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TOPOLOGICAL SIZE OF SCRAMBLED SETS

BY

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Abstract. A subset S of a topological dynamical system (X,f) containing at least two points is called a *scrambled set* if for any $x,y\in S$ with $x\neq y$ one has

$$\liminf_{n\to\infty} d(f^n(x),f^n(y))=0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x),f^n(y))>0,$$

d being the metric on X. The system (X,f) is called *Li–Yorke chaotic* if it has an uncountable scrambled set.

These notions were developed in the context of interval maps, in which the existence of a two-point scrambled set implies Li–Yorke chaos and many other chaotic properties. In the present paper we address several questions about scrambled sets in the context of topological dynamics. There the assumption of Li–Yorke chaos, and also stronger ones like the existence of a residual scrambled set, or the fact that X itself is a scrambled set (in these cases the system is called residually scrambled or completely scrambled respectively), are not so highly significant. But they still provide valuable information.

First, the following question arises naturally: is it true in general that a Li–Yorke chaotic system has a Cantor scrambled set, at least when the phase space is compact? This question is not answered completely but the answer is known to be yes when the system is weakly mixing or Devaney chaotic or has positive entropy, all properties implying Li–Yorke chaos; we show that the same is true for symbolic systems and systems without asymptotic pairs, which may not be Li–Yorke chaotic. More generally, there are severe restrictions on Li–Yorke chaotic dynamical systems without a Cantor scrambled set, if they exist.

A second set of questions concerns the size of scrambled sets inside the space X itself. For which dynamical systems (X,f) do there exist first category, or second category, or residual scrambled sets, or a scrambled set which is equal to the whole space X?

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While reviewing existing results, we give examples of systems on arcwise connected continua in the plane having maximal scrambled sets with any prescribed cardinalities, in particular systems having at most finite or countable scrambled sets. We also give examples of Li–Yorke chaotic systems with at most first category scrambled sets. It is proved that minimal compact systems, graph maps and a large class of symbolic systems containing subshifts of finite type are never residually scrambled; assuming the Continuum Hypothesis, weakly mixing systems are shown to have second category scrambled sets. Various examples of residually scrambled systems are constructed. It is shown that for any minimal distal system there exists a non-disjoint completely scrambled system. Finally, various other questions are solved. For instance, a completely scrambled system may have a factor without any scrambled set, and a triangular map may have a scrambled set with non-empty interior.

Contents

1. INTRODUCTION

There have been several attempts to give a mathematical definition of chaos. Let us only mention those of Li and Yorke in 1975 [36] and Devaney in 1988 [16], because we are addressing the first, while the second is mentioned several times. They are not the only ones even in topological dynamics; of course specialists from other fields have their own, different, views of chaos. If there is a possibility of describing chaos mathematically we doubt that this can be done in a few lines. Nevertheless, the definitions we quote, and the ones we do not as well, have been very stimulating and still are, because they single out properties that are relevant in their respective fields.

The Li–Yorke definition of chaos [36] consists in the existence of a so-called *scrambled set* with uncountable cardinality. Scrambled sets are the central topic of this article. While reviewing existing results about them and introducing new ones, we attempt to understand their significance, and particularly that of their size, in the field of topological dynamics.

By a (discrete) dynamical system we mean a pair (X, f) where X is endowed with the metric d and $f : X \to X$ is a continuous map. Most of the time we assume that X is compact or at least Polish; in any case this is supposed to be stated explicitly. A subset S of X containing at least two points is called a *scrambled set* of the system (or of the map f) if for any $x, y \in S$ with $x \neq y$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \tag{1.1}$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0. \tag{1.2}$$

The origin of this notion is in Li and Yorke's article [36], where S is additionally assumed to be uncountable and to satisfy $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ for any $x \in S$ and any periodic point p (i.e., no $x \in S$ is *asymptotically periodic*). However, it is easy to see that a scrambled set contains at most one asymptotically periodic point, which makes the extra condition unnecessary.

Note that we require that a scrambled set contain at least two points, though for some authors a singleton is also a scrambled set. The system (or the map f) is called *Li–Yorke chaotic* if it has an uncountable scrambled set. This definition, implicitly contained in [36] in the setting of interval dynamical systems, may look strange and was criticized. The first objection is that chaos in this sense may not be "physically" observable. Still, Li and Yorke's idea for defining chaos makes sense at least on the interval because it turns out to be the minimal requirement for a continuous self-map of an interval to have "complex" behaviour: in [44] Smítal proved that any interval map has one of the following two mutually exclusive properties:

(i) f is Li–Yorke chaotic.
(ii) All trajectories of f are *approximable by cycles*, i.e., for any x and $\varepsilon > 0$ there is a periodic point p with $\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \varepsilon$.

The second, more formal, objection against the definition of Li–Yorke chaos is: "why uncountable and not, say, infinite or topologically large or something else?" Again, at least on the interval the objection fails, since later Kuchta and Smítal proved that if an interval map has a scrambled set with two points then it also has an uncountable, in fact Cantor, scrambled set [33]. Nevertheless, for *general* systems the existence of a scrambled set does *not* imply the existence of an uncountable scrambled set (see, e.g., [18, 8]). This fact is a good reason for studying the size of scrambled sets for transformations with various properties acting on various spaces.

So, remember that

> a scrambled set contains at least two points and is defined by (1.1) and (1.2); Li–Yorke chaos is defined as the existence of an uncountable scrambled set.

Regardless of whether one agrees with Li and Yorke's definition of chaos, the size of scrambled sets of a system is definitely one of the many tools for investigating the structure of the system and perhaps evaluate its complexity. The size can be considered in the sense of cardinality or in the topological sense or in the measure-theoretic sense; in this article we adopt the first two points of view only.

Results on the cardinality of scrambled sets are presented in Section 4. The main aim of the present paper is to study the size of uncountable scrambled sets in metric, most of the time compact, spaces from the *topological* point of view. To explain the situation from which we start, recall known results. The research in this direction was started by Bruckner and Hu [12]. They showed that the tent map has a scrambled set of Borel type $G_{\delta\sigma}$ with cardinality c (cardinality of the continuum) in every interval. Furthermore, if f is a continuous self-map of $I = [0, 1]$ with f^2 topologically transitive then, under the Continuum Hypothesis, f has a scrambled set which is of second category in every subinterval of I. Finally, if a scrambled set of an interval map has the Baire property then it is first category. This implies that interval maps have no residual scrambled sets; their scrambled sets cannot even be residual in a subinterval of I—Bruckner and Hu in fact considered only scrambled sets for which $\limsup = 1$ in (1.2), but in [22] Gedeon extended that result to all scrambled sets. Recently Mai proved that on finite graphs scrambled sets have empty interior [38].

In a different direction, as far as we know, whenever a system has been shown to be Li–Yorke chaotic the proof implied the existence of a Cantor or Mycielski scrambled set. A *Mycielski set* is a countable union of Cantor sets. The extra significance of a Mycielski set is that it may be dense in X, whereas a Cantor proper subset cannot; for more on this matter see [1]. It was shown by Kan [30] that Li–Yorke chaotic interval maps always have a Cantor scrambled set; this result was strengthened in [33]. Applying topological theorems of Kuratowski and Mycielski [41], Iwanik proved that when a Polish system is weakly mixing it has a dense Mycielski scrambled set [28]; later Huang and Ye obtained Cantor scrambled sets for compact scattering or Devaney-chaotic systems [26], and Blanchard, Glasner, Kolyada and Maass showed that there are also Cantor scrambled sets in compact positive-entropy systems [9].

This suggests two fields of investigation.

First, is it true in general that a Li–Yorke chaotic system has a Cantor scrambled set, at least when the phase space is compact or Polish? Without answering this question completely we show that, in addition to the above-mentioned cases, this is true in several other classes, for instance among symbolic systems, and that there are severe restrictions on Li–Yorke chaotic Polish dynamical systems that do not possess this property.

The second set of questions concerns the size of scrambled sets inside the space X itself. For which dynamical systems (X, f) do there exist first category, or second category, or residual scrambled sets, or a scrambled set which is equal to the whole space X? In the last two cases the system is called *residually scrambled* and *completely scrambled* respectively. When saying that *a system is scrambled* we simply have in mind that it has at least one scrambled pair.

We already mentioned some negative results in the case of interval maps. On the other hand, there are various examples of completely scrambled systems. Ceder [13] constructed a discontinuous map $f : I \to I$ having the whole compact interval I as a scrambled set. Mai [37] then showed that on $X = (0,1)^n$, $n \geq 2$, one can construct a completely scrambled homeomorphism; he gave further examples (replacing $(0,1)$ by $(0,\infty)$ or $(-\infty,\infty)$ and "homeomorphism" by "C^∞ diffeomorphism", now also including the case $n = 1$) in [38]. Finally, Huang and Ye [25, 27] showed that many metric compacta can be equipped with completely scrambled homeomorphisms and that the resulting dynamical systems may even be weakly mixing.

There is another interesting way of addressing the second set of questions, by asking on which topological spaces one can find transformations having only finite, or at most first category, scrambled sets, etc. The article mentions or proves many results of this kind.

The main body of results concerns the second field of research. A metric system which is an extension of a distal system (Y, g) with Y perfect is shown to have at most first category scrambled sets; examples are given. We establish sufficient conditions for some compact systems not to be residually scrambled: for instance minimal systems, graphs maps and a large class of symbolic systems are never residually scrambled; assuming the Continuum Hypothesis we additionally prove that Polish weakly mixing systems have second category scrambled sets. Various examples of residually scrambled systems, some of them strongly mixing or with positive entropy, some with one fixed point and some with two, are given. It turns out that positive-entropy systems may have first category scrambled sets only, or second category but no residual scrambled sets, or residual scrambled sets (by [10] a completely scrambled system always has zero entropy).

A superficial glance at their definition suggests that the structure of completely scrambled systems ought to be altogether different from that of minimal distal systems. But this is not true. We show that for any compact minimal distal system one can construct a non-disjoint completely scrambled system.

In the end we address various questions concerning factors and extensions. The existence of residual scrambled sets, or the fact of being completely scrambled, are conjugacy invariants, but are they preserved under factor maps or extensions? It is easy to find counter-examples in the case of extensions, but what about almost one-to-one extensions? Here both questions are answered in the negative. In particular we construct a completely scrambled system having a perfect factor without scrambled sets; this proves simultaneously that neither the property of being completely scrambled nor that of having residual scrambled sets are preserved under factor maps.

A related question is also answered. It is proved in [9] that if a compact dynamical system has a scrambled set then so does any extension. We show that this property of dynamical systems is not truly pair-wise, that is, a scrambled pair may have no scrambled preimage in the extension. Last, on what kinds of spaces do there exist maps with scrambled sets having non-empty interior? We already know that this never happens on finite graphs [38] and in particular on the interval but we give an example of a triangular map in the square, that is, an interval extension of an interval map, having this property.

The question of the *measure* of scrambled sets is out of the scope of the present paper. Let us recall only that the first result on this topic was that the standard tent map has no scrambled set with positive Lebesgue measure but, under the Continuum Hypothesis, it has a scrambled set with full outer Lebesgue measure [43]. For a survey of what is known on the Lebesgue measure of scrambled sets of continuous maps $I \to I$ (and $I^n \to I^n$) where I is a real compact interval, we refer the reader to the survey paper [6].

In the literature there are also many results concerning not scrambled sets in general but scrambled pairs. Reviewing them all would have drawn us too far. We just mention two relevant notions, generic chaos and strong scrambled pairs, without telling much about them.

Previous results and those that are proved here leave a host of questions open, some of them not even mentioned in this article. Still, we may risk a very tentative conclusion. First, a system that is not Li–Yorke chaotic can hardly be called chaotic; this is not our discovery, but just a consequence of previous research. So, it is important to know whether a system has an uncountable scrambled set.

On the other hand, when a system is Li–Yorke chaotic the size of its scrambled sets is clearly an important feature but does not give very precise information about its dynamics. When addressing Li–Yorke chaos and scrambled sets for the first time one may have the impression that the bigger its scrambled sets the more chaotic the system is. This is not completely false but it is not true either. Completely scrambled systems have zero entropy [10]. By Proposition 57 many scrambled systems are not disjoint from all minimal distal systems. Systems like the tent map or mixing subshifts of finite type are usually considered to be among the most chaotic ones, but they are not residually scrambled.

The article is divided into eight sections. After some background notions and preliminary results in Section 2, each of the sections covers one of the points that have been emphasised above. Each one begins with an outline of its topic; after that we develop whatever original results we have found and finish with questions, when there are any (in Sections 2 and 6 questions are found at the end of subsections).

The existence of Cantor scrambled sets is addressed in Section 3. Section 4 deals with possible sets of cardinalities of all maximal scrambled sets of a system and develops, among others, examples of systems with only finite or countable scrambled sets. Section 5 deals with systems with only first category scrambled sets. The subject of Section 6 is systems without residual scrambled sets. Section 7 is devoted to systems having residual scrambled sets, among them completely scrambled systems. Finally, results about factors and extensions, preimages of scrambled pairs, and an example of a triangular map having a scrambled set with non-empty interior, are gathered in Section 8.

2. PRELIMINARIES AND FIRST OBSERVATIONS

Throughout the paper by a dynamical system we mean a pair (X, f) where X is equipped with a metric d and f is a continuous map $X \to X$. Whenever X is additionally required to be Polish or compact, this is explicitly mentioned.

The notions of distal, proximal, scrambled and asymptotic pairs, together with derived properties, are used throughout the article. Since they are closely related it is natural to introduce them simultaneously.

In the introduction two properties of a pair of points (x, y) in a dynamical system (X, f) were considered: $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$; when both are simultaneously true the pair (x, y) is said to be *scrambled*. In the literature scrambled pairs are often called Li–Yorke pairs. Since a scrambled pair is just a scrambled set of cardinality two we believe that it is simpler—and also more descriptive—to call them scrambled pairs. The set of scrambled pairs of (X, f) is denoted by $\mathrm{SR}(X, f)$. The notation corresponds to the fact that $\mathrm{SR}(X, f)$ is a relation in X^2; we call it the scrambled relation. Put $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$. When $\mathrm{SR}(X, f) = X \setminus \Delta$ the system is *completely scrambled*.

Other related families of pairs were studied in topological dynamics before the interest for scrambled pairs was aroused. A pair (x, y) is called *distal* if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0;$$

if this property holds for all $x, y \in X$, $x \neq y$, the system (X, f) is called *distal*. On the contrary, a *proximal* pair (x, y) is one such that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0;$$

the set of proximal pairs of (X, f) is denoted by $\mathrm{PR}(X, f)$, and a system for which all pairs are proximal is called *proximal*. Scrambled pairs are proximal; proximal pairs that are not scrambled, that is, pairs such that

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

are called *asymptotic*, and the set of all such pairs is denoted by $\mathrm{AR}(X, f)$. The asymptotic relation is an equivalence relation. One can thus consider the *asymptotic equivalence class* $W^s(x)$ of a point x: $W^s(x) = \{y \in X : (x, y) \in \mathrm{AR}(X, f)\}$. Since $\Delta \subseteq \mathrm{AR}(X, f)$, asymptotic pairs (x, y) with $x \neq y$ are called *proper asymptotic*.

For $\delta > 0$, a pair $(x, y) \in X^2$ is said to be *δ-scrambled* if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta.$$

By definition a scrambled pair is δ-scrambled for some $\delta > 0$. A set $S \subseteq X$ with at least two points is called *δ-scrambled* if any pair $(x, y) \in S \setminus \Delta$ is δ-scrambled. A system is said to be *δ-Li–Yorke chaotic* if it contains an uncountable δ-scrambled set. The situation for scrambled sets is not the same as that for pairs, since a scrambled set is not necessarily δ-scrambled for any $\delta > 0$.

Some global properties of dynamical systems are especially significant when X is compact, but they make sense in the non-compact case too. The following ones are used in this paper, some of them repeatedly. (X, f) is said to be

- *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f^n(x), f^n(y)) < \varepsilon$ for all $n > 0$;
- *minimal* if X contains no proper closed f-invariant subset;
- *transitive* if whenever $U, V \subseteq X$ are non-empty open there exists $n > 0$ such that $U \cap T^{-n}V \neq \emptyset$;
- *weakly mixing* when its Cartesian square $(X \times X, T \times T)$ is transitive;
- *strongly mixing* if whenever $U, V \subseteq X$ are non-empty open there exists $n_0 > 0$ such that $n \geq n_0$ implies $U \cap T^{-n}V \neq \emptyset$;
- *uniquely ergodic* when there is a unique f-invariant Borel probability measure on X.

If the orbit of a point $x \in X$ is dense in X the point x is called a *transitive point*. Transitive metric systems may not contain transitive points. Nevertheless, if the system is separable and second category then transitivity implies that transitive points form a dense G_δ set. For a surjective continuous map on a compact metric space equicontinuity implies distality.

2.1. Basic properties of scrambled sets. Let S be a scrambled set of a dynamical system (X, f). Then the following elementary properties hold (the proofs are left to the reader):

- (S-1) S contains at most one asymptotically periodic point.
- (S-2) $f|_S$ is injective.
- (S-3) For any $n > 0$, $f^n(S)$ is also a scrambled set of f.
- (S-4) If f maps a set $S^* \subseteq X$ injectively into S then S^* is also a scrambled set of f, provided it has at least two points.
- (S-5) If f is uniformly continuous, in particular, if X is compact, then S is also a scrambled set of f^n for any $n > 0$. Moreover, given $\delta > 0$ there are $\delta_n > 0$, $n = 0, 1, \ldots$, such that if S is δ-scrambled for f then it is δ_n-scrambled for f^n.
- (S-6) Let (X, f) and (Y, g) be topologically conjugate with a conjugating homeomorphism h. Let $S \subseteq X$ and $T \subseteq Y$ with $h(S) = T$. If h is a uniform homeomorphism (in particular if the spaces are compact) then S is a scrambled set of f if and only if T is a scrambled set of g. Moreover, given $\delta > 0$ there is $\delta' > 0$ such that if S is δ-scrambled for f then T is δ'-scrambled for g.
- (S-7) If X is compact then S intersects at most one minimal set of f.
- (S-8) The union of any increasing sequence of scrambled sets is a scrambled set. The same is true for "δ-scrambled" instead of "scrambled", provided $\delta > 0$ is fixed.

Note that the existence of a scrambled set is not an invariant of topological conjugacy if the phase spaces are not compact, as in Example 6. In other words, in (S-6) it is important that h be uniformly continuous. Similarly, one can easily show that in (S-5) the uniform continuity of f is essential (for a continuous map f on the real line one may have $|f^{2k}(x) - f^{2k}(y)| = 1$, $k = 0, 1, \ldots$, for some x, y while $\lim_{k\to\infty} |f^{2k+1}(x) - f^{2k+1}(y)| = 0$; hence $\{x, y\}$ is a scrambled set for f but not for f^2).

In (S-8), δ must be fixed. For instance, by [25] there is an infinite countable compact completely scrambled system. It is not δ-scrambled for any $\delta > 0$ by Proposition 5 below though it is the union of an increasing sequence of finite sets $X_1 \subseteq X_2 \subseteq \cdots$ and, due to the finiteness, X_n is δ_n-scrambled for some $\delta_n > 0$.

2.2. Maximal scrambled sets. Since every subset of a scrambled set is scrambled when not a singleton, one is naturally interested in maximal scrambled sets with respect to inclusion. Every scrambled set is a subset of a maximal scrambled set; every δ-scrambled set is a subset of a maximal δ-scrambled set.

A completely scrambled dynamical system has a unique maximal scrambled set, namely X itself. On the other hand, there are systems whose family of maximal scrambled sets has the same cardinality as the family of all subsets of the phase space. For instance

PROPOSITION 1. *Let (X, f) be a dynamical system with* $\operatorname{card} X = \kappa$. *Suppose it has a scrambled set S with* $\operatorname{card} S = \kappa$ *such that the set* $\{y \in f(S) : \operatorname{card} f^{-1}(y) \geq 2\}$ *has cardinality κ. Then (X, f) has 2^κ maximal scrambled sets with cardinality κ. When S is δ-scrambled they are maximal δ-scrambled sets.*

Proof. Assume that S is maximal. Since f is injective on S, the set $f(S)$ has cardinality κ. The family $\{f^{-1}(y) : y \in f(S)\}$ is a decomposition of $f^{-1}(f(S))$ into κ non-empty sets. Obviously, any choice set for this decomposition (any set containing a unique point from each of these sets) is a scrambled set of f with cardinality κ (δ-scrambled if S is δ-scrambled). Moreover, it is a maximal scrambled (δ-scrambled) set since S is maximal. By the assumption, κ of the sets $f^{-1}(y)$, $y \in f(S)$, contain at least two points, and so there are at least 2^κ such choice sets. On the other hand, there are only 2^κ subsets of X. ∎

For the tent map $f(x) = 1 - |2x - 1|$, $x \in I$, this implies the existence of 2^c maximal δ-scrambled sets with cardinality c because it has a δ-scrambled set with cardinality c ([30], [33]) and every point different from 1 has two preimages. Here is another example. Let (B, φ) be a completely scrambled system with cardinality κ and $X = B \times \{0, 1\}$. Define $f : X \to X$ by $f(x, y) = (\varphi(x), 1)$. The set $B \times \{0\}$ is scrambled and we can apply Proposition 1 to get the existence of 2^κ maximal scrambled sets. Here is their list: $(A \times \{0\}) \cup ((B \setminus A) \times \{1\})$ for any $A \subseteq B$.

The tent map example shows that

(S-9) Two different maximal scrambled (or δ-scrambled) sets may not be disjoint.

(S-10) A maximal scrambled (or δ-scrambled) set may not be analytic.

To get (S-10), one can simply use the fact that in a Polish space with cardinality c (say, in I) there are only c analytic sets. Here is another way to check that the tent map has non-analytic maximal scrambled sets. As shown under the Continuum Hypothesis in [43], it has a scrambled set S with outer Lebesgue measure 1; on the other hand, every Lebesgue measurable scrambled set of the tent map has measure 0. Thus a maximal scrambled set $S_{\max}$ containing S is not measurable. Since analytic sets are universally measurable, $S_{\max}$ is not analytic.

It is easy to find scrambled pairs both in the one-sided and two-sided full shifts. Later we use some elementary facts on maximal scrambled sets of the full shift over two symbols.

PROPOSITION 2. *For the full shift over two symbols there is $\delta > 0$ such that all scrambled sets are δ-scrambled and every maximal scrambled set has cardinality c.*

Proof. Consider the one-sided shift (Σ_2, σ); the proof for the two-sided shift is similar. The δ-scrambledness is obvious from the definition of the metric in Σ_2: if $\bar{x}, \bar{y} \in \Sigma_2$ are not asymptotic then there are infinitely many integers i such that $x_i \neq y_i$. To prove the claim on the cardinalities, let S be a scrambled set of (Σ_2, σ) with $\operatorname{card} S < c$. We construct a scrambled set S^* with $S \subseteq S^*$ and $\operatorname{card} S^* = c$; since $\operatorname{card} \Sigma_2 = c$, the result follows.

Fix a sequence $\mathcal{N} = (n_i)_{i=0}^{\infty}$ of non-negative integers with $n_i \nearrow +\infty$ and $n_{i+1} - n_i \to +\infty$. Write $\bar{a} = \bar{b} \bmod \mathcal{N}$ if $a_j = b_j$ for all $j \notin \mathcal{N}$. Note that if $\bar{x}, \bar{y} \in \Sigma_2$ are proximal and $\bar{u} = \bar{x} \bmod \mathcal{N}$, $\bar{v} = \bar{y} \bmod \mathcal{N}$ then $\bar{u}, \bar{v}$ are also proximal.

Two elements $\bar{x}, \bar{y} \in \Sigma_2$ are asymptotic if and only if they coincide from some coordinate on. Thus each asymptotic equivalence class $W^s(x)$ is (infinite) countable, so there are c such classes. Call them *asymptotic cells* for short.

For every $\bar{s} \in S$ put $\bar{s}_{|\mathcal{N}} = s_{n_0} s_{n_1} s_{n_2} \ldots$. Let A be a choice set for the family of all those asymptotic cells which do not contain any of the elements $\bar{s}_{|\mathcal{N}}$, $\bar{s} \in S$ (i.e., A contains just one element from each such asymptotic cell and no other elements). Since there are c asymptotic cells and $\operatorname{card} S < c$, we get $\operatorname{card} A = c$. Choose one element $\bar{t} \in S$ and for every $\bar{a} \in A$ denote by $\bar{t}_{\bar{a} \to \mathcal{N}}$ the element from Σ_2 obtained from $\bar{t}$ by replacing its $n_0, n_1, \ldots$-coordinates by the coordinates of $\bar{a}$ (i.e., the jth coordinate of $\bar{t}_{\bar{a} \to \mathcal{N}}$ is a_i if $j = n_i$ and t_j otherwise). Now put $S^* = S \cup \{\bar{t}_{\bar{a} \to \mathcal{N}} : \bar{a} \in A\}$. Then $\operatorname{card} S^* = c$ and it is easy to check that S^* is scrambled. ■

By Propositions 2 and 1 there are 2^c maximal scrambled sets in the one-sided shift and so most of them are non-analytic. The same can also be easily shown for the two-sided full shift (if S is a maximal scrambled set for the one-sided shift then for instance $\{\ldots s_0 s_0 s_0 \dot{s}_0 s_1 s_2 \cdots : s_0 s_1 s_2 \cdots \in S\}$ is a maximal scrambled set for the two-sided shift). In Section 4 we use the following

EXAMPLE 3. Let $f : I \to I$ be defined by

$$f(x) = \begin{cases} 3x, & x \in [0, 1/3], \\ 1, & x \in [1/3, 2/3], \\ 3 - 3x, & x \in [2/3, 1], \end{cases}$$

respectively. The set C of all $x \in I$ whose trajectories never enter the open plateau $(1/3, 2/3)$ is an f-invariant Cantor set. The system $(C, f|_C)$ is topologically conjugate to the full one-sided shift. On the other hand, every point from $I \setminus C$ is eventually mapped to the fixed point 0 and so every scrambled set contains at most one such point and, if this is the case, if we replace it by the point $0 \in C$ we get again a scrambled set (with the same cardinality), now a subset of C. In view of Proposition 2 and (S-6) the system (I, f) has scrambled sets, there is $\delta > 0$ such that every scrambled set is δ-scrambled, and every maximal scrambled set has cardinality c. Notice that $\delta = 1/3$ (= the length of the plateau) works.

