouplike and by Proposition 2.11(b) every clopen subset is refinable. Hence, μ is refinable.

(b)⇒(c). Obvious.

(c)⇒(a). If μ is weakly refinable then it admits a partition basis $\mathcal{B}$ of refinable clopen sets. If, in addition, $S(\mu)$ is grouplike then by Proposition 2.11(a) all of the sets in $\mathcal{B}$ are good. Since $\mathcal{B}$ is a partition basis of good sets, Theorem 2.7 implies that the measure is good. ∎

Corollary 2.13. *Let μ be a measure on a Cantor space X and U be a nonempty clopen subset of X.*

(a) *If μ is weakly refinable and U is a refinable subset for μ then μ_U is weakly refinable.*
(b) *If μ is refinable then μ_U is refinable.*

Proof. (a) If $\mathcal{B}$ is a partition basis for X of μ-refinable clopens with $X \in \mathcal{B}$ then $\mathcal{B}_U := \{U\} \cup \{V \in \mathcal{B} : V \subset U\}$ is a partition basis for U consisting of sets which are μ_U-refinable by Proposition 2.11(c).

(b) Apply the proof of (a) with $\mathcal{B}$ the set of all clopen subsets of X. ∎

In general, we do not know whether weak refinability and refinability are distinct properties.

$S(\mu)$ is not a complete invariant for the homeomorphism equivalence of measures (Proposition 5.8 will show $S(\beta(1/3)) = S(\beta(1/3, 2/3))$, while Theorem 1.1 verifies that $\beta(1/3, 2/3)$ is not good) but it is a complete invariant among good measures. This result extends to weakly refinable measures. Theorem 9 of Dougherty *et al.* (2007) is stated for Bernoulli measures but, as they point out, the same proof, extended a bit below, yields the result for weak refinabile measures in general.

Theorem 2.14. *Let μ and ν be weakly refinable measures on Cantor spaces X and Y. If $S(\mu) = S(\nu)$ then μ is homeomorphic to ν. In fact, if $\mathcal{X} = \{X_1, \ldots, X_M\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_M\}$ are partitions of X and Y respectively with $Y_1, \ldots, Y_M$ refinable and such that*

$$\mu(X_i) = \nu(Y_i) \quad \text{for } i = 1, \ldots, M, \tag{2.26}$$

then there exists a homeomorphism $h : X \to Y$ with $h_*(\mu) = \nu$ and such that $h(X_i) = Y_i$ for $i = 1, \ldots, M$.

Proof. Since there are only countably many clopen sets, we can index those of X as $\{C_0, C_1, \ldots\}$ and those of Y as $\{D_0, D_1, \ldots\}$.

The proof is a back and forth construction beginning with the partitions $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{Y}_0 = \mathcal{Y}$, and with ϱ_0 the bijection from $\mathcal{X}$ to $\mathcal{Y}$ which sends X_i

to Y_i. When no such partitions are specified we use $\mathcal{X}_0=\{X\}$, $\mathcal{Y}_0=\{Y\}$, and $\varrho_0 : X\mapsto Y$. Notice that Y is refinable, by the definition of weak refinability.

Suppose that we have defined partitions $\mathcal{X}_{2k}$ and $\mathcal{Y}_{2k}$ of X and Y respectively, so that elements of $\mathcal{Y}_{2k}$ are refinable. Suppose also that we have defined a bijection $\varrho_{2k} : \mathcal{X}_{2k} \to \mathcal{Y}_{2k}$ so that

$$\nu(\varrho_{2k}(A)) = \mu(A) \quad \text{for all } A \in \mathcal{X}_{2k}.$$

Let

$$\widehat{\mathcal{X}}_{2k} = \{A \cap C_k : A \in \mathcal{X}_{2k}\} \cup \{A - C_k : A \in \mathcal{X}_{2k}\}.$$

Now let $\mathcal{X}_{2k+1}$ be a refinement of $\widehat{\mathcal{X}}_{2k}$ into refinable sets, removing the empty set if it appears. (This is possible since X is weakly refinable.)

Fix some $B \in \mathcal{Y}_{2k}$. Those elements of $\mathcal{X}_{2k+1}$ which are subsets of $\varrho_{2k}^{-1}(B)$ form a partition of $\varrho_{2k}^{-1}(B)$. Denote this partition of $\varrho_{2k}^{-1}(B)$ as $\{A_1,\dots,A_n\}$. Observe that $\mu(A_1),\dots,\mu(A_n)$ is a list of μ-clopen values (and hence ν-clopen values) which sums to $\mu(\varrho_{2k}^{-1}(B)) = \nu(B)$. Since B is refinable, we can partition B into clopen sets $\{B_1,\dots,B_n\}$ so that $\nu(B_i) = \mu(A_i)$ for $i = 1,\dots,n$. Let $\mathcal{Y}_{2k+1}$ include $\{B_1,\dots,B_n\}$, and let ϱ_{2k+1} map A_i to B_i for $i = 1,\dots,n$. After doing this for each B in $\mathcal{Y}_{2k}$, we obtain $\mathcal{Y}_{2k+1}$, a refinement of $\mathcal{Y}_{2k}$, and a bijection ϱ_{2k+1} from $\mathcal{X}_{2k+1}$ to $\mathcal{Y}_{2k+1}$ so that

$$\nu(\varrho_{2k+1}(A)) = \mu(A) \quad \text{for all } A \in \mathcal{X}_{2k+1}.$$

Further, we find that every element of $\mathcal{X}_{2k+1}$ is refinable. Finally, ϱ_{2k+1} is a refinement of ϱ_{2k} in the obvious sense.

We are now in the same situation as before, with the roles of X and Y reversed. We may therefore repeat the above construction to define $\mathcal{X}_{2k+2}$, $\mathcal{Y}_{2k+2}$, and ϱ_{2k+2}, by first refining the Y side using the set D_k, then refining into refinable sets, and finally using refinability of the sets on the X side to produce a corresponding refinement on the X side.

We have now defined $\mathcal{X}_k$, $\mathcal{Y}_k$, and ϱ_k for all $k \geq 0$. For each $x \in X$, and for each $k \geq 0$, there is a unique $A_k \in \mathcal{X}_k$ so that $x \in A_k$, since $\mathcal{X}_k$ is a partition of X. By construction, the sequence $\{A_k\}_{k\geq 0}$ will be a nested sequence of clopen sets. Define a map ϱ from X to Y by letting $\varrho(x)$ be an element of $\bigcap_{k=0}^{\infty} \varrho_k(A_k)$, another nested sequence. This element is unique since the sets in $\mathcal{Y}_{2k+2}$ separate points in D_k from points not in D_k, and every clopen set is one of these D_k's. It is similarly easy to verify that this map ϱ is a homeomorphism. Since ϱ preserves measure on a partition basis (namely $\bigcup_k \mathcal{X}_k$) it must preserve the measure of every clopen set, and hence every measurable set. ∎

Remark. In his exposition of good measures, Glasner (2002) proved essentially the same theorem. His Proposition 1.4 proves the result for good measures by introducing the property of refinability without naming it.

COROLLARY 2.15. *Let μ be a refinable measure on a Cantor space X. If U and V are clopen subsets of X with $\mu(U) = \mu(V)$ then there exists $h \in H(X, \mu)$ such that $h(U) = V$. In particular, μ_U and μ_V are homeomorphic measures, and so $S(\mu_U) = S(\mu_V)$.*

Proof. Let $\mathcal{X} = \{U, X \setminus U\}$ and $\mathcal{Y} = \{V, X \setminus V\}$. By Theorem 2.14 there exists an automorphism of (X, μ) which maps U to V. The restriction of h defines a homeomorphism from μ_U to μ_V. Finally, homeomorphic measures have the same clopen values sets. ∎

3. Bernoulli type measures. The class of zero-dimensional, nonempty, compact metrizable spaces is closed under countable products. If any one factor is perfect or if there are infinitely many factors each with at least two points then the product is perfect and so it is a Cantor space. The case of products of infinite alphabets was considered by Oxtoby (1970), who showed that any two full nonatomic probability measures on $\mathbb{N}^{\mathbb{N}}$ are homeomorphic. We therefore restrict our attention to the finite alphabet case.

Given a finite set $A = \{a_1, \ldots, a_N\}$ with $|A| = N \geq 2$, we regard A as an alphabet with N letters and for $k = 1, 2, \ldots$ we regard the elements of the product A^k as words of length k.

To describe a measure on A we use the probability vector $p : A \to (0, 1)$ which lists the measures of the individual points. That is,

$$\sum_{i=1}^{N} p_i = 1 \quad \text{with } p_i > 0 \text{ for } i = 1, \ldots, N, \tag{3.1}$$

where we write p_i for $p(a_i)$ when the alphabet A is understood.

With $\mathbb{N} = \{0, 1, \ldots\}$ we call the associated product measure on the Cantor space $A^{\mathbb{N}}$ the *Bernoulli measure* associated with p and denote it by $\beta(p)$, or $\beta(p_1, \ldots, p_N)$ when it is convenient to simply list the values of p. We denote by $\beta(1/N)$ the Bernoulli measure on $A^{\mathbb{N}}$ with $p_i = 1/N$ for $i = 1, \ldots, N$. This is the Haar measure obtained by regarding $A^{\mathbb{N}}$ as either the product of cyclic groups of order N or the additive group of the ring of N-adic integers.

If measures on finite sets A and B are defined by probability vectors p and q, respectively, then the product measure on $A \times B$ is defined by the probability vector $p \times q : A \times B \to (0, 1)$ with

$$(p \times q)(a, b) := p(a) \cdot q(b) \quad \text{for } (a, b) \in A \times B. \tag{3.2}$$

The k-fold product $p^k : A^k \to (0, 1)$ defines the marginal distribution for $\beta(p)$ on the words of length k with respect to the projection $\pi : A^{\mathbb{N}} \to A^k$.

We regard the obvious homeomorphism from $A^{\mathbb{N}} \times B^{\mathbb{N}}$ to $(A \times B)^{\mathbb{N}}$ as an identification which induces the identification of measures:

$$\beta(p) \times \beta(q) = \beta(p \times q). \tag{3.3}$$

By grouping together runs of length k we obtain the obvious homeomorphism from $A^{\mathbb{N}}$ to $(A^k)^{\mathbb{N}}$, which we also regard as an identification which induces the identification of measures:

$$\beta(p^k) = \beta(p). \tag{3.4}$$

The words induce a partition basis on $A^{\mathbb{N}}$:

$$\mathcal{B}_A := \{[w] : w \in A^k \text{ with } k = 1, 2, \ldots\}. \tag{3.5}$$

We call the clopen set $[w] := \pi^{-1}(w)$ the *cylinder set* associated with the word w. We say that w and the associated cylinder set $[w]$ have *length* k when $w \in A^k$.

There is a natural homeomorphism between $[w]$ and the entire space $A^{\mathbb{N}}$ given by $x \mapsto wx$ for all $x \in A^{\mathbb{N}}$. This homeomorphism clearly maps the Bernoulli measure $\beta(p)$ to the relative measure $\beta(p)_{[w]}$. In particular, for every $w \in A^k$ we have

$$S(\beta(p)_{[w]}) = S(\beta(p)). \tag{3.6}$$

Definition 3.1. Let μ be a full, nonatomic measure on a Cantor space X.

(a) A nonempty clopen subset U of X is called a clopen set of μ *type* (or just a clopen of μ type) when $S(\mu_U) = S(\mu)$.

(b) The measure μ is called a measure of *Bernoulli type* when there is a partition basis $\mathcal{B}$ for X consisting of clopen sets of μ type.

Clearly, a Bernoulli measure is of Bernoulli type with the cylinder sets providing the required partition basis. On the other hand, there exist measures of Bernoulli type which are not homeomorphic to Bernoulli measures. For example, let μ be the good measure on a Cantor space X for which $S(\mu) = \mathbb{Q} \cap [0,1]$. Then μ cannot be homeomorphic to $\beta(p)$ for some probability vector p, as this would require that $S(\beta(p)) = \mathbb{Q} \cap [0,1]$, but all elements of $S(\beta(p))$ have denominators a power of the LCD of the entries of p.

Theorem 3.2. *Let μ be a measure of Bernoulli type on a Cantor space X. Then the clopen values set $S(\mu)$ is multiplicative and for every nonempty clopen subset U of X we have*

$$S(\mu) \subset S(\mu_U). \tag{3.7}$$

Also, if $\alpha, \beta \in S(\mu)$ then $\alpha + \beta - \alpha \cdot \beta \in S(\mu)$.

Proof. Let $\alpha, \beta \in S(\mu)$ with $\beta = \mu(U)$ for some clopen subset U of X. We can assume β is nonzero and so that U is nonempty. Let $\{B_1, \ldots, B_k\}$ be a partition of U by sets of μ type. Since $\alpha \in S(\mu) = S(\mu_{B_i})$ there exists a clopen subset V_i of B_i with relative measure α and so $\mu(V_i) = \alpha \cdot \mu(B_i)$. With V the union of the V_i's we obtain a clopen set with measure equal to $\alpha \cdot \beta$.

Thus, $S(\mu)$ is multiplicative. Since V is a subset of U we have $\mu_U(V) = \alpha$. Thus, $S(\mu) \subset S(\mu_U)$.

For the last result, observe that

$$\alpha + \beta - \alpha \cdot \beta = 1 - (1 - \alpha) \cdot (1 - \beta). \tag{3.8} \blacksquare$$

REMARK. It is often convenient to use the above equation in the following form: If U is a clopen subset of X and $\alpha \in S(\mu)$ with μ of Bernoulli type, then there exists a clopen $V \subset U$ with $\mu(V) = \alpha \cdot \mu(U)$.

