COLLOQUIUM MATHEMATICUM

VOL. 110 2008 NO. 2

FINITARY ORBIT EQUIVALENCE AND MEASURED BRATTELI DIAGRAMS

BY

T. HAMACHI (Fukuoka), M. S. KEANE (Middletown, CT, Eindhoven, and Yokohama), and M. K. ROYCHOWDHURY (Fort Collins, CO)

Abstract. We prove a strengthened version of Dye's theorem on orbit equivalence, showing that if the transformation structures are represented as finite coordinate change equivalence relations of ergodic measured Bratteli diagrams, then there is a finitary orbit equivalence between these diagrams.

1. Introduction. Two ergodic measure preserving transformations of nonatomic Lebesgue probability spaces are, by a classical theorem of Dye [D], *orbit equivalent*: there exists a measure preserving isomorphism between the underlying spaces taking orbits to orbits on an invariant set of full measure. In this article, we prove a strengthened version of Dye's theorem, showing that if the transformation orbit structures are represented by the procedure introduced initially by Vershik [V] as equivalence relations of ergodic measured Bratteli diagrams, then there is a *finitary* orbit equivalence between these diagrams. In other words, after a change of the transformation on invariant sets of measure zero, the transformation spaces acquire a Cantor-like topology from the Bratteli diagram representations, and the orbit equivalence is then given by a homeomorphism after removal of sets of measure zero in each space.

We remark that an interesting result concerning orbit equivalence has been obtained by N. Ormes [O]. He shows that any ergodic measure preserving transformation S of a probability Lebesgue space (X, μ) is measure conjugate to a homeomorphism T' whose orbits are the same as the T-orbits for a given Cantor minimal system (Y, T). However, in the case where (X, S) is also a Cantor system with an ergodic invariant measure μ, it is not clear that the measure conjugacy from X to Y in his construction is a homeomorphism after removal of sets of measure zero.

The structure of the article is as follows. First we recall the Vershik procedure in the simple setting we need in order to proceed—this seems to be an easier version than the original one in [V], but the ideas come from [V].

2000 *Mathematics Subject Classification*: 28D05, 37A05, 37A20.

Key words and phrases: orbit equivalence, ergodic, finitary, Bratteli diagram.

This part has also been known precisely in [O] as remarked above. Next, we describe our finitary construction leading to finitary orbit equivalence of ergodic measured Bratteli diagrams, which is similar to the isomorphism theorem for Bernoulli schemes as found in Keane–Smorodinsky ([KS1], [KS2]); earlier versions for odometers have been given in Hamachi–Keane [HK] and for irrational rotations in Roychowdhury [R]. We remark that this proof differs from other proofs of Dye's theorem and seems conceptually and calculationally simpler. In a subsequent article, which will be published elsewhere [HKY], we solve a problem of [GW] by obtaining a dynamical proof of the topological orbit equivalence theorem of Giordano, Putnam, and Skau [GPS].

2. Vershik's procedure. Let $(X,\mathcal{A},\mu,S)$ be an ergodic measure preserving system with $\mu(X)=1$; we suppose that $(X,\mathcal{A},\mu)$ is a nonatomic Lebesgue space, so that the transformation S is aperiodic. We will see that S admits a subset $B\in\mathcal{A}$ with return times 2 or 3. By an easy argument, there exists a set $A_0\in\mathcal{A}$ with $\mu(A_0)>0$ such that

$$A_0\cap SA_0=\emptyset \quad\text{and}\quad \bigcup_{i=0}^{\infty} S^iA_0 = X,$$

since otherwise we would have $Sx=x$ for some set $E\in\mathcal{A}$ of positive measure and for every $x\in E$. We put

$$\begin{aligned} A_1 &= SA_0,\\ A_n &= S^nA_0\setminus \bigcup_{i=0}^{n-1} S^iA_0, \qquad n\ge 2, \end{aligned}$$

and

$$B = S(A_1\cup A_3\cup A_5\cup\cdots).$$

Since the set $S(A_1\cup A_3\cup A_5\cup\cdots)$ is disjoint from $A_1\cup A_3\cup A_5\cup\cdots$,

$$B\cap SB=\emptyset.$$

Moreover,

$$B\cup SB\cup S^2B = X.$$

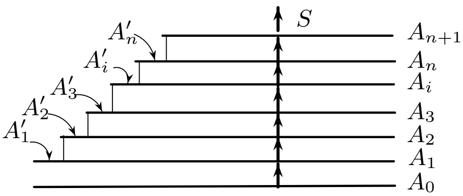
Indeed, put

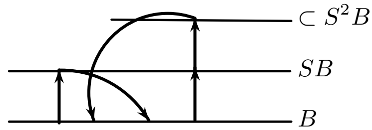
$$A'_n = A_n\setminus S^{-1}A_{n+1}, \qquad n\ge 1.$$