QUESTION. Find conditions under which a compact system has all maximal scrambled sets of the same cardinality.

2.3. Scrambled systems versus δ-scrambled systems. The differences between scrambled sets and δ-scrambled sets, Li–Yorke chaos and δ-Li–Yorke chaos, are significant. For the tent map and the full shift we already know that there exist 2^c maximal δ-scrambled sets for some $\delta > 0$. The same is true for any compact weakly mixing or Devaney chaotic or positive-entropy system, as can be seen in the proofs of [28, Theorem 1], [26, Theorem 4.1], [9, Theorems 2.1 and 2.3].

On the other hand, Floyd's minimal system (see e.g. [5, pp. 24–27]) is Li–Yorke chaotic but not δ-Li–Yorke chaotic for any $\delta > 0$. Another family of examples illustrates the difference between scrambled and δ-scrambled sets. Mai [38], and later Huang and Ye [25], showed that various spaces, among them compact ones, admit completely scrambled homeomorphisms. The resulting dynamical systems can be δ-Li–Yorke chaotic, but the maximal δ-scrambled sets are always smaller than the set X. Before proving this in Proposition 5, one must recall some results about proximality.

For a dynamical system (X, f), $U \subseteq X$ and $y \in X$ let $N(y, U) = \{n \in \mathbb{N} : f^n(y) \in U\}$ be the set of times at which the orbit of y hits U. A subset of $\mathbb{N}$ is called *thick* when it contains arbitrarily long intervals. By $B(q, \varepsilon)$ we denote the open ball with radius ε centred at q.

PROPOSITION 4. *Let (X, f) be a compact dynamical system.*

(1) *If $(x, f(x)) \in \mathrm{PR}(X, f)$ for some $x \in X$, there is a fixed point in X.*
(2) *(X, f) is proximal if and only if X contains a fixed point q which is its unique minimal subsystem. In this case the set $N(x, B(q, \varepsilon))$ is thick whenever $x \in X$ and $\varepsilon > 0$.*

Part (1) is easy. The proof of (2) can be found in [4].

PROPOSITION 5. *Let (X, f) be a compact dynamical system. Then X is not a δ-scrambled set for any $\delta > 0$.*

Proof. Suppose that X is δ-scrambled for some $\delta > 0$. Then by the definition of scrambled sets X has at least two points and (X, f) is completely scrambled, hence proximal. By Proposition 4, X contains a fixed point q and for any $x \in X$ and any $\varepsilon > 0$ the set $N(x, B(q, \varepsilon))$ is thick.

Set $U = B(q, \delta/2)$ and let $y \in X$ with $y \neq q$. We know that $N(y, U) = \{n \in \mathbb{N} : f^n(y) \in U\}$ is thick but since $\limsup_{n\to\infty} d(f^n(y), q) \geq \delta$ the set $N(y, U^c)$ is infinite. This implies that there exists a sequence $\{l_k\} \subseteq N(y, U^c)$ such that $\{l_k + 1, l_k + 2, \dots, l_k + k\} \subseteq N(y, U)$. Without loss of generality assume that $\lim_{k\to\infty} f^{l_k}(y) = y_0$. Then obviously $y_0 \in U^c$, while $f^n(y_0) = \lim_{k\to\infty} f^{l_k+n}(y) \in \mathrm{cl}(U)$, and so $d(f^n(y_0), q) \leq \delta/2$ for each $n \geq 1$. Thus

$$\limsup_{n\to\infty} d(f^n(y_0), f^n(q)) = \limsup_{n\to\infty} d(f^n(y_0), q) \leq \delta/2 < \delta,$$

so (y_0, q) is not δ-scrambled, which contradicts the initial assumption. ∎

EXAMPLE 6. The assumption of compactness in Proposition 5 is essential. There is a continuous self-map $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R}$ is a δ-scrambled set for any $\delta > 0$. Let $\{a_i\}_{i\in\mathbb{Z}}$ satisfy:

(1) $\cdots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots$ (set $I_j = [a_j, a_{j+1}]$ for $j \in \mathbb{Z}$).
(2) $\lim_{i\to-\infty} a_i = -\infty$ and $\lim_{i\to\infty} a_i = +\infty$.
(3) In the sequence $I_0, I_1, I_2, \dots$ there are arbitrarily long intervals, as well as, for any $n \in \mathbb{N}$, a block of n consecutive intervals whose total length is less than $1/n$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the map which sends a_j to a_{j+1} for any $j \in \mathbb{Z}$ and is linear on each I_j. Then f is an increasing homeomorphism and it is not hard to verify that for any $x \neq y$ we have $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| = +\infty$. Hence $\mathbb{R}$ is a δ-scrambled set of $(\mathbb{R}, f)$ for any $\delta > 0$. Notice that the system is topologically conjugate to the translation $x \mapsto x + 1$ which has no scrambled sets at all.

2.4. New scrambled systems from old ones via retraction. It is probably hopeless to try to characterize all spaces that admit maps with scrambled sets or Li–Yorke chaotic maps. However, the question is significant and what follows is worth mentioning.

Let (J, f) be a compact system with a scrambled set S such that J is an absolute retract (e.g., an n-dimensional closed cube I^n, the Hilbert cube or a dendrite). Assume that a compact metric space X contains a homeomorphic copy J^* of J and denote by h the homeomorphism $J \to J^*$. Then X admits a continuous self-map g for which $S^* = h(S)$ is a scrambled set. In fact, it is sufficient to take $g = (h \circ f \circ h^{-1}) \circ r$ where $r : X \to J^*$ is a retraction. Since h is a uniform homeomorphism, the set S^* is really a scrambled set for g (δ'-scrambled if S is δ-scrambled for f).

In particular, taking into account Example 59, a triangular map in the square with an open scrambled set, we can see that the following is true: if a compact metric space X contains a homeomorphic copy U of an open two-dimensional disk such that U is an open set in X, there exists a continuous self-map g of X such that (X, g) has an open scrambled set $\emptyset \neq V \subseteq U$. For instance, all surfaces admit systems with open scrambled sets. This should also work in dimensions greater than 2 since one can probably find analogues of Example 59 in higher dimensions.

3. EXISTENCE OF CANTOR SCRAMBLED SETS

In many situations a system with an *uncountable* scrambled set actually contains a *Cantor* scrambled set. This is the case for all interval maps [33]. Moreover, when proving that weakly mixing systems [28], Devaney chaotic systems [26] or positive-entropy systems [9] are Li–Yorke chaotic, one exhibits a Cantor scrambled set.

Then it is natural to ask whether in every system with an uncountable scrambled set there is also a Cantor scrambled set. Here we show that the answer is yes for further classes of systems, for instance subshifts and systems having no proper asymptotic pairs. The general question is still unsolved, which means that no counter-example is known; finding one does not look an easy task, supposing it is possible. Regarding the *space*, the only thing we know is that a Li–Yorke chaotic interval map always has a Cantor scrambled set.

A complete separable metric space is called *Polish*. In this section dynamical systems are usually assumed to be Polish.

We must recall several definitions and properties that are more or less classical in topological dynamics, and prove technical results.

DEFINITION 7. Let (X, f) be a dynamical system and $\varepsilon > 0$. A closed subset $K \subseteq X$ is called *ε-sensitive* if for any $x, y \in K$ and $\delta > 0$, there exist $x', y' \in K$ such that

$$d(x, x') < \delta, \quad d(y, y') < \delta \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x'), f^n(y')) \geq \varepsilon.$$

Sensitivity is usually defined as Lyapunov instability at all points, in the following way: for $\varepsilon > 0$, a closed subset K of X is called *classically ε-sensitive* if for any $x \in K$ and $\delta > 0$ there exists $y \in K$ such that $d(x, y) < \delta$ and $d(f^n x, f^n y) > \varepsilon$ for some $n \in \mathbb{Z}_+$. Definition 7, which concerns pairs of points, is equivalent to the classical definition of sensitivity. It is clear that an ε-sensitive closed subset of X is also ε'-sensitive in the classical sense for any $\varepsilon' < \varepsilon/2$. Conversely, we have the following

FACT. *In a Polish dynamical system, a classically ε-sensitive closed set K is $\varepsilon/2$-sensitive in the sense of Definition* 7.

Here is why. Let $K(\varepsilon,N)=\{(x,y)\in K\times K : d(f^n(x),f^n(y))\le\varepsilon/2\ \forall n\ge N\}$. Then $K(\varepsilon,N)$ is a nowhere dense closed subset of $K\times K$. It is clear that $K(\varepsilon,N)$ is closed. If there exist non-empty open subsets U,V of K such that $U\times V\subseteq K(\varepsilon,N)$, then for any $x_1,x_2\in U$ and $x_3\in V$,

$$d(f^nx_1,f^nx_2)\le d(f^nx_1,f^nx_3)+d(f^nx_2,f^nx_3)\le\varepsilon\quad\forall n\ge N.$$

In particular, $\operatorname{diam}(f^n(U))\le\varepsilon$ for $n\ge N$. Restricting oneself to some smaller non-empty open subset of U if necessary, one can assume that $\operatorname{diam}(f^n(U))\le\varepsilon$ for all $n\ge 0$. But this contradicts the classical ε-sensitivity of K since for any $x,y\in U$ and $n\ge 0$, $d(f^nx,f^ny)\le\varepsilon$.

Then $\bigcup_{N=1}^{\infty}K(\varepsilon,N)$ is first category in $K\times K$, so its complement is second category in the Baire space $K\times K$. The complement is contained in $\{(x,y)\in K\times K:\limsup_{n\to\infty}d(f^n(x),f^n(y))\ge\varepsilon/2\}$. This implies that K is $\varepsilon/2$-sensitive. ∎

A closed set without isolated points is called *perfect*. It is easy to see that an ε-sensitive set, being closed by definition, must be perfect. Let (X,f) be a dynamical system, $\varepsilon>0$ and $K\subseteq X$ be a closed set. Put

$$R(K,\varepsilon)=\{(x,y)\in K\times K:\limsup_{n\to\infty}d(f^n(x),f^n(y))\ge\varepsilon\}.$$

Proposition 8. *Let (X,f) be a Polish dynamical system, $\varepsilon>0$ and let $K\subseteq X$ be a closed set. Then*:

(1) *$R(K,\varepsilon)$ is a G_δ subset of $K\times K$.*
(2) *K is ε-sensitive if and only if $R(K,\varepsilon)$ is a dense G_δ subset of $K\times K$.*

Proof. For $m,l\in\mathbb{N}$, let

$$R(K,m,l;\varepsilon)=\Big\{(x,y)\in K\times K:\sup_{n\ge m}d(f^n(x),f^n(y))>\frac{l}{l+1}\,\varepsilon\Big\}.$$

Then $R(K,m,l;\varepsilon)$ is an open subset of $K\times K$. Thus their intersection $R(K,\varepsilon)=\bigcap_{l,m=1}^{\infty}R(K,m,l;\varepsilon)$ is a G_δ subset of $K\times K$. So (1) is true; (2) is obvious by (1) and the definition of ε-sensitive sets. ∎

Definition 9. A dynamical system (X,f) is called *positively expansive* if there exists $\delta>0$ such that $\sup_{n\ge0}d(f^n(x),f^n(y))\ge\delta$ for any $x\ne y\in X$; δ is called the *expansive constant* of (X,f).

Proposition 10. *Let (X,f) be a positively expansive Polish system with expansive constant $\delta>0$ and let K be a perfect subset of X. Then K is $\delta/2$-sensitive.*

Proof. Since $R(K,\delta/2)=\bigcap_{l,m=1}^{\infty}R(K,m,l;\delta/2)$, by Proposition 8 we need only prove that $R(K,m,l;\delta/2)$ is dense in $K\times K$ for each $m,l\in\mathbb{N}$. If this is not true, there exist non-empty open subsets U,V of K and $m,l\in\mathbb{N}$

such that $(U \times V) \cap R(K, m, l; \delta/2) = \emptyset$. For $x \in U$,

$$\sup_{n \geq m} d(f^n(x), f^n(y)) \leq \frac{l}{l+1} \frac{\delta}{2}$$

for any $y \in V$, hence $\sup_{n \geq m} \operatorname{diam}(f^n(V)) \leq l\delta/(l+1)$. Given $y_0 \in V$, by continuity of f there exists a smaller neighbourhood $V_{y_0} \subseteq V$ of y_0 in K such that $\sup_{n \geq 0} \operatorname{diam}(f^n(V_{y_0})) \leq l\delta/(l+1)$. Since K is perfect, there exists $y_1 \in V_{y_0} \setminus \{y_0\}$. Then $\sup_{n \geq 0} d(f^n(y_1), f^n(y_0)) \leq l\delta/(l+1)$, which contradicts δ-expansiveness. ∎

Definition 11. Let (X, f) be a dynamical system. A subset $K \subseteq X$ is called *regionally proximal* if $\liminf_{n \to \infty} d(f^n(U), f^n(V)) = 0$ for any two non-empty sets $U, V \subseteq K$ that are open in the relative topology of K.

Given (X, f) and a (not necessarily f-invariant) set $K \subseteq X$, put $\mathrm{PR}(K, f) := \mathrm{PR}(X, f) \cap (K \times K)$.

Proposition 12. *Let (X, f) be a Polish dynamical system and $K \subseteq X$ be a non-empty closed set. The following statements are equivalent*:

(1) *K is regionally proximal.*
(2) *For any $x, y \in K$ and $\delta > 0$, there exist $x', y' \in K$ and $n \in \mathbb{N}$ such that $d(x, x') < \delta$, $d(y, y') < \delta$ and $d(f^n(x'), f^n(y')) < \delta$.*
(3) $\mathrm{PR}(K, f)$ *is a dense subset of $K \times K$.*
(4) $\mathrm{PR}(K, f)$ *is a dense G_δ subset of $K \times K$.*

Proof. The implications (4)⇒(3)⇒(1)⇒(2) are obvious. To prove (2)⇒(4) assume (2) and note that in this case the open sets

$$R(K, n, f) = \{(x, y) \in K \times K : \text{there exists } k \in \mathbb{N} \text{ such that } d(f^k(x), f^k(y)) < 1/n\}$$

are dense for all n. Then the set $\mathrm{PR}(K, f) = \bigcap_{n=1}^{\infty} R(K, n, f)$ is a dense G_δ in the Baire space $K \times K$. ∎

Every separable metric space, hence every Polish space, has cardinality at most $c = 2^{\aleph_0}$. Now an uncountable Polish space has cardinality c: indeed, a Polish space X can be uniquely written as a disjoint union $X = P \cup S$ where S is countable and P is closed (hence Polish) and perfect (it is the perfect kernel of X). Since any non-empty perfect Polish space contains a Cantor set, this implies that any uncountable Polish space contains a Cantor set, so its cardinality is at least c.

Let X be a Polish space. A *Mycielski set* is the union of countably many Cantor sets. A Mycielski subset of X always contains a Cantor subset, but it may be dense in X. The following lemma is a rewriting of Mycielski's theorem [41, Theorem 1].

LEMMA 13. *Let X be a perfect Polish space. For any dense G_δ subset R of $X \times X$, there exists a dense Mycielski set $M \subseteq X$ having the property that whenever x_1, x_2 are distinct elements of M one has $(x_1, x_2) \in R$, i.e., $M \times M \setminus \Delta \subseteq R$.*

PROPOSITION 14. *Let (X, f) be a Polish dynamical system and $\delta > 0$. If a closed subset K of X is δ-sensitive and regionally proximal, then there exists a Mycielski set $S \subseteq K$ such that S is δ-scrambled and dense in K.*

Proof. If the closed subset K of X is δ-sensitive and regionally proximal, then $R(K,\delta)$ and $\mathrm{PR}(K, f)$ are dense G_δ subsets of $K \times K$ by Propositions 8 and 12. This implies that $R := R(K,\delta) \cap \mathrm{PR}(K, f)$ is a dense G_δ subset of $K \times K$. Since K is Polish and, being δ-sensitive, also perfect, by Lemma 13 there is a dense Mycielski set $S \subseteq K$ such that $S \times S \subseteq R \cup \{(x, x) : x \in K\}$. As every pair $(x, y) \in R$ is δ-scrambled, S is a δ-scrambled set. ∎

Let X be Polish. For a non-trivial weakly mixing system (X, f) the set of scrambled pairs is dense in $X \times X$; in [28] this is shown to imply that (X, f) is Li–Yorke chaotic, that the uncountable scrambled set is dense in X and δ-scrambled for some $\delta > 0$. These facts can also be deduced from the last result: Since $(X \times X, f \times f)$ is non-trivial and transitive, we infer that X is perfect, regionally proximal and δ-sensitive for $\delta = \mathrm{diam}(X) > 0$. So (X, f) is δ-Li–Yorke chaotic by Proposition 14. That a Polish Devaney chaotic system is δ-Li–Yorke chaotic for some $\delta > 0$ is shown in [39]; Proposition 14 also provides another proof.

For any set $D \subseteq X$ we denote by D_u the set of *condensation points* of D, i.e.,

$$D_u = \{x \in X : V_x \cap D \text{ is uncountable for any neighbourhood } V_x \text{ of } x\}.$$

LEMMA 15. *If D is an uncountable subset of the Polish space X, then*

(1) *D_u is a non-empty closed subset of X;*
(2) *$(D_u \cap D)_u = D_u$ and D_u is perfect.*

Proof. Use the well known fact that if D is a subset of a separable space (hence D itself is separable) then all but countably many points of D are condensation points of D. Hence (1) follows.

Now we show (2). $(D_u \cap D)_u \subseteq D_u$ is obvious. Conversely, for $x \in D_u$ and $\varepsilon > 0$, $L = D \cap B(x, \varepsilon)$ is an uncountable subset of X, so $L \cap L_u$ is uncountable by what was said above; thus $(D \cap D_u) \cap B(x, \varepsilon) \supset L \cap L_u$ is uncountable too, which implies by the way that D_u is perfect. Since ε is arbitrary, it follows that $x \in (D \cap D_u)_u$. ∎

THEOREM 16. *Let (X, f) be a Polish dynamical system and $\delta > 0$. If (X, f) is δ-Li–Yorke chaotic, then there exists a Cantor δ-scrambled set $S \subseteq X$.*

Proof. Let $C \subseteq X$ be an uncountable δ-scrambled set. Then, by Lemma 15, C_u is a perfect subset of X and $(C_u \cap C)_u = C_u$; in particular $C_u \cap C$ is a dense subset of C_u. Since $(C_u \cap C) \times (C_u \cap C) \setminus \Delta \subseteq R(C_u, \delta) \cap \mathrm{PR}(C_u, f)$ and $(C_u \cap C) \times (C_u \cap C) \setminus \Delta$ is dense in $C_u \times C_u$, C_u is δ-sensitive and regionally proximal by Propositions 8 and 12. By Proposition 14, there exists a Cantor δ-scrambled set $S \subseteq C_u$. ∎

COROLLARY 17. *If (X, σ) is a subshift there exists $\delta > 0$ such that every scrambled set $S \subseteq X$ is δ-scrambled. In particular, if (X, σ) is Li–Yorke chaotic, then it contains a Cantor δ-scrambled set.*

PROPOSITION 18. *Let (X, f) be a positively expansive Polish system with expansive constant $\delta > 0$. If (X, f) is Li–Yorke chaotic, then it contains a Cantor $\delta/2$-scrambled set.*

Proof. Let C be an uncountable scrambled set of (X, f). Then C_u is a perfect set and $(C_u \cap C)_u = C_u$. Since $(C_u \cap C) \times (C_u \cap C) \in \mathrm{PR}(C_u, f)$ and $(C_u \cap C) \times (C_u \times C)$ is dense in $C_u \times C_u$, C_u is regionally proximal. By Proposition 10, C_u is $\delta/2$-sensitive. Then by Proposition 14 there exists a Cantor $\delta/2$-scrambled set $S \subseteq C_u$. ∎

Let (X, f) be a dynamical system and K be a closed subset of X. Define

$$\mathrm{Eq}(K, f) = \{x \in K : \forall \varepsilon > 0\ \exists \text{ neighbourhood } V_x \text{ of } x \text{ such that } \sup_{n \geq 0} \mathrm{diam}(f^n(V_x \cap K)) \leq \varepsilon\}.$$

$\mathrm{Eq}(K, f)$ is the set of all points of K that are *equicontinuous in* K (not necessarily in X!). We remind the reader that the *asymptotic class* of x is $W^s(x) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$.

LEMMA 19. *Let (X, f) be a dynamical system and K be a closed subset of X. Then*:

(1) $\mathrm{Eq}(K, f)$ *is a G_δ subset of K.*
(2) *If there exists $x \in X$ such that $Eq(K, f) \subseteq W^s(x)$, then for any compact subset S of* $\mathrm{Eq}(K, f)$ *one has* $\lim_{n\to\infty} \mathrm{diam}(f^n(S)) = 0$.

Proof. Put

$$\mathrm{Eq}(K, f; m) = \{x \in K : \exists \text{ neighbourhood } V_x \text{ of } x \text{ such that } \sup_{n \geq 0} \mathrm{diam}(f^n(V_x \cap K)) \leq 1/m\}.$$

Then (1) results from the facts that $\mathrm{Eq}(K, f; m)$ is open in K and $\mathrm{Eq}(K, f) = \bigcap_{m=1}^{\infty} \mathrm{Eq}(K, f; m)$.

Let us show (2). Fix some $\varepsilon > 0$; for any $y \in \mathrm{Eq}(K, f)$ there is an open neighbourhood V_y of y such that $\sup_{n\geq 0} \mathrm{diam}(f^n(V_y \cap K)) \leq \varepsilon/2$. If S is a compact subset of $\mathrm{Eq}(K, f)$, there is a finite subset $\{x_1, \ldots, x_l\}$ of S such that $\bigcup_{i=1}^{l}(V_{x_i}\cap S)=S$. By the assumption of (2), $\{x_1,\dots,x_l\}\subseteq W^s(x)$, so

$$\lim_{n\to\infty}\operatorname{diam}(f^n(\{x_1,\dots,x_l\}))=0.$$

Combining this fact and $\sup_{n\geq 0}(f^n(V_y\cap S))\leq\varepsilon/2$ for any $y\in S$, one gets

$$\limsup_{n\to\infty}\operatorname{diam}(f^n(V_{x_i}\cap S))\leq\varepsilon,\qquad i=1,\dots,l.$$

Since ε was arbitrary, $\lim_{n\to\infty}\operatorname{diam}(f^n(S))=0$. ∎

Before we prove the main result of this section, Theorem 21, the reader must be reminded of the Kuratowski–Ulam theorem (see [1, 34, 42]):

PROPOSITION 20. *If X,Y are two Polish spaces and R is a dense G_δ set in $X\times Y$, then there exists a dense G_δ set $A\subseteq X$ such that for every $x\in A$ the set $A_x:=\{y\in Y:(x,y)\in R\}$ is a dense G_δ set.*

THEOREM 21. *Let (X,f) be a Polish dynamical system. If (X,f) is Li–Yorke chaotic, then at least one of the following properties holds:*

1. *There exists a Cantor scrambled set $K\subseteq X$.*
2. *There exist an uncountable scrambled set S of (X,f) and $x_0\in X$ such that $\operatorname{cl}(S)=S_u$, $\operatorname{Eq}(\operatorname{cl}(S),f)\subseteq W^s(x_0)$ and $\operatorname{Eq}(\operatorname{cl}(S),f)$ is a dense G_δ subset of $\operatorname{cl}(S)$.*

Proof. Let C be an uncountable scrambled set of (X,f). By Lemma 15, C_u is a non-empty perfect subset of X and $(C_u\cap C)_u=C_u$. For $m,n\in\mathbb{N}$ put

$$U_{m,n}=\{(x,y)\in C_u\times C_u:\exists k\geq m\text{ such that }d(f^k(x),f^k(y))>1/n\}.$$

As a union of open subsets of $C_u\times C_u$, $U_{m,n}$ is open in $C_u\times C_u$. Put $F_{m,n}=\operatorname{cl}(U_{m,n})$, $V_n=\bigcap_{m=1}^{\infty}U_{m,n}$ and $F_n=\bigcap_{m=1}^{\infty}F_{m,n}$. Since $F_{m,n}\setminus U_{m,n}$ is a nowhere dense closed subset of $C_u\times C_u$ for each $n,m\in\mathbb{N}$, $F_n\setminus V_n\subseteq\bigcup_{m=1}^{\infty}(F_{m,n}\setminus U_{m,n})$ is a first category F_σ subset of $C_u\times C_u$ for each $n\in\mathbb{N}$.