We will frequently use the following result, which we will call the *Two Implies Three Lemma.*

LEMMA 3.3. *For a measure μ on a Cantor set X and a clopen subset U of X any two of the following conditions imply the third.*

(a) *U is a clopen subset good for μ.*
(b) *U is a clopen subset of μ type.*
(c) *The measure of U is a divisor for $S(\mu)$, i.e. $\mu(U) \in \mathrm{Div}(S(\mu))$.*

Proof. This is obvious from a restatement of each of these conditions in terms of clopen values sets:

$$\begin{aligned} &\text{(a)} \ \Leftrightarrow \ \mu(U) \cdot S(\mu_U) = S(\mu) \cap [0, \mu(U)]; \\ &\text{(b)} \ \Leftrightarrow \ S(\mu_U) = S(\mu); \\ &\text{(c)} \ \Leftrightarrow \ \mu(U) \cdot S(\mu) = S(\mu) \cap [0, \mu(U)]. \ \blacksquare \end{aligned} \tag{3.9}$$

THEOREM 3.4. *Let μ be a full, nonatomic measure on a Cantor space X.*

(a) *If μ is a good measure then for every clopen $U \subset X$ of μ type the value $\mu(U)$ is a divisor of $S(\mu)$.*
(b) *If X admits a partition basis $\mathcal{B}$ such that each $U \in \mathcal{B}$ is of μ type with $\mu(U) \in \mathrm{Div}(S(\mu))$ then μ is a good measure of Bernoulli type.*
(c) *If μ is a good measure of Bernoulli type then $S(\mu) + \mathbb{Z}$ is a subring of $\mathbb{R}$ and every positive element of $S(\mu) + \mathbb{Z}$ is a sum of positive units of the ring.*

Proof. (a) For a good measure every clopen subset is good and so if U is a clopen of type μ its value is a divisor by the Two Implies Three Lemma.

(b) The partition basis consists of sets of μ type and so the measure is of Bernoulli type by definition. Since the values are divisors, the members of $\mathcal{B}$ are good by Two Implies Three again. Hence, the measure is good by Theorem 2.7.

(c) Since the measure is good, $S(\mu)$ is grouplike, and since the measure is of Bernoulli type, $S(\mu)$ is multiplicative by Theorem 3.2. By Proposition 2.1(c), $S(\mu) + \mathbb{Z}$ is a ring. Since every clopen can be partitioned by members of $\mathcal{B}$, every element of $S(\mu) \setminus \{0\}$ can be written as a sum of values of elements of $\mathcal{B}$. These are divisors of $S(\mu)$ by Two Implies Three again and so they are units of the ring by Proposition 2.1(d). Since 1 is a positive unit it follows that every positive element of $S(\mu) + \mathbb{Z}$ can be written as a sum of positive units. ■

We have the following converse of Theorem 3.4(c).

THEOREM 3.5. *Let μ be a good measure on a Cantor space X with $G = S(\mu) + \mathbb{Z}$ the group generated by the clopen values set.*

(a) *The measure μ is of Bernoulli type, i.e. the clopens of μ type form a partition basis for X, iff G is a subring of $\mathbb{R}$ such that every positive element of G is a sum of positive units of G.*
(b) *Every nonempty clopen subset of X is a clopen of μ type iff G is a subfield of $\mathbb{R}$.*

Proof. Since every clopen subset is good, the Two Implies Three Lemma says that a clopen U is of μ type iff $\mu(U)$ is a divisor for $S(\mu)$ or, equivalently when G is a ring, a unit of G. In particular, Proposition 2.1(e) implies that every nonempty clopen is of μ type iff $S(\mu)$ is fieldlike and so iff G is a field. This proves (b). The necessity of the units condition in (a) for a measure of Bernoulli type is Theorem 3.4(c).

By Corollary 2.12 a good measure is refinable and so if $\alpha_1, \ldots, \alpha_k \in S(\mu)$ with sum equal to $\mu(U)$ then U can be partitioned by clopen sets whose measures are exactly these values. Hence, if $\mu(U)$ is a sum of divisors then U can be partitioned by sets of μ type. Hence, for a good measure the units condition implies Bernoulli type. ■

REMARKS. (a) It does not suffice in (a) above that the ring be generated as a group by the units, i.e. that every element of the ring be a sum of units. For example, let r be a transcendental number in the unit interval and let G be the rational vector space generated by the powers, both positive and negative, of r. Since r is transcendental, the powers are linearly independent over $\mathbb{Q}$. Hence, every nonzero element of the ring G can be written uniquely as $r^n \cdot P(r)$ where $P(r)$ is a polynomial in r with rational coefficients and a nonzero constant term. Under product the degrees of the polynomials add and so the element is a unit of the ring exactly when it is a constant times a power of r. In particular, $r - r^2$ is a member of $G \cap (0, 1)$ which cannot be written as a sum of positive units. Hence, the associated good measure (see Theorem 2.3(d)) is not of Bernoulli type.

(b) If μ is a good Bernoulli type measure but $S(\mu)$ is not fieldlike then there will exist clopen sets U with $\mu(U)$ not a divisor. By the Two Implies Three Lemma such a set is not of μ type and hence the inclusion $S(\mu) \subset S(\mu_U)$ proved in Theorem 3.2 is proper.

The condition that every clopen is of μ type yields some special results.

Definition 3.6. We say that a measure μ on a Cantor space X satisfies the *Quotient Condition* when every nonempty clopen subset U of X is of μ type, that is, $S(\mu_U) = S(\mu)$.

Theorem 3.7. *Assume that μ is a measure on a Cantor space X which satisfies the Quotient Condition, i.e. every nonempty clopen subset of X is a clopen of μ type. Then μ is a refinable measure of Bernoulli type and for every nonempty clopen $U \subset X$ the relative measure μ_U is homeomorphic to μ. The rationals $\mathbb{Q} \cap I$ are contained in $S(\mu)$ and if $\alpha, \beta \in S(\mu) \setminus \{0\}$ then*

$$(\alpha + \beta)/2, \alpha/(\alpha + \beta) \in S(\mu). \tag{3.10}$$

Furthermore, the following conditions on μ are equivalent:

(i) *The measure μ is good.*
(ii) *$S(\mu)$ is fieldlike.*
(iii) *$S(\mu)$ is grouplike.*
(iv) *$1/2 \in \mathrm{Div}(S(\mu))$ or, equivalently, for every $\alpha \in S(\mu)$,*

$$\alpha < \frac{1}{2} \;\Rightarrow\; 2 \cdot \alpha \in S(\mu). \tag{3.11}$$

(v) *For every $\alpha \in S(\mu)$,*

$$\alpha < \frac{1}{2} \;\Rightarrow\; \frac{\alpha}{1 - \alpha} \in S(\mu). \tag{3.12}$$

Proof. Since every clopen is of μ type, μ is of Bernoulli type.

Next we show that $\mathbb{Q} \cap I \subset S(\mu)$, following Akin (2005), Theorem 3.4.

Let $\{V_1, \ldots, V_n\}$ be a partition of cardinality n by nonempty clopens. By Theorem 3.2 and the Remark thereafter, there exist clopens $U_i \subset V_i$ with $\mu(U_i) = \mu(V_1) \cdot \ldots \cdot \mu(V_n)$ for $i = 1, \ldots, n$. Now, $\{U_1, \ldots, U_n\}$ is a partition of $U = U_1 \cup \cdots \cup U_n$ with $\mu_U(U_i) = 1/n$ for $i = 1, \ldots, n$. Hence, $k/n \in S(\mu_U) = S(\mu)$ for $k = 1, \ldots, n$.

Furthermore, if $\alpha_1, \ldots, \alpha_n \in S(\mu)$ then we can choose clopens $W_i \subset U_i$ with $\mu_{U_i}(W_i) = \alpha_i$ and so $\mu_U(W_i) = \alpha_i/n$ for $i = 1, \ldots, n$. Let $W = W_1 \cup \cdots \cup W_n$ so that $\mu_U(W) = \gamma/n$ with $\gamma = \alpha_1 + \cdots + \alpha_n$. Hence, $\mu_W(W_1 \cup \cdots \cup W_k) = (\alpha_1 + \cdots + \alpha_k)/\gamma$ for $k = 1, \ldots, n$. Since U and W are of μ type, these ratios are in $S(\mu)$.

Applying this with $n = 2$ we obtain (3.10).

Now if the sum γ is itself in the multiplicative set $S(\mu)$ then $\alpha_1 + \cdots + \alpha_k \in S(\mu)$ for $k = 1, \ldots, n$.

To prove refinability we use induction. Assume that G is a clopen with $\gamma = \mu(G)$. We may assume $n > 1$ and that the α_i's are nonzero. We have just seen that $\alpha_1 + \alpha_2 \in S(\mu)$ and so by inductive hypothesis we can find a clopen partition $\{G_{1,2}, G_3, \ldots, G_n\}$ of G with $\mu(G_{1,2}) = \alpha_1 + \alpha_2$ and with $\mu(G_i) = \alpha_i$ for $i > 2$. Now apply (3.10) to find a clopen subset G_1 of $G_{1,2}$

such that $\mu_{G_{1,2}}(G_1) = \alpha_1/(\alpha_1+\alpha_2)$ and so $\mu(G_1) = \alpha_1$. Let $G_2 = G_{1,2} \setminus G_1$. Thus, $\{G_1, \ldots, G_n\}$ is the required partition.

This argument applies to every relative measure μ_U with U clopen and so they are all refinable. Since they all have the same clopen values set, they are all homeomorphic by Theorem 2.14.

Finally, (i)⇒(ii) follows from Theorem 3.5(b); (ii)⇒(iii) and (v) is obvious, as is (iii)⇒(iv).

(iv)⇒(i). If (3.11) is true then from (3.10) we see that $\alpha, \beta \in S(\mu)$ implies $\alpha + \beta \in S(\mu)$ provided that the sum is less than 1. So Proposition 2.1(b) implies that $S(\mu)$ is grouplike and so μ is good by Corollary 2.12.

(v)⇒(iv). Assume $\mu(U) = \alpha < 1/2$ for some clopen U. Let $V = X \setminus U$ so that $\mu(V) = 1 - \alpha$. Since $\alpha/(1-\alpha) \in S(\mu) = S(\mu_V)$, there exists a clopen subset U_1 of V with this relative measure and so with $\mu(U_1) = \alpha$. The clopen set $U \cup U_1$ has measure 2α. ■

For a measure μ on X we will denote by μ^n the product measure on X^n and by $\mu^{\mathbb{N}}$ the product measure on the infinite product $X^{\mathbb{N}}$.

DEFINITION 3.8. We say that a measure μ on a Cantor space X satisfies the *Product Condition* when the product measure $\mu^{\mathbb{N}}$ on $X^{\mathbb{N}}$ is homeomorphic to μ on X.

PROPOSITION 3.9. *Let μ be a measure on a Cantor space X.*

(a) *The product measure $\mu^{\mathbb{N}}$ satisfies the Product Condition.*
(b) *Assume that the product measure $\mu \times \mu = \mu^2$ on $X \times X$ is homeomorphic to μ on X. For every positive integer n the product μ^n is homeomorphic to μ and $S(\mu^{\mathbb{N}}) = S(\mu)$. The clopen values set $S(\mu)$ is multiplicative. Furthermore, if ν_1 and ν_2 are measures on compact spaces Y_1 and Y_2, then μ maps to ν_1 and to ν_2 iff it maps to the product $\nu_1 \times \nu_2$ on $Y_1 \times Y_2$.*
(c) *If μ satisfies the Product Condition then $\mu \times \mu$ is homeomorphic to μ.*
(d) *If μ is a finite or countably infinite product of measures each of which satisfies the Product Condition then μ satisfies the Product Condition.*
(e) *Assume that μ is a good measure. Then $\mu^{\mathbb{N}}$ is a good measure with $S(\mu^{\mathbb{N}}) + \mathbb{Z}$ the subring of $\mathbb{R}$ generated by the group $S(\mu) + \mathbb{Z}$. Furthermore, the following conditions are equivalent:*

   (i) *μ satisfies the Product Condition.*
   (ii) *$\mu \times \mu$ is homeomorphic to μ.*
   (iii) *$S(\mu \times \mu) = S(\mu)$.*
   (iv) *$S(\mu)$ is multiplicative.*
   (v) *$S(\mu)$ is ringlike.*

Proof. Clearly the product of a finite or countably infinite number of copies of $\mu^{\mathbb{N}}$ is a measure homeomorphic to $\mu^{\mathbb{N}}$. This proves (a) and (c). Similarly, (d) is obvious.

Let $h_1 : X \to X \times X$ be a homeomorphism mapping μ to μ^2. Define $h_n = 1_{X^{n-1}} \times h_1 : X^n \to X^{n+1}$. Clearly, the composition $h^n := h_n \circ h_{n-1} \circ \cdots \circ h_1 : X \to X^{n+1}$ is a homeomorphism mapping μ to μ^{n+1}. Define the "limit" $h^\infty : X \to X^{\mathbb{N}}$ by $h^\infty(x)_i = h^n(x)_i$ provided $n \geq i$. Thus, the composition of h^∞ with the projection to X^n agrees with the composition of h^n with the projection to X^n. It follows from compactness that h^∞ is surjective. Furthermore, it maps μ to $\mu^{\mathbb{N}}$. On the other hand, each coordinate projection maps $\mu^{\mathbb{N}}$ to μ. Hence, the two measures have the same clopen values set. It is not clear that h^∞ is injective and so we do not know whether, in general, the Product Condition always holds whenever $\mu \times \mu$ is homeomorphic to μ.

Clearly, the projections map $\nu_1 \times \nu_2$ to ν_1 and to ν_2. If μ maps to each of the factors then $\mu \times \mu$ maps to $\nu_1 \times \nu_2$. These yield the final results in (b).

If U, V are clopens with $\mu(U) = \alpha$ and $\mu(V) = \beta$, then $(\mu \times \mu)(U \times V) = \alpha\beta \in S(\mu \times \mu)$. Hence, $S(\mu \times \mu) = S(\mu)$ implies $S(\mu)$ is multiplicative. This completes the proof of (b) and shows that (iii)⇒(iv) in (e) as well. Also, (i)⇒(ii) by (c) and (ii)⇒(iii) is clear. If μ is good and so $S(\mu)$ is grouplike then (iv)⇒(v) by Proposition 2.1(c).

If μ is good then the product $\mu^{\mathbb{N}}$ is good by Corollary 2.9. By (i)⇒(iv) applied to $\mu^{\mathbb{N}}$ we see that $S(\mu^{\mathbb{N}})$ is ringlike and since $\mu^{\mathbb{N}}$ projects to μ the ring $S(\mu^{\mathbb{N}}) + \mathbb{Z}$ contains the group $S(\mu) + \mathbb{Z}$. On the other hand, the rectangular clopens in $X^{\mathbb{N}}$ form a partition basis with measures products of elements of $S(\mu)$. Hence, $S(\mu^{\mathbb{N}})$ is contained in the ring generated by $S(\mu)$.

Finally, if $S(\mu)$ is ringlike then $S(\mu) = S(\mu^{\mathbb{N}})$. As both measures are good, they are homeomorphic because the clopen values set is a complete invariant for good measures. Hence, (v)⇒(i). ∎

Remark. While we do not know whether every measure of Bernoulli type satisfies the Product Condition, it is clear that any Bernoulli measure does satisfy the Product Condition.