Then $B=A_2\cup A_4\cup A_6\cup\cdots\cup SA'_1\cup SA'_3\cup SA'_5\cup\cdots$, and $SB\supset A_3\cup A_5\cup A_7\cup\cdots\cup SA'_2\cup SA'_4\cup SA'_6\cup\cdots$. So $A_0=\bigcup_{i=1}^{\infty} SA'_i\subset B\cup SB$ and $A_1=SA_0\subset SB\cup S^2B$. Therefore, $B\cup SB\cup S^2B\supset A_0\cup A_1\cup A_2\cup A_3\cup\cdots=X$. Consequently, we see that if

$$\tau(x)=\min\{n\ge 1: S^n(x)\in B\},$$

then for each $x \in B$, $\tau(x)$ is equal to either 2 or 3.





Next, consider the induced transformation S_B defined by

$$S_B(x) := S^{\tau(x)}(x) \qquad (x \in B)$$

on B, and apply the same argument. Using induction, this yields a cutting-and-stacking procedure whose stacks at the nth stage are of height at least 2^n; taking further a sequence $B_1, B_2, \ldots$ of sets in $\mathcal{A}$ which generate $\mathcal{A}$ and refining the stacks at each stage n so that the sets of this stage include B_n, we obtain a cutting-and-stacking procedure which fully represents the original transformation $(X, \mathcal{A}, \mu, S)$, up to a set of measure zero in the way that each interval of all of the stacks has length equal to the measure of the corresponding measurable subset of X. Now define the measured Bratteli diagram (see next section for definitions) associated with this procedure by letting the vertex set V_n at stage n be the set of stacks at level n, and by drawing an edge in E_n from a stack $u \in V_{n-1}$ to a stack $v \in V_n$ for each piece of the stack u which is cut by the procedure and placed in the stack v. Each finite path $a = a_1 \ldots a_n$ of the diagram starting from the root and consisting of n edges a_i corresponds to an interval appearing in the cutting-and-stacking procedure. a is equipped with the same measure as the length of the interval, and all the other finite paths $b = b_1 \ldots b_n$ starting from the root and of the same length n and the same range vertex, $r(b_n) = r(a_n)$, have the same measure as a does. Thus we have given a brief proof sketch of:

2.1. Theorem (Vershik [V]). *Let $(X, \mathcal{A}, \mu, S)$ be an ergodic measure preserving transformation of a nonatomic Lebesgue probability space. Then there exists a measured Bratteli diagram and an isomorphism of $(X, \mathcal{A}, \mu)$ with the path space of this diagram which takes S-orbits in X to full equivalence classes of the diagram.*

3. Bratteli diagram and finite coordinate change relation. As is well known (see [B], [HPS]), a *Bratteli diagram* is a countable graph $G = (V, E)$, whose vertex set V and edge set E satisfy the following conditions:

(1) V and E are countable unions of pairwise disjoint finite sets V_n, $n \geq 0$, and E_n, $n \geq 1$, respectively, where V_0 is a singleton set.

(2) E is equipped with source maps $s = s_n : E_n \to V_{n-1}$, and range maps $r = r_n : E_n \to V_n$. Call $s(a)$ and $r(a)$ for $a \in E$ the *source vertex* and the *range vertex* of a.
(3) $s^{-1}(v)$ is nonempty for all $v \in V$ and $r^{-1}(v)$ is nonempty for all $v \in V \setminus V_0$.

The graph G is associated with a one-sided infinite edge space $X = X_G$:

$$X = \{(a_n)_{n\geq 1} : a_n \in E_n,\ r(a_n) = s(a_{n+1})\ (n \geq 1)\}.$$

It is a compact space with the topology induced from the product topology of the infinite product space of the sets E_n for $n \geq 1$, each equipped with the discrete topology. We let $\mathcal{A}$ be the σ-algebra consisting of all Borel subsets of X.