Put

$$\operatorname{NA}(C_u,f)=\{(x,y)\in C_u\times C_u:\limsup_{n\to\infty}d(f^n(x),f^n(y))>0\},$$
$$\operatorname{AR}(C_u,f)=\{(x,y)\in C_u\times C_u:\lim_{n\to\infty}d(f^n(x),f^n(y))=0\}.$$

Then $\operatorname{NA}(C_u,f)=\bigcup_{n=1}^{\infty}V_n$ and $\operatorname{AR}(C_u,f)=C_u\times C_u\setminus\operatorname{NA}(C_u,f)$.

Let $W=C_u\times C_u\setminus\bigcup_{n=1}^{\infty}F_n$ and $A=\operatorname{cl}(W)$. Then $W\subseteq\operatorname{AR}(C_u,f)$ and $W=\bigcap_{n=1}^{\infty}(C_u\times C_u\setminus F_n)$ is a dense G_δ subset of A. There are two cases:

CASE 1: A is a nowhere dense closed subset of $C_u\times C_u$, i.e., A considered as a subset of $C_u\times C_u$ has no interior points.

In this case, let $D_0=C_u\times C_u\setminus(A\cup\bigcup_{n=1}^{\infty}(F_n\setminus V_n))$. Since A and $F_n\setminus V_n$, $0<n<\infty$, are first category F_σ subsets of $C_u\times C_u$, D_0 is a dense G_δ subset of $C_u \times C_u$. Note that

$$C_u \times C_u \setminus \mathrm{NA}(C_u, f) = \Big(C_u \times C_u \setminus \bigcup_{n=1}^{\infty} F_n\Big) \cup \Big(\bigcup_{n=1}^{\infty} F_n \setminus \bigcup_{m=1}^{\infty} V_m\Big)$$
$$\subseteq A \cup \bigcup_{n=1}^{\infty} (F_n \setminus V_n) = C_u \times C_u \setminus D_0,$$

so that $D_0 \subseteq \mathrm{NA}(C_u, f)$; in other words, $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ for any $(x, y) \in D_0$.

Set $R = D_0 \cap \mathrm{PR}(C_u, f)$; then R is a dense G_δ subset of $C_u \times C_u$ by Proposition 12. By Lemma 13, there exists a Cantor set S such that $S \times S \subseteq R \cup \{(x, x) : x \in C_u\}$. Since (x, y) is a scrambled pair for any $(x, y) \in R$, S is a scrambled set.

Case 2: $\mathrm{int}_{C_u \times C_u}(A) \neq \emptyset$, i.e., there exist non-empty open subsets U, V of C_u such that

$$\mathrm{cl}(U) \times \mathrm{cl}(V) \subseteq A.$$

Obviously $\mathrm{cl}(U) \times \mathrm{cl}(V) \cap W$ is a dense G_δ subset of $\mathrm{cl}(U) \times \mathrm{cl}(V)$. For $x \in \mathrm{cl}(U)$ put

$$E(x; U, V) = \{y \in C_u : (x, y) \in \mathrm{cl}(U) \times \mathrm{cl}(V) \cap W\}.$$

Applied to $\mathrm{cl}(U) \times \mathrm{cl}(V) \cap W$, the Kuratowski–Ulam theorem (Proposition 20) implies that there exists $x_0 \in \mathrm{cl}(U)$ such that $E(x_0; U, V)$ is a dense G_δ subset of $\mathrm{cl}(V)$.

Since $(C_u \cap C)_u = C_u$ and $V \subseteq C_u$, we have $(V \cap C)_u = \mathrm{cl}(V)$. Then $S = V \cap C$ is an uncountable scrambled set of (X, f) and $\mathrm{cl}(S) = \mathrm{cl}(V) = S_u$. Since $W \subset \mathrm{AR}(C_u, f)$, it follows that $E(x_0; U, V) \subseteq W^s(x_0) \cap \mathrm{cl}(S)$.

Next we show that $E(x_0; U, V) \subseteq \mathrm{Eq}(\mathrm{cl}(S), f)$; this directly implies that $\mathrm{Eq}(\mathrm{cl}(S), f)$ is a dense G_δ subset of $\mathrm{cl}(S)$ because $E(x_0; U, V)$ is dense in $\mathrm{cl}(S)$ and by Lemma 19(1). The reason is the following. Fix $y \in E(x_0; U, V)$. For $k \in \mathbb{N}$, since

$$(x_0, y) \in W \subseteq C_u \times C_u \setminus F_k = \bigcup_{m=1}^{\infty} (C_u \times C_u \setminus F_{m,k}),$$

there exists $m(y, k) \in \mathbb{N}$ such that $(x_0, y) \in C_u \times C_u \setminus F_{m(y,k),k}$. As $F_{m(y,k),k}$ is closed we can find an open neighbourhood $V_{y,k}$ of y in C_u such that $\{x_0\} \times V_{y,k} \subseteq C_u \times C_u \setminus F_{m(y,k),k}$.

Note that $\sup_{n \geq m(y,k)} d(f^n(x_0), f^n(y')) \leq 1/k$ for $y' \in V_{y,k}$, so

$$\sup_{n \geq m(y,k)} \mathrm{diam}(f^n V_{y,k}) \leq 2/k.$$

Then by continuity one can choose a smaller neighbourhood $V'_{y,k}$ of y in C_u such that

$$\sup_{n\geq 0} \operatorname{diam}(f^n V'_{y,k}) \leq 2/k.$$

Since k is arbitrary, $y \in \mathrm{Eq}(\mathrm{cl}(S), f)$.

It remains to show that $\mathrm{Eq}(\mathrm{cl}(S), f) \subseteq W^s(x_0)$. We have already proved that $E(x_0; U, V) \subseteq W^s(x_0)$. Let $y \in \mathrm{Eq}(\mathrm{cl}(S), f)$. For any $\varepsilon > 0$, there exists a neighbourhood V_y of y in $\mathrm{cl}(S)$ such that $\sup_{k\geq 0} \operatorname{diam}(f^k(V_y)) \leq \varepsilon$. Since $E(x_0; U, V)$ is dense in $\mathrm{cl}(S)$, $E(x_0; U, V) \cap V_y \neq \emptyset$. Take $y' \in E(x_0; U, V) \cap V_y$. Then $\lim_{k\to\infty} d(f^k(y'), f^k(x_0)) = 0$. Moreover,

$$\begin{aligned}\limsup_{k\to\infty} d(f^k(y), f^k(x_0)) &\leq \limsup_{k\to\infty} (d(f^k(y), f^k(y')) + d(f^k(y'), f^k(x_0)) \\ &\leq \limsup_{k\to\infty} d(f^k(V_y)) + \lim_{k\to\infty} d(f^k(y'), f^k(x_0)) \leq \varepsilon.\end{aligned}$$

Since ε was arbitrary, $\lim_{k\to\infty} d(f^k(y), f^k(x_0)) = 0$, i.e., $y \in W^s(x_0)$. ■

We do not know any dynamical system for which property (2) above holds. Let us point out that for a discrete dynamical system (X, f) satisfying (2), $\mathrm{cl}(S) = S_u$ is a perfect subset of X, there exists a set $K = \bigcup_{n=1}^{\infty} C_n$, where each C_n is a Cantor set, such that $K \subseteq \mathrm{Eq}(\mathrm{cl}(S), f) \subseteq W^s(x)$ and K is dense in $\mathrm{cl}(S)$. Moreover, by Lemma 19, $\lim_{k\to\infty} \operatorname{diam}(f^k(\bigcup_{n=1}^{m} C_n)) = 0$ for any $m \in \mathbb{N}$.

COROLLARY 22. *Let (X, f) be a Li–Yorke chaotic Polish dynamical system. If $W^s(x)$ is countable for every $x \in X$, then there exists a Cantor scrambled set in (X, f).*

For a transitive non-sensitive system (X, f), $\mathrm{Eq}(X, f)$ is just the set of all transitive points of (X, f) and there exists a sequence $\{n_i\}$ of natural numbers such that

$$\lim_{i\to\infty} \sup_{x\in X} d(f^{n_i}(x), x) = 0.$$

This is proved in [24, 2] (in [24] the standing hypothesis is that the set X is compact metric but the proof does not use compactness). This implies that $W^s(x) = \{x\}$ for $x \in X$. So if a transitive, non-sensitive, Polish dynamical system is Li–Yorke chaotic it contains a Cantor scrambled set. The same remark applies to a larger class of systems introduced in [46]. Call (X, f) *doubly recurrent* if $(X \times X, f \times f)$ is recurrent (a system is recurrent if all points are recurrent). In a doubly recurrent system, unless $x = y$ the pair (x, y) is never asymptotic, because there must exist a subsequence (k_n) such that $(f^{k_n}(x), f^{k_n}(y)) \to (x, y)$ as $n \to \infty$. So $W^s(x) = \{x\}$.

Another result is obtained by the same methods:

THEOREM 23. *Let (X, f) be a Polish dynamical system, C be a scrambled set of (X, f) and P be a perfect subset of X. If $\mathrm{cl}(C \cap P) = P$, then at least one of the following two cases happens:*

(1) *There exists a Mycielski scrambled set* $K \subseteq P$ *with* $\mathrm{cl}(K) = P$.
(2) *There exist a non-empty open subset* V *of* P (*in the relative topology*) *and* $x \in X$ *such that* $\mathrm{Eq}(\mathrm{cl}(V), f) \subseteq W^s(x)$ *and* $\mathrm{Eq}(\mathrm{cl}(V), f)$ *is a dense* G_δ *subset of* $\mathrm{cl}(V)$.

Proof. The proof is the same as that of Theorem 21, replacing C_u by P. ■

Corollary 24. *Let* (X, f) *be a transitive Polish dynamical system which does not reduce to a periodic orbit. If there exists a scrambled set* C *of* (X, f) *with* $\mathrm{int}(\mathrm{cl}(C)) \neq \emptyset$, *then there exists a Mycielski scrambled set* K *of* (X, f) *with* $\mathrm{cl}(K) \supseteq \mathrm{int}(\mathrm{cl}(C))$.

Proof. Let $W = \mathrm{int}(\mathrm{cl}(C))$ and $P = \mathrm{cl}(W)$. Since (X, f) is a non-periodic transitive system, X is perfect. Hence P is also perfect and $\mathrm{cl}(P \cap C) = P$. By Theorem 23, if there does not exist any Mycielski scrambled set $K \subseteq P$ with $\mathrm{cl}(K) = P$, then there exist a non-empty open subset V of P and $x \in X$ such that $\mathrm{Eq}(\mathrm{cl}(V), f) \subseteq W^s(x)$ and $\mathrm{Eq}(\mathrm{cl}(V), f)$ is a dense G_δ subset of $\mathrm{cl}(V)$.

Since V is an open subset of P, we can find an open subset U of X such that $U \cap P = V$. Let $W_1 = U \cap W$. Clearly, W_1 is a non-empty open subset of X, $W_1 \subseteq V$ and $\mathrm{cl}(W_1) = \mathrm{cl}(V)$. Let $\mathrm{Tran}(f)$ be the set of all transitive points of (X, f). Since $W_1 \cap \mathrm{Tran}(f)$ is a dense G_δ subset of W_1, $W_1 \cap \mathrm{Tran}(f)$ is also a dense G_δ in $\mathrm{cl}(W_1) = \mathrm{cl}(V)$. Thus $W_1 \cap \mathrm{Tran}(f) \cap \mathrm{Eq}(\mathrm{cl}(V), f)$ is a dense G_δ in $\mathrm{cl}(V)$. Fix $x_0 \in W_1 \cap \mathrm{Tran}(f) \cap \mathrm{Eq}(\mathrm{cl}(V), f)$, $x_0 \neq x$.

For any $m \in \mathbb{N}$, there exists an open neighbourhood U'_m of x_0 in X such that $\sup_{n \geq 0} \mathrm{diam}(f^n(U'_m \cap \mathrm{cl}(V))) \leq 1/m$, as $x_0 \in \mathrm{Eq}(\mathrm{cl}(V), f)$. Let $U_m := U'_m \cap W_1$. Since $x_0 \in W_1$, U_m is an open neighbourhood of x_0 and $\sup_{n \geq 0} \mathrm{diam}(f^n(U_m)) \leq 1/m$. Since $x_0 \in \mathrm{Tran}(f)$, there exists $i_m \geq m$ such that $f^{i_m} x_0 \in U_m$. Hence $\sup_{n \geq 0} d(f^{i_m + n}(x_0), f^n(x_0)) \leq 1/m$. Moreover, $\sup_{z \in X} d(f^{i_m}(z), (z)) \leq 1/m$, as $\mathrm{cl}(\{x_0, f(x_0), \dots\}) = X$. Letting $m \to \infty$, one gets $\lim_{m \to \infty} \sup_{z \in X} d(f^{i_m}(z), z) = 0$. Since $\lim_{m \to \infty} i_m = +\infty$, every point $z \in X$ is recurrent whence $W^s(y) = \{y\}$ for any $y \in X$. In particular $W^s(x) = \{x\}$. This contradicts the fact that $x \neq x_0 \in \mathrm{Eq}(\mathrm{cl}(V), f) \subseteq W^s(x)$. ■

Corollary 24 implies for instance that whenever a non-periodic Polish system is transitive and contains a dense scrambled set, it contains a dense Mycielski scrambled set. Known examples of this situation are Polish weakly mixing systems [28] and compact transitive systems with a fixed point [26].

The existence of an uncountable scrambled set may imply that of a Cantor scrambled set for the same system. Nevertheless, the following proposition shows that if S is an uncountable scrambled set of (X, f), then in general there is no chance to find a Cantor scrambled set *as a subset of* S.

For proving this let us recall the notion of a Bernstein set. A set B in a space X is called a *Bernstein set* if neither B nor $X \setminus B$ contains a non-empty perfect subset of X. Hence neither B nor $X \setminus B$ contains a Cantor set. A theorem due to Bernstein (see [31] or, for a stronger result, [35, p. 261]) says that the Axiom of Choice implies the existence of a Bernstein set B with $\mathrm{card}(B) = \mathrm{card}(X \setminus B) = c$ in every uncountable Polish space X.

PROPOSITION 25. *Let X be Polish and let S be an uncountable scrambled set in (X, f).*

(i) *If S is analytic then it contains a Cantor set.*
(ii) *There exists an uncountable scrambled set $S^* \subseteq S$ which does not contain any Cantor set.*

Proof. (i) In a Polish space every uncountable analytic set contains a Cantor set.

(ii) Suppose that S contains a Cantor set C. Since C is a Polish space, it contains a Bernstein set B (in C). Put $S^* = B$. ∎

So, a Polish system contains a Cantor scrambled set if and only if it contains an *analytic* uncountable scrambled set.

QUESTIONS. 1. Let X be a Polish or a compact metric space, and let (X, f) be a dynamical system. Can one prove that if (X, f) is Li–Yorke chaotic then X contains a Cantor scrambled set?

2. It is shown in [33] that for interval maps, the existence of just one scrambled pair is enough to imply the existence of a δ-scrambled Cantor set. Is this also true for graph maps?

4. CARDINALITIES OF SCRAMBLED SETS. SYSTEMS WITH ONLY FINITE OR COUNTABLE SCRAMBLED SETS

Of course a system may have no scrambled pairs at all. Compact metric surjective systems having no scrambled pairs are called *almost distal*. Distal systems obviously belong to this class. Some properties of almost distal systems, established in [9], are similar to those of distal systems: they have entropy 0 and they are minimal when transitive; moreover, minimal almost distal systems are disjoint from all weakly mixing systems (disjointness is defined below). Examples are given in the same article.

In [3] the authors introduced a striking generalization of almost distal systems; a pair (x, y) is said to be *strong Li–Yorke*, or, consistently with our nomenclature, *strong scrambled*, if it is scrambled and recurrent (meaning that (x, y) is in the closure of its own positive orbit); a subset S of X containing at least two points is called a *strong scrambled set* of the system if for any $x, y \in S$ with $x \neq y$, (x, y) is a strong scrambled pair; a system is called *semi-distal* if it contains no strong scrambled pairs, and a system is called *strong Li–Yorke chaotic* if it has an uncountable strong scrambled set. Semi-distal systems have the same three properties we mentioned for almost distal systems; in addition they form a more natural class from the point of view of topological dynamics. Floyd's system [5] is semi-distal but not almost distal, actually it is Li–Yorke chaotic. So it turns out that the features of the set of strong scrambled pairs may be at least as significant as those of the set of scrambled pairs in the theory of chaos. It seems to us that this point should be given some attention in future research. Here is an example: since the set of all recurrent points of a dynamical system is G_δ, when a Polish dynamical system (X, f) is strong Li–Yorke chaotic it contains a Cantor strong scrambled set. The proof relies on the results of the previous section.

A *joining* J between two dynamical systems (X, f) and (Y, g) is a closed $f \times g$-invariant subset of the Cartesian product $X \times Y$ such that its projections to X and Y are onto. (X, f) and (Y, g) are said to be *disjoint* if $X \times Y$ is their unique joining. The significance of the disjointness results mentioned above for almost distal and semi-distal systems is that minimal compact metric systems which have no scrambled pairs, or even no strong scrambled pairs, share with minimal distal systems the property of being remote from weakly mixing systems structurally. If we consider weak mixing as a chaotic property, and distality as a deterministic property, this heuristically means that the existence of scrambled pairs is a kind of basic requirement for chaoticity.

Among the self-maps of a real compact interval, all maps of Sharkovsky type 2^n for all $n \in \{0\} \cup \mathbb{N}$ and also some of those of type 2^∞ have no scrambled pairs. On the other hand, some of the maps of type 2^∞ and all the maps of type greater than 2^∞ have Cantor scrambled sets. But when the space X is not an interval, the situation is far from being so clear-cut.

If a system has a scrambled set with two points then, in contrast to the interval case, it may happen that it has no scrambled set with three points. An example of a triangular map in the square with this property is found in [18]. In [21] it was shown that the Cantor set and the Warsaw circle also admit continuous self-maps with this property. Examples of systems with only boundedly finite, or countable, scrambled sets were given in [8]: symbolic systems generated by primitive constant-length substitutions have at most finite scrambled sets; some have no scrambled sets at all and others have finite scrambled sets. Systems with infinite but only countable scrambled sets are obtained as inverse limits of a sequence of constant-length substitution systems.

To complete this picture of systems having only finite or countable scrambled sets, we further develop an idea from [21] to answer a question posed there, what are possible (sets of) cardinalities of (maximal) scrambled sets of compact dynamical systems. Theorem 27 states that these sets of cardinalities may be *arbitrary* subsets of $\{2,3,\dots\} \cup \{\aleph_0\} \cup \{c\}$. This provides new examples of systems with only finite or countable scrambled sets. We start with the following

LEMMA 26. *For every $\delta > 0$ and every cardinal number $2 \le m \le \aleph_0$ there is a compact system (X_m, f_m) such that it has scrambled sets, all being δ-scrambled, and all its maximal scrambled sets have cardinality m. Moreover, X_m can be chosen to be an arcwise connected one-dimensional planar continuum, a subset of a rectangle with height 2 whose left and upper sides belong to X_m and all their points are fixed for f_m. Finally, the horizontal size of the rectangle can be made arbitrarily small and the map f_m can be, in the metric of uniform convergence, as close to the identity as we wish.*

Proof. It is sufficient to prove this for $\delta = 1$. Call a sequence $s = (s_i)_{i=1}^{\infty}$ of positive real numbers *admissible* if $s_i \searrow 0$ and $\sum s_i = +\infty$. Given such a sequence, put $h(0) = 0$, $h(1) = 1+s_1, \dots , h(1+\sum_{i=1}^{n} s_i) = 1+\sum_{i=1}^{n+1} s_i, \dots,$ and let h be affine on each of the intervals $[0,1]$, $[1+\sum_{i=1}^{n} s_i, 1+\sum_{i=1}^{n+1} s_i]$, $n = 0,1,\dots.$ Then obviously h is a homeomorphism of $[0,\infty)$, 0 is a fixed point of h and the trajectory of 1 is $1, 1+s_1, 1+s_1+s_2, \dots.$ Notice that $\varrho_{\sup}(h, \mathrm{Id}) = s_1$, $\varrho_{\sup}$ being the supremum metric. It is not difficult to show that in the dynamical system $([0,\infty), h)$ any two *positive* real numbers are asymptotic and their nth images tend to $+\infty$. This is in fact true for any continuous h such that $h(0) = 0$, $h(x) > x$ for all $x > 0$, $\lim_{x\to\infty}(h(x) - x) = 0$ and h is increasing on some interval $[x_0,\infty)$.

Consider the zigzag curve G in the plane determined uniquely by the following set of conditions:

- G is a subset of the rectangle $[0,1] \times [-1,1]$.
- $G = \bigcup_{n=0}^{\infty} J_n$ where each J_n is a straight line segment with length 2.
- J_0 has endpoints $\langle 1,1\rangle$ and $\langle 1,-1\rangle$.
- For $k = 1,2,\dots,$ the segments J_{2k-1} and J_{2k} have one endpoint in common, the other endpoint of J_{2k-1} (resp. J_{2k}) being $\langle 1/(2k-1), -1\rangle$ (resp. $\langle 1/(2k), -1\rangle$).

The segments J_n are called *laps* of G. The endpoints of laps whose second coordinate is -1 are called *lower turning points* of G and the other endpoints of laps *upper turning points* of G.

There is a unique homeomorphism h^* of the ray $[0,\infty)$ onto G such that for any $x, y \in [0,\infty)$, $x \neq y$, one has $|x-y| = \varrho(h^*(x), h^*(y))$, where $\varrho(a,b)$ denotes the length of the arc in G with endpoints a, b. Now carry over the dynamics from $([0,\infty), h)$ onto G via this homeomorphism h^*. Then add the arc of fixed points formed by two straight line segments: the vertical segment V with endpoints $\langle 0,-1\rangle$ and $\langle 0,1\rangle$ and the horizontal segment H

with endpoints $\langle 0,1\rangle$ and $\langle 1,1\rangle$. We thus get a system (W,g) where $W = G\cup V\cup H$ is homeomorphic to the so-called Warsaw circle and g is such that if we put $G^\circ = G\setminus\{\langle 1,1\rangle\}$, then all points from $W\setminus G^\circ$ are fixed and all points $x,y\in G^\circ$ are asymptotic and have as their ω-limit sets the vertical segment V. Under h the point $1^* := h^*(1)$ makes "jumps" with lengths $s_1,s_2,\dots$ along the zigzag curve G, towards the segment V. Call 1^* the *distinguished point* of the system.

We have thus assigned a system (W,g_s) to the admissible sequence $s = (s_i)_{i=1}^\infty$. Notice that any point from G° forms a scrambled pair with any point from V and there are no other scrambled pairs in this system.

Since s is admissible, the sequences $s^{(k)}$, $k=0,1,\dots$, defined by $s^{(k)} = (s_i/2^k)_{i=1}^\infty$ are also admissible (here $s^{(0)}=s$) and so for any $k=0,1,\dots$ we can make the same construction as above, to get a system $(W,g_{s^{(k)}})$ with a distinguished point 1_k^* making jumps with lengths $s_1/2^k, s_2/2^k,\dots$. Instead of G use now the symbol G_k. For $k\to\infty$ the maps $g_{s^{(k)}}$ uniformly converge to the identity on W.

We can now define a system $(X_{\aleph_0}, f_{\aleph_0})$ with the required properties but, for the moment, in the 3-dimensional Euclidean space (with coordinates x,y,z) rather than in the plane. Put

$$X_{\aleph_0} := (W_0\cup W_1\cup\cdots)\cup W_\infty$$

where $W_0 := W\times\{0\}$ (W being the Warsaw circle defined above) and for $k=1,2,\dots$ the set W_k is the image of W_0 under rotation of $(\pi/2)(1-1/2^k)$ around the y-axis, and W_∞ is that of W_0 under rotation of $\pi/2$ around the y-axis (so all the points from W_∞ have zero x-coordinates and nonnegative z-coordinates). Note that any two of the sets $W_\infty, W_0, W_1,\dots$ intersect just in the segment $\{0\}\times[-1,1]\times\{0\} = V\times\{0\}$. Obviously, $X_{\aleph_0}$ is one-dimensional and arcwise connected.

Endow the phase space $X_{\aleph_0}$ with a dynamics. Let $f_{\aleph_0}$ be the identity on W_∞ and let $f_{\aleph_0}|_{W_k}$ be just the isometric copy of $g_{s^{(k)}}$ (i.e., $(W_k, f_{\aleph_0}|_{W_k})$ is topologically conjugate to $(W,g_{s^{(k)}})$ via the isometry between W_k and W).