Now, beginning with any measure μ on a Cantor space X, we describe the *Quotient Construction* on μ which yields a measure $\mu^{\mathbb{Q}}$ satisfying the Quotient and Product Conditions.

Note that for each n, there are only countably many clopen subsets of X^n, and hence the set of all pairs (n, U) with U a clopen subset of X^n is countable. There is, therefore, a sequence $U(i)_{i=1}^\infty$ so that each $U(i)$ is a nonempty clopen subset of $X^{n(i)}$ for some $n(i) \geq 1$, and so that for any n, and any nonempty clopen subset U of X^n, the subset U appears in this sequence infinitely often. For each i let μ_i be the relative measure $(\mu^{n(i)})_{U(i)}$

on $U(i)$. Define

$$\mu^{\mathbb{Q}} := \prod_i \mu_i \quad \text{on} \quad X^{\mathbb{Q}} := \prod_i U(i). \tag{3.13}$$

It is clear that a different choice of counting function results in a rearrangement of the factors in the product. Hence the Quotient Construction is well-defined up to homeomorphism. In fact, it is easy to check that if each clopen is counted only once, instead of infinitely often, then a homeomorphic measure results. Hint: for each clopen $U \subset X^n$ consider the products $U^k \subset X^{nk}$ for $k = 1, 2, \ldots$.

THEOREM 3.10. *Let μ be a measure on a Cantor space X.*

(a) *The Quotient Construction measure $\mu^{\mathbb{Q}}$ satisfies the Product Condition and the Quotient Condition and*

$$S(\mu^{\mathbb{Q}}) = \{\mu^n(V)/\mu^n(U) : V \subset U \text{ are clopens in } X^n \text{ with } U \text{ nonempty}, \text{ for } n = 1, 2, \ldots\}. \tag{3.14}$$

Furthermore, if $\mu \times \mu$ is homeomorphic to μ, e.g. if μ satisfies the Product Condition, then

$$S(\mu^{\mathbb{Q}}) = \{\mu(V)/\mu(U) : V \subset U \text{ are clopens in } X \text{ with } U \text{ nonempty}\}. \tag{3.15}$$

(b) *μ is homeomorphic to $\mu^{\mathbb{Q}}$ iff μ satisfies the Product Condition and the Quotient Condition.*

(c) *The Quotient Construction on μ and on $\mu^{\mathbb{N}}$ yield homeomorphic results. That is, $(\mu^{\mathbb{N}})^{\mathbb{Q}}$ is homeomorphic to $\mu^{\mathbb{Q}}$.*

(d) *Assume that μ is a good measure. Then $\mu^{\mathbb{Q}}$ is a good measure with $S(\mu^{\mathbb{Q}}) + \mathbb{Z}$ the subfield of $\mathbb{R}$ generated by the group $S(\mu) + \mathbb{Z}$. Furthermore, the following conditions are equivalent:*

  (i) *μ satisfies the Quotient Condition.*
  (ii) *$S(\mu)$ is fieldlike.*
  (iii) *μ is homeomorphic to $\mu^{\mathbb{Q}}$.*

Proof. (a) Since each $(\mu^n)_U$ factor occurs infinitely often in $\mu^{\mathbb{Q}}$, the Product Condition is obvious.

If V is a nonempty clopen subset of $X^{\mathbb{Q}}$ then there exists a positive integer I and a clopen subset $\widetilde{V}$ of $\widetilde{U} := \prod_{i=1}^{I} U(i)$ so that $V = \widetilde{V} \times \prod_{i>I} U(i)$. If $N = \sum_{i=1}^{I} n(i)$ then $\widetilde{U}$ is a clopen subset of X^N and the product measure $\prod_{i=1}^{I} \mu_i$ equals $(\mu^N)_{\widetilde{U}}$. Furthermore

$$(\mu^{\mathbb{Q}})_V = ((\mu^N)_{\widetilde{U}})_{\widetilde{V}} \times \prod_{i>I} \mu_i. \tag{3.16}$$

In general, if $A \subset B$ have positive ν-measure then

$$\nu_A = (\nu_B)_A. \tag{3.17}$$

Hence, we have

$$(\mu^{\mathbb{Q}})_V = (\mu^N)_{\widetilde{V}} \times \prod_{i>I} \mu_i, \tag{3.18}$$

which is clearly homeomorphic to $\mu^{\mathbb{Q}}$. Hence, $\mu^{\mathbb{Q}}$ satisfies the Quotient Condition.

In addition,

$$\mu^{\mathbb{Q}}(V) = \mu^N(\widetilde{V})/\mu^N(\widetilde{U}), \tag{3.19}$$

from which (3.14) follows. If $\mu \times \mu$ is homeomorphic to μ then, by Proposition 3.9(b), each μ^N is homeomorphic to μ and so (3.15) follows in that case.

This completes the proof of (a). A similar analysis shows that if U is a clopen subset of $(X^{\mathbb{N}})^n$ then there exists a positive integer N and a clopen subset $\widetilde{U}$ of X^N so that, up to rearrangement of coordinates, $U = \widetilde{U} \times X^{\mathbb{N}}$ and $(\mu^{\mathbb{N}})_U = (\mu^N)_{\widetilde{U}} \times \mu^{\mathbb{N}}$. From this (c) follows easily.

If μ is homeomorphic to $\mu^{\mathbb{Q}}$ then it satisfies the Product and the Quotient Conditions by (a). For the converse assume first that $\mu \times \mu$ is homeomorphic to μ and that μ satisfies the Quotient Condition. Each $\mu^{n(i)}$ is homeomorphic to μ by Proposition 3.9(b). So each μ_i is homeomorphic to μ_U for U some nonempty clopen subset of X. So by the Quotient Condition each μ_i is homeomorphic to μ. (Clearly μ_i also satisfies the Quotient Condition. By Theorem 3.7, both these measures are refinable. They have the same clopen values sets, so by Theorem 2.14, they are homeomorphic.) Hence, $\mu^{\mathbb{Q}}$ is homeomorphic to a countably infinite product of copies of μ, i.e. to $\mu^{\mathbb{N}}$. If μ satisfies the Product Condition then $\mu^{\mathbb{N}}$ is homeomorphic to μ, proving (b).

Now assume that μ is good. By Corollary 2.9 and Theorem 2.3(b) each μ_i is a good measure and so by Corollary 2.9 again the product $\mu^{\mathbb{Q}}$ is good as well.

Now (iii)⇒(i) by part (a), and (i)⇒(ii) follows from Theorem 3.7.

By using (i)⇒(ii) for $\mu^{\mathbb{Q}}$ we see that $S(\mu^{\mathbb{Q}}) + \mathbb{Z}$ is a subfield of $\mathbb{R}$ which contains the group $S(\mu) + \mathbb{Z}$. From (3.14) it follows that $S(\mu^{\mathbb{Q}}) + \mathbb{Z}$ is contained in the subfield generated by $S(\mu)$.

In particular, if $S(\mu)$ is fieldlike then μ and $\mu^{\mathbb{Q}}$ have the same clopen values set. Since both measures are good, they are homeomorphic to each other. Thus, (ii)⇒(iii). ■

In Section 6 we will use the Quotient Construction to produce a family of measures which satisfy the Quotient and Product Conditions but which are not good.

LEMMA 3.11. *Let $\{\mu_0, \mu_1, \ldots\}$ be a finite or infinite sequence of measures on nonempty zero-dimensional spaces $\{X_0, X_1, \ldots\}$ with X_0 a Cantor space.*

Let μ denote the product measure on the Cantor space X which is the product of these spaces.

(a) *If $S(\mu_0)$ is ringlike or, equivalently, grouplike and multiplicative, and $S(\mu_i) \subset S(\mu_0)$ for $i = 1, \ldots,$ then $S(\mu) = S(\mu_0)$.*

(b) *Assume that μ_0 is a good measure of Bernoulli type and assume that for $i = 1, 2, \ldots$ there exists a partition basis $\mathcal{B}_i$ for X_i such that $U \in \mathcal{B}_i$ implies $\mu_i(U) \in \mathrm{Div}(S(\mu_0))$ or, equivalently, $\mu_i(U)$ is a unit of the ring generated by $S(\mu_0)$. The measure μ on X is homeomorphic to μ_0 on X_0 and so it is a good measure of Bernoulli type.*

Proof. (a) The 0 coordinate projection maps μ to μ_0 and so $S(\mu_0) \subset S(\mu)$. For $i = 0, 1, \ldots$ let $\mathcal{B}_i$ be any partition basis for X_i. Define the partition basis $\mathcal{B}$ on the product X by choosing members of $\mathcal{B}_i$ for finitely many coordinates i and then pulling back their product to X via the projection map. Since $S(\mu_0)$ is multiplicative and $S(\mu_i) \subset S(\mu_0)$ for all i, it follows that the value of μ on each element of $\mathcal{B}$ lies in $S(\mu_0)$. Since $\mathcal{B}$ is a partition basis and $S(\mu_0)$ is grouplike it follows that $S(\mu) \subset S(\mu_0)$.

(b) Since μ_0 is of Bernoulli type we can choose a partition basis $\mathcal{B}_0$ for X_0 consisting of clopens of μ_0 type. Since the measure is good all these clopens are good and the Two Implies Three Lemma implies that $\mu_0(U) \in \mathrm{Div}(S(\mu_0))$ for all $U \in \mathcal{B}_0$. By Theorem 3.4(c), $S(\mu_0)$ is ringlike, and by Proposition 2.1(d), $\alpha \in (0, 1]$ is a divisor of $S(\mu_0)$ iff it is a unit of the ring $S(\mu_0) + \mathbb{Z}$ generated by $S(\mu_0)$. With the partition bases $\mathcal{B}_i$ $(i = 1, 2, \ldots)$ given in the hypothesis of (b) the values, all divisors of $S(\mu_0)$, lie in $S(\mu_0)$ which is grouplike, so $S(\mu_i) \subset S(\mu_0)$ for all i. By part (a), $S(\mu) = S(\mu_0)$. Since the clopen values set is a complete invariant for good measures, we finish the proof by showing directly that μ is good.

From the sequence $\{\mathcal{B}_i : i = 0, 1, \ldots\}$ of our partition bases we construct the product partition base $\mathcal{B}$ as in part (a). A typical element $U \in \mathcal{B}$ is a product of $U_i \in \mathcal{B}_i$ for i's in some finite set I and $U_i = X_i$ for the remaining factors. So $\mu(U)$ is the product of the $\mu_i(U_i)$ for $i \in I$ and so it is a divisor of $S(\mu_0) = S(\mu)$. We show that U is a clopen of μ type. By the Two Implies Three Lemma U is good for μ. As we have a partition basis of X consisting of clopens good for μ it follows from Theorem 2.8 that μ is a good measure.

To show that U is a clopen of μ type we must show that $S(\mu_U) = S(\mu)$. The measure μ_U is the product measure obtained by using $(\mu_i)_{U_i}$ on the factor U_i. For $i = 0$ the clopen $U_0 \in \mathcal{B}_0$ is assumed to be of μ_0 type and so $S((\mu_0)_{U_0}) = S(\mu_0)$. For $i = 1, 2, \ldots$ the numbers in $S((\mu_i)_{U_i})$ are obtained by taking elements of $S(\mu_i)$ which are smaller than $\mu_i(U_i)$ and then dividing by $\mu_i(U_i)$. These are elements of $S(\mu_0)$ divided by a larger number which is a divisor of $S(\mu_0)$. So we have $S((\mu_i)_{U_i}) \subset S(\mu_0) = S((\mu_0)_{U_0})$ for all i.

Applying part (a) to this product measure we obtain $S(\mu_U) = S((\mu_0)_{U_0}) = S(\mu_0) = S(\mu)$. Thus, U is a clopen of μ type. ∎

Given any measure μ on a Cantor space, let $G = G(\mu)$ be the group generated by $S(\mu)$, so that

$$S(\mu) + \mathbb{Z} \subset G(\mu) \tag{3.20}$$

with equality iff $S(\mu)$ is grouplike. Since $G \cap [0,1]$ is grouplike, Theorem 2.3 implies there is a good measure μ_{g}, unique up to homeomorphism, such that $S(\mu_{\mathrm{g}}) = G \cap [0,1]$. We call μ_{g} a *good measure envelope* of μ.

THEOREM 3.12. *Let μ be a measure of Bernoulli type on a Cantor space X and let μ_{g} be a good measure envelope of μ. That is, μ_{g} is a good measure whose clopen values set generates the same group $G(\mu)$ as does μ. Then $G(\mu)$ is a ring and so $S(\mu_{\mathrm{g}})$ is ringlike. The product measure $\mu \times \mu_{\mathrm{g}}$ satisfies*

$$S(\mu) \subset S(\mu_{\mathrm{g}}) = S(\mu \times \mu_{\mathrm{g}}). \tag{3.21}$$

If the measure μ_{g} is of Bernoulli type or, equivalently, if the ring $G(\mu)$ is such that every positive element of $G(\mu)$ is a sum of positive units, then the following conditions are equivalent:

(a) *$\mu \times \mu_{\mathrm{g}}$ is homeomorphic to μ_{g}.*
(b) *$\mu \times \mu_{\mathrm{g}}$ is a good measure of Bernoulli type.*
(c) *$\mu \times \mu_{\mathrm{g}}$ is a good measure.*
(d) *For every clopen $U \subset X$ of μ type the value $\mu(U)$ is a unit of the ring $G(\mu)$.*
(e) *X admits a partition basis $\mathcal{B}$ such that each $U \in \mathcal{B}$ is of μ type with $\mu(U)$ a unit of the ring $G(\mu)$.*

When these conditions hold then μ is homeomorphic to μ_{g} iff $S(\mu)$ is grouplike.

Proof. $G(\mu)$ is a ring because $S(\mu)$ is multiplicative by Theorem 3.2. The equation between clopen values sets then follows from Lemma 3.11(a). Since all Cantor spaces are homeomorphic we can assume that the measure μ_{g} is on X. The equivalence between the ring condition and the assumption that μ_{g} is of Bernoulli type comes from Theorem 3.5(a).

(b)⇒(c) is obvious. (c)⇔(a) because μ_{g} is good and the clopen values set is a complete invariant for good measures.