On $(X, \mathcal{A})$ one defines a *finite coordinate change equivalence relation* $\mathcal{S} = \mathcal{S}_X$ by letting, for x and y in X, $x \sim y$ if there is an integer $n \geq 1$ such that $x_i = y_i$ for $i \geq n$. For $x \in X$ we denote by $\mathcal{S}(x)$ the equivalence class $\{y \in X : x \sim y\}$, and call these classes $\mathcal{S}$-*orbits*.

For $n \geq 1$, we denote by $\mathcal{S}_n$ the subrelation of $\mathcal{S}$, $x \overset{n}{\sim} y$, defined by $x_i = y_i$ for all $i > n$. We denote by $\mathcal{S}_n(x)$ the $\mathcal{S}_n$-equivalence class $\{z \in X : x \overset{n}{\sim} z\}$. For simplicity we identify the cylinder

$$[a_1 \ldots a_n] = \{x \in X : x_i = a_i,\ 1 \leq i \leq n\}$$

of X with the word $a_1 \ldots a_n$. If the cylinder is nonempty, the word is said to be *admissible.* For $m \leq n$, $\mathcal{S}_m$ is also considered to be an equivalence relation on the set of all admissible words or cylinders $a_1 \ldots a_n$ of length n, namely

$$a_1 \ldots a_n \overset{m}{\sim} b_1 \ldots b_n \quad \text{if } a_i = b_i \text{ for all } m < i \leq n.$$

That is, a word $a_1 \ldots a_n$ corresponds to a *path* in the graph G from the root to level n, and two words of length n are m-equivalent ($\overset{m}{\sim}$) if the corresponding paths terminate at level m at the same vertex in V_m and coincide from level $m+1$ to n. When $m = n$, we also write $a_1 \ldots a_n \sim b_1 \ldots b_n$ if $a_1 \ldots a_n \overset{n}{\sim} b_1 \ldots b_n$.

4. Measured Bratteli diagram. Given a Bratteli diagram $G = (V, E)$, let $X = X_G$ and $\mathcal{S}$ be the finite coordinate change relation on $(X, \mathcal{A})$. In what follows we will just write X instead of $(X, \mathcal{A})$. A probability measure μ on X is said to be $\mathcal{S}$-*invariant* if for all $n \geq 1$ and all admissible cylinders $a_1 \ldots a_n$ and $b_1 \ldots b_n$ with $a_1 \ldots a_n \overset{n}{\sim} b_1 \ldots b_n$,

$$\mu(a_1 \ldots a_n) = \mu(b_1 \ldots b_n).$$

μ (or $\mathcal{S}$) is said to be *ergodic* if any $\mathcal{S}$-invariant set $E \in \mathcal{A}$ (i.e. for a.e. $x \in E$, $x' \sim x$ implies $x' \in E$) has measure 0 or 1.

4.1. PROPOSITION. *Any G admits an $\mathcal{S}$-invariant probability measure.*

Proof. Fix $x \in X$, and for $n \geq 1$ let μ_n^x be the probability measure on X defined by

$$\mu_n^x = \frac{1}{\sharp \mathcal{S}_n(x)} \sum_{y \in \mathcal{S}_n(x)} \delta_y,$$

where δ_y means a Dirac measure. Then μ_n^x is $\mathcal{S}_n$-invariant. Actually for $a = a_1 \cdots a_n$,

$$\mu_n^x(a) = 1/A$$

where A is the number of words of length n ending at the vertex $r(a_n)$. Since the set of all probability measures on a compact metric space is weak*-compact, one obtains a sequence of integers $n_1 < n_2 < \cdots$ such that

$$\mu^{(x)} = w^*\text{-} \lim_{k \to \infty} \frac{1}{\sharp \mathcal{S}_{n_k}(x)} \sum_{y \in \mathcal{S}_{n_k}(x)} \delta_y$$

exists. Then $\mu^{(x)}$ is an $\mathcal{S}$-invariant probability measure. ∎

We remark two things on $\mathcal{S}$-invariant probability measures μ. The one is that the measure $\mu(a_1 \ldots a_n)$ of $a_1 \cdots a_n$ only depends on $r(a_n)$. The other is that there is a subgraph $G' = (V', E')$ of G such that μ has support on the closed subset $X' = X_{G'}$ of X on which μ is regular. The Bratteli diagram G' is obtained by removing all the edges $e \in E$ together with the range vertex $r(e)$ if $e \in E_n$ and there is a path $e_1 \ldots e_{n-1} e$ in G such that