Now put $S := \{1_k^* : k=0,1,\dots\}\cup\{p\}$ where $p\in V\times\{0\}$, say $p = \langle 0,1,0\rangle$. We show that S is a maximal 1-scrambled set of $f_{\aleph_0}$. For any k, the points p and 1_k^* form obviously a 2-scrambled pair. Now fix nonnegative integers $a<b$. The points 1_a^* and 1_b^* move along different curves G_a and G_b (subsets of W_a and W_b, respectively). Since G_a and G_b are isometric to G and both the points 1_a^* and 1_b^* approach the same segment $V\times\{0\}$ (which is their ω-limit set), to show that they form a 1-scrambled pair it is obviously sufficient to show that they would form a 1-scrambled pair if they moved along the same curve G. So, without changing the notation we think of 1_a^* and 1_b^* as points moving along G (one phase space, two different dynamics). The two points start from the same position (at time zero $1_a^* = 1_b^*$) and they

make jumps of $s_i/2^a$ and $s_i/2^b$ respectively ($i = 1, 2, \dots$). Thus each jump of 1_a^* is 2^{b-a} times longer than the corresponding jump of 1_b^*. Denote by a_i and b_i the position of the faster point 1_a^* and that of the slower point 1_b^* at time i, respectively.

We are ready to show that 1_a^* and 1_b^* form a 1-scrambled set in the plane. It is easy to see that $\liminf_{i\to\infty} \varrho_{\text{euc}}(a_i, b_i) = 0$ where ϱ_{euc} denotes the Euclidean distance. In fact, for all i large enough we have the following situation: the points a_i and b_i are already in a very small neighbourhood of V and when looking at their projections on V, each of these points makes very small jumps along V, say up several times, then down several times, again up several times etc. Thus one can see that there are times when the projections on V are very close to each other. Since our points are also close to V, it follows that their Euclidean distances at these times are also very small.

To prove that $\limsup_{i\to\infty} \varrho_{\text{euc}}(a_i, b_i) \geq 1$, it is sufficient to show that $\limsup_{i\to\infty} |\pi_2(a_i) - \pi_2(b_i)| \geq 1$, π_2 being the projection onto V. Let $i_0 \in \mathbb{N}$ be given. We may assume that b_{i_0} (hence also a_{i_0}) is in that part of G (very close to V) where the laps of G have second projections with lengths very close to 2, the length of V. It is now sufficient to show that there exists $i \geq i_0$ such that $|\pi_2(a_i) - \pi_2(b_i)| \geq 1$ or at least $|\pi_2(a_i) - \pi_2(b_i)| \approx 1$. Take $j \geq i_0$ such that $s_j/2^a$ (hence also $s_j/2^b$) is very small and $\pi_2(b_j) < \pi_2(b_{j+1}) \approx -1$, i.e., the slower of our two points is very close to a lower turning point and goes up. If $\pi_2(a_j) \geq 0$ one can put $i = j$. So let $\pi_2(a_j) < 0$. Since the jumps of the faster point are 2^{b-a} times as long as the jumps of the slower point, we have the following: when the slower point reaches the small vicinity of the closest upper turning point (i.e., when its 2nd projection increases approximately by 2), the 2nd projection of the faster point approximately does not change, i.e., it is approximately the same as the 2nd projection of a_j (we use the fact that 2^{b-a} is an *even* positive integer). Hence there is a time $i > j$ with $|\pi_2(a_i) - \pi_2(b_i)| > 1$ or ≈ 1.

We have thus proved that the infinite countable set S is 1-scrambled for $f_{\aleph_0}$, and $X_{\aleph_0}$ is arcwise connected. On the other hand, in the system $(X_{\aleph_0}, f_{\aleph_0})$ there is no uncountable scrambled set since every scrambled set contains at most one fixed point and for any k it intersects the set G_k° in at most one point (any two points from G_k° being asymptotic). This argument also shows that S is a maximal scrambled set.

The first claim of the lemma is proved for $m = \aleph_0$. When $m \geq 2$ is finite it is sufficient to take the subsystem (X_m, f_m) where $X_m = W_0 \cup \dots \cup W_{m-2}$ and $f_m = f_{\aleph_0}|_{X_m}$.

Further, it is not hard to check that there is a homeomorphism of the set $X_{\aleph_0}$ (or X_m) into the plane $z = 0$ which fixes the segment $V \times \{0\}$ and also gives all the required geometry of the planar image of our set (the fact that X_m can be as "narrow" as we wish is obvious since only the length of the vertical segment V is important for 1-scrambledness). Finally, we may assume that $\varrho_{\sup}(f_m, \mathrm{Id}|_{X_m})$ is as small as we wish: just take an admissible sequence s with sufficiently small first term s_1. ∎

THEOREM 27. *For every non-empty subset A of the set $\{2, 3, \dots\} \cup \{\aleph_0\} \cup \{c\}$ there is a compact system (X, f) such that the set of cardinalities of all maximal scrambled sets of this system coincides with A. Given any $\delta > 0$, all scrambled sets of (X, f) may be assumed to be δ-scrambled. Finally, X can be chosen to be an arcwise connected one-dimensional planar continuum.*

Proof. It is sufficient to prove this for $\delta = 1$. Given a cardinality α, a system (Y, g) is said to be a *standard α-system in the rectangle* $[p, q] \times [u, v]$ if Y is an arcwise connected one-dimensional planar continuum in $[p, q] \times [u, v]$, all the points in $(\{p\} \times [u, v]) \cup ([p, q] \times \{v\})$ are fixed points of g, the set of cardinalities of all its maximal scrambled sets coincides with $\{\alpha\}$ and all its scrambled sets are 1-scrambled.

Let $A \neq \emptyset$. We show how to construct the system for $A = \{2, 3, \dots\} \cup \{\aleph_0\} \cup \{c\}$; after that the construction for smaller sets A is obvious. By Lemma 26, for every finite $m \geq 2$ in the rectangle $[1/(2m + 1), 1/(2m)] \times [-1, 1]$ there exists a standard m-system (X_m, f_m). Again by Lemma 26 we may assume that the supremum distance between f_m and the identity on X_m goes to zero as $m \to \infty$; so when later adding to our picture the vertical segment $\{0\} \times [-1, 1]$ of fixed points, we do not destroy continuity. Again by Lemma 26, in the rectangle $[1/3, 1/2] \times [-1, 1]$ there exists a standard $\aleph_0$-system $(X_{\aleph_0}, f_{\aleph_0})$ and, by Example 3, in the segment $X_c = \{1\} \times [-2, 1]$ one can define a continuous map f_c such that one of the endpoints, say $\langle 1, 1\rangle$, is fixed, the system has scrambled sets, all being 1-scrambled, and all its maximal scrambled sets have cardinality c (the segment has length 3 since in Example 3 we had only $\delta = 1/3$). Construct (X, f) as the union of all the systems described and add two segments $\{0\} \times [-1, 1]$ and $[0, 1] \times \{1\}$ of fixed points of f to make X arcwise connected. The system (X, f) has all the required properties. ∎

QUESTION. Does the conclusion of Theorem 27 hold if X is required/additionally required to be a locally connected continuum?

5. SYSTEMS WITH ONLY FIRST CATEGORY SCRAMBLED SETS

We have just addressed the case of dynamical systems that are not Li–Yorke chaotic. Among those that are Li–Yorke chaotic, can one find systems having uncountable scrambled sets, all of which are first category? In this short section we gather some common-sense observations around this question.

First, the answer to the question is yes—the system from Example 3 has uncountable scrambled sets, all being nowhere dense. Another trivial example of a Li–Yorke chaotic system with only nowhere dense scrambled sets is the direct product of a system with uncountable scrambled sets and the system given by the identity on an interval. Floyd's system [5] also has the required properties. The next result is inspired by it.

A set A is called *somewhere dense* if it is not nowhere dense, i.e., if its closure has non-empty interior. A system is called *trivial* if it is a singleton.

PROPOSITION 28. *Let X be a metric space and (X, f) be a dynamical system. Let $\phi : (X, f) \to (Y, g)$ be a factor map.*

(1) *Suppose (Y, g) is distal but not trivial. Then (X, f) has no dense scrambled set.*
(2) *Suppose that (X, f) is transitive, (Y, g) is distal and Y has no isolated point. Then every scrambled set of (X, f) is nowhere dense, thus first category.*

Proof. (1) Obviously a factor map can project a scrambled pair to a proximal pair only, and there are no proximal pairs in the distal system (Y, g) except diagonal ones, so if $(x, x') \in \mathrm{SR}(X, f)$ one has $\phi(x) = \phi(x')$. This implies that if S is scrambled it is contained in one fibre of ϕ, in other words, there is $y \in Y$ such that $S \subseteq \phi^{-1}(y)$. But the set $\phi^{-1}(y)$ is not dense unless $Y = \{y\}$.

(2) Let S be a scrambled set of (X, f). We already know from the proof of (1) that $S \subseteq \phi^{-1}(y)$ for some y. Suppose that $\phi^{-1}(y)$ is somewhere dense. Then, being a closed set, it contains an open set $U \neq \emptyset$. By transitivity there is $n > 0$ such that $U \cap f^{-n}(U) \neq \emptyset$, so $U \cap f^n(U) \neq \emptyset$; considering the fact that $\phi(U) = \{y\}$ and $\phi(f^n(U)) = \{g^n(y)\}$ this implies that $g^n(y) = y$. Moreover, one easily checks that y is a transitive point of (Y, g) because its preimage contains a non-empty open set of X and (X, f) is transitive. As (Y, g) contains a transitive periodic point, it reduces to a periodic orbit, which contradicts the fact that it contains no isolated points. Then $\phi^{-1}(y)$, hence S, is nowhere dense in X. ∎

A by-product of the above proposition is that a transitive system having a somewhere dense scrambled set has a maximal distal factor which is a single periodic orbit. When the system has a dense scrambled set its maximal distal factor is trivial.

Before proving a stronger result for minimal compact systems, let us remind the reader of a classical result (McMahon's proof can be found for instance in [5]).

Proposition 29. *A minimal compact system is weakly mixing if and only if its maximal equicontinuous factor is trivial.*

The *regionally proximal relation* $\mathrm{RPR}(X,f)$ of (X,f) is defined as follows: $(x,y)\in \mathrm{RPR}(X,f)$ if for any $\varepsilon>0$ one can find x', $y'\in X$ and a natural integer n such that $d(x,x')<\varepsilon$, $d(y,y')<\varepsilon$ and $d(f^n(x'),f^n(y'))<\varepsilon$. In a compact system the maximal equicontinuous factor is obtained by collapsing $\mathrm{RPR}(X,f)$, which in the minimal case is a closed equivalence relation [5]. Also recall that a minimal equicontinuous system is distal.

Proposition 30. *Let (X,f) be a minimal compact system. The following statements are equivalent*:

(1) *There exists a scrambled set S which is somewhere dense* (*i.e.*, $\mathrm{int}(\mathrm{cl}(S))\neq\emptyset$).
(2) *There exists a Mycielski scrambled set S which is somewhere dense.*
(3) *The maximal distal factor of (X,f) is periodic and (X,f) is not periodic.*
(4) *There exist $n\in\mathbb{N}$ and pairwise disjoint closed subsets $X_0,\dots,X_{n-1}$ of X such that $\bigcup_{i=0}^{n-1}X_i=X$, $f(X_i)=X_{i+1\,(\mathrm{mod}\,n)}$ and $f^n|_{X_i}$ is non-trivial weakly mixing.*

Proof. By Corollary 24, (1) is equivalent to (2). (2)⇒(3) results from Proposition 28.

Now if (3) is true the maximal equicontinuous factor (Y,g) of (X,f), which is a factor of the maximal distal factor, is also periodic. Let $\phi:(X,f)\to(Y,g)$ be the corresponding factor map. Set $n=\mathrm{card}\,Y$ and $y=\phi(x)$ for some $x\in X$. Then $Y=\{y,g(y),\dots,g^{n-1}(y)\}$. Let $X_i=\pi^{-1}(g^i(y))$ for $i=0,1,\dots,n-1$. Then $X_0,X_1,\dots,X_{n-1}$ are pairwise disjoint closed subsets of X, $\bigcup_{i=0}^{n-1}X_i=X$ and $f(X_i)=X_{i+1\,(\mathrm{mod}\,n)}$. It is not hard to check that the system (X_i,f^n) is minimal for $i=0,1,\dots,n-1$. Its maximal equicontinuous factor is trivial; by the assumption the sets X_i are the equivalence classes of $\mathrm{RPR}(X,f)$. Now $\mathrm{RPR}(X,f^n)=\mathrm{RPR}(X,f)$, so $\mathrm{RPR}(X_i,f^n)=X_i$; it follows that the maximal equicontinuous factor of (X_i,f^n) is trivial, so it is weakly mixing by Proposition 29. Since (X,f) is not periodic, (X_i,f^n) is non-trivial, and (4) is true.

Finally, (4)⇒(1) because a non-trivial Polish weakly mixing system contains a dense scrambled set (see the remarks after Proposition 14). ∎

There are classical examples of minimal Li–Yorke chaotic extensions of distal systems that are not Cartesian products, for instance among Toeplitz symbolic systems. Some of them have positive entropy [47, 23, 14, 45]. By the above proposition these systems have only nowhere dense scrambled sets.

Question. The scrambled sets of a minimal compact system are somewhere dense when its maximal distal factor is periodic, and of course not conjugate to the whole system; on the other hand, its scrambled sets are nowhere dense whenever its maximal distal factor is not periodic. Is the situation as clear-cut when the system is not minimal? Or else, can one construct *transitive* compact systems with uncountable but at most first category scrambled sets, having a trivial maximal distal factor?

6. NON-RESIDUAL SCRAMBLED SETS AND SECOND CATEGORY SCRAMBLED SETS

Given a compact metric space X, a set $S \subseteq X$ is called residual in an open set $U \subseteq X$ if the set $S \cap U$ is residual in U (in the relative topology). A set S is said to be *nowhere residual* if there is no open set $U \neq \emptyset$ such that S is residual in U.

It has been known since 1987 that when an interval map is Li–Yorke chaotic its scrambled sets are nowhere residual [12, 22]. In this section we prove that various other assumptions on dynamical systems imply that scrambled sets cannot be residual, or even are nowhere residual. In the whole section we are considering compact spaces, except in Subsection 6.4 where X is only assumed to be Polish.

In Subsection 6.1 assumptions concern the dynamics of the map f on X, with compactness as a standing hypothesis. The main result is that for a large class of compact systems, including minimal systems, scrambled sets are nowhere residual.

In Subsection 6.2 the space X is a graph, f being arbitrary. Our result simply generalizes Bruckner's and Hu's and Gedeon's results on interval maps [12, 22]: scrambled sets of graph maps are nowhere residual.

The two assumptions in Subsection 6.3 are of a mixed nature: the space is a Cantor set but there are also dynamical conditions. When the system is symbolic an assumption on so-called contexts implies that scrambled sets are not residual (sometimes even nowhere residual). This concerns a wide class of non-minimal subshifts. When the system is a cellular automaton, an assumption on the rule permits to prove the same property. In both cases X is a symbolic space, but there are further conditions on the dynamics.

In the last subsection, assuming the Continuum Hypothesis (CH), we show that a generically chaotic system (one with a residual set of scrambled pairs in $X \times X$) on a Polish space has a scrambled set which is everywhere second category, i.e., second category in every open ball. This has several applications, always assuming CH: all weakly mixing graph maps and all non-trivial weakly mixing subshifts of finite type have everywhere second category nowhere residual scrambled sets. Furthermore, for non-trivial compact minimal systems, weak mixing is equivalent to the existence of an everywhere second category scrambled set. Finally, the maximal possible size of scrambled sets in minimal compact systems is completely characterized: nowhere dense when their maximal distal factor is not periodic, second category (but nowhere residual) otherwise. Let us emphasize the fact that the nowhere residuality of scrambled sets for compact minimal systems and for graph maps is proved without assuming CH.

6.1. General non-residuality results. Recall that Proposition 28 above implies that a compact system having a non-trivial distal factor cannot be residually scrambled.

The first result in this subsection provides a necessary condition for a class of compact dynamical systems (including all invertible ones) to be residually scrambled. This condition implies that various compact systems are never residually scrambled. In particular, the scrambled sets of minimal systems are nowhere residual.

PROPOSITION 31. *Let (X, f) be a compact dynamical system with f such that the preimage of any residual set is second category. If X contains a residual scrambled set it also contains a fixed point.*

Proof. Let $S \subseteq X$ be a residual scrambled set. According to the assumption, $f^{-1}(S)$ is second category and so $S \cap f^{-1}(S) \neq \emptyset$. In other words, there is $x \in S$ such that $f(x) \in S$ too. Since S is scrambled, the pair $(x, f(x))$ is proximal. By compactness there exists an increasing sequence (k_n), $n \in \mathbb{N}$, such that the sequences $(f^{k_n}(x))$ and $(f^{k_n+1}(x))$ converge to the same point p, which must then be fixed. ∎

We do not know whether the existence of a residual scrambled set implies the existence of a fixed point; all known examples contain one. At least Example 54 below shows that a residually scrambled system may have more than one (by Proposition 4 this implies that the system is not proximal).

Proposition 31 implies that on compact spaces homeomorphisms without fixed points, for instance minimal ones, have no residual scrambled sets. We now derive a deeper consequence from the same observations. A map is called *feebly open* if the image of any non-empty open set has non-empty interior (such maps are sometimes called *semi-open* or *quasi-interior*). The following characterization proves useful. It seems to be folklore (see the preprint [40] for stronger results) and at least in one direction it was already used in dynamics [32]; here is a short proof for completeness.

LEMMA 32. *A continuous self-map f of a compact metric space X is feebly open if and only if the f-preimage of every residual set is residual.*

Proof. Let f be a feebly open continuous self-map of a space X; in this direction compactness is not required. Let S be residual in X, i.e., $X \setminus S$ is a union of countably many nowhere dense sets. To prove that $f^{-1}(S)$ is residual it is therefore sufficient to show that the preimage of any nowhere dense set A is nowhere dense. Moreover, since the closure of a nowhere dense set is nowhere dense, assume that A is closed. Then, by continuity, the set $f^{-1}(A)$ is closed and since f is feebly open, this set has empty interior. Hence $f^{-1}(A)$ is nowhere dense.

Conversely, let X be compact metric and let f be a transformation of X such that any residual set has residual preimage. If U is a non-empty open set of X, let V be a non-empty open set such that $V \subseteq \mathrm{cl}(V) \subseteq U$. By compactness $f(\mathrm{cl}(V))$ is closed in X. Since $f^{-1}(X \setminus f(\mathrm{cl}(V))) \subseteq X \setminus \mathrm{cl}(V)$ and the last set is not residual, $X \setminus f(\mathrm{cl}(V))$ is not residual in X. So the closed set $f(\mathrm{cl}(V))$ has non-empty interior, and the same is true for $f(U)$. This shows that f is feebly open. ∎

Thus Proposition 31 applies in particular to all feebly open maps. Since a singleton is not considered to be a scrambled set and, by [5, 32], any minimal self-map of a compact metric space is feebly open, the scrambled sets of minimal systems cannot be residual, even when the map is not a homeomorphism. Proposition 33 tells more: the scrambled sets of a minimal dynamical system on a compact metric space are nowhere residual.

Recall that a system (X, f) is *non-wandering* or *regionally recurrent* if for every non-empty open set $U \subseteq X$ there exists $k \geq 1$ with $f^k(U) \cap U \neq \emptyset$ (equivalently, $U \cap f^{-k}U \neq \emptyset$). In such a case we also say that the map f itself is non-wandering.

PROPOSITION 33. *Let (X, f) be a compact dynamical system with the following three properties*:

(1) *f is feebly open*;
(2) *f is non-wandering*;
(3) *f has no periodic points.*

Then any scrambled set of (X, f) is nowhere residual. In particular minimal compact dynamical systems have only nowhere residual scrambled sets (*if any*).

Proof. By (1) and Lemma 32 the preimage of a residual set of X is residual in X; $f^{-1}(A)$ is also residual in $f^{-1}(B)$ whenever B is open and $A \subseteq B$ is residual in B.

Suppose U is a non-empty open subset of X, and $S \subseteq U$ is residual in U and scrambled. Since f is non-wandering, there exists $k > 0$ such that $U \cap f^{-k}U \neq \emptyset$. It follows that $f^{-k}S$ is residual in $f^{-k}U$, hence also residual in the non-empty open set $U \cap f^{-k}U$. Since S is also residual in this set,

$S \cap f^{-k}S \neq \emptyset$. Thus there is a point $x \in S$ such that $f^k(x) \in S$. Then using (S-5) from Subsection 2.1 and by Proposition 4(1) applied to f^k the set X contains a fixed point of f^k, which contradicts (3).

When (X, f) is minimal, f is feebly open by [5, 32]; this implies that the preimage of a residual set is residual. A minimal system is non-wandering. Finally, a minimal system may contain a periodic point but then X consists of just one periodic orbit and so has no scrambled set. The conclusion follows. ∎

Notice that (2) implies the surjectivity of f when X is compact. Transitivity implies (2).

In the way of examples, some transitive non-minimal systems satisfy all the assumptions of Proposition 33.

EXAMPLE 34. Suppose (X, f) is minimal and infinite and (Y, g) is a weakly mixing, non-minimal system with g feebly open (say, the tent map on the interval). The Cartesian product $(X \times Y, f \times g)$ is also feebly open and contains no periodic points because its factor (X, f) contains none. Being the product of a minimal system and a weakly mixing system, it is transitive and so non-wandering. Finally, since Y is not minimal neither is $X \times Y$.

The interval map $f(x) = \min\{1, 2x\}$ on the compact interval $[0, 1]$ does not satisfy any of these conditions and still all its scrambled sets are nowhere residual (it has none). Thus none of the conditions (1), (2) and (3) is necessary. One can also consider less trivial examples. Say, Example 49 shows that a non-feebly open system may have only nowhere residual scrambled sets. Full shifts have dense periodic points; nevertheless their scrambled sets are nowhere residual.

Thus, for a better understanding of Proposition 33 it is important to know whether it remains true when dropping one of the three assumptions. We can answer this question only partially, which means that the status of the proposition is still far from clear.

There are examples showing that Proposition 33 is no longer true in general when (1) and (2) hold but not (3). For instance (X, σ) in the proof of Proposition 55 is residually scrambled; σ is invertible, hence open, and the system is transitive, but X contains one periodic point. The situation is similar when (1) and (3) hold but not (2):

EXAMPLE 35. Let $\Sigma = \prod_{i=1}^{\infty}\{0, 1, 2\}$ be equipped with the standard totally disconnected product topology. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two elements of Σ their sum $x \oplus y = (z_1, z_2, \dots)$ is defined as follows. If $x_1 + y_1 < 3$, then $z_1 = x_1 + y_1$ and the carry is 0; if $x_1 + y_1 = 3$, then $z_1 = x_1 + y_1 - 3 = 0$ and we carry 1 to the next position. The other terms $z_2, \dots$ are successively determined in the same fashion. Let $T : \Sigma \to \Sigma$ be defined by $T(z) = z \oplus 1$ for $z \in \Sigma$, where $1 = (1, 0, 0, \cdots)$. T is known to be minimal. It is called an *adding machine*.

For $x \in \Sigma$, let $n(x) = \#\{i \in \mathbb{N} : x_i \neq 1\}$ and $h(x) = 2^{-n(x)}$, assuming $2^{-\infty} = 0$. Put $X = \bigcup_{x \in \Sigma} \{x\} \times [0, h(x)]$. Then X is a closed subset of $\Sigma \times [0, 1]$. Now define $f : X \to X$ by

$$f(x, th(x)) = (Tx, th(Tx)) \quad \text{for any } x \in \Sigma \text{ and } t \in [0, 1].$$

As a homeomorphism, f is feebly open; (3) holds because there exists a non-trivial minimal factor. The set $U = \{(1, 1, 1, \cdots)\} \times (1/2, 1]$ of X is open. It is a scrambled set of (X, f): the iterates of f map it to intervals of positive length over points of Σ, but the lengths oscillate between 0 and 1/4. On the other hand, (2) does not hold since no point $z \in U$ ever returns to its neighbourhood U under the action of f.

Question. Will Proposition 33 work if one removes the assumption (1)? Note that Example 54 is residually scrambled but none of (1), (2) and (3) holds.

6.2. Graph maps. Recall that on a real compact interval only nowhere residual scrambled sets may exist [12, 22]. What is the situation on graphs? By [38] scrambled sets have empty interiors. Using this result we prove a stronger result, Theorem 45: on graphs scrambled sets are nowhere residual. In contrast, recall that it is shown in [25] that there exists a completely scrambled compact dynamical system (X, f) with X a dendrite.

First recall that a *graph* is a metric continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints; in other words, it is a 1-dimensional compact connected polyhedron.

Recall that a point x in a graph G is called a *branch point* or *endpoint* if its order satisfies $\mathrm{ord}_G(x) \geq 3$ or $\mathrm{ord}_G(x) = 1$, respectively. Let $x \in G$ with $\mathrm{ord}_G(x) = v$. Then a closed neighbourhood $\overline{U}_x$ of x is said to be a *canonical neighbourhood* of x if it is of the form $J_1 \cup \cdots \cup J_v$ where for every i the set J_i is an arc for which x is an endpoint, J_i does not contain any branch point of G different from x (x may or may not be a branch point), and for $i \neq j$, $J_i \cap J_j = \{x\}$. In other words, if $v \geq 3$ or $v \leq 2$ then, respectively, $\overline{U}_x$ is just a simple v-od with vertex x or an arc which is a neighbourhood of x; in both cases $\overline{U}_x$ must be sufficiently small not to contain any branch point different from x.