(c)⇒(d). If U is a clopen subset of X of μ type then $S(\mu_U) = S(\mu)$ generates the group $G(\mu)$ and so μ_{g} is a good measure envelope for μ_U. Since $(\mu \times \mu_{\mathrm{g}})_{(U \times X)} = \mu_U \times \mu_{\mathrm{g}}$, the above equation applied to μ_U implies that $U \times X$ is a clopen subset of $X \times X$ of $\mu \times \mu_{\mathrm{g}}$ type. By (c) the latter is a good measure and so Two Implies Three shows that $\mu(U) = (\mu \times \mu_{\mathrm{g}})(U \times X)$ is a divisor of $S(\mu \times \mu_{\mathrm{g}})$ and so it is a unit of the ring $G(\mu)$.

(d)⇒(e) is obvious.

(e)$\Rightarrow$(b). This is where we need the assumption that the measure μ_{g} is of Bernoulli type. Since μ_{g} is a good measure of Bernoulli type, X admits a partition basis $\mathcal{B}_{\mathrm{g}}$ of clopens of μ_{g} type. The measures of the elements of $\mathcal{B}_{\mathrm{g}}$ are all divisors of the ring $G(\mu)$ by Two Implies Three. Consider the partition basis $\mathcal{B}\times\mathcal{B}_{\mathrm{g}}$ on $X\times X$. A typical clopen $U\times V$ has relative measure $\mu_U \times (\mu_{\mathrm{g}})_V$. Because μ_{g} is good, $(\mu_{\mathrm{g}})_V$ is good, and since V is of μ_{g} type, $(\mu_{\mathrm{g}})_V$ is homeomorphic to μ_{g}. As in the proof of (c)$\Rightarrow$(d), $U \in \mathcal{B}$ of μ type implies that μ_{g} and hence $(\mu_{\mathrm{g}})_V$ are good measure envelopes for μ_U. Hence, $S(\mu_U\times(\mu_{\mathrm{g}})_V) = S((\mu_{\mathrm{g}})_V) = S(\mu_{\mathrm{g}}) = S(\mu\times\mu_{\mathrm{g}})$. Thus, each $U\times V$ is a clopen of $\mu\times\mu_{\mathrm{g}}$ type. Furthermore, $(\mu\times\mu_{\mathrm{g}})(U\times V) = \mu(U)\cdot\mu_{\mathrm{g}}(V)$ is a product of units of the ring $G(\mu)$ and so it is itself a unit. As $S(\mu\times\mu_{\mathrm{g}}) = S(\mu_{\mathrm{g}})$ is ringlike these are divisors of the clopen values set. By Theorem 3.4(b), the product measure is a good measure of Bernoulli type.

Finally, if $S(\mu)$ is grouplike and hence ringlike then condition (e) and Proposition 2.1(d) imply that μ satisfies the conditions of Theorem 2.3(c) and so it is a good measure. Since $S(\mu) = S(\mu_{\mathrm{g}})$ in the grouplike case, μ is homeomorphic to μ_{g}. ∎

We conclude with a little result sharpening the notion of weak refinability for measures of Bernoulli type.

The definition of a weakly refinable measure includes the requirement that X be a refinable set. This addition is necessary, for example, in the proof of Theorem 2.14. For measures of Bernoulli type this demand is superfluous.

LEMMA 3.13. *Let μ be a measure on a Cantor space X with $S(\mu)$ multiplicative*, *e.g. a measure of Bernoulli type. If there exists a partition of X by refinable clopen sets then X is refinable.*

Proof. Let $\{U_1,\dots,U_N\}$ be a partition of X by refinable clopen sets and let $\alpha_1,\dots,\alpha_k \in S(\mu)$ with $\alpha_1+\dots+\alpha_k = 1$. Since $S(\mu)$ is multiplicative, $\alpha_i\mu(U_j) \in S(\mu)$ for $i = 1,\dots,k$ and $j = 1,\dots,N$. Since U_j is refinable, there exists a partition $\{U_{1j},\dots,U_{kj}\}$ of U_j with $\mu(U_{ij}) = \alpha_i\mu(U_j)$ for $i = 1,\dots,k$ and $j = 1,\dots,N$. Let $V_i = \bigcup_j U_{ij}$ for $i = 1,\dots,k$ to define the required partition of X. ∎

4. Bernoulli measures. Recall that a Bernoulli measure is defined via a probability vector $p : A \to (0,1)$ on a finite alphabet $A = \{a_1,\dots,a_N\}$ with cardinality $|A| = N$ at least 2. That is, the $p(a)$'s are positive real numbers which sum to 1. When we list the elements we will write p_i for $p(a_i)$.

The Bernoulli measure $\beta(p)$ on $A^{\mathbb{N}}$ is the product of copies of the measure associated with p on A, which we will call the *p-measure* on A. If $q : B \to (0,1)$ is another such probability vector, we will say that $f : A \to B$ *maps p to q* if $p = q\circ f$. This is equivalent to saying that f maps the p-measure on

A to the q-measure on B. If μ is a measure on a space X then a *p-partition* on (X,μ) is a clopen partition $\{U_1,\dots,U_N\}$ of X with $N=|A|$ such that

$$\mu(U_i)=p_i \quad \text{for } i=1,\dots,N. \tag{4.1}$$

PROPOSITION 4.1. *If $p:A\to(0,1)$ and $q:B\to(0,1)$ are probability vectors on finite alphabets, then the following conditions are equivalent*:

(1) *The measure $\beta(p)$ maps to the measure $\beta(q)$.*
(2) *For some positive integer k the probability vector p^k on A^k maps to q.*
(3) *There exists a q-partition of $(A^{\mathbb{N}},\beta(p))$.*

Proof. (1)⇒(3). The words of length one, i.e. the letters of B, define a cylindrical q-partition of $(B^{\mathbb{N}},\beta(q))$. If $f:A^{\mathbb{N}}\to B^{\mathbb{N}}$ maps $\beta(p)$ to $\beta(q)$ then the preimage under f of any q-partition is a q-partition.

(3)⇒(2). Assume that $\{U_1,\dots,U_N\}$ is a q-partition of $(A^{\mathbb{N}},\beta(p))$. There exists a positive integer k such that each U_i is a union of cylindrical sets of length k. Map the words in U_i to b_i to define the map $f:p^k\to q$.

(2)⇒(1). If $f:p^k\to q$ then the product of copies of f is a map from $(A^k)^{\mathbb{N}}\to B^{\mathbb{N}}$ taking $\beta(p^k)$ to $\beta(q)$. Furthermore, $\beta(p^k)$ is naturally homeomorphic to $\beta(p)$ by grouping the words in $A^{\mathbb{N}}$ into blocks of length k. ∎

REMARK. While it follows from Proposition 3.9(b), it is easy to check directly that for probability vectors q_1 and q_2, the measure $\beta(p)$ maps to $\beta(q_1)$ and to $\beta(q_2)$ iff it maps to the product $\beta(q_1\cdot q_2)\approx\beta(q_1)\times\beta(q_2)$.

We obtain special results in the two letter case ($|A|=2$). We will always use the alphabet $2=\{0,1\}$ for this case. The probability vector is determined by the value r on 0 with $1-r$ the value on 1. We will write $\beta(r,1-r)$ for the Bernoulli measure on $2^{\mathbb{N}}$.

COROLLARY 4.2. *Let $p:A\to(0,1)$ be a probability vector and $r\in(0,1)$. The following conditions are equivalent*:

(1) $r\in S(\beta(p))$.
(2) $S(\beta(r,1-r))\subset S(\beta(p))$.
(3) *$\beta(p)$ maps to $\beta(r,1-r)$.*

In particular, if $r,s\in(0,1)$ we have

$$\beta(r,1-r)\sim\beta(s,1-s) \;\Leftrightarrow\; S(\beta(r,1-r))=S(\beta(s,1-s)). \tag{4.2}$$

Proof. (3)⇒(2)⇒(1) is clear. If $r\in S(\beta(p))$ then there exists a clopen set U with $\beta(p)(U)=r$. Hence, $\{U,A^{\mathbb{N}}\setminus U\}$ is an $(r,1-r)$-partition for $(A^{\mathbb{N}},\beta(p))$. By the above proposition $\beta(p)$ maps to $\beta(r,1-r)$. ∎

REMARK. In general, $S(\beta(q))\subset S(\beta(p))$ implies $q_i\in S(\beta(p))$ for $i=1,\dots,|B|$. However, as we will see below, the converse need not hold.

The above corollary is a special case of a more general result.

THEOREM 4.3. *Let μ be a measure of Bernoulli type on a Cantor space X and let $r \in (0,1)$. The following conditions are equivalent*:

(1) $r \in S(\mu)$.
(2) $S(\beta(r,1-r)) \subset S(\mu)$.
(3) μ *maps to* $\beta(r,1-r)$.

Proof. Again (3)⇒(2)⇒(1) is clear.

If $r \in S(\mu)$ then there exists a clopen set U_0 with $\mu(U) = r$. Hence, $\{U_0, U_1\}$ is an $(r,1-r)$-partition for (X,μ). With $A = \{0,1\}$ this defines a map, $\pi_1 : X \to A$. Inductively, we define coherent maps $\pi_k : X \to A^k$ such that for every word $w \in A^k$ the preimage $U_w := \pi_k^{-1}(w)$ is a clopen set with $\mu(U_w) = r^a(1-r)^b$ where a, b are the numbers of occurrences of $0, 1$ respectively in the word w. At the next stage, for each such $w \in A^k$, Theorem 3.2 implies that $S(\mu) \subset S(\mu_{U_w})$ and so there exists an $(r,1-r)$-partition $\{U_{w0}, U_{w1}\}$ of (U_w, μ_{U_w}). These partitions define the map π_{k+1}. From this inverse system of maps we get the inverse limit map $\pi : X \to A^{\mathbb{N}}$ such that the preimage of every cylinder set has μ-measure its $\beta(r,1-r)$-value. This directly constructs the map from μ to $\beta(r,1-r)$. ∎

Notice that the values of $\beta(p)$ on the cylinder sets are all products of the $p(a)$'s. Hence, the subring of $\mathbb{R}$ generated by $\{p(a) : a \in A\}$ contains $S(\beta(p))$ and so it is the ring generated by $S(\beta(p))$. We will denote this subring $\mathbb{Z}[p]$ or $\mathbb{Z}[p_1, \ldots, p_N]$. Of course, $\mathbb{Z}[p]$ contains $S(\beta(p)) + \mathbb{Z}$ with equality iff $S(\beta(p))$ is grouplike. When $N = 2$ and $p_1 = r$, the ring is just $\mathbb{Z}[r]$, the ring of polynomials in r with integer coefficients.

Now we apply the results of the preceding section to Bernoulli measures.

THEOREM 4.4. *Let $p : A \to (0,1)$ be a positive probability vector on a finite alphabet and let $\beta(p)$ be the associated Bernoulli measure on $A^{\mathbb{N}}$. The following conditions are equivalent*:

(a) $\beta(p)$ *is a good measure.*
(b) *For each* $a \in A$, $p(a) \in \mathrm{Div}(S(\beta(p)))$.
(c) *The clopen values set* $S(\beta(p))$ *is grouplike and for each* $a \in A$, $p(a)$ *is a unit in the ring* $\mathbb{Z}[p]$.

Proof. (a)⇒(c). For a good Bernoulli measure Theorem 3.4(c) implies that $S(\mu)$ is ringlike and values of the measure on the cylinders are divisors by the Two Implies Three Lemma and hence are units of the ring by Proposition 2.1(d).

(c)⇒(b). As the measure is Bernoulli, Theorem 3.2 implies that the clopen values set is multiplicative and so by Proposition 2.1(c) it is ringlike when it is grouplike. By Proposition 2.1(d) the values $p(a)$ are divisors of $S(\beta(p))$ when they are units of the ring.

(b)⇒(a). The measure of every cylinder set is a product of the $p(a)$'s which are assumed to be divisors. Since $\mathrm{Div}(S(\beta(p)))$ is multiplicative, the cylinder sets are clopens of μ type with divisor values. By Two Implies Three again the cylinders are good clopens. By Theorem 3.4(b), the measure is good. ∎

In particular, we obtain the following for the 2-letter alphabet case.

Corollary 4.5. *For $r \in (0,1)$, r and $1-r$ are units of the ring $\mathbb{Z}[r]$ iff $1/(r(1-r))$ is an algebraic integer. The measure $\beta(r, 1-r)$ is good iff the clopen values set $S(\beta(r,1-r))$ is grouplike and $1/(r(1-r))$ is an algebraic integer.*

Proof. Recall that a complex number is an algebraic integer iff it is the root of a monic polynomial equation with integer coefficients. Furthermore, the algebraic integers are a subring of $\mathbb{C}$.

The number r is a unit for $\mathbb{Z}[r]$ iff the reciprocal is in the ring, that is, iff there exists a polynomial $P(r)$ with integer coefficients such that $1 - r \cdot P(r) = 0$. If the polynomial is of degree d then we can divide by r^{d+1} to obtain a monic polynomial with root $1/r$. This argument is reversible and so if $1/r$ is an algebraic integer then r is a unit. Similarly, for $1-r$. So if r and $1-r$ are units, $1/(r(1-r))$ is a product of algebraic integers and so it is an algebraic integer.

On the other hand, if $1/(r(1-r))$ is an algebraic integer, then just as before there is a polynomial such that $1 - r(1-r) \cdot P(r(1-r)) = 0$. Since $P(r(1-r))$ is a polynomial in r it follows that $r(1-r)$ is a unit for the ring. But if the product of two ring elements is a unit then each factor is a unit. Hence, r and $1-r$ are units.

The ring generated by $S(\beta(r,1-r))$ is just $\mathbb{Z}[r] = \mathbb{Z}[1-r]$ and so the final result follows from Theorem 4.4. ∎

Thus, the problem of deciding whether a Bernoulli measure is good is split into two pieces. As we will see in the examples below, it is usually easy to compute the ring generated by the $p(a)$'s and hence to decide whether or not the $p(a)$'s are units in the ring. The hard part is to decide whether the clopen values set is grouplike. By the following result, it is usually enough to recognize divisors.