$$\mu[e_1 \ldots e_{n-1} e] = 0.$$

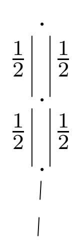
We note that then all the other edges f in E_n ending at $r(e)$ are also removed, because any path $f_1 \ldots f_{n-1} f$ in G also satisfies

$$\mu[f_1 \ldots f_{n-1} f] = \mu[e_1 \ldots e_{n-1} e] = 0.$$

Thus the rest G' of G also satisfies the conditions (1)–(3) of the previous section, and the measure μ is an $\mathcal{S}'$-invariant probability measure on $X' = X_{G'}$, where $\mathcal{S}'$ is the finite coordinate change equivalence relation, and takes positive measure for all cylinders of X'. So whenever a Bratteli diagram $G = (V, E)$ and an $\mathcal{S}$-invariant probability measure μ on $X = X_G$ are given, we may and will assume that μ is regular.

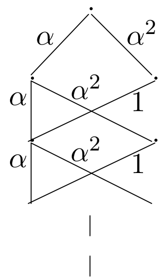
Here are examples of measured Bratteli diagrams:

4.2. Example (Binary odometer).



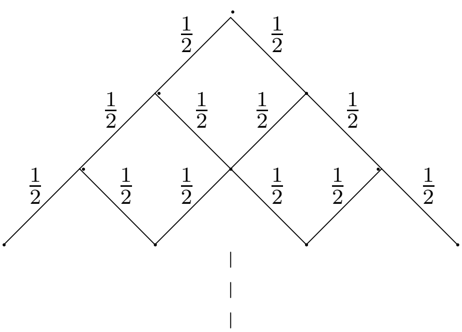
The finite coordinate change relation $\mathcal{S}$ arising from the measured Bratteli diagram is called the *binary odometer*.

4.3. EXAMPLE (Irrational rotation).



The finite change relation arising from the diagram is called the *irrational rotation* by $\alpha = (\sqrt{5}-1)/2$.

4.4. EXAMPLE (Binomial relation).



The finite coordinate change relation arising from the diagram is called the *binomial relation*.

5. Hopf equivalence. Let $G = (V, E)$ be a Bratteli diagram and μ an ergodic, nonatomic, $\mathcal{S}$-invariant probability measure on $X = X_G$. We assign in arbitrary manner the linear order from 0 to $k-1$ to the set of edges $e \in E_n$ ending at the vertex $v \in V_n$, for each $n \geq 1$ and $e \in E_n$, where k is the number of these edges. We denote by $\lambda(e)$ the order of the edge e. Call the edges e and f the *minimal* and *maximal* edges if $\lambda(e) = 0$ and $\lambda(f) = k-1$. Write

$$\begin{aligned}
X_{\min} &= \{x \in X : x_n \text{ is minimal for all } n \geq 1\},\\
X_{\max} &= \{x \in X : x_n \text{ is maximal for all } n \geq 1\},\\
X^{(n)}_{\min} &= \{x \in X : x_i \text{ is minimal for all } 1 \leq i \leq n\},\\
X^{(n)}_{\max} &= \{x \in X : x_i \text{ is maximal for all } 1 \leq i \leq n\}.
\end{aligned}$$

The clopen sets $X_{\min}^{(n)}$ and $X_{\max}^{(n)}$ are decreasing and satisfy

$$\bigcap_{n\geq 1} X_{\min}^{(n)} = X_{\min} \quad \text{and} \quad \bigcap_{n\geq 1} X_{\max}^{(n)} = X_{\max}.$$

Note that $X_{\min}$ and $X_{\max}$ are nonempty and $\mu(X_{\min}^{(n)}) = \mu(X_{\max}^{(n)})$, and hence $\mu(X_{\min}) = \mu(X_{\max})$. If $z, z' \in X_{\min}$ and $z \sim z'$ then $z = z'$. Since μ is ergodic and nonatomic, this means that

$$\mu(X_{\min}) = \mu(X_{\max}) = 0.$$

Now we define the one-to-one map $S : X_{\max}^c \to X_{\min}^c$ by setting

$$Sx = x', \qquad x \in X_{\max}^c,$$

where x' is next to x in the sense that if we let

$$n = \min\{i \geq 1 : x_i \text{ is not maximal}\}$$

then

$$\begin{aligned} \lambda(x'_n) &= \lambda(x_n) + 1, && i < n, \\ \lambda(x'_i) &= 0, && i < n, \\ x'_i &= x_i, && i > n. \end{aligned}$$

S is continuous on the open set $X_{\max}^c$ and so also is S^{-1} on $X_{\min}^c$, and

$$\mathcal{S}(x) = \{S^n x : n \in \mathbb{Z}\}, \qquad x \notin \mathcal{S}(X_{\max}) \cup \mathcal{S}(X_{\min}).$$

So, S is a measure preserving automorphism of X, and is called an *adic transformation* ([V]). We define the $\mathcal{S}$-invariant measurable subset

$$X_0 = \{x \notin \mathcal{S}(X_{\max}) \cup \mathcal{S}(X_{\min}) : x \text{ is not periodic under } S\}$$

of full measure.