The sets J_i are called *x-rays* (of $\overline{U}_x$). If J is an x-ray there is a (surjective) homeomorphism $J \to [0, 1]$ sending x to 0. When $f : G \to G$, $f(x) = y$ and an x-ray J is mapped by f into an y-ray, the claim that $f|_J$ is strictly increasing has the obvious sense (think of x-rays as linearly ordered sets with x being their minimum).

Obviously there are arbitrarily small canonical neighbourhoods of x. In the family of all x-rays of all canonical neighbourhoods of x define the relation $J \sim J' \Leftrightarrow J \subseteq J'$ or $J' \subseteq J$. This is an equivalence relation with $v = \mathrm{ord}_G(x)$ classes of equivalence which will be called *directions from* x. The direction from x containing the x-ray J is denoted by $[J]$. So, given a canonical neighbourhood $J_1 \cup \cdots \cup J_v$ of x, the directions from x are the classes $[J_1], \ldots, [J_v]$.

Let $f : G \to G$, $p, q \in G$ and $f(p) = q$. Let P be a direction from p and Q be a direction from q. We say that P is mapped by f to Q and we write $f(P) = Q$ if there are canonical neighbourhoods $\overline{U}_p$ and $\overline{U}_q$ of p and q, respectively, such that the p-ray of $\overline{U}_p$ belonging to the direction P is mapped *onto* that q-ray of $\overline{U}_q$ which belongs to the direction Q. Moreover, if the p-ray is mapped onto the q-ray in a strictly increasing way we say that the direction P is mapped to the direction Q in a strictly increasing way.

It may happen that a direction is not sent by f to any direction; however, f never sends a direction to two or more different directions. So f induces a function (denoted again by f) defined on a (possibly empty) subfamily of all directions, whose values are directions.

Let $f : G \to G$, $p, q \in G$ and $f(p) = q$. It is possible that no direction from p is mapped to a direction from q. We say that *p is mapped to q regularly* if every direction from p is mapped to a direction from q in a strictly increasing way and different directions from p are mapped to different directions from q. Note that in such a case $\mathrm{ord}_G(q) \geq \mathrm{ord}_G(p)$ and there are canonical neighbourhoods $\overline{U}_p$ and $\overline{U}_q$ such that $f|_{\overline{U}_p}$ is a homeomorphism $\overline{U}_p \to f(\overline{U}_p) \subseteq \overline{U}_q$.

The next two results concern graph maps that are one-to-one on residual subsets. Observe that a map is always one-to-one on a scrambled set.

LEMMA 36 ([12]). *Let I, J be real compact intervals and $f : I \to J$ be continuous. If f is one-to-one on a residual set $S \subseteq I$ then $f(S)$ is residual in $f(I)$ and f is strictly monotone on I (hence a homeomorphism $I \to f(I)$).*

Proof. The first part is proved in Proposition 4.1 in [12], the second part is then trivial (see the beginning of the proof of Theorem 4.2 in [12] or Lemma 3 in [22]). ∎

Of course this still holds when replacing the intervals by homeomorphic images, i.e., arcs. We still refer to Lemma 36 when having this analogue in mind.

LEMMA 37. *Let H, H' be two graphs and let $f : H \to H'$ be continuous. Let f be one-to-one on a residual set $S \subseteq H$. Then*:

(1) *If I_1 and I_2 are two non-overlapping arcs in H then $f(I_1) \cap f(I_2)$ has empty interior in H'.*

(2) *$f(I)$ is an arc in H' whenever I is an arc in H with sufficiently small diameter. In such a case $f|_I$ is a homeomorphism $I \to f(I)$.*
(3) *$f(S)$ is residual in $f(H)$.*

Proof. (1) If the interior of $f(I_1) \cap f(I_2)$ is nonempty then it contains an arc J such that for some (non-overlapping) arcs $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ one has $f(J_1) = f(J_2) = J$. By Lemma 36, the sets $f(S \cap J_1)$ and $f(S \cap J_2)$ are residual in J and hence intersect in a residual subset of J. This contradicts the injectivity of f on S.

(2) By uniform continuity of f, if $\operatorname{diam} I$ is small enough then $f(I)$ does not contain any circle and does not contain more than one branch point of H'. Then $f(I)$ is an arc or an n-od, $n \geq 3$. In the latter case some subarc J of that n-od is covered at least twice, i.e., there are non-overlapping arcs $I_1, I_2 \subseteq I$ with $f(I_1) = f(I_2) = J$; a contradiction with (1).

(3) By (2), one can cover H by finitely many arcs $I_1, \ldots, I_k$ such that for every i, $f|_{I_i} : I_i \to f(I_i)$ is a homeomorphism. Then $f(S) = \bigcup_{i=1}^k f(S \cap I_i)$ is residual in $\bigcup_{i=1}^k f(I_i) = f(H)$. ■

LEMMA 38. *Let H, H' be two graphs and let $f : H \to H'$ be continuous. Then*:

(1) *If f is one-to-one on a set S which is residual in a neighbourhood V of a point p, then p is mapped to $f(p)$ regularly and so, in particular,* $\operatorname{ord}_H(p) \leq \operatorname{ord}_{H'}(f(p))$.
(2) *If $H = H'$ and f is one-to-one on a set S which is residual in the whole graph H then the function* $\operatorname{ord}_H : H \to \mathbb{N}$ *is nondecreasing along the trajectories of f.*
(3) *If f is one-to-one on a set S which is residual in the whole graph H then even, for every $y \in H'$,*

$$\sum_{x \in f^{-1}(y)} \operatorname{ord}_H(x) \leq \operatorname{ord}_{H'}(y);$$

it follows that f is only finite-to-one.

Proof. (1) Write $q = f(p)$ and consider canonical neighbourhoods $\overline{U}_p \subseteq V$ and $\overline{U}_q$ of p and q respectively such that $f(\overline{U}_p) \subseteq \overline{U}_q$. Let J be a p-ray of $\overline{U}_p$. Since f is one-to-one on S, $f(J)$ is a nondegenerate connected subset of $\overline{U}_q$ containing the point q. In fact $f(J)$ is an arc with endpoint q, hence a subarc of one of the q-rays. To see this, suppose that $f(J)$ contains points b_1, b_2 different from q and lying in two different q-rays of $\overline{U}_q$. Then, moving with an argument along J, we start at the point p (which is mapped to q), then we get to a point which is mapped to, say, b_1 and then to a point which is mapped to b_2. Hence one can find two non-overlapping arcs A_1, A_2 in J which are mapped onto the same arc B (a subset of a q-ray).

This contradicts Lemma 37(1). Therefore $f(J)$ is a subset of a q-ray. By Lemma 36, f is strictly increasing on J.

(2) trivially follows from (1).

(3) Note first that by (1) any preimage of y is mapped to y regularly. Further, let $x_1 \neq x_2$ be preimages of y and $\overline{U}_{x_1}$, $\overline{U}_{x_2}$ and $\overline{U}_y$ be their canonical neighbourhoods with $\overline{U}_{x_1} \cap \overline{U}_{x_2} = \emptyset$ and $f(\overline{U}_{x_i}) \subseteq \overline{U}_y$, $i = 1, 2$. Let J_1 be any of the x_1-rays and J_2 be any of the x_2-rays in $\overline{U}_{x_1}$ and $\overline{U}_{x_2}$, respectively. Then we already know that $f(J_1)$ and $f(J_2)$ are subsets of y-rays in $\overline{U}_y$. Moreover, these two y-rays are different; otherwise there would be two disjoint arcs with the same image, which would create a contradiction as in the proof of (1). ∎

Easy examples show that if $f : H \to H'$ and $H \neq H'$ the assumptions of Lemma 37 do not imply that f is a homeomorphism. Nevertheless, we have

LEMMA 39. *Let G be a graph and $f : G \to G$ be continuous. Let f be one-to-one on a residual set $S \subseteq G$. Then f is a homeomorphism of G onto $f(G)$.*

Proof. It is sufficient to show that some iterate of f (hence also f itself) is injective on G.

If G is an arc use Lemma 36. If G is a simple closed curve then the injectivity of f follows from the inequality in Lemma 38(3) (in this case f is a homeomorphism of G onto $f(G) = G$ since the circle does not contain a homeomorphic image different from itself).

Finally, suppose that G has branch points. By Lemma 38(2) the function ord is non-decreasing along the trajectories of f. Therefore every branch point is mapped to a branch point and, as there are only finitely many branch points, eventually periodic. Since ord is constant along a periodic orbit, by Lemma 38(3) no periodic point has a preimage outside its orbit and so all branch points are in fact periodic. Thus there exists $k \geq 1$ such that all branch points are fixed for $g = f^k$. Fix a branch point b. The map g is injective on S and by Lemma 38(1), g maps b to b regularly. This means that f permutes the directions from b. Since there are only finitely many branch points and each of them has finite order, there is n such that all branch points and also all directions from all branch points are fixed under the map $h = g^n$.

We show that the map $h = f^{kn}$ which is injective on S is in fact injective on G.

Let B be the non-empty set of branch points of G. Each point of B is fixed for h and so $h(B) = B$. Applying the inequality in Lemma 38(3) to points of B it follows that $h(G \setminus B) \subseteq G \setminus B$. Let K be a connected component of $G \setminus B$. Since $\overline{K} \setminus K$ consists of one or two branch points and every branch point is mapped by h regularly to itself, $h(K) \cap K \neq \emptyset$. This together with the fact that $h(K)$ is a connected subset of $G \setminus B$ implies that $h(K) \subseteq K$ and $h(\overline{K}) \subseteq \overline{K}$. The set $\overline{K}$ is an arc (at least one of whose endpoints is a branch point) or a simple closed curve (containing exactly one branch point) and so, by what we have already proved at the very beginning of this proof, h maps $\overline{K}$ homeomorphically to (not necessarily onto) itself. Since branch points are fixed, K is also mapped homeomorphically to itself by h. Therefore h is injective on G; hence so is f. ∎

LEMMA 40. *Let G be a finite graph and $f : G \to G$ be continuous. Then no scrambled set of (G, f) is residual in G.*

Proof. By [22] this is true if G is an arc. Assume that G is a simple closed curve, say the circle $\mathbb{S}^1$, and that f has a residual scrambled set, on which it is one-to-one. By Lemma 39, f is a homeomorphism of $\mathbb{S}^1$ onto itself and by Proposition 31 it has a fixed point. Then f^2 is a degree 1 homeomorphism with a fixed point and so it can be considered to be an interval map. Now homeomorphisms on the interval do not have scrambled sets.

If the set B of branch points of G is not empty, let K be a component of $G \setminus B$. By Lemma 39, f is a homeomorphism and as in the proof of Lemma 39 one has $f^n(\overline{K}) \subseteq \overline{K}$ for an appropriate iterate. Since $\overline{K}$ is an arc or a simple closed curve, no scrambled set of f^n can be residual in $\overline{K}$. ∎

We want to strengthen Lemma 40 by proving that scrambled sets on graphs are in fact nowhere residual. To do this we use two weaker results: Lemma 40 and Mai's result that scrambled sets on graphs have empty interiors [38]. Without them it does not seem easy to find a direct proof. We start by proving that a scrambled set of a graph map cannot be residual in a circle. Equivalently, we have

LEMMA 41. *Let G be a finite graph and $f : G \to G$ be continuous. Let W be a subgraph of G such that the graph W has no endpoint. Then no scrambled set of (G, f) is residual in W.*

Proof. Consider the (obviously finite) sets

$$\mathcal{A} = \{W : W \text{is a subgraph of } G \text{ and } W \text{ has no endpoint as a graph}\},$$

$$\mathcal{A}_s = \{W \in \mathcal{A} : \text{there is a scrambled set } R \text{ of } (G, f) \quad \text{such that } R \cap W \text{ is residual in } W\}.$$

Assume on the contrary that $\mathcal{A}_s \neq \emptyset$. Fix $W \in \mathcal{A}_s$. Then there is a residual set S in W which is scrambled for (G, f). Hence the set $f(W)$ is non-degenerate and obviously it is a subgraph of G. Since $\mathrm{ord}_W(x) \geq 2$ for all $x \in W$, it follows from Lemma 38(1) applied to $f|_W : W \to f(W)$ that $f(W) \in \mathcal{A}$. By Lemma 37(3) the set $f(S)$ is residual in $f(W)$ and since $f(S)$ is scrambled for (G, f), we see that $f(W) \in \mathcal{A}_s$. This shows that $\mathcal{A}_s$ is a non-empty finite f-invariant set, and there exist $V \in \mathcal{A}_s$ and $n \in \mathbb{N}$ such that $f^n(V) = V$. Then $(V, f^n|_V)$ is a residually scrambled system, which contradicts Lemma 40. ■

We must still show that a scrambled set of a graph map cannot be residual in an arc. First we prove the following lemma (notice that it is not sufficient to assume that f is one-to-one on S):

LEMMA 42. *Let G be a finite graph and $f : G \to G$ be continuous. Let I be an arc in G and let the scrambled set $S \subseteq I$ be residual in I. Then $f(I)$ is an arc and $f|_I : I \to f(I)$ is a homeomorphism.*

Proof. We may think of I as the interval $[0, 1]$. Applying Lemma 37(2) to the map $f|_I : I \to G$ we find that if $\delta > 0$ is small enough then $f|_{[0,\delta]}$ maps $[0, \delta]$ homeomorphically onto an arc in G. Let δ_0 be the supremum of all such δs. Then f is one-to-one on $[0, \delta_0] \setminus \{\delta_0\}$. If f is one-to-one on $[0, \delta_0]$ the lemma follows; if $f(\delta_0) = f(x)$ for some $0 \le x < \delta_0$ then $f([x, \delta_0])$ is a circle K in G and by Lemma 37(3) the set $f(S) \cap K$ is residual in K. Since $f(S) \cap K$ is a scrambled set of f, this contradicts Lemma 41. ■

LEMMA 43. *Let C be a circle and $g : C \to C$ be a homeomorphism. Then g has no scrambled pair.*

Proof. It is sufficient to show that $f := g^2$ has no scrambled pair. Here f is an orientation-preserving, i.e., degree one, homeomorphism. If its rotation number is rational it is known that every non-periodic point is asymptotic to a periodic point and so there is no scrambled pair. If the rotation number of f is irrational then by Poincaré's classification theorem either f is topologically conjugate to an irrational rotation (hence without scrambled pairs), or f has a factor which is an irrational rotation; the factor map can be chosen to be monotone. In the latter case there is a nowhere dense set E such that $f|_E$ is conjugate to an irrational rotation and the lengths of contiguous (wandering) intervals go to zero under iterates of f. This shows that, again, there is no scrambled pair. ■

LEMMA 44. *Let G be a finite graph and $f : G \to G$ be continuous. Let J be an arc in G with endpoints c, d. Suppose that a scrambled set S of f is residual in J. Then for any increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers, if $\lim_{i\to\infty} d(f^{n_i}(c), f^{n_i}(d)) = 0$ then $\lim_{i\to\infty} \operatorname{diam}(f^{n_i}(J)) = 0$.*

Proof. Since S is scrambled also for all iterates of f, by Lemma 42 $f^n|_J : J \to f^n(J)$ is a homeomorphism for any $n \in \mathbb{N}$. Hence

(1) for any $n \in \mathbb{N}$, $f^n(J)$ is an arc with endpoints $f^n(c)$ and $f^n(d)$ and contains a scrambled set of f residual in $f^n(J)$,
(2) for every $r, s = 0, 1, 2, \ldots$, the map $f^s|_{f^r(J)} : f^r(J) \to f^{r+s}(J)$ is a homeomorphism.

To prove the lemma, assume that $\operatorname{diam}(f^{n_i}(J))$ does not converge to zero. Passing to a subsequence if necessary, assume that there exists $\varepsilon > 0$ such that $\operatorname{diam}(f^{n_i}(J)) \geq \varepsilon$ for all n_i. On the other hand, note that for the distances of the endpoints of the arcs $f^{n_i}(J)$ we have $d(f^{n_i}(c), f^{n_i}(d)) \to 0$. The last two properties are contradictory if G is a tree. So G contains circles and it is not hard to see that for i large enough there exist circles $C_i \subseteq G$ with

(3) $C_i \cap f^{n_i}(J)$ is an arc,
(4) $\lim_{i\to\infty} \operatorname{diam}(C_i \setminus f^{n_i}(J)) = 0$.

Without loss of generality suppose that this holds for all i (and not only for i large enough). Obviously

(5) the set $f^{n_i}(S)$ is residual in $C_i \cap f^{n_i}(J)$ and scrambled for f.

Since there are only finitely many circles in G, again by passing to a subsequence of (n_i) if necessary, we may assume that $C_1 = C_2 = \cdots$. By (2), for any $m \in \mathbb{N}$ the map f^m, when restricted to any of the arcs $C_1 \cap f^{n_i}(J)$, is a homeomorphism onto its range. We claim that

(6) there exists a circle D such that it contains a scrambled set of f which is residual in some arc on this circle, and moreover, for all m, f^m is one-to-one on D (hence a homeomorphism $D \to f^m(D)$).

Let us prove (6). The sets $C_1 \setminus f^{n_i}(J)$, $i = 1, 2, \ldots$, are arcs in C_1 by (2) and their diameters tend to zero by (4). Again passing to a subsequence, assume that these arcs converge to a point $x_0 \in C_1$. In view of (2) this implies that all the iterates f^m are one-to-one at least on $C_1 \setminus \{x_0\}$: any two points different from x_0 belong to a set $C_1 \cap f^{n_i}(J)$ for some i. Now distinguish two cases.

Case 1. For infinitely many of the numbers n_i the set $C_1 \cap f^{n_i}(J)$ contains x_0. Then the same simple argument as above shows that all the iterates f^m are one-to-one on the whole circle C_1 and we are done by putting $D = C_1$ and taking (1) into account.

Case 2. The set $C_1 \cap f^{n_i}(J)$ contains x_0 for finitely many i only. Then we may assume that *none* of the sets $C_1 \cap f^{n_i}(J)$ contains x_0. When i tends to infinity, the arc $f^{n_i}(J)$ covers C_1 except a smaller and smaller neighbourhood of x_0. The branch points different from x_0 (x_0 may or may not be a branch point) have positive distance from x_0 and so for i large enough $f^{n_i}(J) \subseteq C_1$ (because $f^{n_i}(J)$ is an arc). We may assume that this is the case for all i.

Suppose that for some m_0, the map f^{m_0} is not injective on C_1. Since it is injective on $C_1 \setminus \{x_0\}$, $f^{m_0}(x_0) = f^{m_0}(x) =: z$ for some $x \in C_1 \setminus \{x_0\}$. Notice that $f^{m_0}(C_1)$ is a figure eight, i.e., two circles having in common just the point z. Let D be one of these two circles. Now we come to the key point: since *all* the iterates of f are injective on $C_1 \setminus \{x_0\}$, all the iterates of f are injective on D (for if f^k were not injective on D then f^{m_0+k} would not be injective on $C_1 \setminus \{x_0\}$). To show that D contains a scrambled set of f which is residual in some arc on D, use (1). This proves (6).

A consequence is that $D, f(D), f^2(D), \dots$ are circles containing somewhere residual scrambled sets of f. Since there are only finitely many circles in G, there exist $m_1 < m_2$ such that $f^{m_1}(D) = f^{m_2}(D) =: C$. This implies $f^k(C) = C$ for $k = m_2 - m_1$. Let $g = f^k|_C$. Then C is a circle, g is a homeomorphism of C, and there exists a scrambled set E of g that is residual in some arc on C. This contradicts Lemma 43. ■

At last we are ready to prove

THEOREM 45. *Let G be a finite graph and $f : G \to G$ be continuous. Then any scrambled set of (G, f) is nowhere residual.*

Proof. Suppose on the contrary that there exist a scrambled set S of (G, f) and a non-empty open set U of G such that S is residual in U. Let B be the set of branch points of G. Then there exists an arc I such that $I \subseteq U \setminus B$ and the two endpoints c, d of I belong to S. The set S is obviously residual in I.

Since $c, d \in S$ form a proximal pair of (G, f), by Lemma 44 any two different points in I are proximal. On the other hand, two different points $x, y \in I$ cannot be asymptotic: otherwise again by Lemma 44 any two different points between x and y would also be asymptotic, which contradicts the residuality of S in I. Then I is a scrambled set of (G, f). This is impossible: graph maps may only have scrambled sets with empty interior (see [37]). ■

6.3. Symbolic systems and cellular automata. We show that a large class of invertible symbolic systems are not residually scrambled; in many transitive cases scrambled sets are nowhere residual. The class contains subshifts of finite type, sofic systems, synchronising systems and more. Results are easy to transpose to the one-sided versions of the subshifts. This is not completely surprising, considering the fact that the tent map is an interval map but has a very satisfactory coding by the one-sided full 2-shift. What could not be foreseen is that the result extends to a considerably larger class.

In the end we show that some cellular automata are not residually scrambled. The arguments are essentially the same as for subshifts, with one major difference: distality is used instead of asymptoticity.

Here are some definitions and notation for symbolic systems. Let A be a finite alphabet, and let A^* be the set of all finite sequences of symbols of A, or words on A, including the empty word; denote by A^+ the set of all non-empty words; write $|w|$ for the length of the word w. If $x \in A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$ is a symbolic sequence on A, denote by x_i the ith coordinate of x; by $x_{i,j}$ the block formed by the coordinates of x from i to j, $-\infty \leq i < j \leq +\infty$ ($0 \leq i < j \leq +\infty$ if $x \in A^{\mathbb{N}}$). The shift σ is defined on $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$ by $(\sigma(x))_i = x_{i+1}$ for i belonging to the suitable set of indices. $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ are called the *two-sided* and the *one-sided full shifts* on A respectively.

A *subshift* on A is a closed shift-invariant subset of the compact spaces $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. If X is a subshift define $L(X)$ as the set of words occurring as blocks of coordinates in elements of X. Given a word $u \in L(X)$ one defines the cylinder set associated to u starting at i as $[u]_i = \{x \in X : x_{i,i+|u|} = u\}$; by the definition of $L(X)$ this subset of X is not empty.

Let $X \subseteq A^{\mathbb{Z}}$ be a subshift and let $w \in L(X)$. Given $x, y \in A^{\mathbb{Z}}$ denote by $z = x_{-\infty,0}.(w)_0.y_{|w|,+\infty}$ the point of $A^{\mathbb{Z}}$ such that $z_{-\infty,0} = x_{-\infty,0}$, $z_{0,|w|-1} = w$ and $z_{|w|,+\infty} = y_{|w|,+\infty}$. A *context* of w is a pair of infinite blocks $(x_{-\infty,0}, y_{|w|,+\infty})$ with $x, y \in X$ such that the point $x_{-\infty,0}.(w)_0.y_{|w|,+\infty}$ belongs to X. In other words, a context of w consists of one left-infinite string and one right-infinite string such that if one inserts w between them, the result forms a point of X. Denote by $C(w)$ the set of all contexts of w; the cylinder set $[w]$ is equal to the set $\{x_{-\infty,0}.(w)_0.y_{|w|,+\infty} : (x_{-\infty,0}, y_{|w|,+\infty}) \in C(w)\}$. The set $C(w)$ depends on X, but in this article there is no risk of confusion.

When $X \subseteq A^{\mathbb{Z}}$, a right (left) context of u is defined analogously as a right-infinite (left-infinite) string that can be concatenated to u in X. So are the sets $C_r(u)$ and $C_l(u)$ of right and left contexts of u. Right contexts can be defined for $X \subseteq A^{\mathbb{N}}$, which is not the case of contexts.

The proof of the following result contains the basic tools for this subsection.

PROPOSITION 46. *Let (X, σ) be a subshift of $A^{\mathbb{Z}}$ (resp. $A^{\mathbb{N}}$). If there are two distinct words $u, v \in L(X)$ with $|u| = |v|$ and $C(u) = C(v)$ (resp. $C_r(u) = C_r(v)$), then a scrambled set cannot be residual in $[u]$ and $[v]$, and (X, σ) is not residually scrambled.*

Proof. The proof is given for a two-sided subshift; there is hardly any difference in the one-sided case.

Define a map $\phi : X \to X$ as follows: when $x \in [u]$ change $x_{0,|u|} = u$ to v, if $x \in [v]$ change $x_{0,|u|} = v$ to u, and leave all other coordinates of x unchanged; outside these two cylinder sets let ϕ be the identity. Because u and v have the same sets of contexts, $\phi(x) \in X$ on $[u] \cup [v]$. The map ϕ depends only on a finite set of coordinates, which implies that it is continuous; as an involution it is a homeomorphism; finally, when $x \in [u] \cup [v]$, ϕ changes finitely many coordinates and at least one, so that in this case the pair $(x, \phi(x))$ is properly asymptotic.

Suppose $A \in X$ is residual. Then it is residual in the two open sets $[u]$ and $[v]$; because ϕ is a homeomorphism, $\phi(A \cap [v])$ is residual in $[u]$, so that it intersects $A \cap [u]$. Let x belong to the intersection; then $x = x_{-\infty,0}.(u)_0.x_{|u|,+\infty}$ and $\phi x = x_{-\infty,0}.(v)_0.x_{|v|,+\infty}$ both belong to A. They are properly asymptotic, so A is not scrambled. ∎

Proposition 46 is the most general result we could obtain with the present method. Its assumption holds for a large class of subshifts, a claim we illustrate presently.