Theorem 4.6. *Let μ be a measure of Bernoulli type on a Cantor space X. If there exist two disjoint clopen sets $U_1, U_2 \subset X$ such that $\mu(U_1), \mu(U_2) \in \mathrm{Div}(S(\mu))$ then $S(\mu)$ is grouplike.*

Proof. Let $\delta_1 = \mu(U_1)$, $\delta_2 = \mu(U_2)$. By Proposition 2.1(b) it suffices to show that if $\alpha, \beta \in S(\mu)$ with $\alpha + \beta < 1$ then $\alpha + \beta \in S(\mu)$. Because the measure is of Bernoulli type, Theorem 3.2 implies that $\alpha \cdot \delta_2 \in S(\mu_{U_1})$ and

$\beta \cdot \delta_1 \in S(\mu_{U_2})$. Thus, we can choose clopen subsets $V_1 \subset U_1$ and $V_2 \subset U_2$ such that

$$\mu(V_1) = \alpha \cdot \delta_2 \cdot \delta_1, \quad \mu(V_2) = \beta \cdot \delta_1 \cdot \delta_2. \tag{4.3}$$

The union V has measure $(\alpha+\beta)\delta_1\delta_2$. Since $\alpha+\beta < 1$ and the product $\delta_1\delta_2$ is a divisor, it follows that $\alpha+\beta \in S(\mu)$. ■

This does not help too much in practice because while it is not too hard to recognize ring units, the equivalence between ring units and divisors only works after we know that the clopen values set is grouplike. However, we do have the following amusing result:

COROLLARY 4.7. *Let μ be a measure of Bernoulli type on a Cantor space X. Assume that there exists a clopen set $U \subset X$ with $\mu(U) = 1/2$. The clopen values set is grouplike iff the following condition holds*:

$$\alpha \in S(\mu) \text{ and } \alpha < 1/2 \ \Rightarrow\ 2\alpha \in S(\mu). \tag{4.4}$$

Proof. The existence of U simply says $1/2 \in S(\mu)$ and the latter condition says that that $1/2 \in \mathrm{Div}(S(\mu))$. Hence, U and its complement will do as the disjoint clopens required to apply Theorem 4.6. ■

We conclude by considering the action of $H(X, \beta(p))$, the automorphism group, on the space X. Recall that if a group H acts on a space X then the action is called *minimal* if for every $x \in X$ the orbit $Hx = \{h(x) : h \in H\}$ is dense in X. The action is called *transitive* if $Hx = X$ for every $x \in X$, or equivalently, the entire space consists of a single orbit.

If ξ is a permutation of $\mathbb{N}$ then ξ^* is the homeomorphism on $X = A^{\mathbb{N}}$ defined by permuting the coordinates using ξ:

$$\xi^*(x)_i := x_{\xi(i)} \quad \text{for all } i \in \mathbb{N}. \tag{4.5}$$

Clearly, ξ^* preserves every Bernoulli measure, so it is in every $H(X, \beta(p))$.

If $x \in A^{\mathbb{N}}$ and $a \in A$ we define

$$N(x, a) := \{i \in \mathbb{N} : x_i = a\} \tag{4.6}$$

and, with $X = A^{\mathbb{N}}$,

$$X_\infty = \{x \in X : |N(x, a)| = \infty \text{ for all } a \in A\}, \tag{4.7}$$

where $|N| \in \mathbb{N} \cup \{\infty\}$ denotes the cardinality of the set N.

Thus, $x \in X_\infty$ iff every letter of A occurs infinitely often in x. Clearly, if $x, y \in X_\infty$ then there exists a permutation ξ such that $\xi^*(x) = y$. Hence,

$$x \in X_\infty \ \Rightarrow\ H(X, \beta(p))x \supset X_\infty. \tag{4.8}$$

By Theorem 2.3(a) the automorphism group $H(X, \mu)$ of a good measure μ acts transitively on X. Conversely, Theorem 2.15 of Akin (2005) says that a measure μ on a Cantor space X is a good measure if $H_\mu(X)$ acts minimally on X and, in addition, if for every nonempty clopen $U \subset X$ the clopen values set $S(\mu_U)$ is grouplike. For Bernoulli measures we obtain sharper results.

If a is a letter in a finite alphabet A then we will write $\overline{a}$ for the point of $A^{\mathbb{N}}$ all of whose coordinates equal a.

THEOREM 4.8. *Let $p : A \to (0,1)$ be a positive probability vector on a finite alphabet, $A = \{a_1, \ldots, a_N\}$, and let $\beta(p)$ be the associated Bernoulli measure on $X = A^{\mathbb{N}}$. The following conditions are equivalent*:

(1) *The group $H(X, \beta(p))$ acts transitively on X.*
(2) *The group $H(X, \beta(p))$ acts minimally on X.*
(3) *For $i, j = 1, \ldots, N$ there exists $h_{ij} \in H(X, \beta(p))$ such that*

$$h_{ij}(\overline{a}_i) \in [a_j]. \tag{4.9}$$

Proof. (1)⇒(2). Obvious.

(2)⇒(3). A minimal action takes the point $\overline{a}_i$ into the nonempty clopen cylinder $[a_j]$.

(3)⇒(1). By (4.8) all of the points of X_∞ are contained in a single orbit. So it suffices, given $x \in X$, to construct $h \in H(X, \beta(p))$ such that $h(x) \in X_\infty$. At least one of the sets $N_i := N(x, a_i)$ is infinite. Without loss of generality suppose N_1 is infinite. We will construct h by using the automorphisms h_{1j} of (4.9) with the choice $h_{11} =$ identity. Notice that the letter a_j occurs in $h_{1j}(\overline{a}_1)$ at least once.

Since we have assumed N_1 is infinite, N_1 can be partitioned into a countable class of infinite subsets. Letting Z be a countably infinite index set, we may first partition N_1 into infinite sets, $\{I_{zj} : z \in Z, j = 1, \ldots, N\}$, and then fixing some $e \in Z$, we adjust the set I_{e1} by adding in all elements of $\mathbb{N} \setminus N_1$. The result of this is that $\{I_{zj} : z \in Z, j = 1, \ldots, N\}$ is a partition of $\mathbb{N}$, so that each I_{zj} is an infinite set, and each of them except I_{e1} consists only of coordinates at which x has the symbol a_1. Using this partition of $\mathbb{N}$ we can identify:

$$X = A^{\mathbb{N}} = \prod_{z,j} A^{I_{zj}}, \tag{4.10}$$

and this identification relates the Bernoulli measure $\beta(p)$ on $A^{\mathbb{N}}$ to the product of the Bernoulli measures $\beta(p)$ on $A^{I_{zj}}$. Use the natural bijection from $\mathbb{N}$ to I_{zj} obtained by counting the elements of the latter subset of $\mathbb{N}$ and we can define $h_{zj} \in H(A^{I_{zj}}, \beta(p))$ to be a copy of h_{1j} via this identification. Let

$$h := \prod_{z,j} h_{zj}. \tag{4.11}$$

In an obvious way, h contains infinitely many copies of each h_{1j} and at x each of these except for $e1$ is acting on a copy of $\overline{a}_1$. Hence, $h(x)$ contains infinitely many copies of each letter a_j. That is, $h(x) \in X_\infty$. ■

Corollary 4.9. *If the group $H(A^{\mathbb{N}}, \beta(p))$ acts minimally on the space $A^{\mathbb{N}}$ then for each $a \in A$, $p(a)$ is a unit in the ring $\mathbb{Z}[p]$. If, in addition, the clopen values set $S(\beta(p))$ is grouplike then the measure $\beta(p)$ is good.*

Proof. By (4.9) there exist automorphisms h_{ij} such that $h_{ij}(\overline{a}_i) \in [a_j]$ for $i, j = 1, \dots, N$. By continuity, there exists a positive integer M such that for all i, j, $h_{ij}([a_i^M]) \subset [a_j]$ where a^M is the word of length M with all entries equal to a. Since the measure of $[a_i^M]$ equals p_i^M it follows that $p_i^M/p_j \in S(\beta(p)_{[a_j]}) = S(\beta(p))$.

When we expand the right side of the equation $1 = (p_1 + \dots + p_N)^{MN}$ by using the Binomial Theorem we see that every term contains some p_i raised to a power at least M. Hence, it follows that $1/p_j$ is in the group generated by $S(\beta(p))$, which is the ring generated by $p_1, \dots, p_N$. Thus, each p_j is a unit of the ring.

If, in addition, the clopen values set is grouplike then the measure is good by Theorem 4.4. ∎

Proposition 4.10. *Let $a \in A$. If $h(\overline{a}) \neq \overline{a}$ for some $h \in H(A^{\mathbb{N}}, \beta(p))$, then for some $b \in A$ with $b \neq a$ and some positive integer M,*

$$p(a)^M/p(b) \in S(\beta(p)). \tag{4.12}$$

Proof. If $h(\overline{a}) \neq \overline{a}$ then some coordinate of $h(\overline{a})$ is a letter $b \in A$ distinct from a. By composing with a permutation of coordinates we can assume that the first coordinate is b. That is, $h(\overline{a}) \in [b]$. As shown in the proof of Corollary 4.9, this implies that for some positive integer M, (4.12) holds. ∎

Theorem 4.11. *Let $p : A \to (0,1)$ be a positive probability vector on a finite alphabet, $A = \{a_1, \dots, a_N\}$, and let $\beta(p)$ be the associated Bernoulli measure on $X = A^{\mathbb{N}}$. If the measure $\beta(p)$ is refinable then the following conditions are equivalent:*

(1) *The group $H(X, \beta(p))$ acts transitively on X.*
(2) *The group $H(X, \beta(p))$ acts minimally on X.*
(3) *For $i, j = 1, \dots, N$ there exists $M_{ij} > 0$ such that*

$$p_j - p_i^{M_{ij}} \in S(\beta(p)). \tag{4.13}$$

(4) *There exists $M_0 > 0$ such that for all $M \geq M_0$ and for all $i, j = 1, \dots, N$, $p_j - p_i^M \in S(\beta(p))$.*

Proof. (1)⇔(2). Apply Theorem 4.8.

(2)⇒(4). As in the proof of Corollary 4.9 there exists M_0 such that for $M \geq M_0$, $h_{ij}([a_i^M]) \subset [a_j]$. The measure of the complementary set is $p_j - p_i^M$ and so this number is a clopen value.

(4)⇒(3). Obvious.

(3)⇒(2). By Theorem 4.8 it suffices for each $i, j = 1, \dots, N$ to construct a homeomorphism h_{ij} satisfying (4.9). For $i = j$ we can use the identity and so we assume that $i \neq j$. Let $M = M_{ij}$ be chosen to satisfy (4.13).

Let $X_1 = [a_i^M]$, which has measure p_i^M. Note that p_i^M and $p_j - p_i^M$ are clopen values which sum to p_j, which is the measure of $[a_j]$. Since $\beta(p)$ is refinable by assumption, there exists a clopen subset X_2 of $[a_j]$ with $\beta(p)(X_2) = p_i^M = \beta(p)(X_1)$. Let $X_3 = X \setminus (X_1 \cup X_2)$ so that $\mathcal{X} = \{X_1, X_2, X_3\}$ is a partition of X. Now exchange X_1 and X_2. Let $Y_1 = X_2$, $Y_2 = X_1$ and $Y_3 = X_3$. Hence, $\mathcal{Y} = \{Y_1, Y_2, Y_3\}$ is a clopen partition with $\beta(p)(X_k) = \beta(p)(Y_k)$ for $k = 1, 2, 3$. Since the measure is assumed to be refinable, all clopen sets are refinable. By Theorem 2.14 there exists $h_{ij} \in H(X, \beta(p))$ such that $h_{ij}(X_k) = Y_k$ for $k = 1, 2, 3$. Since $\overline{a}_i \in X_1$ and $Y_1 = X_2 \subset [a_j]$ we obtain (4.9). ■

5. Rational Bernoulli measures. Now we consider the special case of a Bernoulli measure associated with a probability vector $p : A \to (0, 1)$ such that all the values $p(a)$ are rational. Recall that when $A = \{a_1, \dots, a_N\}$ we write p_i for $p(a_i)$, $i = 1, \dots, N$.

For any integer $n > 1$ we define the set

$$S(n) := \{i/n^k : i = 0, \dots, n^k \text{ and } k = 0, 1, \dots\}. \tag{5.1}$$

Recall that $\beta(1/n)$ denotes the Bernoulli measure obtained by using uniform weights on an alphabet of n symbols. That is, $\beta(1/n) = \beta(p)$ where $p = (1/n, \dots, 1/n)$. This is a Haar measure and so it is good. Clearly,

$$S(n) = S(\beta(1/n)). \tag{5.2}$$

In particular, $S(n)$ is ringlike. The associated ring, $S(n) + \mathbb{Z}$, is just $\mathbb{Z}[1/n]$, the ring of rationals which can be written with denominator a power of n.

In considering these sets there is a useful partial order on the positive integers.

DEFINITION 5.1. Given positive integers a, b we say that *a semi-divides* b (and write $a \prec b$) if every prime which divides a also divides b, or equivalently, if a divides some positive power of b. Thus, *a has the same prime divisors as b* iff each semi-divides the other (in which case we write $a \simeq b$).

For subsets A, B of $\mathbb{R}$ we write $A \cdot B = \{a \cdot b : a \in A,\, b \in B\}$, just the products, not the sums of products. In particular, if A and B are multiplicative then $A \cdot B$ is. However, even if A and B are ringlike, $A \cdot B$ need not be grouplike. For example, if A_2 and A_3 are the subgroups generated by $\{1, \sqrt{2}\}$ and $\{1, \sqrt{3}\}$, which are subrings, then the subgroup (and subring) generated by the products consists of all numbers which can be written as $(1\ \ \sqrt{2})K(1\ \ \sqrt{3})^T$ with K any 2×2 integer matrix. It is easy to check that the products, i.e. the elements of $A_2 \cdot A_3$, are exactly those with $\det(K) = 0$.

LEMMA 5.2. *Let a, b, n, n_1, n_2 be positive integers.*

(a) *If a and b are relatively prime and with $a < b$ then $a/b \in S(n)$ iff b semi-divides n.*
(b) *$S(n_1) \subset S(n_2)$ iff n_1 semi-divides n_2. In particular, $S(n_1) = S(n_2)$ iff n_1 and n_2 have the same prime divisors.*
(c) *If $n = n_1 \cdot n_2$ then $S(n_1) \cdot S(n_2) = S(n)$.*

Proof. (a) Clearly, the fraction in lowest terms a/b lies in $S(n)$ iff b divides some power of n.

(b) Obvious from (a).

(c) Since $S(n)$ is multiplicative, we get the inclusion $S(n_1)\cdot S(n_2) \subset S(n)$ from (b).

If $S(n_1) = S(n_2)$ then by (b) this common set is $S(n)$ as well and the equality is clear. If this is not true then by (b) the ratio

$$r := \log(n_2)/\log(n_1) \tag{5.3}$$

is irrational and so by Kronecker's Theorem translation by r on the reals mod 1 is a minimal homeomorphism.