5.1. Definition. Let C be an open subset of X. We call a countable set $\mathcal{C} = \{C_\xi : \xi \in \Lambda\}$ of cylinders a *cylinder partition* of C if the cylinders are pairwise disjoint subsets of C and

$$\sum_{\xi\in\Lambda} \mu(C_\xi) = \mu(C).$$

5.2. Definition. Let O_1 and O_2 be open subsets of X. We say that they are *Hopf equivalent* to each other if there are cylinder partitions $\{C_{1,i} : i \geq 1\}$ of C_1 and $\{C_{2,i} : i \geq 1\}$ of C_2 such that

$$C_{1,i} \sim C_{2,i}, \qquad i \geq 1.$$

For any open set $O \subset X$ the return time τ under S to O is defined, because $\mu(O) > 0$ and S is conservative. For any point $x \in X_0$ and any integer $k \geq 1$, the points $S^i x$, $0 \leq i < k$, are all different. Moreover, one then obtains an integer $n \geq 1$ such that

$$(S^i x)_j = x_j, \qquad 0 \leq i < k,\ j \geq n.$$

One also sees that for any $x' \in X$ with $x_j = x'_j$, $1 \le j < n$,

$$(S^i x')_j = x'_j, \quad 0 \le i < k,\, j \ge n,$$

and

$$(S^i x)_j = (S^i x')_j, \quad 0 \le i < k,\, 1 \le j < n.$$

This means that for the cylinder $C = [x_1 \dots x_{n-1}]$, the return time $\tau(x)$ to C satisfies

$$\tau(x) \ge k, \quad x \in C.$$

5.3. Lemma. *Let $C \subset X$ be a clopen set. Then the return time τ to C satisfies, for each integer $k \ge 1$,*

$$\mu(C^k) = \mu(\operatorname{Int} C^k),$$

where $C^k = \{x \in C : \tau(x) = k\}$ and Int *denotes interior, and hence τ is a continuous function on C up to a null set.*

Proof. C is the union of a finite number of pairwise disjoint cylinders $E^r = c_{r,1} \dots c_{r,q}$, $1 \le r \le p$, of the same length q for some $q \ge 1$. It is enough to show that for any $1 \le r \le p$ and any $k \ge 1$ with $\mu(C^k) > 0$, any point $x \in X_0 \cap E^r \cap C^k$ is an interior point of C^k. One has an integer $n \ge q$ satisfying the following conditions:

(i) The words $(S^i x)_1 \dots (S^i x)_n$, $0 \le i < k$, are all different.
(ii) Any word in (i) except $x_1 \dots x_n$ has no prefix in $\{c_{r,1} \dots c_{r,q} : 1 \le r \le p\}$.
(iii) $(S^i x)_j = x_j$, $j > n$, $1 \le i \le k$.
(iv) $(S^k x)_1 \dots (S^k x)_q \in \{c_{r,1} \dots c_{r,q} : 1 \le r \le p\}$.

Let x' be any point in X such that

$$x'_j = x_j, \quad 1 \le j \le n.$$

Then

$$(S^i x')_1 \dots (S^i x')_n = (S^i x)_1 \dots (S^i x)_n, \quad 1 \le i \le k,$$
$$(S^i x')_j = x'_j, \quad j > n,\ 1 \le i \le k,$$
$$(S^k x')_j = (S^k x)_j, \quad 1 \le j \le q,$$

that is, $x' \in C^k$. ∎

5.4. Remark. The proof of Lemma 5.3 shows that there is a cylinder partition $\{C_{k,i,j} : i, k \ge 1,\, 0 \le j < k\}$ of X satisfying:

1. $C_{k,i,0} \subset C^k$, $i, k \ge 1$.
2. $\sum_{i \ge 1} \mu(C_{k,i,0}) = \mu(C^k)$, $k \ge 1$.
3. $C_{k,i,j} = S^j(C_{k,i,0})$, $0 < j < k$, $i, k \ge 1$.
4. $C_{k,i,0} \sim C_{k,i,1} \sim \dots \sim C_{k,i,k-1}$, $i, k \ge 1$.