Let $X \subset A^{\mathbb{Z}}$ be a subshift. The word $w \in L(X)$ is said to be a *synchronising word for* X if

$$\forall u, v \in L(x) \text{ such that } uw \in L(X) \text{ and } wv \in L(X), \text{ one has } uwv \in L(X);$$

this definition is also valid for one-sided subshifts. In the full shifts $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ any word is synchronising.

Assuming again $X \subseteq A^{\mathbb{Z}}$, by compactness this property is equivalent to

$$\forall x^- \in C_l(w),\ y^+ \in C_r(w), \text{ one has } x^- w y^+ \in X.$$

A symbolic system is said to be a *synchronising system* if it is transitive and has a synchronising word. Full shifts, transitive subshifts of finite type, transitive sofic systems are synchronising, but there are many more.

Here by a *periodic* symbolic system we mean a finite union of periodic orbits. It is not hard to check that a synchronising system has a dense set of periodic points, and that if it is not periodic it has positive topological entropy. These properties imply that a non-periodic synchronising system is Li–Yorke chaotic.

COROLLARY 47. *Let (X, σ) be a non-periodic synchronising subshift, and let $A \subseteq X$ be scrambled. Then A is nowhere residual.*

Proof. The proof is given for a two-sided subshift; it is also valid for a one-sided subshift. We have to prove that A cannot be residual in any cylinder set $[u]_j$ for $u \in L(X)$ and $j \in \mathbb{Z}$. So suppose that there exist $u \in L(X)$ and $j \in \mathbb{Z}$ such that $A \cap [u]_j$ is residual in $[u]_j$.

Let w be a synchronising word of (X, σ). Since (X, σ) is transitive, there exists $v \in L(X)$ such that $wvw \in L(X)$. Set $l = |vw|$. Put $x \in X$ with $x_{kl,kl+l-1} = vw$ for any $k \in \mathbb{Z}$. Then $\sigma^l(x) = x$, i.e., $\mathrm{orb}(x, \sigma)$ is a periodic orbit. Let $L(X, x) = \{x_{i,j} : -\infty < i \le j < \infty\}$. Since (X, σ) is non-periodic, $L(X) \setminus L(X, x) \neq \emptyset$. Take $e \in L(X) \setminus L(X, x)$. Since (X, σ) is transitive, there exist $c_1, c_2, c_3 \in L(X)$ such that $wc_1uc_2ec_3w \in L(X)$. Put $v_1 = c_1uc_2ec_3$ and $k = |v_1w|$. Then $v_1 \in L(X) \setminus L(X, x)$ and $wv_1w \in L(X)$.

Since w is synchronising and $wvw, wv_1w \in L(X)$, the two words $a := wv_1w(vw)^k$ and $b := wv_1w(v_1w)^l$ belong to $L(X)$. Obviously $|a| = |b|$, $[a]_i \cup [b]_i \subseteq [u]_j$, where $i = j - (|w| + |c_1|)$. Since $v_1 \notin L(X, x)$ and $(vw)^k \in L(X, x)$,

$a \neq b$. Since $|a| = |b|$ and the synchronising word w appears at both ends of a and b, $C(a) = C(b)$. Since A is residual in $[u]_j$, A is also residual in both non-empty open sets $[a]_i$ and $[b]_i$. As $C(a) = C(b)$ and $a \neq b$, one can define an involution ϕ on $[a]_i \cup [b]_i$ as in the proof of Proposition 46. Then $A \cap [a]_i$ and $\phi(A) \cap [a]_i$ have non-empty intersection, and this implies that A contains a proper asymptotic pair. Therefore a scrambled set A cannot be residual in $[u]_j$. ∎

The synchronising property is much stronger than the assumption of Proposition 46. Obviously transitivity is not necessary: when X is the union of two non-periodic synchronising systems its scrambled sets are nowhere residual; with the help of elementary techniques from symbolic dynamics it is not hard to prove that scrambled sets of non-transitive subshifts of finite type or sofic systems are also nowhere residual. A deeper reason is that the existence of two different words having the same length and contexts is a weaker assumption than that of a synchronising word that connects to itself in two different ways after n iterations. Together, the following statement and example illustrate this fact.

PROPOSITION 48. *The shift action on a one-sided synchronising subshift $X \subseteq A^{\mathbb{N}}$ is feebly open* (*so*, *the preimage of a residual set under σ is residual*).

Proof. Let $u = au' \in L(X)$, where $a \in A$. By assumption there exist v and a synchronising word w with $uvw \in L(X)$; this implies in particular that the set of points $y \in X$ that can be concatenated to uvw is the same as the corresponding set for $u'vw$, in other words, $\sigma([uvw]) = [u'vw]$, which means that $\sigma([u])$ contains the non-empty open set $[u'vw]$. ∎

In contrast with this result, there are Li–Yorke chaotic subshifts having no residual scrambled sets, while in general the preimage of a residual set under the map is not residual. Not being feebly open, these systems are not synchronising.

EXAMPLE 49. The construction is in two steps. We first construct a one-sided subshift X_E on $\{0,1\}$, containing a residual subset whose preimage under σ is not residual (in view of Lemma 32 this is equivalent to finding an open set having an image with empty interior; in this case it is easy to find one). Then we construct a symbolic extension Y_E of X_E having the same property, and which contains no residual scrambled sets. Both X_E and Y_E are Li–Yorke chaotic because they are weakly mixing.

For $E \subseteq \mathbb{N}$ put

$$X_E = \{x \in \{0,1\}^{\mathbb{N}} : x_i = 1,\, x_j = 1,\, i < j \Rightarrow j - i \in E\}.$$

(X_E, σ) is a dynamical system because X_E is closed and shift-invariant.

Observe that X_E is never empty (it always contains the fixed point 0); the shift acts surjectively on X_E (if $x \in X_E$ then $0.x \in X_E$ is in the preimage of x); and that $L(X_E) = \{w \in \{0,1\}^* : w_i = 1,\, w_j = 1,\, i < j \Rightarrow j - i \in E\}$, since if w satisfies this condition, the cylinder set $[w]_0$ contains the point $w.0^\infty$. Let us mention without details that if E is thick, i.e., contains intervals of unbounded length, then X_E is weakly mixing. This relies upon a well-known characterization of weak mixing in [19].

Suppose E is thick, i.e., contains intervals of unbounded length, but its complement E^c is infinite. We construct a dense open subset U of X_E, the preimage of which is of course open but not dense. This means that the preimage of at least one residual subset of X_E under the shift is not residual.

Consider $w \in L(X_E)$ and let N be the right endpoint of a finite interval $I \subseteq E$ such that $|I| \geq |w|$, that is, $N \in E$ but $N+1 \notin E$; such intervals I and endpoints N always exist because E is thick but not cofinite. Then $w' = w0^{N-|w|}1 \in L(X_E)$, because the distance between any occurrence of 1 in the prefix w of w' and its final 1 is always in I. But $1w' \notin L(X_E)$ since $N+1 \notin E$. This means that $\sigma^{-1}([w']_0) \cap [1]_0 = \emptyset$. Now let w vary in $L(X_E)$. Put $U = \bigcup_{w \in L(X_E)} [w']_0$. All the sets $[w']_0$ are open and so is their union U. The density of U results from the facts that $[w']_0 \subseteq U \cap [w]_0$ is never empty for $w \in L(X_E)$, and that the sets $[w]_0$ form a base of neighbourhoods for X_E. On the other hand,

$$\sigma^{-1}(U) \cap [1]_0 = [1]_0 \cap \bigcup_{w \in L(X_E)} \sigma^{-1}[w']_0 = \bigcup_{w \in L(X_E)} (\sigma^{-1}[w']_0 \cap [1]_0) = \emptyset,$$

which implies that $\sigma^{-1}(U)$ is not dense. This completes the first step.

Let again $E \subseteq \mathbb{N}$ be thick but not cofinite and consider the one-sided subshift (Y_E, σ) on the alphabet $\{r, b, 1\}$ with the following properties: collapsing r and b to 0 sends Y_E onto X_E, so (X_E, σ) is a factor of (Y_E, σ) and all preimages of forbidden cylinders of X_E are forbidden in Y_E; moreover the words rb and br are forbidden in Y_E. Thus to obtain a point of Y_E, one takes a point of X_E and paints each of its strings of 0s *uniformly* red (r) or blue (b) in an arbitrary way. The two cylinder sets $[1r^i1]$ and $[1b^i1]$ are not empty whenever $i+1 \in E$, and one easily checks that $C([1r^i1]) = C([1b^i1])$ so that Proposition 46 applies and (Y_E, σ) is not residually scrambled (it is not hard to adapt the proof to deduce that scrambled sets are nowhere residual). On the other hand, Y_E contains a dense open set, the preimage of which under σ is not dense; the proof is the same as for (X_E, σ), up to evident changes. The fact that (Y_E, σ) is weakly mixing can be deduced easily from the similar property of (X_E, σ).

With the same arguments it is not hard to show that scrambled sets of (Y_E, σ) are nowhere dense.

Here is a further example. Let E be a thick subset of $\mathbb{N}$ that is not cofinite; then X_E is weakly mixing. If (Y, σ') is a non-trivial minimal strongly mixing subshift the product system $(X_E \times Y, \sigma \times \sigma')$ contains, like (X_E, σ), a residual set with non-residual preimage; being the product of one transitive and one strongly mixing system, it is transitive. Finally, by Proposition 33 the scrambled sets of (Y, σ') are nowhere residual; neither are those of the product system.

Finally, it can be proved that some cellular automata have no residual scrambled sets. The method of the proof is the same as for Proposition 46, except that the absence of residual scrambled sets is established with the help of distal, instead of asymptotic, pairs. A one-dimensional *cellular automaton* (CA) is a continuous shift-commuting self-map of $A^{\mathbb{Z}}$, where A is a finite set. CA are exactly those transformations of $A^{\mathbb{Z}}$ such that there exist an integer $r \geq 0$ and a so-called *local map* $f : A^{2r+1} \to A$ with

$$(F(x))_i = f(x_{-r+i}, \ldots, x_i, \ldots, x_{r+i}) \quad \text{for } x \in A^{\mathbb{Z}}.$$

PROPOSITION 50. *Let A be a finite group, and let F be a CA with local map f of the form $f(x_0, \ldots, x_r) = x_0 + g(x_1, \ldots, x_r)$, where g is any map from A^r to A. Then $(A^{\mathbb{Z}}, F)$ is not residually scrambled.*

Proof. Define $\phi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ as the continuous homeomorphism such that $(\phi x)_0 = x_0 + a$, where $a \in A$, $a \neq 0$, and $(\phi x)_i = x_i$, $i \neq 0$.

Obviously by induction $(F^n(\phi x))_0 = (F^n(x))_0 + a$, so $d(F^n(\phi x), F^n x) \geq \delta > 0$ for all n, which means that the pair $(x, \phi(x))$ is distal for any x. When $S \subset A^{\mathbb{Z}}$ is residual, S and $\phi(S)$ have non-empty intersection, which implies that S contains a distal pair. ∎

This class of CA contains some one-sided linear CA (those for which f is a linear combination of a finite number of coordinates), but also many non-linear ones, for instance the Coven cellular automata [15, 11], which exhibit various dynamical properties.

QUESTION. Example 49 and others that we cannot describe here suggest that there are large classes of transitive, not necessarily synchronising, symbolic systems for which one can prove that their scrambled sets are nowhere residual. Can one describe such a class? Proposition 55 shows that a transitive symbolic system may be residually scrambled: this sets a limit to what can be expected in this direction.

6.4. Second category scrambled sets. In view of Proposition 33, Theorem 45 and Proposition 46 it is natural to ask whether a minimal map, a graph map or a synchronising subshift may have second category scrambled sets. In what follows we answer these questions positively, but for this we have to assume the Continuum Hypothesis. The following is obvious:

LEMMA 51. *Let X be a Baire space. Suppose that $S \subseteq X$ is nowhere residual and has the Baire property. Then S is a first category set.*

Thus if S is a scrambled set for a minimal map in a compact metric space and S has the Baire property then S is first category. The next lemma is implicitly contained in [12].

LEMMA 52. *Let X be a Polish space without isolated points and let $R \subseteq X^2$ be residual in X^2 and symmetric (i.e., $(y,x) \in R$ whenever $(x,y) \in R$). Assuming the Continuum Hypothesis, there exists an everywhere second category set $S \subseteq X$ such that $S \times S \setminus \Delta \subseteq R$.*

Proof. Since R is a residual subset of the square of a Polish space, by Proposition 20 there is a residual set $A \subseteq X$ such that for every $x \in A$ there is a residual set $A_x \subseteq X$ such that $\{x\} \times A_x \subseteq R$. As A and A_x are residual sets, so is $A_x \cap A$. There is no loss of generality in supposing $A_x \subseteq A$.

Assume the Continuum Hypothesis. Let Ω be the first uncountable ordinal and let

$$G_0, G_1, \ldots, G_\alpha, \ldots \qquad (\alpha < \Omega)$$

be the second category sets of type G_δ in X. Since A is residual and G_0 second category, there is $x_0 \in G_0 \cap A$. Since X has no isolated point, every singleton in X is nowhere dense, $A_{x_0} \setminus \{x_0\}$ is residual and we can choose $x_1 \in G_1 \cap A_{x_0} \setminus \{x_0\}$. Similarly choose $x_2 \in G_2 \cap (A_{x_0} \cap A_{x_1}) \setminus \{x_0, x_1\}$. Suppose that for all $\beta < \alpha$ we have chosen $x_\beta \in G_\beta \cap \bigcap_{\gamma<\beta} A_{x_\gamma} \setminus \{x_\gamma : \gamma < \beta\}$. To find x_α, consider the set

$$S_\alpha = \Big(G_\alpha \cap \bigcap_{\gamma<\alpha} A_{x_\gamma}\Big) \setminus \{x_\gamma : \gamma < \alpha\} .$$

As a countable intersection of dense G_δ sets, S_α is a dense G_δ. Choose $x_\alpha \in S_\alpha$ and put $S = \{x_\alpha : \alpha < \Omega\}$. The fact that S intersects every second category G_δ set implies that S is everywhere second category. Indeed, suppose that S is first category in an open ball B; then $B \setminus S$ is residual in B and contains a G_δ set C, dense in B. Since B is open and non-empty and $C \subseteq B$ is second category and G_δ in B, C is also second category and G_δ in X but S is disjoint from C, which is a contradiction. Finally, fix $(x,y) \in S \times S \setminus \Delta$. Then $x = x_\alpha$ and $y = x_\beta$ for some $\alpha \neq \beta$. If $\alpha < \beta$ then $x_\beta \in A_{x_\alpha}$ and so $(x_\alpha, x_\beta) \in R$. If $\alpha > \beta$ then similarly $(x_\beta, x_\alpha) \in R$ and by symmetry of R again $(x,y) \in R$. ∎

A system (X,f) is called *generically chaotic* if scrambled pairs are residual in $X \times X$.

PROPOSITION 53. *Let (X,f) be generically chaotic with X Polish, and assume the Continuum Hypothesis. Then there exists an everywhere second* *category scrambled set for f. In particular*, *in the class of* compact (metric) *systems we have*, *under CH*:

(1) *Non-trivial weakly mixing systems have everywhere second category scrambled sets.*
(2) *A non-trivial minimal system has an everywhere second category scrambled set if and only if it is weakly mixing.*
(3) *All scrambled sets of a minimal non-periodic system (X, f) are nowhere dense if and only if the maximal distal factor of (X, f) is not periodic.*

Proof. Generic chaos implies that X has no isolated points. Put $R = \mathrm{SR}(X, f)$ and apply Lemma 52.

(1) Weak mixing for non-trivial systems implies generic chaos (pairs (x, y) that are transitive for $f \times f$ are residual in $X \times X$ and scrambled)

(2) For one implication use (1). Conversely, if a compact metric minimal system (X, f) has an everywhere second category scrambled set, by Proposition 28 its maximal distal factor (Y, g) has an isolated point. Since (Y, g) is minimal, this implies that it is just a periodic orbit. Suppose that its period is at least 2. Then (X, f) consists of several non-intersecting clopen sets which are permuted by f. This contradicts the assumption that (X, f) has *everywhere* second category scrambled set. We have thus proved that the maximal distal factor, hence the maximal equicontinuous factor, of (X, f) is trivial. It follows that (X, f) is weakly mixing by Proposition 29.

(3) This follows from Proposition 30. ∎

This result also implies existence of positive-entropy systems having second category but no residual scrambled sets. Any weakly mixing system with positive entropy which is not residually scrambled has this property, for instance minimal weakly mixing systems with positive entropy (by Proposition 33), full shifts (by Proposition 46) or the tent map.

7. SYSTEMS WITH RESIDUAL SCRAMBLED SETS

In Section 6 above we gathered all we know about dynamical systems that cannot be residually scrambled. Here we describe some residually scrambled systems.

As mentioned above a variety of completely scrambled dynamical systems are known to exist [37, 38, 25, 27]: by definition a completely scrambled system (X, f) is one such that X itself is a scrambled set. A completely scrambled system is proximal, so by Proposition 4 its unique minimal subset is a singleton when X is compact. It is invertible, therefore feebly open. A symbolic system is never completely scrambled; indeed, a symbolic system contains proper asymptotic pairs unless it is the union of finitely many periodic orbits (then it is distal). For a similar reason the entropy of a completely scrambled compact system is 0; Proposition 3 in [10] states that when X is compact and $h(f) > 0$ the set of points $x \in X$ such that there is an asymptotic pair (x, y) with $y \neq x$ has measure 1 for any positive-entropy measure. Finally, a completely scrambled, weakly mixing system is constructed in [27] but no strongly mixing example is known.

Call a system *residually scrambled* if it has a residual scrambled set. To our knowledge this weaker notion has not been previously investigated. Here we give various examples. Some of them show that unlike completely scrambled systems, residually scrambled systems are not necessarily proximal, invertible or even feebly open; they can be strongly mixing or have positive entropy; there are symbolic residually scrambled systems. Finally, the technique used for constructing a residually scrambled positive-entropy system also permits one to construct, given any minimal distal system, a completely scrambled system that is not disjoint from it.

Let us start with a rather simple example:

EXAMPLE 54. Let (X, f) be completely scrambled; f is thus a homeomorphism and has a unique fixed point q. Let $S = \mathbb{N} \cup \{p\}$ be disjoint from X and endowed with the topology such that each point of $\mathbb{N}$ is isolated and p is the limit of the sequence (n) as $n \to \infty$. The map g is defined on $Y = X \cup S$ as follows: $g(x) = f(x)$ for $x \in X$; $g(p) = p$; $g(n) = n - 1$ for $n > 1$, and $g(1)$ is some point $x_1 \in X \setminus \{q\}$.

(Y, g) contains the two fixed points p and q. One checks easily that the set $Y \setminus (\{p\} \cup \{f^{-n}(x_1) : n \in \mathbb{N}\})$ is scrambled and residual. The preimage of a residual set is not necessarily residual: let $E \subseteq Y$ be such that $\mathbb{N} \subset E$ and $E \cap X$ is residual in X but does not contain x_1; then $g^{-1}(E)$ does not intersect the open singleton $\{1\}$. Since none of the forward images of the open set $\{n\}$, intersects $\{n\}$ the system (Y, g) is not regionally recurrent.

The example above satisfies none of the assumptions of Proposition 33; in particular the preimage of a residual set is not always residual. This shows that residually scrambled systems may have more than one fixed point and therefore not be proximal (see also Proposition 31).

Section 6.3 exhibits almost endless classes of symbolic systems in which scrambled sets are never residual: minimal subshifts and more (Proposition 33); by Proposition 46, many subshifts containing a dense set of periodic points (like synchronising systems) and probably others; and also subshifts which do not satisfy the assumptions of Proposition 46. So why should this not be a general property of symbolic systems? Well, it is not. We now construct a residually scrambled symbolic system. In contrast with Example 54 it is proximal, which implies it has a unique fixed point, and invertible, therefore feebly open.

PROPOSITION 55. *There exists a proximal, residually scrambled, symbolic system (X, σ). It is strongly mixing and there is a non-atomic ergodic measure μ on (X, σ).*

Proof. We are using the notation for symbolic systems introduced in Subsection 6.3.

The symbolic system (X, σ) is defined on the alphabet $\{0, 1\}$ by a classical k-block construction. For each non-negative integer k the k-block is a word on $\{0, 1\}$ defined inductively by the formula

$$B_{k+1} = B_k.B_k.1 \quad \text{with } B_0 = 0.$$

The length of B_k is denoted by ℓ_k; it is easy to check that $\ell_k > k$ for any $k \in \mathbb{N}$. The closed σ-invariant set $X \subseteq \{0, 1\}^{\mathbb{Z}}$ is the set of all sequences x such that any block of coordinates of x is contained as a subword in one of the blocks B_k, $k \in \mathbb{N}$.

S. Bailey and K. Petersen pointed out that (X, σ) is the symbolic system generated by the non-primitive substitution $\tau : \{0, 1\} \to \{0, 1\}^*$ defined by $\tau(0) = 001$ and $\tau(1) = 1$. This remark provides tools for a deeper investigation of the properties of this subshift.

The following elementary observations are used repeatedly below. For $0 \le m \le n$, B_n always begins with an occurrence of B_m; the word 01^n always occurs at the end of an n-block and only there. An occurrence of B_k is either preceded or followed by another occurrence of B_k, and whenever B_k occurs in $x \in X$ it is always inside an occurrence of B_{k+1}. Finally, given $k > 0$ and $x \in X$, any coordinate of x which is not included in an occurrence of B_k has value 1.

CLAIM 1. *The system (X, σ) is the union of three shift-invariant sets*:

(1) *the fixed point p on the letter* 1;
(2) *the orbits of the two points z and z' such that $z_i = 1$ for $i < 0$, $z'_i = 1$ for $i \ge 0$, $z_0 = z'_{-1} = 0$, and there are infinitely many positive (resp. negative) coordinates i with $z_i = 0$ (resp. $z'_i = 0$)*;
(3) *all points $x \in X$ with infinitely many* 0*s among both negative and positive coordinates.*

Proof of the claim. The length of 1-strings is unbounded in X so p belongs to X by closure. Suppose $z \in X$ is such that $z_n = 1$ for $n < 0$ and $z_0 = 0$: then z contains a 1-block starting at time 0. Since 1-blocks occur in pairs and all negative coordinates are 1 there is a second 1-block just after the first, therefore a 2-block starting at 0. By induction $z_{0,\ell_k} = B_k$ for any k, which determines all non-negative coordinates of z, hence z itself; z' is also completely determined when $z'_n = 1$ for $n \ge 0$ and $z'_{-1} = 0$. One easily checks that z and z' belong to X. All points of X that do not belong to their orbits have infinitely many occurrences of 0 among positive and negative coordinates. ■

For $x \in X$, define $r_k(x)$ as the greatest negative coordinate of x at which a k-block begins (when it exists). The integer $r_k(x)$ is defined if and only if there is an occurrence of 0 in the negative coordinates of x; then it is defined for any k. The sequence $(r_k(x))$, $k \in \mathbb{N}$, is non-increasing: the last k-block starting in the negatives coordinates is contained in a $k+1$-block which starts either at the same coordinate (then $r_{k+1}(x) = r_k(x)$) or strictly before $r_k(x)$.

Claim 2. *Suppose $x, y \in X$ form an asymptotic pair and have infinitely many* 0*s among their positive coordinates. Then $x = y$.*

Proof of the claim. Suppose $x \in X$ contains infinitely many 0s in the positive coordinates and (x, y) is an asymptotic pair; these two properties are preserved by shifting x and y the same number of times. Thus we can assume that there is $n_0 < 0$ such that $x_i = y_i$ for $n \geq n_0$ and that $x_{n_0} = y_{n_0} = 0$.

Then both sequences $r_n(x)$ and $r_n(y)$, $n \in \mathbb{Z}$, exist. They are the same: as $x_{n_0} = y_{n_0} = 0$ the n_0 coordinate of x and that of y belong to a k-block which terminates at the end of the first occurrence of 01^k to the right of n_0 in x and y respectively; but since $x_i = y_i$ for $n \geq n_0$, 01^k occurs in the same place for x and y, and the k-blocks containing the n_0 coordinate of x and y also begin in the same position.

It is impossible for x and y to have an infinite string of 1s in the negative coordinates: by Claim 1 they would have to belong to the orbit of z; two asymptotic points belonging to the same symbolic orbit are eventually periodic, and the only periodic orbit in X is $\{p\}$, which means that the two points would be asymptotic to p. This contradicts the fact that they have infinitely many 0s among their positive coordinates.

Now we prove $r_n(x) \to -\infty$ as $n \to -\infty$. Supposing this is not true, there is k such that $r_n(x) = r_k(x) = s_0$ for $n > k$. Then for any $n > k$ there is an n-block starting at the coordinate s_0 for x and y, and since in points of X there is always an occurrence of 1^n before an n-block, this means that $x_j = y_j = 1$ for $j < s_0$, that is, there is an infinite string of 1s in the negative coordinates of x and y, which we just proved to be false.