Given $a/n^k < 1$, choose ε positive but small enough that $n^k/a > n_1^{\varepsilon}$. By Kronecker's Theorem we can choose positive integers k_1, k_2 larger than k so that

$$k_1 < k_2 r < k_1 + \varepsilon, \tag{5.4}$$

and so

$$1 < n_2^{k_2}/n_1^{k_1} < n_1^{\varepsilon}. \tag{5.5}$$

Thus, we have

$$a/n^k = \alpha \cdot \beta \quad \text{with} \quad \alpha = (a/n^k)(n_2^{k_2}/n_1^{k_1}) \text{ and } \beta = n_1^{k_1}/n_2^{k_2}. \tag{5.6}$$

By construction $0 < \alpha, \beta < 1$ and β is clearly in $S(n_2)$. Since $k_2 > k$ the n_2^k factor in n^k is canceled out in α and so $\alpha \in S(n_1)$. ■

We will call a probability vector $p : A \to (0,1)$ *rational* when the values $p(a)$ are rational for all $a \in A$. We will call the associated Bernoulli measure $\beta(p)$ a *rational Bernoulli measure.*

PROPOSITION 5.3. *If $p : A \to (0,1)$ is a rational probability vector with $A = \{a_1, \ldots, a_N\}$ then there exist unique positive integers $n, m_1, \ldots, m_N$ such that*

- *$p_i = m_i/n$ for $i = 1, \ldots, N$,*
- *the greatest common divisor of $\{m_1, \ldots, m_N\}$ is 1.*

We then say that $(m_1/n, \ldots, m_N/n)$ is p in reduced form with LCD n.

The measure $\beta(1/n)$ maps to $\beta(p)$ and so $S(\beta(p)) \subset S(n)$. The additive group generated by $S(\beta(p))$, which is the ring generated by $\{m_i/n : i = 1, \dots, N\}$, is $\mathbb{Z}[1/n]$ and so $S(\beta(p)) = S(n)$ iff $S(\beta(p))$ is grouplike.

A positive fraction a/b with a and b relatively prime is an element of the ring $\mathbb{Z}[1/n]$ iff b semi-divides n and it is then a unit of the ring iff a semi-divides n as well.

Proof. The number n is the LCD of the fractions $p_i,\ i = 1, \dots, N$. Since

$$\sum_{i=1}^{N} m_i = n, \tag{5.7}$$

any common divisor of $m_1, \dots, m_N$ would be a divisor of n and so n would not be the *least* common denominator. Hence, the additive subgroup of $\mathbb{Z}$ generated by $m_1, \dots, m_N$ is $\mathbb{Z}$ itself and so the additive subgroup generated the p_i's is $(1/n) \cdot \mathbb{Z}$ and the associated subring is $\mathbb{Z}[1/n] = S(n) + \mathbb{Z}$. Hence, $S(\beta(p)) = S(n)$ iff $S(\beta(p))$ is grouplike.

In any case, since the Haar measure $\beta(1/n)$ is good and its clopen values set $S(n)$ contains the fractions m_i/n, it admits a p-partition and so maps to $\beta(p)$ by Proposition 4.1.

By Lemma 5.2(a), a fraction in lowest terms is in $\mathbb{Z}[1/n]$ iff the denominator semi-divides n. It is a unit, i.e. its reciprocal is in the subring, iff the numerator semi-divides n as well. ∎

Definition 5.4. Let $p = (m_1/n, \dots, m_N/n)$ be a probability vector in reduced form. We say that p satisfies the *Divisibility Condition* when m_i semi-divides n for $i = 1, \dots, N$. We say that p satisfies the *Weak Divisibility Condition* when for every prime z there exist a, b with $1 \le a < b \le N$ such that z is not a factor of either m_a or m_b.

To check the Weak Divisibility Condition we need only consider primes x which divide some m_i, because $N \ge 2$.

Lemma 5.5. *The Divisibility Condition implies the Weak Divisibility Condition.*

Proof. We prove the contrapositive. Let z be a prime for which the Weak Divisibility Condition fails. Because $m_1, \dots, m_N$ are relatively prime, there must be some m_a which does not have z as a factor. Since the Weak Divisibility Condition fails for z we must have $z \mid m_j$ for all $j \neq a$. Since n is the sum of the m_i's, it then follows that n is congruent to m_a mod z. Thus, the m_j with $j \neq a$ do not semi-divide n, so the Divisibility Condition does not hold. ∎

Failure of the Weak Divisibility Condition for any prime greater than 2 implies that the clopen values set of the associated Bernoulli measure is not grouplike.

Theorem 5.6. *Let $p = (m_1/n, \dots, m_N/n)$ be a probability vector in reduced form. Assume that z is a prime number such that $z \,|\, m_j$ for $j = 2, \dots, N$. The measure $\beta(p)$ does not map to $\beta(1/M)$ for any positive integer M. If $z > 2$ or if $4 \,|\, m_j$ for $j = 2, \dots, N$ then the clopen values set $S(\beta(p))$ is not grouplike.*

Proof. As shown in the proof of Lemma 5.5, z is not a divisor of m_1 and $m_1 \equiv n \bmod z$. In particular, n is a unit in the field $\mathbb{Z}_z$ of integers mod z. Hence, the projection map from $\mathbb{Z}$ to $\mathbb{Z}_z$ extends to a ring homomorphism π from $\mathbb{Z}[1/n]$ onto $\mathbb{Z}_z$.

If a is the letter of the alphabet A with weight m_1/n then except for the word a^k every word of length k has weight a fraction with numerator congruent to zero mod z. On the other hand, a^k has weight a fraction with numerator congruent to the denominator mod z. It follows that the image under π of the clopen values set $S(\beta(p))$ is exactly $\{0, 1\} \subset \mathbb{Z}_z$. If $z > 2$ then this is a proper subset of $\mathbb{Z}_z$. Since π is surjective on $S(n) = \mathbb{Z}[1/n] \cap I$ it follows that $S(\beta(p))$ is a proper subset of $S(n)$. By Proposition 5.3, $S(\beta(p))$ is not grouplike.

For any clopen partition $\mathcal{U}$ of $X = A^{\mathbb{N}}$ there exists a positive integer k such that the partition by cylinder sets of length k refines $\mathcal{U}$. So the measure of the element of $\mathcal{U}$ which contains the cylinder set $[a^k]$ is mapped to 1 by π. The measures of the remaining members of $\mathcal{U}$ are mapped to 0 by π. In particular, the measure $\beta(p)$ does not admit a $(1/M, \dots, 1/M)$ partition for any positive integer M. Proposition 4.1 implies that $\beta(p)$ does not map to $\beta(1/M)$.

If $4 \,|\, m_j$ for $j = 2, \dots, N$ then we obtain a ring homomorphism π from $\mathbb{Z}[1/n]$ onto the ring $\mathbb{Z}_4$ because n is a unit of $\mathbb{Z}_4$ which is congruent to m_1 mod 4. Again π maps $S(\beta(p))$ to $\{0, 1\}$, a proper subset of $\mathbb{Z}_4$, and so $S(\beta(p))$ is not grouplike. ∎

A number of examples for the exceptional prime $z = 2$ case can be obtained from the following.

Proposition 5.7. *Let $(m_1, \dots, m_N)$ be a vector of positive integers. For $k = 1, \dots, N+1$ define $n_k = \sum_{i<k} m_i$ with the empty sum $n_1 = 0$ and $n = n_{N+1} = \sum_{i=1}^{N} m_i$. The following two conditions are equivalent*:

(i) *For $i = 1, \dots, N$, $n_i \geq m_i - 1$.*
(ii) *For $k = 2, \dots, N+1$, every integer j with $0 \leq j \leq n_k$ can be written as a sum using each m_i with $i < k$ at most once, i.e. for $i < k$ we can choose ε_i equal to 0 or 1 so that $j = \sum_{i<k} \varepsilon_i \cdot m_i$.*

If $(m_1, \dots, m_n)$ satisfies these conditions then $p = (m_1/n, \dots, m_N/n)$ is a probability vector in reduced form with $S(\beta(p)) = S(n)$ and so $S(\beta(p))$ is grouplike.

Proof. (i)⇒(ii). From (i) we have $1 = n_1 + 1 \geq m_1$ and so $m_1 = 1$ and $n_2 = m_1 = 1$. So condition (ii) is obvious for $k = 2$. Proceed by induction on k. If $0 \leq j \leq m_k - 1$ then, by (i), $j \leq n_k$ and so by induction hypothesis $j = \sum_{i<k} \varepsilon_i \cdot m_i$. Choose $\varepsilon_k = 0$. If $m_k \leq j \leq n_{k+1} = m_k + n_k$ then by induction hypothesis $j - m_k = \sum_{i<k} \varepsilon_i \cdot m_i$. Choose $\varepsilon_k = 1$.

(ii)⇒(i). If (i) fails because $n_i < m_i - 1$ then $j = m_i - 1$ satisfies $0 \leq j \leq n_{i+1}$ but if we try to write $j = \sum_{a<i+1} \varepsilon_a \cdot m_a$ then $\varepsilon_i = 0$ because $j < m_i$. Then $\sum_{a<i+1} \varepsilon_a \cdot m_a \leq n_i < j$. Hence, (ii) fails.

Now suppose that $(r_1, \dots, r_M)$ is a vector of positive integers with $s_m = \sum_{j<m} r_j$ for $m = 1, \dots, M+1$. If both $(r_1, \dots, r_M)$ and $(m_1, \dots, m_N)$ satisfy the above conditions then we construct the product vector of length NM with entries $m_i \cdot r_j$ ordered lexicographically. That is, the predecessors of $m_i \cdot r_j$ are all $m_a \cdot r_b$ with either $a < i$ or $a = i$ and $b < j$. In particular the sum of the predecessors is

$$(5.8) \qquad \begin{aligned} n_i \cdot s_{M+1} + m_i \cdot s_j &\geq n_i + m_i \cdot s_j \\ &\geq m_i - 1 + m_i \cdot (r_j - 1) = m_i \cdot r_j - 1. \end{aligned}$$

Thus, the product vector so ordered satisfies condition (i).

By induction the vector $(m_1, \dots, m_N)^k$ ordered lexicographically satisfies condition (i) and so condition (ii) as well. This implies that by using only cylinders of length k we see that $\{j/n^k : 0 \leq j \leq n^k\} \subset S(\beta(p))$. Thus, $S(\beta(p)) = S(n)$ and so it is grouplike. Notice that $m_1 = 1$ implies that the probability vector is in reduced form. ■

It is easy to check condition (i) for $(1, 2, 2, \dots, 2)$, for $(1, 2, 2^2, 2^3, \dots$ $\dots, 2^{N-1})$ and for $(1, 2, 2, 6)$. These yield probability vectors for which the Weak Divisibility Condition fails, but only for the prime $z = 2$. Nonetheless, the associated clopen values set is grouplike.

We illustrate these results by describing the $N = 2$ case.

Proposition 5.8.

(a) *$\beta(1/2)$ is a good measure.*
(b) *$S(\beta(1/3, 2/3)) = S(3)$ but $\beta(1/3, 2/3)$ does not map to $\beta(1/3)$.*
(c) *If $p = (m_1/n, m_2/n)$ is a probability vector in reduced form with $n > 3$ the $S(\beta(p))$ is not grouplike.*

Proof. Let $p = (m_1/n, m_2/n)$ be a probability vector in reduced form. We can assume that $m_1 \leq m_2$.When $n = 2, m_1 = m_2 = 1$. The vector satisfies the Divisibility Condition and $\beta(p)$ is the good measure $\beta(1/2)$.

Now assume $n > 2$. Since m_1 and m_2 are relatively prime, we have $m_1 < m_2$. The Weak Divisibility Condition fails for any prime divisor of m_2. If $n = 3$ then $m_1 = 1$ and $m_2 = 2$ and (b) follows from Theorem 5.7.

If $n > 3$ then $m_2 \geq 3$. Thus, either 4 or an odd prime divides m_2 and so, by Theorem 5.6, $S(\beta(p))$ is not grouplike. ∎

Remark. We have $S(\beta(1/4, 3/4)) \subset S(4) = S(\beta(1/4))$ but the inclusion is strict because the first set is not grouplike. On the other hand, $1/4 \in S(\beta(1/4, 3/4))$ and so this does not suffice to yield $S(\beta(1/4)) \subset S(\beta(1/4, 3/4))$. This is an example which justifies the remark after Corollary 4.2. Notice also that $\beta(1/2) = \beta(1/4)$ but $1/2 \notin S(\beta(1/4, 3/4))$.

Our major result in this section is a strong converse of Theorem 5.6.

Theorem 5.9. *Let $p = (m_1/n, \ldots, m_N/n)$ be a probability vector in reduced form. The vector p satisfies the Weak Divisibility Condition iff $\beta(p) \sim \beta(1/n)$, i.e. each measure maps to the other. In particular, if p satisfies the Weak Divisibility Condition, then the clopen values set $S(\beta(p))$ equals $S(n)$ and so it is grouplike.*

Proof. By Proposition 5.3, $\beta(1/n)$ always maps to $\beta(p)$. If $\beta(p)$ maps to $\beta(1/n)$ then by Theorem 5.6 the Weak Divisibility Condition holds.

It remains to show that if p satisfies the Weak Divisibility Condition then $\beta(p)$ maps to $\beta(1/n)$. By Proposition 4.1 and the remark thereafter it suffices to find for each prime divisor z of n a positive integer k such that the vector p^k maps onto $(1/z, \ldots, 1/z)$. With z fixed we will construct the map for a suitable k.

Each word of length k has weight a fraction with denominator n^k. The numerator is a *monomial* of the form $m_1^{e_1} ... m_N^{e_N}$ with exponents $e_1, \ldots, e_N$ nonnegative integers which sum to k, which we call a *monomial of type* $e_1, \ldots, e_N$. A *power monomial* has some $e_i = k$ and so the remaining exponents are 0. Otherwise, we call the expression a *nonpower monomial.* Observe that two monomials are of the same type exactly when the exponents all agree. So monomials of different types may nonetheless have the same value. For example, even if $m_1 = m_2$ we still regard the power monomials m_1^k and m_2^k to be of different type. With this convention the number of words of length k with weight n^{-k} times the monomial of type $e_1 \ldots e_N$ is given by the multinomial coefficient $\binom{k}{e_1 \ldots e_N}$. We will regard these as different monomials of the same type. Recall that for each nonpower monomial type $e_1 \ldots e_N$ we have

$$(5.9)\qquad \binom{k}{e_1 \ldots e_N} \geq k \quad \text{and} \quad z \,\Big|\, \binom{k}{e_1 \ldots e_N} \text{ when } k \text{ is a power of } z.$$

Eventually, we will choose k to be a large power of z. Our task is to distribute the N^k monomials of degree k into z different boxes so that each box has the same weight. All of the power monomials will be put in the first box. Then the trick is to balance this weight by placing monomials in each of the remaining $z-1$ boxes in a way which uses up exactly a multiple of z of each type required. By (5.9) any types which remain occur with some multiple of z. Hence, we can distribute each of the remaining types equally among the z boxes.