5.5. Proposition. *Let $A, B \subset X$ be clopen sets. If $\mu(A) = \mu(B)$, then A and B are Hopf equivalent to each other.*

Proof. Let $0 < \varepsilon < \mu(A)/2$. By choosing a clopen subset $A' \subset A$ such that $\mu(A') > \mu(A) - \varepsilon$, it is enough to construct finitely many pairwise disjoint, and the same number of cylinders $[a_i]$ and $[b_i]$ satisfying the following conditions:

(i) $[a_i] \subset A$, $[b_i] \subset B$,
(ii) $[a_i] \sim [b_i]$, $i \geq 1$,
(iii) $\sum_i \mu([a_i]) > \mu(A') - \varepsilon$.

It follows from the ergodic theorem that there exists a measurable subset $E \subset X$ with $\mu(E) < \varepsilon$ and an integer $N \geq 1$ such that if $n \geq N$ and $x \notin E$ then

$$\sharp\{0 \leq k < n : S^k x \in A'\} < \sharp\{0 \leq k < n : S^k x \in B\},$$
$$\sharp\{0 < k \leq n : S^{-k} x \in A'\} < \sharp\{0 < k \leq n : S^{-k} x \in B\}.$$

Take an integer $M \geq 2/\varepsilon$ and a cylinder $C \subset X$ such that the return time τ to C satisfies

$$\tau(x) \geq MN, \qquad x \in C.$$

Then by Remark 5.4 one has a cylinder partition $\{C_{k,i,j} : 0 \leq j < k,\ i \geq 1,\ k \geq MN\}$ of X satisfying:

1. $C_{k,i,0} \subset \{x \in C : \tau(x) = k\}$, $i \geq 1$, $k \geq MN$.
2. $\sum_{i \geq 1} \mu(C_{k,i,0}) = \mu(\{x \in C : \tau(x) = k\})$, $k \geq MN$.
3. $C_{k,i,j} = S^j(C_{k,i,0})$, $0 \leq j < k$, $i \geq 1$, $k \geq MN$.
4. $C_{k,i,0} \sim C_{k,i,1} \sim \cdots \sim C_{k,i,k-1}$, $i \geq 1$, $k \geq MN$.

By decomposing the cylinder $C_{k,i,0}$ into finitely many cylinders for each k and each i if necessary, we may assume that the set A' (resp. B) is the union of a countable number of the cylinders $C_{k,i,j}$ up to a null set.

Set

$$\Gamma = \{(k,i) \in [MN,\infty) \times [1,\infty) : \exists j \text{ with } N \leq j < k - N \text{ such that } C_{k,i,j} \cap E^c \neq \emptyset\},$$

and

$$H = \bigcup_{(k,i)\in\Gamma} \bigcup_{j=N}^{k-N-1} C_{k,i,j}.$$

Then $\mu(H) > 1 - 2\varepsilon$ and for each $(k,i) \in \Gamma$, the number of cylinders in the collection $\{C_{k,i,j} : 0 \leq j < k\}$ that are contained in A' is smaller than the number of those contained in B. Making a maximal number of one-to-one matchings between these cylinders completes the proof. ■

6. Tower partition. Let us use some definitions similar to those in [HK]. Let $G = (V, E)$ be a Bratteli diagram, $X = X_G$, and μ an $\mathcal{S}$-invariant probability measure on X.

6.1. Definition. Let C be an open set of X. A *tower partition* of C is a cylinder partition $\mathcal{C}$ of C endowed with an equivalence relation such that each equivalence class consists of cylinders having the same length and the same range vertex, and in each equivalence class, a specified cylinder called a *bottom cylinder* is designated. The union of the bottom cylinders is called the *bottom of the tower partition*, and is denoted by $B(\mathcal{C})$.