For $n > 0$ one has

$$x_{r_n(x), r_n(x)+\ell_n-1} = y_{r_n(x), r_n(x)+\ell_n-1} = B_n;$$

owing to the fact that the sequence r_k is non-increasing, all coordinates x_i and y_i are thus determined and equal for $i \leq r_0(x)$. But we supposed that $x_{n_0} = y_{n_0} = 0$, which implies that $r_0(x) = r_0(y) \geq n_0$, and that $x_i = y_i$ for $i \geq n_0$. Therefore x and y coincide at all coordinates and $y = x$, which finishes the proof of Claim 2. ■

Claim 3. *The system (X, σ) is proximal.*

Proof of the claim. Let us check that given n there is n_0 such that if $w \in L(X)$ has length greater than or equal to n_0 there is an occurrence of 1^n in w. This property obviously implies that occurrences of 1^n are syndetic in any $x \in X$. This in its turn implies that $\{p\} \subseteq X$ and that X contains no other minimal system. By Proposition 4 the last property is equivalent to proximality.

Fix $n > 0$ and let w with $|w| = 3\ell_n$ occur in B_k for some $k > n+1$. The word B_k can be decomposed into occurrences of B_n separated by strings of 1s. If w contains an occurrence of B_n it contains an occurrence of 1^n. Otherwise w contains at most one strict suffix of B_n to the left and one strict prefix to the right with more than ℓ_n occurrences of 1 in between. As $\ell_n > n$ this implies that any word of length $n_0 = 3\ell_n$ contains an occurrence of 1^n, which implies proximality as above. ∎

Claim 4. *The system (X, σ) contains a residual scrambled set.*

Proof of the claim. (X, σ) was just shown to have no distal pairs. We claim that it contains a unique, countable asymptotic class.

Suppose first that $x \in X$ is asymptotic to the fixed point p. Then $x_n = 1$ for $n \geq n_0$; by shifting x if necessary put $n_0 = 0$ and assume $x_{-1} = 0$. By Claim 1, $x_n = 1$ for $n \geq 0$ and $x_{-1} = 0$ imply $x = z'$. Thus the equivalence class $W^s(p)$ of all points asymptotic to p consists of p and the orbit of z'; it is a countable set.

Next suppose that the pair (x, y), $x \neq y$, is asymptotic but x and y are not asymptotic to p, i.e., have infinitely many 0s in the positive coordinates. Then by Claim 2 one has $x = y$.

Thus the countable set $W^s(p)$ is the unique non-degenerate asymptotic class in X. Since (X, σ) is proximal, $X \setminus W^s(p)$ is a residual scrambled set. ∎

Claim 5.

(1) *(X, σ) is strongly mixing.*
(2) *(X, σ) is not uniquely ergodic*; *there exists a non-atomic*, *σ-ergodic measure on X.*

Proof of the claim. (1) By definition we must prove that if U, V are two non-empty open subsets of X, there is $n_0 > 0$ such that $n > n_0$ implies $U \cap \sigma^{-n} V \neq \emptyset$. It is sufficient to check this property for cylinder sets, so that finally a necessary and sufficient condition is that given $u, v \in L(X)$ there is $n_0 > 0$ such that for $n > n_0$ one can find $w \in L(X)$ with $|w| = n$ and $uwv \in L(X)$. This is easy to check. There is an integer $k > 0$ such that u and v occur in the k-block, so that $B_k = suw_0 = w_0' vs'$ with $|w_0 w_0'| = n_0$. Now if $n > 0$ the $n+k$-block terminates with a k-block followed by 1^n;

thus in the two consecutive k-blocks of the $k+1$-block there appears the word $suw_0 1^n w'_0 vs'$, in which u and v are connected by a word of length n_0+n.

(2) The Dirac measure on $\{p\}$, δ_p, is an ergodic measure on (X,σ). There exists at least another one. Consider the point $z \in X$ introduced in Claim 1, and the sequence of measures

$$\mu_n = \frac{1}{\ell_n} \sum_{i=1}^{\ell_{n-1}} \delta_{\sigma^i z},$$

where ℓ_k is the length of the k-block used in the definition of X. The measures μ_n belong to the set of all probability measures on X, which is compact for the topology of weak* convergence. Take a converging subsequence μ_{i_n}; its limit μ' is obviously σ-invariant. Moreover,

$$\mu_n([1]_0) = \frac{1}{\ell_n} \sum_{i=1}^{\ell_{n-1}} \delta_{\sigma^i z}([1]_0) = \frac{1}{\ell_n} \sum_{i=1}^{\ell_{n-1}} z_i;$$

thus $\mu_n([1]_0)$ is equal to the proportion of 1s in the n-block. Denoting by s_n the number of 1s in the n-block, one has $s_{n+1} = 2s_n + 1$ with $s_0 = 0$, so $s_n = 2^{n-1} - 1$, and $\ell_{n+1} = 2\ell_n + 1$ with $\ell_0 = 1$, so $\ell_n = 2^n - 1$; thus $\mu_n([1]_0) \to 1/2$ as $n \to \infty$.

Then $\mu'([1]_0) = \lim_{n\to\infty} \mu_{i_n}([1]_0) = \lim_{n\to\infty} \mu_n([1]_0) = 1/2$. The measure μ' is not necessarily ergodic but it is a barycentre of ergodic measures; $\mu'([1]_0) = 1/2$ implies that there is an ergodic μ with $\mu([1]_0) \le 1/2$. Since $\delta_p([1]_0) = 1$ this means that $\mu \neq \delta_p$, and since p is the unique periodic point in X, μ is non-atomic. ∎

This finishes the proof of the proposition. ∎

The easy job of checking that the one-sided version of (X,σ) in the last proposition is also residually scrambled and feebly open is left to the reader.

REMARKS.

1. In the construction above the word B_0 may be chosen arbitrarily, provided it contains one 0 at least.
2. (X,σ) has zero topological entropy. This can be proved by counting words, but it is also an easy consequence of the result from [10] that is quoted at the beginning of this section.

Starting from the example (X,σ) from Proposition 55 we now build a residually scrambled, positive-entropy compact system. The proof relies on three properties of X: its proximality, the fact that it is residually scrambled, and the existence of a non-Dirac invariant measure.

First recall some definitions and properties. Let (X,f) be a compact dynamical system. A point $x \in X$ is called a *distal point* if for any $y \in X$

with $y \neq x$, $\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0$. Denote the set of all distal points of (X, f) by $D(X, f)$. A compact system (X, f) is called *point distal* if it is minimal and $D(X, f) \neq \emptyset$. It is well known that for a point distal system (X, f), $D(X, f)$ is a dense G_δ set of X [17]. Toeplitz systems are point distal and some of them have positive entropy [47].

The definition of entropy pairs is given in [7], together with the two properties used here: that a system has a non-empty set of entropy pairs if and only if its topological entropy is positive; and that unless a factor map $\phi : (X, f) \to (Y, g)$ collapses an entropy pair (x, y), the image pair $(\phi(x), \phi(y))$ is an entropy pair of (Y, g). Moreover, it is proved in [22, Theorem 3(5)] that when (X, f) and (Y, g) are compact systems with $h(X, f) = 0$ and $h(Y, g) > 0$, then

$$E(X \times Y) \supseteq \{((x, y_1), (x, y_2)) : x \in \operatorname{supp}(\mu),\ (y_1, y_2) \in E(Y, g)\},$$

where $E(W, \theta)$ is the set of all entropy pairs of the system (W, θ) and μ is an f-invariant Borel probability measure on X.

PROPOSITION 56. *Let (X, f) be a proximal, residually scrambled, system with X compact. If there is an invariant measure μ for (X, f) which is not Dirac, then (X, f) is a factor of a residually scrambled system (Z, h) with positive entropy. In particular, there is a residually scrambled system with positive entropy.*

Proof. Since (X, f) is proximal, there is a fixed point p of (X, f) which is the unique minimal set of (X, f). Let (Y, g) be a compact point distal system with positive entropy. Since (Y, g) is minimal, $\{p\} \times Y$ is the unique minimal subsystem of the product system $(X \times Y, f \times g)$.

Consider the relation $\backsim$ on $X \times Y$ with $(x_1, y_1) \backsim (x_2, y_2)$ if and only if $(x_1, y_1) = (x_2, y_2)$ or $x_1 = x_2 = p$. Clearly $\backsim$ is an $f \times g$-invariant closed equivalence relation on $X \times Y$ and induces a factor map $\pi : (X \times Y, f \times g) \to (Z, h)$, where $Z = X \times Y/\backsim$ and $h(\pi(x_1, y_1)) = \pi(f(x_1), g(y_1))$ for each $(x_1, y_1) \in X \times Y$. Likewise (Z, h) has a unique minimal subsystem which is a fixed point, which by Proposition 4 is equivalent to proximality.

The map $\pi_1 : Z \to X$ with $\pi_1(\pi(x_1, y_1)) = x_1$ for $(x_1, y_1) \in X \times Y$ is well defined, continuous, surjective and $\pi_1 : (Z, h) \to (X, f)$ is a factor map.

In what follows we show that (Z, h) has the required properties.

First we prove that (Z, h) has positive entropy. There is an f-invariant Borel probability measure μ on X such $\operatorname{supp}(\mu)$ is different from $\{p\}$, so that it is not a singleton. By [22],

$$E(X \times Y) \supseteq \{((x, y_1), (x, y_2)) : x \in \operatorname{supp}(\mu),\ (y_1, y_2) \in E(Y, g)\}$$

is not empty, since $E(Y, g) \neq \emptyset$. Moreover,

$$\begin{aligned} E(Z,h) &\supseteq \pi\times\pi(E(X\times Y, f\times g))\setminus\{(z,z): z\in Z\} \\ &\supseteq \{(\pi(x,y_1),\pi(x,y_2)) : x\in \mathrm{supp}(\mu)\setminus\{p\} \text{ and } (y_1,y_2)\in E(Y,g)\} \\ &\neq \emptyset \quad (\text{as } \mathrm{supp}(\mu)\setminus\{p\}\neq\emptyset) \end{aligned}$$

and as $\mathrm{supp}(\mu)$ is not equal to the singleton $\{p\}$ the relation $\backsim$ does not collapse all entropy pairs of $(X\times Y, f\times g)$. Then by [7], $h_{\mathrm{top}}(Z,h)>0$.

Let A be a dense G_δ scrambled set of (X,f). If $W^s(p)$ is the asymptotic class of p, since $\#(W^s(p)\cap A)\le 1$ and X is perfect, A can be supposed not to contain any point of $W^s(p)$. Let $B=D(Y,g)$. Then $A\times B$ is a dense G_δ subset of $X\times Y$. Since $p\notin A$, we may assume that $A\times B=\bigcap_{n=1}^\infty U_n$, where U_n is a dense open subset of $X\times Y$ and $U_n\cap(\{p\}\times Y)=\emptyset$ for $n\in\mathbb{N}$. For these reasons and as $\pi^{-1}(\pi(U_n))=U_n$, $\pi(U_n)$ is a dense open subset of Z. This implies that $C=\pi(A\times B)=\bigcap_{n=1}^\infty \pi(U_n)$ is a dense G_δ subset of Z.

We claim that C is a scrambled set of (Z,h). Let $z_1,z_2\in C$, $z_1\neq z_2$. Since (Z,h) is proximal, $\liminf_{n\to\infty} d(h^n(z_1),h^n(z_2))=0$. Let $z_i=\pi(x_i,y_i)$, where $(x_i,y_i)\in A\times B$, $i=1,2$. There are two cases.

Case 1: $x_1\neq x_2$. Since $x_1,x_2\in A$, (x_1,x_2) is a scrambled pair of (X,f), therefore not asymptotic. Then (z_1,z_2) is not an asymptotic pair of (Z,h), as $\pi_1(z_i)=x_i$, $i=1,2$. So it is a scrambled pair.

Case 2: $x_1=x_2$, so $y_1\neq y_2$. Since $W^s(p)\cap A=\emptyset$, $x_1=x_2\notin W^s(p)$. Thus there exists an infinite sequence $\{n_j\}$ of natural numbers such that $\lim_{j\to\infty} f^{n_j}(x_i)=x$ for some $x\in X\setminus\{p\}$, and $\lim_{j\to\infty} g^{n_j}(y_i)=y_i'$ for some $y_i'\in Y$, $i=1,2$. Since y_1, $y_2\in D(Y,g)$ and $y_1\neq y_2$, the points y_1' and y_2' are not equal. Moreover, $\pi(x,y_1')\neq\pi(x,y_2')$, as $x\neq p$. Then $\lim_{j\to\infty} h^{n_j}(z_i)=\pi(x,y_i')$, $i=1,2$, which implies that (z_1,z_2) is not an asymptotic pair of the proximal system (Z,h). Therefore (z_1,z_2) is scrambled, and since $(z_1,z_2)\in C$ is arbitrary, C is a scrambled set.

The existence of a positive-entropy, residually scrambled system results from choosing for (X,f) the system constructed in the proof of Proposition 55. ∎

The construction of (Z,h) in the last proof can be exploited in various other ways. One instance is the following result; there is another one at the beginning of the next section. Recall that disjointness is defined at the beginning of Section 4.

Proposition 57. *Let (Y,g) be a compact distal minimal system. There exists a completely scrambled system (Z,h) which is not disjoint from (Y,g).*

Proof. Suppose (X,f) is completely scrambled with X compact (no assumption on measures is needed here). Then the system (Z,h) constructed from (X,f) and (Y,g) as in the proof of Proposition 56 is completely scrambled, since $A' = X$ is scrambled and the set $D(Y,g)$ of distal points of (Y,g) is Y itself.

Moreover, (Z,h) is not disjoint from (Y,g): keeping the notation of the last proof, let K be the closed, $f\times g\times g$-invariant set $X\times \Delta_Y \subseteq X\times Y\times Y$, and define the factor map $\overline{\pi} : (X\times Y\times Y, f\times g\times g) \to (Z\times Y, h\times g)$ by setting $\overline{\pi}(x,y,y') = (\pi(x,y),y')$. Then $J = \overline{\pi}(K)$ is a closed, $h\times g$-invariant proper subset of $Z\times Y$; inside J any point $z\in Z$ such that $z=\pi(x,y)$ for $x\neq p$ is joined to y only, while the unique fixed point $\pi(p,y)$ is joined to the whole space Y. ■

The statement is also true when (Y,g) is not minimal, but it only makes sense in the minimal case (two non-minimal systems are never disjoint).

Heuristically the meaning of Proposition 57 is that, given any minimal distal system (Y,g), one can construct a system which preserves most of the structure of (Y,g) while being completely scrambled. The property of being completely scrambled is not as powerful as one would think at first sight.

Questions

1. Is there a non-proximal dynamical system (X,f) with f feebly open, such that X contains a residual scrambled set? In other words, if f is feebly open, does the existence of a residual scrambled set imply proximality?
2. May a triangular map have a residual scrambled set? See Example 59 below.
3. There are residually scrambled systems that are weakly (even strongly) mixing, *or* have positive entropy, but our construction does not permit to obtain the three properties simultaneously. Are they compatible?

8. FACTORS AND EXTENSIONS FROM THE POINT OF VIEW OF SCRAMBLED SETS

The properties of being completely or residually scrambled are conjugacy invariants for compact systems. In this section we show that they are not invariant under factor maps or almost one-to-one extensions. We also show that triangular maps, which are particular extensions of interval maps, may have scrambled sets with non-empty interior; for interval maps scrambled sets are always nowhere residual [12], [22]. Finally, in [9] it is proved that if $\phi : (X,f) \to (Y,g)$ is a factor map and g has a scrambled pair then f has a scrambled pair too. Here we show that this property of systems is not true pairwise, i.e., a scrambled pair of (Y,g) may have no element of its preimage that is a scrambled pair for (X,f).

All the systems described in this section are on compact spaces.

An extension of a residually scrambled system may not be residually scrambled—consider the Cartesian product of a completely scrambled system and a distal system as an extension of its completely scrambled factor.

An *almost one-to-one extension* is an extension that is one-to-one on a dense G_δ set. If we additionally assume that (X,F) is a transitive almost one-to-one extension of a residually scrambled system (Z,h) then still (X,F) may not have a residual scrambled set. In Proposition 57 consider the Cartesian product $(X\times Y, f\times g)$ where X is perfect, (X,f) is completely scrambled and (Y,g) is non-trivial distal, and its completely scrambled factor (Z,h). Since the factor map collapses only pairs (p,y) where p is the unique fixed point of f, $(X\times Y, f\times g)$ is an almost one-to-one extension of (Z,h). But it is also an extension of the non-trivial distal system (Y,g), so by Proposition 28(1) the scrambled sets of $(X\times Y, f\times g)$ are not dense. To ensure transitivity for $(X\times Y, f\times g)$ assume in addition that (X,f) is weakly mixing [27] and (Y,g) is minimal; the product of a minimal system and a weakly mixing system is always transitive [19].

The case of factors is not so easy. In the following example a factor of a completely scrambled system has no scrambled sets. This is due to the fact that the image of a scrambled pair may be asymptotic: here (Z,G) is completely scrambled but its factor (Y,h) is asymptotic, in the sense that all pairs of Y are asymptotic.

PROPOSITION 58. *There exist two systems (Z,G) and (Y,h) and a factor map $(Z,G)\to(Y,h)$ such that Z and Y are perfect compact metric spaces and*

(1) *(Z,G) is completely scrambled*;
(2) *(Y,h) has no scrambled sets.*

Proof. Let (X,f) be the completely scrambled system constructed in [25]. (X,f) has the following properties:

- X is a countable infinite set and f is a homeomorphism.
- There exists $x_0\in X_{\rm is}$ such that $\{f^j(x_0): j\in\mathbb{Z}\}=X_{\rm is}$, where $X_{\rm is}$ is the set of all isolated points of X ($X_{\rm is}$ is dense in X but this does not imply that the *forward* orbit of x_0 is dense in X).
- (X,f) has a unique fixed point p which is the unique minimal set of (X,f) and $p\notin X_{\rm is}$.

In the first step we embed (X,f) into a system (X_1,f) where X_1 is a perfect compact subset of the interval (we still write (X_1,f) instead of a more precise (X_1,f_1)). The system (X_1,f) is not completely scrambled, actually $\bigcup_{x\in X} W^s(x)=X_1$. As a consequence any scrambled set of (X_1,f) has at most countably many points.

To do this embed X into the open interval $(0,1)$ as a compact subset. For each $x \in X_{\mathrm{is}}$ put

$$\varepsilon_x = \frac{d(x,(X\setminus\{x\})\cup\{1\})}{4} \quad \text{and} \quad I_x = [x, x+\varepsilon_x].$$

Clearly $\varepsilon_x > 0$, $I_x \subseteq (0,1)$ and $d(I_x, ((X\setminus\{x\})\cup\bigcup_{y\in X_{\mathrm{is}}\setminus\{x\}} I_y)) > \varepsilon_x$ for $x \in X_{\mathrm{is}}$. The set $X_1 = X \cup \bigcup_{x\in X_{\mathrm{is}}} I_x \subseteq [0,1]$ is closed. Extend the map f to X_1 as follows: $f(x+s\varepsilon_x) = f(x) + s\varepsilon_{f(x)}$ for $x \in X_{\mathrm{is}}$ and $s \in [0,1]$. It is clear that X_1 is a perfect compact metric space and as $f : X \to X$ is a homeomorphism, so is $f : X_1 \to X_1$. Since $\lim_{j\to\infty} \operatorname{diam}(f^j(I_x)) = \lim_{j\to\infty} \varepsilon_{f^j(x)} = 0$ for $x \in X_{\mathrm{is}}$ and (X,f) is completely scrambled, we have

$$W^s(x) = \begin{cases} I_x & \text{when } x \in X_{\mathrm{is}}, \\ \{x\} & \text{when } x \in X\setminus X_{\mathrm{is}}. \end{cases} \tag{8.1}$$

So $X_1 = \bigcup_{x\in X} W^s(x)$.

The system (X_1, f) just defined has a factor (Y,h) and an extension (Z,G) with the required properties. The factor (Y,h) is very easy to describe. The set X is a closed f-invariant subset of X_1. By collapsing X to a fixed point one creates a factor (Y,h) of (X_1,f). Observe that any two points in Y are asymptotic; this implies that (Y,h) has no scrambled sets.

The really hard part of the proof consists in constructing (Z,G) as an extension of (X_1,f). Consider the set $X_1 \times S^1$, where S^1 is the unit circle of the complex plane, and the point $x_0 \in X$. Since (x_0,p) is a scrambled pair of (X,f), hence of (X_1,f), there exist $x^* \in X_1$ and a sequence $\{n_j\}_{j\in\mathbb{Z}_+}$ of non-negative integers with $\lim_{j\to\infty} f^{n_j}(x_0) = x^* \neq p$. We may assume that $n_0 = 0$ and $n_{j+1} - n_j \geq j$ for $j \in \mathbb{N}$.

For $j \in \mathbb{Z}_+$, when $j \in [n_{2k}, n_{2k+1})$ for some $k \in \mathbb{Z}_+$, define $m(j) = n_{2k+1} - n_{2k}$ and

$$r_j(s,z) = z\cdot e^{\pi i s/m(j)} \quad \text{for } s \in I \text{ and } z \in S^1;$$

when $j \in [n_{2k+1}, n_{2(k+1)})$ for $k \in \mathbb{Z}_+$, define $m(j) = n_{2(k+1)} - n_{2k+1}$ and

$$r_j(s,z) = z\cdot e^{-\pi i s/m(j)} \quad \text{for } s \in I \text{ and } z \in S^1.$$

For all j the map r_j is continuous from $I \times S^1$ to S^1. Set

$$C_j = \max_{(s,z)\in I\times S^1} |r_j(s,z) - z| \quad \text{for } j \in \mathbb{Z}_+.$$

Since $\lim_{j\to\infty} m(j) = +\infty$,

$$\lim_{j\to\infty} C_j = \lim_{j\to\infty} |e^{\pi i/m(j)} - 1| = 0. \tag{8.2}$$

Define $g : X_1 \times S^1 \to S^1$ by

$$g(x,z)=\begin{cases} z & \text{when } (x,z)\in (X_1\setminus \bigcup_{j=0}^{\infty} I_{f^j(x_0)})\times S^1,\\ r_j\left(\dfrac{x-f^j(x_0)}{\varepsilon_{f^j(x_0)}},z\right) & \text{when } (x,z)\in I_{f^j(x_0)}\times S^1 \text{ for some } j\in\mathbb{Z}_+.\end{cases}$$

Considering (8.2) and by continuity of r_j it is not hard to see that g is a continuous map on $X_1\times S^1$. Let $F(x,z)=(f(x),g(x,z))$ for $(x,z)\in X_1\times S^1$. Then F is a homeomorphism on $X_1\times S^1$, as f is a homeomorphism, g is continuous and for fixed $x\in X_1$, $g(x,\cdot):S^1\to S^1$ is a homeomorphism.

For $x\in I_{x_0}$ write $x=x_0+s\varepsilon_{x_0}$ with $s\in I$. Then with the same s we have $f^m(x)=f^m(x_0)+s\varepsilon_{f^m(x_0)}$ for $m\in\mathbb{Z}_+$. Therefore, for $j\in\mathbb{Z}_+$ and $x\in I_{x_0}$ we are able to compute

$$(8.3)\qquad F^j(x,z)=\begin{cases} (f^j(x), z\cdot e^{\pi i s\frac{j-n_{2k}}{n_{2k+1}-n_{2k}}}) \\ \qquad\qquad \text{when } j\in[n_{2k},n_{2k+1}) \text{ for some } k\in\mathbb{Z}_+,\\ (f^j(x), z\cdot e^{\pi i s\frac{n_{2(k+1)}-j}{n_{2(k+1)}-n_{2k+1}}}) \\ \qquad\qquad \text{when } j\in[n_{2k+1},n_{2(k+1)}) \text{ for some } k\in\mathbb{Z}_+.\end{cases}$$

Consider the relation $\backsim$ on $X_1\times S^1$ with $(x_1,z_1)\backsim(x_2,z_2)$ if and only if $(x_1,z_1)=(x_2,z_2)$ or $x_1=x_2=p$. It is a closed F-invariant equivalence relation on $X_1\times S^1$, which induces naturally a factor map $\pi:(X_1\times S^1,F)\to(Z,G)$, where $Z=X_1\times S^1/\backsim$ and $G(\pi(x,z))=\pi(F(x,z))$ for each $(x,z)\in X_1\times S^1$. The situation is the same as in the proof of Proposition 56: Z is a perfect compact metric space and (Z,G) has a unique minimal subsystem $\{\pi(\{p\}\times S^1)\}$ which is a fixed point; by Proposition 4, (Z,G) is proximal. The map $\pi_1:Z\to X_1$ with $\pi_1(\pi(x,z))=x$ for $(x,z)\in X_1\times S^1$ is well-defined, continuous and surjective and $\pi_1:(Z,G)\to(X_1,f)$ is a factor map.

It remains to show that the proximal system (Z,G) is completely scrambled; for this purpose we need only prove that for any $y_1\neq y_2\in Z$, (y_1,y_2) is not asymptotic for G. Suppose that there exist $(x_i,z_i)\in X_1\times S^1$, $i=1,2$, such that for $y_i=\pi(x_i,z_i)$, $i=1,2$, the pair (y_1,y_2) is proper asymptotic for G. Since $\pi_1(y_i)=x_i$, $i=1,2$, the pair (x_1,x_2) is asymptotic. Then, by (8.1) and the fact that $X_{\text{is}}=\{f^j(x_0):j\in\mathbb{Z}\}$, either $x_1=x_2\in X\setminus X_{\text{is}}$ or $x_1,x_2\in I_{f^j(x_0)}$ for some $j\in\mathbb{Z}$.