Our procedure will require a small initial normalization.

Assume that the values are written in nonincreasing order,

$$m_1 \geq m_2 \geq \cdots \geq m_N. \tag{5.10}$$

We will need to assume that the maximum value is unique, i.e.

$$m_1 > m_2. \tag{5.11}$$

Suppose instead that $m_1 = m_2$ and that this common value is associated with letters a and b of the alphabet A of size N. Consider the probability vector p^2 on the alphabet $A \times A$ of size N^2. Let B be the alphabet $(A \times A \setminus \{ab, ba\}) \cup \{*\}$. Let q be the probability vector which assigns weight $2m_1m_2/n^2$ to the new symbol $*$ and which otherwise agrees with p^2. Clearly, p^2 maps to q. If a prime w does not divide m_i and m_j then it does not divide m_i^2 and m_j^2. So the Weak Divisibility Condition for p implies the same for q, which is also, therefore, in reduced form with denominator n^2. Furthermore, the new maximum value, $2m_1^2$, occurs uniquely at the symbol $*$. Since z is a prime factor of n^2 our construction below will apply to q. If q^k maps to $(1/z, \ldots, 1/z)$ then so does p^{2k}.

Our construction will use the following *Successive Divisions Lemma*:

LEMMA 5.10. *Assume that $a_1, \ldots, a_s$ are positive integers with greatest common divisor equal to 1. Let r be a positive integer and for $i = 0, \ldots, r-1$ and $u = 2, \ldots, s$ let a_{ui} be a positive integer which semi-divides a_u. For every integer t there exist integers c_{ui} for $i = 0, \ldots, r-1$ and $u = 2, \ldots, s$ and an integer t_r such that*

$$t = t_r \cdot a_1^r + \sum_{i=0}^{r-1} \Big(\sum_{u=2}^{s} c_{ui} \cdot a_{ui} \Big) a_1^i \tag{5.12}$$

with $0 \leq c_{ui} < a_1$ for $i = 0, \ldots, r-1$ and $u = 2, \ldots, s$.

Proof. For each i, the gcd of $a_1, a_{2i}, \ldots, a_{si}$ is 1. So any integer can be written as a sum of integer multiples of these. Hence, there exist integers $c_{20}, \ldots, c_{s0}$ such that $t - \sum_u c_{u0}a_{u0}$ is congruent to 0 mod a_1. Since this remains true when each c_{u0} is varied within its congruence class, we can assume that each c_{u0} is between 0 and $a_1 - 1$. Now let t_1 be the integer $(t - \sum_u c_{u0}a_{u0})/a_1$ and similarly choose c_{u1} so that $t_2 = (t_1 - \sum_u c_{u1}a_{u1})/a_1$ is an integer. Iterate this procedure r times (or use induction on r). ∎

At last we are ready.

First, choose d a positive integer such that

$$(m_2/m_1)^d < (m_1 - m_2)/(zN^2 m_1^2). \tag{5.13}$$

Next, choose j a positive integer such that

$$(j+d)N m_2^{d+1} < (m_1/m_2)^j. \tag{5.14}$$

These use conditions (5.10) and (5.11).

Let $r = j + d$. By the Weak Divisibility Condition $\gcd(m_2, \dots, m_N) = 1$. Apply the Successive Divisions Lemma with $t = m_1^j$ to obtain

$$m_1^j = t_r \cdot m_2^r + \sum_{i=0}^{r-1} \Big(\sum_{u=3}^{N} c_{ui} \cdot m_u^{r-i} \Big) m_2^i \tag{5.15}$$

with $0 \leq c_{ui} < m_2$ for $i = 0, \dots, r-1$ and $u = 3, \dots, N$. By (5.10) the double sum is bounded by rNm_2^{r+1}. By assumption (5.14) this is smaller than m_1^j. Hence, the remaining coefficient t_r is positive as well.

Let $M_0 = \max(t_r, m_2 - 1)$. We have written m_1^j as a sum of monomials of degree $j + d$ each with exponent $e_1 = 0$ and such that each type is used at most M_0 times.

Now let

$$M_1 = m_1 + M_0 N m_1^d. \tag{5.16}$$

Let k be a power of z such that $k > d + j$.

By Fermat's Little Theorem $a \equiv a^z \mod z$ for any integer a. Hence, $\sum_i m_i^k \equiv \sum_i m_i = n \equiv 0 \mod z$. That is,

$$T := \frac{\sum_{i=1}^{N} m_i^k}{z} \tag{5.17}$$

is a positive integer.

By the Weak Divisibility Condition again the set $\{m_1\} \cup \{m_a \cdot m_b : 2 \leq a < b \leq N\}$ has gcd equal to 1.

Let $R = k - d$. We apply the Successive Divisions Lemma again to this list to obtain

$$T = t_R \cdot m_1^R + \sum_{i=0}^{R-1} \Big(\sum_{2 \leq a < b \leq N} c_{abi} \cdot m_a^{k-i-1} m_b \Big) m_1^i \tag{5.18}$$

with $0 \leq c_{abi} < m_1$ for $i = 0, \dots, r-1$ and $2 \leq a < b \leq N$. Notice that the monomials in the double sum have degree k while the initial term has degree $R = k - d$. The double sum is bounded by

$$\begin{aligned} N^2 m_1 m_2^d \Big(\sum_{i=0}^{R-1} m_2^{R-i} m_1^i \Big) &< N^2 m_1 m_2^d m_1^R (1 - (m_2/m_1))^{-1} \\ &= m_1^k \cdot (m_2/m_1)^d \cdot (N^2 m_1^2/(m_1 - m_2)). \end{aligned} \tag{5.19}$$

On the one hand, $T > m_1^k/z$. So by our choice of d in (5.13), the coefficient t_R is positive. On the other hand, $T < Nm_1^k$ implies

$$0 < t_R < Nm_1^d. \tag{5.20}$$

Recall that we have expressed m_1^j as a sum of monomials of degree $j+d$ each with exponent $e_1 = 0$ and such that each type is used at most M_0 times. By assumption $k > d+j$ and so $R > j$. When we multiply the sum by $t_R m_1^{R-j}$ and substitute into (5.18) we achieve our goal: We have written T as a sum of nonpower monomials of degree k and each monomial is used at most $M_1 = Nm_1^d M_0 + m_1$ times.

We choose k to be the first power of z which is greater than $d+j$ and which is also greater than $z(z-1)M_1$. By (5.9) each nonpower monomial type occurs in p^k at least k times.

Now we proceed as described at the beginning. Place all of the power monomials in the first box. Then select a list of types of degree k nonpower monomials which sum to T taking care to use any one type at most M_1 times. Now repeat this same list $z(z-1)$ times. In each of the $z-1$ boxes after the first, place z copies of the list. Thus, each of these contains monomials which sum to $z \cdot T = \sum_{i=1}^N m_i^k$. Furthermore, we have used up the power monomials and a multiple of z from each nonpower type. The remaining types each occur as a multiple of z. Hence, each remaining type can be equally distributed among the z boxes. ■

THEOREM 5.11. *Let $p = (m_1/n, \ldots, m_N/n)$ be a probability vector in reduced form for an alphabet A of size N. Let $X = A^{\mathbb{N}}$. The following conditions are equivalent*:

(i) *The vector p satisfies the Divisibility Condition.*
(ii) *The group $H(X, \beta(p))$ acts minimally on X.*
(iii) *The group $H(X, \beta(p))$ acts transitively on X.*
(iv) *The measure $\beta(p)$ is good.*
(v) *The measure $\beta(p)$ is homeomorphic to $\beta(1/n)$.*

Proof. (v)⇒(iv). The measure $\beta(1/n)$ is good.

(iv)⇒(iii). See Theorem 2.3(a).

(iii)⇒(ii). Obvious.

(ii)⇒(i). By Corollary 4.9 each m_i/n is a unit in the ring $\mathbb{Z}[1/n]$. Suppose $m_i/n = a/b$ with a, b relatively prime. By Proposition 5.3, a semi-divides n. We obtained a from m_i by canceling some common factors with n. Hence, m_i semi-divides n as well. Thus, p satisfies the Divisibility Condition.

(i)⇒(v). By Proposition 5.3 each m_i/n is a unit of the ring $\mathbb{Z}[1/n]$, and $\mathbb{Z}[1/n] = \mathbb{Z}[p]$, the group generated by $S(\beta(p))$. By Lemma 5.5 and Theorem 5.9, $S(\beta(p))$ is grouplike. By Theorem 4.4 the measure $\beta(p)$ is good. By Proposition 5.3, $\beta(p)$ is homeomorphic to $\beta(1/n)$. ■

Thus, the question of whether $\beta(p)$ is good reduces to the easy-to-answer question of whether p satisfies the Divisibility Condition, in which case $\beta(p) \approx \beta(1/n)$. The question of whether $S(\beta(p))$ is grouplike (or equivalently, whether $S(\beta(p)) = S(n)$) is in general somewhat harder. We do know $\beta(p) \sim \beta(1/n)$ iff p satisfies the Weak Divisibility Condition. Also, if the Weak Divisibility Condition fails for some odd prime then $S(\beta(p))$ is not grouplike and so it is a proper subset of $S(n)$. The remaining cases are when the Weak Divisibility Condition fails only for the prime 2, and these may or may not produce grouplike clopen values sets. For example, Proposition 5.8 implies that $S(\beta(1/3, 2/3))$ is grouplike, but $S(\beta(1/5, 4/5))$ is not.

The simplest illustrations of good measures generated by the above theorem have $N = 3$ and $n = 2^q + 1$ so that

$$\beta\left(\frac{1}{2}, \frac{1}{2n}, \frac{2^{q-1}}{n}\right) \approx \beta\left(\frac{1}{2n}\right) = \beta\left(\frac{1}{2}\right) \times \beta\left(\frac{1}{n}\right). \tag{5.21}$$

For example, with $q = 8$,

$$\beta\left(\frac{1}{2}, \frac{1}{514}, \frac{128}{257}\right) \approx \beta\left(\frac{1}{514}\right) = \beta\left(\frac{1}{2}\right) \times \beta\left(\frac{1}{257}\right). \tag{5.22}$$

6. Bernoulli measures on $2^{\mathbb{N}}$. The rational case is almost completely described by the results of the previous section. Thanks to Dougherty *et al.* (2007) and Yingst (2008) the case $N = 2$ is also very well understood. We will describe here (mostly without proof) the relevant results.

Throughout this section we will use $A = 2 = \{0, 1\}$ so that $X = 2^{\mathbb{N}}$. With $p = (r, 1 - r)$ we have $p(0) = r$ and $p(1) = 1 - r$.

We have seen that as U varies over the clopen subsets of X, the numbers $\beta(r, 1 - r)(U)$ constitute the clopen values set $S(\beta(r, 1 - r))$. Instead, if we fix the clopen set U and vary r then we obtain the restriction to $(0, 1)$ of a polynomial function $P(r)$ with integer coefficients. For example, if $U = [w]$ for some word $w \in 2^k$ then $P(r) = r^m(1 - r)^{k-m}$ where m is the number of 0's in the word w. Following Austin (2007) we will call these the *partition polynomials.*

The cylinder set $[w]$ of length k is the union of the two cylinder sets $[w0]$ and $[w1]$ of length $k + 1$. By using this splitting, any clopen set U can be written as a union of cylinders all of the same length. Thus, a partition polynomial is exactly a polynomial P with integer coefficients which can be expressed, for some $k \geq 0$, as

$$P(t) = \sum_{i=0}^{k} c_i t^i (1 - t)^{k-i} \quad \text{with} \quad 0 \leq c_i \leq \binom{k}{i}. \tag{6.1}$$

We will call (6.1) the *partition expression of degree k* for P.

By further splitting we can write U as the union of cylinders with common length any integer greater than k. In polynomial terms this corresponds to multiplying by $1 = (r + (1 - r))^m$ for some positive m. Thus, P admits a partition expression of any degree greater than k as well. However, since the set of polynomials $\{t^i(1 - t)^{k-i} : i = 0, \dots, k\}$ is linearly independent, the partition expression for any given degree is unique.

Using a theorem of Hausdorff, Dougherty *et al.* provide an elegant and useful characterization of partition polynomials:

THEOREM 6.1. *A polynomial P is a partition polynomial iff it is either the constant* 0 *or* 1, *or a polynomial with integer coefficients such that*

$$0 < t < 1 \ \Rightarrow\ 0 < P(t) < 1. \tag{6.2}$$

Proof. See Dougherty *et al.* (2007), Theorem 6, or Yingst (2008), Theorem 2.6. ∎

Notice that this criterion allows us to recognize a partition polynomial even when it is not written in the form (6.1). For example, the polynomial $3t(1-t)$ and its complement $1-3t(1-t) = t^2-t(1-t)+(1-t)^2$ are partition polynomials with partition expressions of degree 3 obtained by multiplying through by $1 = t + (1 - t)$.

Define the group

$$H(2^{\mathbb{N}}, \beta) := \bigcap_{0<r<1} H(2^{\mathbb{N}}, \beta(r, 1 - r)). \tag{6.3}$$

So $H(2^{\mathbb{N}}, \beta)$ consists of the homeomorphisms which preserve every Bernoulli measure on $2^{\mathbb{N}}$. For example, the homeomorphism ξ^* obtained from a permutation ξ of $\mathbb{N}$ lies in $H(2^{\mathbb{N}}, \beta)$.

Clearly, if U is a clopen subset of $2^{\mathbb{N}}$ and $h \in H(2^{\mathbb{N}}, \beta)$ then U and $h(U)$ have the same associated partition polynomials. The converse is true as well.