Given a tower partition $\mathcal{C}$, for each equivalence class $\{C_1, \ldots, C_d\}$ of $\mathcal{C}$ and each $1 \le i, j \le d$ we introduce partially defined homeomorphisms $\mathcal{C}_{j,i} : C_i \to C_j$ by setting

$$\mathcal{C}_{j,i}(x) = x', \qquad x \in C_i,$$

where $x' \in C_j,\ x_k = x'_k,\ k > n$, and n is the common length of the cylinders C_i and C_j. Then

$$\mathcal{C}_{i,i} = \mathrm{Id}|_{C_i} \quad \text{and} \quad \mathcal{C}_{k,j}\mathcal{C}_{j,i} = \mathcal{C}_{k,i} \quad (1 \le i, j, k \le d).$$

We call $\mathcal{C}_{j,i}$'s the *associated homeomorphisms* of $\mathcal{C}$. For $x \in C_i$ and $x' \in C_j$, we write

$$x \overset{\mathcal{C}}{\sim} x' \quad \text{if } x' = \mathcal{C}_{j,i}x,$$

and $\mathcal{C}(x) = \{x' : x \overset{\mathcal{C}}{\sim} x'\}$.

For a subset $E \subset C_i$ and j, we let $E' = \mathcal{C}_{j,i}(E)$, and say that $E' \subset C_j$ is induced from E by the $\mathcal{C}$-equivalence of C_i, or briefly that E' is *induced from E by $\mathcal{C}$*, and write

$$E \overset{\mathcal{C}}{\sim} E'.$$

When we discuss an equivalence $A = [a_1 \ldots a_m] \overset{m}{\sim} B = [b_1 \ldots b_m]$ of cylinders, we also say that for a subset $E \subset A$ the subset $E' \subset B$ defined below is induced from E by the equivalence:

$$E' = \{y : y_i = b_i\ (1 \le i \le m),\ y_i = x_i\ (i > m) \text{ for some } x \in E\}.$$

We then write

$$E \overset{m}{\sim} E'.$$

Now we define tower extension and tower refinement, similarly to [HK].

6.2. Definition. An *extension* of an equivalence class of a tower partition is defined to be the set of cylinders constructed by juxtaposing the same word to the cylinders belonging to the equivalence class. That is, for a word w, it is the set of cylinders cw, where c ranges over the equivalence class, and where w can be an empty word.

6.3. Definition. Let $\mathcal{C}$ be a tower partition of an open subset C of X, and $\mathcal{E}$ be a tower partition of the bottom $B(\mathcal{C})$ such that $\mathcal{E}$ is, as a partition, finer than or equal to the partition of $B(\mathcal{C})$ consisting of the bottom cylinders of the equivalence classes of $\mathcal{C}$. We let $\mathcal{C}'$ be the tower partition of C constructed in the way that for every equivalence class of $\mathcal{E}$ the equivalence class of $\mathcal{C}'$ is the collection of cylinders induced from every cylinder in the $\mathcal{E}$-equivalence class by $\mathcal{C}$, where the bottom cylinder of the $\mathcal{E}$-equivalence class is also the bottom cylinder of the $\mathcal{C}'$-equivalence class. $\mathcal{C}'$ is called a *tower refinement* of $\mathcal{C}$, and $\mathcal{E}$ is also said to be *extended* to C' by $\mathcal{C}$. In particular, if $\mathcal{E}$ is trivial as an equivalence relation, we say that the tower refinement $\mathcal{C}'$ is a *tower extension* of $\mathcal{C}$. In other words, each equivalence class of the tower extension is an extension of an equivalence class of $\mathcal{C}$.

A tower refinement $\mathcal{C}'$ of a tower partition $\mathcal{C}$ is, in other words, a refinement of $\mathcal{C}$ as a partition, and as an equivalence relation, $\mathcal{C}$ is a subrelation of $\mathcal{C}'$. Each cylinder $c' \in \mathcal{C}'$ is contained in a unique $c \in \mathcal{C}$, yielding a map $\pi : \mathcal{C}' \to \mathcal{C}$ which preserves the measure μ. Let $\mathcal{E}^{(0)}$ be the cylinder partition of $B(\mathcal{C})$, which is the restriction of $\mathcal{C}'$ to $B(\mathcal{C})$ as a partition, and endow $\mathcal{E}^{(0)}$ with the trivial equivalence relation. The tower refinement of $\mathcal{C}$ extended from the tower partition $\mathcal{E}^{(0)}$ to C is a tower extension of $\mathcal{C}$ and is denoted by $\hat{\mathcal{C}}$. It is a subrelation of $\mathcal{C}'$ and we call the equivalence relation of $\hat{\mathcal{C}}$ the *subrelation of* $\mathcal{C}'$ *coming from* $\mathcal{C}$. If c' and $\tilde{c}'$ are equivalent in $\hat{\mathcal{C}}$, then we write $c' \approx \tilde{c}'$, namely $c' = cu$, $\tilde{c}' = \tilde{c}u$ for some word u, where $\pi(c') = c,\ \pi(\tilde{c}') = \tilde{c},\ c \sim \tilde{c}$.