Case 1: $x_1=x_2=x\in X\setminus X_{\text{is}}$. Since $y_1\neq y_2$, one has $x\neq p$ and $z_1\neq z_2$. Since $f(X\setminus X_{\text{is}})=X\setminus X_{\text{is}}$ and $g(x,z)=z$ for $x\in X\setminus X_{\text{is}}$, $G^n(y_i)=G^n(\pi(x,z_i))=\pi(F^n(x,z_i))=\pi(f^n(x),z_i)$ for $n\in\mathbb{Z}_+$. Since (X,f) is a completely scrambled system and $x\neq p$ there exists a sequence $\{m_l\}$ of natural numbers such that $\lim_{l\to\infty}f^{m_l}(x)=x'$ for some $x'\in X\setminus\{p\}$. So $\lim_{l\to\infty}G^{m_l}(y_i)=\pi(x',z_i)$. As (y_1,y_2) is asymptotic, $\pi(x',z_1)=\pi(x',z_2)$. This, together with $x'\neq p$, implies that $z_1=z_2$ and $y_1=y_2$, a contradiction.

CASE 2: $x_1, x_2 \in I_{f^j(x_0)}$ for some $j \in \mathbb{Z}$. Since $y_1,\ y_2$ are asymptotic if and only if $G^{-j}(y_1),\ G^{-j}(y_2)$ are, assume without loss of generality that $x_1, x_2 \in I_{x_0}$. Then there exist $s_i \in I$ such that $x_i = x_0 + s_i \varepsilon_{x_0}$ for $i = 1, 2$.

For the subsequence $\{n_j\}$ defined above one has $\lim_{j\to\infty} f^{n_j}(x_0) = x^* \neq p$. Since $W^s(x_0) = I_{x_0}$ this implies that $\lim_{j\to\infty} f^{n_j}(x_i) = x^*$, $i = 1, 2$. By (8.3) one deduces that for $i = 1, 2$,

$$(8.4)\qquad \lim_{k\to\infty} G^{n_{2k}}(y_i) = \pi(x^*, z_i) \quad \text{and} \quad \lim_{k\to\infty} G^{n_{2k+1}}(y_i) = \pi(x^*, z_i \cdot e^{\pi i s_i}).$$

Since $y_1 \neq y_2$, $(x_1, z_1) \neq (x_2, z_2)$. This means that either $z_1 \neq z_2$ or $s_1 \neq s_2$. Hence $z_1 \neq z_2$ or, in case $z_1 = z_2 = z$, $ze^{\pi i s_1} \neq ze^{\pi i s_2}$. Then since $x^* \neq p$ the formulas (8.4) show that (y_1, y_2) is not asymptotic, which is a contradiction. ∎

Let us address the second point, scrambled sets with non-empty interior in systems that are not completely scrambled. There are compact metric spaces that do not admit any scrambled set, say finite spaces or rigid continua. On the other hand, there are many spaces admitting a scrambled set with non-empty interior. Here is an example that is not completely scrambled. Let (I, f) be an interval map with a Cantor scrambled set S. Then $Y = (I \times \{0\}) \cup (S \times \{1\})$ is a compact subset of the unit square I^2. Consider the continuous map $g : Y \to Y$ such that $g(x, 0) = (f(x), 0)$ for any $x \in I$ and $g(s, 1) = (f(s), 0)$ for any $s \in S$. Then $S \times \{1\}$ is a clopen scrambled set for (Y, g).

Observe that the system (Y, g) described above is an extension of (I, f) and that f, as an interval map, has only nowhere residual scrambled sets. A more concrete fact is that a system with an open scrambled set can be constructed on the *whole* square I^2 as a triangular map, i.e., as an extension of an interval system for which each fibre equals I. It may be possible to generalize this result to higher dimensions, as suggested in Subsection 2.4.

EXAMPLE 59. There is a triangular map in the square which has a non-empty open scrambled set.

Proof. Again let (X, f) be the completely scrambled system constructed in [25], with its fixed point p. Then for any $x \in X \setminus \{p\}$, $\omega(x) \setminus \{p\} \neq \emptyset$ and $\omega(x) \supset \{p\}$. Construct the system (X_1, f) as in the proof of the last proposition, keeping the corresponding notation.

Extend the map f on X_1 to a continuous map on the whole unit interval $I = [0, 1]$, still denoting the extended map as f. Fix a point $x_0 \in X_{\mathrm{is}}$ and construct a continuous map $h : [0, 1] \to [0, 1/2]$ such that

$$h(x) > 0 \quad \text{for } x \neq p, \qquad h(x) = 1/2 \quad \text{for } x \in I_{x_0} \quad \text{and} \quad h(p) = 0.$$

With the help of h define a continuous map $\theta : I \times I \to I$ by

$$\theta(x, (1-t)i + t \cdot h(x)) = (1-t)i + t \cdot h(f(x)) \quad \text{ for } x, t \in I \text{ and } i = 0, 1;$$

note that $(1-t)i+t\cdot h(x)$ ranges from 0 to $h(x)$ when $i=0$ and from $h(x)$ to 1 when $i=1$, and that when x is fixed, θ is a linear map on each of these two intervals. Also define $h_1,h_2:[0,1]\to[1/2,1]$ by

$$h_1(t)=\begin{cases}1-t/2, & 0\le t\le 1/2,\\ (1+t)/2, & 1/2\le t\le 1,\end{cases}\qquad h_2(t)=\begin{cases}1-t, & 0\le t\le 1/2,\\ t, & 1/2\le t\le 1.\end{cases}$$

One has $h_i(0)=h_i(1)=1$, $i=1,2$, and $h_1(s)\ge h_2(s)$ for $s\in I$.

Since (x_0,p) is a scrambled pair, there exists a sequence $\{n_j\}_{j\in\mathbb{Z}_+}$ of non-negative integers such that $n_0=0$, $n_{j+1}-n_j\ge j$ for $j\in\mathbb{Z}_+$ and $\lim_{j\to\infty}f^{n_j}(x_0)=x^*$ for some $x^*\in X\setminus\{p\}$. For $j\in\mathbb{Z}_+$ define a sequence of maps $r_j:I\times I\to I$ in the following way:

- whenever $j\in[n_{2k},n_{2k+1})$ for some $k\in\mathbb{Z}_+$, put $m(j)=n_{2k+1}-n_{2k}$ and
$$\begin{aligned}&r_j(s,(1-t)i+t\cdot h_1(s)^{1/m(j)})\\ &\qquad=(1-t)i+t\cdot h_2(s)^{1/m(j)}\quad\text{for } s,t\in I \text{ and } i=0,1;\end{aligned}$$
- whenever $j\in[n_{2k+1},n_{2(k+1)})$ for some $k\in\mathbb{Z}_+$, put $m(j)=n_{2(k+1)}-n_{2k+1}$ and
$$\begin{aligned}&r_j(s,(1-t)i+t\cdot h_2(s)^{1/m(j)})\\ &\qquad=(1-t)i+t\cdot h_1(s)^{1/m(j)}\quad\text{for } s,t\in I \text{ and } i=0,1.\end{aligned}$$

Each map r_j is continuous on $I\times I$ and $r_j(0,y)=r_j(1,y)=y$ for any $y\in I$. For $j\in\mathbb{Z}_+$, let $C_j(s)=\max_{y\in I}|r_j(s,y)-y|$ and $C_j=\max_{s\in I}C_j(s)$. Then, taking into account the fact that $h_1(s)\ge h_2(s)\ge 0$, a simple computation shows that

$$\begin{aligned}C_j(s)&=|h_1(s)^{1/m(j)}-h_2(s)^{1/m(j)}|=h_1(s)^{1/m(j)}-h_2(s)^{1/m(j)}\\ &\le 1-\left(\frac{1}{2}\right)^{1/m(j)}.\end{aligned}$$

Since additionally $\lim_{j\to\infty}m(j)=+\infty$, this implies that

$$\lim_{j\to\infty}C_j=0. \tag{8.5}$$

Then define a map $g:I\times I\to I$ by

$$g(x,y)=\begin{cases}\theta(x,y) & \text{for } y\in I \text{ when } x\in I\setminus\bigcup_{j=0}^{\infty}I_{f^j(x_0)},\\ r_j\left(\dfrac{x-f^j(x_0)}{\varepsilon_{f^j(x_0)}},\theta(x,y)\right) & \text{for } y\in I \text{ when } x\in I_{f^j(x_0)},\ j\in\mathbb{Z}_+.\end{cases}$$

Claim 1. *g is a continuous map on $I\times I$.*

Proof of the claim. Since $\lim_{j\to\infty}\operatorname{diam}(I_{f^j(x_0)})=\lim_{j\to\infty}\varepsilon_{f^j(x_0)}=0$, the set

$$A=\bigcup_{j=0}^{\infty}I_{f^j(x_0)}\cup\omega(x_0,f),$$

where $\omega(x_0, f)$ is the set of all ω-limit points of x_0, is a closed subset of I. As $g(x,\cdot) = \theta(x,\cdot)$ when $x \in I \setminus A$, g is continuous on $(I \setminus A) \times I$ because θ is continuous on the open subset $(I \setminus A) \times I$ of $I \times I$.

Assume $(x, y) \in I_{f^j(x_0)} \times I$ for some $j \in \mathbb{Z}_+$. A point of $\omega(x_0, f)$ is not isolated in X so $(f^j(x_0) - \varepsilon_{f^j(x_0)}, f^j(x_0) + 2\varepsilon_{f^j(x_0)}) \cap A = I_{f^j(x_0)}$ by the definition and properties of I_x, $x \in X_{\mathrm{is}}$. Hence

$$g(x,y) = \begin{cases} \theta(x,y) & \text{when } x \in (f^j(x_0) - \varepsilon_{f^j(x_0)}, f^j(x_0)),\ y \in I, \\ r_j\left(\dfrac{x - f^j(x_0)}{\varepsilon_{f^j(x_0)}}, \theta(x,y)\right), & \\ & \text{when } x \in [f^j(x_0), f^j(x_0) + \varepsilon_{f^j(x_0)}],\ y \in I, \\ \theta(x,y) & \\ \quad \text{when } x \in (f^j(x_0) + \varepsilon_{f^j(x_0)}, f^j(x_0) + 2\varepsilon_{f^j(x_0)}),\ y \in I. & \end{cases} \tag{8.6}$$

Since $r_j(0, y) = r_j(1, y) = y$ for $y \in I$, one has

$$\begin{aligned} g(f^j(x_0), y) &= \theta(f^j(x_0), y), \\ g(f^j(x_0) + \varepsilon_{f^j(x_0)}, y) &= \theta(f^j(x_0) + \varepsilon_{f^j(x_0)}, y) \quad \text{for } y \in I. \end{aligned} \tag{8.7}$$

Since r_j and θ are continuous on $I \times I$, g is continous on $I_{f^j(x_0)} \times I$ by (8.6) and (8.7).

We still have to show that g is continuous on $\omega(x_0, f) \times I$. If this is not true, there exist $(x_n, y_n) \in I \times I$ and $(x^*, y^*) \in \omega(x_0, f) \times I$ such that $(x_n, y_n) \to (x^*, y^*)$ but $g(x_n, y_n) \not\to g(x^*, y^*)$. Since $g(x, y) = \theta(x, y)$ when $x \notin \bigcup_{j=0}^{\infty} I^{f^j(x_0)}$, and $g(x^*, y^*) = \theta(x^*, y^*)$ and θ is continuous on $I \times I$, for n large enough one must have $x_n \in \bigcup_{j=0}^{\infty} I_{f^j(x_0)}$. Thus for n large enough there is a unique $t_n \in \mathbb{Z}_+$ such that $x_n \in I_{f^{t_n}(x_0)}$. But $x^* \notin I_{f^j(x_0)}$ for any $j \in \mathbb{Z}_+$ and $x_n \to x^*$, so that $\lim_{n\to\infty} t_n = \infty$. Now

$$\begin{aligned} &\limsup_{n\to\infty} |g(x_n, y_n) - g(x^*, y^*)| \\ &\qquad = \limsup_{n\to+\infty} \left| r_{t_n}\left(\frac{x_n - f^{t_n}(x_0)}{\varepsilon_{f^{t_n}(x_0)}}, \theta(x_n, y_n)\right) - \theta(x^*, y^*)\right| \\ &\qquad \le \limsup_{n\to\infty} \Bigg(|\theta(x_n, y_n) - \theta(x^*, y^*)| \\ &\qquad\qquad + \left| r_{t_n}\left(\frac{x_n - f^{t_n}(x_0)}{\varepsilon_{f^{t_n}(x_0)}}, \theta(x_n, y_n)\right) - \theta(x_n, y_n)\right| \Bigg) \\ &\qquad \le \limsup_{n\to\infty} (|\theta(x_n, y_n) - \theta(x^*, y^*)| + C_{t_n}) \\ &\qquad = \limsup_{n\to\infty} C_{t_n} \quad (\text{as } \theta \text{ is continuous and } (x_n, y_n) \to (x^*, y^*)) \\ &\qquad = 0 \quad (\text{because } \lim_{n\to\infty} t_n = \infty \text{ and } (8.5)), \end{aligned}$$

which contradicts $g(x_n, y_n) \not\to g(x^*, y^*)$. This ends the proof of Claim 1. ∎

We claim that the triangular map $F(x,y) = (f(x), g(x,y))$ for $(x,y) \in I \times I$ has the required properties. By Claim 1, F is continuous on $I \times I$. For $(x,y) \in I_{x_0} \times [0,1/2]$, there exists a unique pair $(s,t) \in [0,1] \times [0,1]$ such that $x = x_0 + s \cdot \varepsilon_{x_0}$ and $y = t \cdot h(x)$, as $h(x) = 1/2$ for $x \in I_{x_0}$.

CLAIM 2. *Let* $(x,y) \in I_{x_0} \times [0,1/2]$. *Then*:

(i) $f^m(x) = f^m(x_0) + s\varepsilon_{f^m(x_0)}$ *for* $m \in \mathbb{Z}_+$.
(ii) *For* $j \in \mathbb{Z}_+$,

$$F^j(x,y) = \begin{cases} \left(f^j(x), t\left(\dfrac{h_2(s)}{h_1(s)}\right)^{\frac{j-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot h(f^j(x))\right) \\ \qquad \textit{when } j \in [n_{2k}, n_{2k+1}) \textit{ for some } k \in \mathbb{Z}_+, \\ \left(f^j(x), t\left(\dfrac{h_2(s)}{h_1(s)}\right)^{\frac{n_{2(k+1)}-j}{n_{2(k+1)}-n_{2k+1}}} \cdot h(f^j(x))\right) \\ \qquad \textit{when } j \in [n_{2k+1}, n_{2(k+1)}) \textit{ for some } k \in \mathbb{Z}_+. \end{cases}$$

Proof of the claim. (i) follows from the definition of f; (ii) is proved by induction on j. For $j = 0$, it is clear that $F^0(x,y) = (x,y) = (x, t \cdot h(f^0(x)))$, i.e., (ii) is true for $j = 0$. Assume that (ii) is true for $j = u$. For $j = u+1$, there are two cases.

CASE 1: $u \in [n_{2k}, n_{2k+1})$ for some $k \in \mathbb{Z}_+$. Since $h_1(s) \geq h_2(s)$ and $\max_{y \in I} h(y) = 1/2 \leq h_1(s)$, one has

$$(\star) \quad t\left(\frac{h_2(s)}{h_1(s)}\right)^a \in I \quad \text{and} \quad t\left(\frac{h_2(s)}{h_1(s)}\right)^a \cdot \frac{h(y)}{h_1(s)^b} \in I \text{ for any } a > 0,\ 0 < b < 1$$

and $y \in I$. Using $(\star)$ repeatedly one gets

$$\begin{aligned} F^{u+1}(x,y) &= F(F^u(x,y)) = F\left(f^u(x), t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot h(f^u(x))\right) \\ &\hfill \text{(by the induction hypothesis)} \\ &= \left(f^{u+1}(x), r_u\left(s, \theta\left(f^u(x), t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot h(f^u(x))\right)\right)\right) \\ &\hfill \text{(as } f^u(x) = f^u(x_0) + s\varepsilon_{f^u(x_0)} \in I_{f^u(x_0)}) \\ &= \left(f^{u+1}(x), r_u\left(s, t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot h(f^{u+1}(x))\right)\right) \\ &\hfill \text{(by } t(\tfrac{h_2(s)}{h_1(s)})^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \in I \text{ and the definition of } \theta) \\ &= \left(f^{u+1}(x), r_u\left(s, t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot \frac{h(f^{u+1}(x))}{h_1(s)^{1/m(u)}} \cdot h_1(s)^{1/m(u)}\right)\right) \end{aligned}$$

$$= \left(f^{u+1}(x), t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot \frac{h(f^{u+1}(x))}{h_1(s)^{1/m(u)}} \cdot h_2(s)^{1/m(u)}\right)$$

(by the definition of r_u; by $(\star)$ $t(\frac{h_2(s)}{h_1(s)})^{\frac{u-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot \frac{h(f^{u+1}(x))}{h_1(s)^{1/m(u)}} \in I$)

$$= \left(f^{u+1}(x), t\left(\frac{h_2(s)}{h_1(s)}\right)^{\frac{u+1-n_{2k}}{n_{2k+1}-n_{2k}}} \cdot h(f^{u+1}(x))\right)$$

(as $m(u) = n_{2k+1} - n_{2k}$).

It follows that (ii) is true for $j = u + 1$ (when $u = n_{2k+1} - 1$, note that $\frac{u+1-n_{2k}}{n_{2k+1}-n_{2k}} = 1 = \frac{n_{2(k+1)}-(u+1)}{n_{2(k+1)}-n_{2k+1}}$).

CASE 2: $u \in [n_{2k+1}, n_{2(k+1)})$ for some $k \in \mathbb{Z}_+$. A computation similar to that of Case 1 shows that (ii) is true for $u + 1$.

We have thus shown that (ii) is always true for $j = u + 1$. This finishes the proof of Claim 2. ■

Now we can show that $C := [x_0, x_0 + \varepsilon_{x_0}/2] \times [0, 1/2]$ is a scrambled set of the triangular map F. For any $(x_1, y_1) \neq (x_2, y_2) \in C$, there exist unique $s_i \in [0, 1/2]$ and $t_i \in [0, 1]$ such that $x_i = x_0 + s_i\varepsilon_{x_0}$ and $y_i = t_ih(x_i)$, as $h(x_i) = 1/2$, where $i = 1, 2$. Because $\lim_{n\to\infty} \varepsilon_{f^n(x_0)} = 0$, one has

$$\lim_{n\to\infty} \max\{|f^n(x_1) - f^n(x_0)|, |f^n(x_2) - f^n(x_0)|\} = 0. \tag{8.8}$$

Since (X, f) is proximal and p is the fixed point of (X, f), there exists an infinite sequence $\{m_l\}$ of natural numbers such that $\lim_{l\to\infty} f^{m_l}(x_0) = p$. By (8.8) this implies that $\lim_{l\to\infty} f^{m_l}(x_i) = p$, $i = 1, 2$. By Claim 2(ii), property $(\star)$ and since $h(p) = 0$, one further obtains $\lim_{l\to\infty} F^{m_l}(x_i, y_i) = (p, 0)$, $i = 1, 2$. The pair $((x_1, y_1), (x_2, y_2))$ is thus proximal.

It cannot be asymptotic. By the choice of the sequence $\{n_j\}$, we have $\lim_{j\to\infty} f^{n_j}(x_0) = x^* \neq p$, which by (8.8) implies that $\lim_{j\to\infty} f^{n_j}(x_i) = x^*$ for $i = 1, 2$. Again by Claim 2(ii) one has

$$\lim_{k\to\infty} F^{n_{2k}}(x_i, y_i) = (x^*, t_ih(x^*)),$$
$$\lim_{k\to\infty} F^{n_{2k+1}}(x_i, y_i) = \left(x^*, t_i\,\frac{h_2(s_i)}{h_1(s_i)}\,h(x^*)\right), \qquad i = 1, 2.$$

As $(s_1, t_1) \neq (s_2, t_2) \in [0, 1/2] \times [0, 1]$, it is easy to see that either $t_1 \neq t_2$ or $t_1h_2(s_1)/h_1(s_1) \neq t_2h_2(s_2)/h_1(s_2)$. As $h(x^*) \neq 0$ ($h(x) = 0$ only when $x = p$ and $x^* \neq p$) one has

$$\lim_{k\to\infty} F^{n_{2k}}(x_1, y_1) \neq \lim_{k\to\infty} F^{n_{2k}}(x_2, y_2)$$

or

$$\lim_{k\to\infty} F^{n_{2k+1}}(x_1, y_1) \neq \lim_{k\to\infty} F^{n_{2k}}(x_2, y_2).$$

So $((x_1, y_1), (x_2, y_2))$ is not asymptotic, which means that it is scrambled. Finally, since (x_1, y_1) and (x_2, y_2) are arbitrary, C is a scrambled set of $(I \times I, F)$ while obviously $\mathrm{int}(C) \neq \emptyset$. ∎

Let $F(x, y) = (f(x), g(x, y))$ be a triangular map in the square. We address the connection between scrambled pairs for f and scrambled pairs for F.

Example 60. There is a triangular map $F(x, y) = (f(x), g_x(y))$ of the unit square $[0, 1]^2$ such that for some x_1, x_2 one has:

> $\{x_1, x_2\}$ is a scrambled pair in the basis but there are no y_1, y_2 such that the pair $\{(x_1, y_1), (x_2, y_2)\}$ is a scrambled pair of F. (Neither does there exist $n \geq 0$ such that for some y_1, y_2, the pair $\{(f^n(x_1), y_1), (f^n(x_2), y_2)\}$ is a scrambled pair of F.)

Endow $[0, 1]^2$ with the distance

$$\underline{d}((x, y), (x', y')) = \sup\{d(x, x'), d(y, y')\}$$

where d is the usual distance on the unit interval. First define the basis map f. Let $f(0) = f(1) = 1/2$, $f(1/4) = 0$, $f(3/4) = 1$ and let f be affine on each of the intervals $[0, 1/4]$, $[1/4, 3/4]$ and $[3/4, 1]$; then f is continuous. There are seven important points: $0 < a < b < 1/2 < c < d < 1$ where $b = 1/4$, $c = 3/4$ and $a < 1/2 < d$ are all fixed points. These seven points divide the basis into six intervals $I_1, \ldots, I_6$ (numbered from left to right). The intervals $[0, 1/2]$ and $[1/2, 1]$ are invariant under f, and it is not hard to show that f is transitive in each of these two intervals.

No matter how we define the fibre maps g_x, the left and right halves of the square are F-invariant. The square can be partitioned into six vertical strips, namely the strips over the above mentioned six intervals. Denote these strips by $S_1, \ldots, S_6$, where $S_i = I_i \times [0, 1]$, from left to right. For $x \in I_1$ put $g_x(y) = 0$ (constant maps), for $x \in I_6$ put $g_x(y) = 1$, for $x \in I_3 \cup I_4$ put $g_x(y) = y$ and for $x \in I_2 \cup I_5$ fix g_x in such a way as to ensure continuity of F.

Let $x_1 \in I_3 \setminus \{1/2\}$ be a transitive point of f restricted to the interval $[0, 1/2]$ and let $x_2 = 1 - x_1 \in I_4 \setminus \{1/2\}$. By symmetry $f^n(x_2) = 1 - f^n(x_1)$. Then $\{x_1, x_2\}$ is a scrambled pair in the basis, owing to the fact that the closer $f^n(x_1)$ is to $1/2$, the closer $f^n(x_2)$ is to $1/2$, hence to $f^n(x_1)$. Fix y_1, y_2 arbitrarily. Notice that the basis map was defined in such a way that the points (except those in the three fixed fibres) move to the right if they are in $S_1 \cup S_4 \cup S_5$ and to the left if they are in $S_2 \cup S_3 \cup S_6$. Therefore $d(f^n(x_1), f^n(x_2)) < d(f^{n-1}(x_1), f^{n-1}(x_2))$ only when $f^{n-1}(x_1)$ and $f^{n-1}(x_2)$ lie in the intervals I_1 and I_6 respectively. Moreover, the points $f^{n-1}(x_1)$ and $f^{n-1}(x_2)$ must be mapped by f to I_3 and I_4 respectively, if one wishes $\underline{d}(F^n(x_1, y_1), F^n(x_2, y_2))$ to get close to 0. But then the distance between $F^n(x_1, y_1)$ and $F^n(x_2, y_2)$ is at least 1, owing to the definition of g_x in the strips S_1 and S_6 (the points $F^n(x_1, y_1)$ and $F^n(x_2, y_2)$ are on the bottom and top sides of the square respectively). Thus

$$\liminf_{n\to\infty} \underline{d}(F^n(x_1, y_1), F^n(x_2, y_2)) > 0,$$

so $\{(x_1, y_1), (x_2, y_2)\}$ is not a scrambled pair of F.

QUESTION. Is it true that "most" scrambled pairs of f must be projections of scrambled pairs of F? Or, conversely, can there exist a factor $(Y, F) \to (X, f)$ and an uncountable scrambled set $E \subseteq X$ which does not contain the image of any uncountable scrambled set of (Y, F)?

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