THEOREM 6.2. *Let U be a clopen subset of $2^{\mathbb{N}}$ with associated partition polynomial P.*

- (a) *If V is a clopen subset of $2^{\mathbb{N}}$ with associated partition polynomial P, then there exists $h \in H(2^{\mathbb{N}}, \beta)$ such that $h(V) = U$.*
- (b) *If Q is a polynomial with integer coefficients such that $0 < Q(r) < P(r)$ for all $r \in (0, 1)$, then there exists a clopen set V contained in U with associated partition polynomial Q.*

Proof. (a) There is a positive integer k large enough that U and V can each be expressed as a union of cylinders corresponding to words of length k and P has a partition expression of degree k. The coefficient of $r^i(1 - r)^{k-i}$ in P is the number of words in U (and in V) with exactly i 0's. Hence, there is a bijection ϱ on 2^k which maps the words of V to those of U and such that for every word $w \in 2^k$, $N(0,w) = N(0,\varrho(w))$. Define h on $2^{\mathbb{N}} = 2^k \times 2^{\mathbb{N}}$ by $h = \varrho \times \mathrm{id}_{2^{\mathbb{N}}}$. Then h is an element of $H(2^{\mathbb{N}},\beta)$ which maps V to U.

(b) By Theorem 6.1, Q and $P-Q$ are partition polynomials. Choose k so that P,Q and $P-Q$ all have partition expressions of degree k and so that U is a union of cylinders of length k. Arguing as in (a) we can select words among those which are contained in U to get terms which add up to Q. The associated cylinders define V. For details see Dougherty *et al.* (2007), Theorem 7. ∎

From this theorem it follows that we can describe the property of refinability (see Definition 2.10) in terms of partition polynomials.

PROPOSITION 6.3. *Let U be a clopen subset of $2^{\mathbb{N}}$ with associated partition polynomial P and let $r \in (0,1)$. The clopen set U is refinable for the measure $\beta(r,1-r)$ iff whenever $Q_1,\dots,Q_k$ are partition polynomials such that $P(r) = Q_1(r)+\dots+Q_k(r)$, then there exist partition polynomials $H_1,\dots,H_k$ such that $H_i(r) = Q_i(r)$ for $i=1,\dots,k$ and $P = H_1+\dots+H_k$.*

Proof. One direction is clear and the other easily follows from Theorem 6.2(b). ∎

In describing Bernoulli measures on $2^{\mathbb{N}}$ we first consider the case when r is transcendental.

THEOREM 6.4. *If $0<r<1$ is transcendental, then*

$$H(2^{\mathbb{N}},\beta(r,1-r)) = H(2^{\mathbb{N}},\beta). \tag{6.4}$$

That is, a homeomorphism which fixes $\beta(r,1-r)$ fixes every Bernoulli measure.

If $h \in H(2^{\mathbb{N}},\beta(r,1-r))$ and $x \in 2^{\mathbb{N}}$ then

$$N(0,h(x)) = N(0,x), \qquad N(1,h(x)) = N(1,x). \tag{6.5}$$

That is, if x has only finitely many 0's (or 1's) then $h(x)$ has the same finite number of 0's (resp. 1's). In particular, the points $\overline{0}$ and $\overline{1}$ are each fixed by every element of the group and the action of the group $H(2^{\mathbb{N}},\beta(r,1-r))$ is not minimal.

The measure $\beta(r,1-r)$ is refinable but not good. $S(\beta(r,1-r))$ is not grouplike.

Proof. If P and Q are polynomials with integer coefficients and $P(r) = Q(r)$ with r transcendental then $P=Q$. So if two clopen sets have the same $\beta(r,1-r)$ measure then they have the same partition polynomial and so have the same $\beta(s,1-s)$ measure for all $s \in (0,1)$. Applying this to U and $h(U)$ for $h \in H(2^{\mathbb{N}},\beta(r,1-r))$ we obtain (6.4). Refinability follows from Proposition 6.3 using $H_i = Q_i$ for all i.

Now let $r \in (0,1)$, $h \in H(2^{\mathbb{N}}, \beta(r,1-r))$ and $x \in 2^{\mathbb{N}}$. Suppose that x has exactly k 1's and that $h(x)$ has more than k 1's. Permuting to move the 1's to the front we can assume that for some $M > 0, h$ maps the cylinder set $[1^k 0^M]$ into the cylinder set $[1^{k+1}]$ (compare with the proof of Proposition 4.10). This implies that there exists a partition polynomial P such that

$$(1-r)^k r^M = (1-r)^{k+1} P(r)$$

and hence $r^M = (1-r)P(r)$. This equation is not a polynomial identity since 1 is not a root of t^M. Hence, r is algebraic. The contrapositive and the analogous argument for the 0's yield (6.5) for r transcendental. Since the action has fixed points it is not minimal. Since the action is not minimal the measure is not good.

Since the measure is refinable but not good, $S(\beta(r,1-r))$ is not grouplike, but this is easy to prove directly.

To prove that the clopen values set is not grouplike it suffices to show that

$$2\min(r, 1-r) \notin S(\beta(r,1-r)). \tag{6.6}$$

Observe that $2\min(r,1-r) < 1$.

By interchanging r and $1-r$ if necessary, we can assume $r < 1-r$ and so $\min(r,1-r) = r$. We show that $2r \in S(\beta(r,1-r))$ implies r is algebraic.

If there is a clopen set U with measure $2r$ then let P be the partition polynomial associated with U. We must have $P(1) \in \{0,1\}$. (This follows from Theorem 6.1 or by noting that $P(1) = \beta(1,0)(U)$, where $\beta(1,0)$ is defined as expected, resulting in a single point mass.) Then $P(1) \neq 2 \cdot 1$ and the equation $P(r) = 2r$ is not a polynomial identity, so r is algebraic. ∎

Using the transcendental case, we obtain via the Quotient Construction of Theorem 3.10 a class of examples of nearly—but not quite—good measures.

THEOREM 6.5. *With $r \in (0,1)$ transcendental, let $\mu = \beta(r,1-r)^{\mathbb{Q}}$ be the measure obtained on $X = (2^{\mathbb{N}})^{\mathbb{Q}}$ by applying the Quotient Construction to the Bernoulli measure $\beta(r,1-r)$. The measure μ is refinable and every nonempty clopen subset of X is of μ type, but μ is not a good measure.*

Proof. By choosing the labeling we can assume that $r < 1/2$. Otherwise, replace r by $1-r$. By Theorem 3.10 the measure μ satisfies the Product and the Quotient Conditions and so by Theorem 3.7 it is refinable. To show that μ is not good we will show that $r/(1-r) \notin S(\mu)$.

Since the Bernoulli measure $\beta(r,1-r)$ satisfies the Product Condition it follows from (3.15) that $S(\beta(r,1-r)^{\mathbb{Q}})$ is

$$(6.7)\quad \left\{\frac{\beta(r,1-r)(V)}{\beta(r,1-r)(U)} : V \subset U \text{ are clopens in } X \text{ with } U \neq \emptyset\right\}$$
$$= \{1\} \cup \{P(r)/Q(r) : P, Q \text{ are partition polynomials}$$
$$\text{with } P < Q \text{ on } (0,1) \text{ and } Q \neq 0\}.$$

Since r is transcendental,

$$(6.8)\qquad \frac{r}{1-r} = \frac{P_V(r)}{P_U(r)} \;\Rightarrow\; \frac{t}{1-t} = \frac{P_V(t)}{P_U(t)}$$

for all $t \neq 1$. However, $P_V \leq P_U$ on $(0,1)$ implies that the ratio $P_V(t)/P_U(t)$ is bounded by 1 as t approaches 1. On the other hand, $t/(1-t)$ approaches infinity as t approaches 1. This contradiction shows that $r/1-r \notin S(\mu)$ and so μ is not good, again by Theorem 3.7. ∎

When p is a positive distribution on a finite alphabet A we can give a fairly explicit description of the result of the Quotient Construction applied to the Bernoulli measure $\beta(p)$.

As described in the proof of (3.15), we need only consider nonempty clopen subsets U of $A^{\mathbb{N}}$. Such a set can always be described via a nonempty set $B \subset A^n$ of words of length n for some $n = 1, 2, \ldots$, for coordinates up to n, and a copy of $A^{\mathbb{N}}$ for the coordinates larger than n. The product distribution p^n on A^n induces a distribution on the set B which we will denote by p_B. The relative measure $\beta(p)_U$ is just the product of the measure p_B on B with a copy of $\beta(p)$. In the Quotient Construction we use this factor infinitely often yielding the product $\beta(p_B) \times \beta(p)$.

It follows that the quotient measure is just the product of Bernoulli measures:

$$(6.9)\qquad \beta(p)^{\mathbb{Q}} = \prod\{\beta(p_B) : \emptyset \neq B \subset A^n \text{ for } n = 1, 2, \ldots\}.$$

On the other hand, beginning with r transcendental, the group generated by $S(\beta(r, 1-r))$ is the polynomial ring $\mathbb{Z}[r]$ which is isomorphic to the polynomial ring $\mathbb{Z}[t]$ in the variable t. There is a unique—up to homeomorphism—good measure μ_r with $S(\mu_r) = \mathbb{Z}[r] \cap I$. By Proposition 3.9, μ_r satisfies the Product Condition. Since the polynomial ring has no units other than ± 1, the only divisor of $S(\mu_r)$ is 1. Since every clopen subset is good, the Two Implies Three Lemma implies that X itself is the only clopen subset of μ_r type. *A fortiori*, the measure μ_r is not of Bernoulli type.

Equation (6.4) says that $\beta(r, 1-r)$ has very few automorphisms when r is transcendental. We turn now to the much richer algebraic case. If r is an algebraic number then there exists a polynomial R of minimum degree among those with integer coefficients having r as a root and with the greatest common divisor of the coefficients equal to 1. By the Euclidean Algorithm, R divides any rational polynomial with root r. Hence, R is uniquely defined up to multiplication by ± 1. Ignoring the sign ambiguity, we call R the *minimal polynomial* for r.

In Yingst (2008) there is a useful description of the clopen values set in the algebraic case.

THEOREM 6.6. *If r is an algebraic number in $(0,1)$, and R is its minimal polynomial, then $S(\beta(r,1-r))$ is the set of all values $Q(r)$ where Q varies over the integer polynomials satisfying $Q(0) \equiv 0$ or 1 modulo $R(0)$, $Q(1) \equiv 0$ or 1 modulo $R(1)$, and $0 < Q(\widetilde{r}) < 1$ for each root $\widetilde{r}$ of R in $(0,1)$.*

Proof. See Yingst (2008), Corollary 6.12. ∎

REMARK. Notice that if $R(0), R(1) \in \{-2,-1,1,2\}$ then the congruence requirements hold for any integer polynomial Q.

THEOREM 6.7. *If r is an algebraic number in $(0,1)$, and R is its minimal polynomial, then $S(\beta(r,1-r))$ is grouplike iff $R(0), R(1) \in \{-2,-1,1,2\}$ and, in addition, r is the only root of R in $(0,1)$.*

Proof. See Yingst (2008), Corollary 6.13. Notice that sufficiency of these conditions for grouplikeness follows easily from Theorem 6.6 and the Remark thereafter. ∎

THEOREM 6.8. *If r is an algebraic number in $(0,1)$, and R is its minimal polynomial, then the following conditions are equivalent*:

(1) *The measure $\beta(r,1-r)$ is refinable.*
(2) *r and $1-r$ are units in the polynomial ring $\mathbb{Z}[r]$.*
(3) *$1/(r(1-r))$ is an algebraic integer.*
(4) *The automorphism group $H(2^{\mathbb{N}},\beta(r,1-r))$ acts minimally on $2^{\mathbb{N}}$.*
(5) *The automorphism group $H(2^{\mathbb{N}},\beta(r,1-r))$ acts transitively on $2^{\mathbb{N}}$.*
(6) *$R(0), R(1) \in \{-1,1\}$.*

Proof. (5)⇒(4). Obvious.

(4)⇒(2). Use Corollary 4.9, noting that $\mathbb{Z}[r] = \mathbb{Z}[r,1-r]$.

(2)⇔(3). See Corollary 4.5.

(2)⇒(6). If r and $1-r$ are both units, then $r(1-r)$ is a unit. Hence, there is some polynomial Q with integer coefficients so that $r(1-r)Q(r) = 1$. If $S(t) = 1-t(1-t)Q(t)$, then r is a root of S and so R divides S with quotient an integer polynomial by Gauss' Lemma. Hence, $R(0)$ divides $S(0) = 1$ and $R(1)$ divides $S(1) = 1$. Since $R(0)$ and $R(1)$ are integers they are each equal to ± 1.

(6)⇒(1). See Dougherty *et al.* (2007), Theorem 10, or Yingst (2008), Theorem 6.4.

(1)⇒(5). By Theorem 4.11, it suffices to show that for sufficiently large N, the numbers $1-r-r^N$ and $r-(1-r)^N$ are in $S(\beta(r,1-r))$. By Theorem 6.6

and the Remark thereafter, it suffices to choose N large enough that

$$1-\widetilde{r}-\widetilde{r}^N>0 \quad \text{and} \quad \widetilde{r}-(1-\widetilde{r})^N>0 \tag{6.10}$$

at each of the (finitely many) roots of R in $(0,1)$. ∎

Corollary 6.9. *If r is an algebraic number in $(0,1)$, and R is its minimal polynomial, then $\beta(r,1-r)$ is a good measure iff $\{R(0),R(1)\}\subset\{-1,1\}$ and, in addition, r is the only root of R in $(0,1)$.*

Proof. By Corollary 2.12 a measure is good iff it is both refinable and has a grouplike clopen values set. So the result follows from the characterizations in Theorems 6.7 and 6.8. See also Dougherty *et al.* (2007), Theorem 15. ∎

In Dougherty *et al.* (2007) the authors observe that

$$R(t)=14t^6-21t^4+8t^2+t-1 \tag{6.11}$$

defines an irreducible polynomial with $R(0)=-1$, $R(1)=1$ and with three roots in the interval $(0,1)$. If r is any of these roots then $\beta(r,1-r)$ is refinable but not good. Hence, $S(\beta(r,1-r))$ is not grouplike but nonetheless the group $H(2^{\mathbb{N}},\beta(r,1-r))$ acts transitively on $2^{\mathbb{N}}$.

Finally, the rational case with $N=2$ is rather trivial. The only good measure is the Haar measure $\beta(1/2)$. Both $\beta(1/3,2/3)$ and its complementary isomorph $\beta(2/3,1/3)$ have grouplike clopen values sets but do not satisfy the Divisibility Condition and so are not refinable. By Proposition 5.7 the rest are not refinable and do not have grouplike clopen values sets.

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Received 30 October 2006;
revised 26 March 2007 (4811)