Clearly, if $\mathcal{C}'$ is a tower refinement of $\mathcal{C}$ and $\mathcal{C}''$ is a tower refinement of $\mathcal{C}'$, then $\mathcal{C}''$ is also a tower refinement of $\mathcal{C}$—in this case, we write $c'' \cong \tilde{c}''$ for the subrelation of $\mathcal{C}''$ coming from $\mathcal{C}$, namely, $c'' = cuv$ and $\tilde{c}'' = \tilde{c}uv$ for some word v, where $c = \pi^2(c''),\ \tilde{c} = \pi^2(\tilde{c}''),\ c \overset{\mathcal{C}}{\sim} \tilde{c}$, and where $\pi(c'') = cu,\ \pi(\tilde{c}'') = \tilde{c}u$.

7. $(1-\varepsilon)$**-**$\mathcal{S}_n$**-invariant tower partition.** We will introduce a notion of $(1-\varepsilon)$-$\mathcal{S}_n$-invariant tower partition, which is quite similar to $(1-\varepsilon)$-cyclic tower partition of odometers [HK]. Let X be the path space associated with a Bratteli diagram $G = (V, E)$ and μ be an $\mathcal{S}$-invariant probability measure.

7.1. Definition. Let $0 < \varepsilon < 1$ and $n \geq 1$. A tower partition $\mathcal{C}$ is said to be $(1-\varepsilon)$-$\mathcal{S}_n$-*invariant* if there exists a union $\mathcal{E}$ of finitely many $\mathcal{C}$-equivalence classes of cylinders of the same length $\geq n$ such that

$$\sum_{c \in \mathcal{E}} \mu(c) > 1 - \varepsilon,$$

and for $c, c' \in \mathcal{E}$,

$$c \overset{n}{\sim} c' \ \Rightarrow\ c \overset{\mathcal{C}}{\sim} c'.$$

In the definition above it is easily seen that there exists a $\mathcal{C}$-invariant set E of positive measure $> 1-\varepsilon$ such that for x and x' in E, if $x \overset{n}{\sim} x'$ then $x \overset{\mathcal{C}}{\sim} x'$.

7.2. PROPOSITION. *Let $\mathcal{C}$ be a tower partition, $n \geq 1$, and $0 < \varepsilon < 1$. There exists a tower refinement $\mathcal{C}'$ of $\mathcal{C}$ which is $(1-\varepsilon)$-$\mathcal{S}_n$-invariant.*

Proof. Take a union $\mathcal{E}$ of finitely many $\mathcal{C}$-equivalence classes so that

$$\sum_{c\in\mathcal{E}} \mu(c) > 1-\varepsilon.$$

Also take an extension $\hat{\mathcal{C}}$ of $\mathcal{C}$ so that every cylinder $\hat{c} \in \hat{\mathcal{C}}$ either is in $\mathcal{C}$, or has constant length $\geq n$ and is contained in a cylinder $c \in \mathcal{E}$. Put

$$\hat{\mathcal{E}} = \{\hat{c} \in \hat{\mathcal{C}} : \hat{c} = cu \text{ for some } c \in \mathcal{E} \text{ and } u\}.$$

We define the tower partition $\mathcal{C}'$ by letting, for $\hat{c}, \hat{c}' \in \hat{\mathcal{C}}$,

$$\hat{c} \overset{\mathcal{C}'}{\sim} \hat{c}'$$

if either $\hat{c}, \hat{c}' \notin \hat{\mathcal{E}}$, $\hat{c}, \hat{c}' \in \mathcal{C}$ and $\hat{c} \overset{\mathcal{C}}{\sim} \hat{c}'$, or $\hat{c}, \hat{c}' \in \hat{\mathcal{E}}$ and there exist a finite number of cylinders $\hat{c}_i \in \hat{\mathcal{E}}$, $1 \leq i < k$ and $\hat{c}'_i \in \hat{\mathcal{E}}$, $2 \leq i \leq k$, such that

$$\begin{aligned} &\hat{c}_i \overset{n}{\sim} \hat{c}'_{i+1}, \quad 1 \leq i < k, \\ &\hat{c}_1 \overset{\hat{\mathcal{C}}}{\sim} \hat{c}, \quad \hat{c}'_k \overset{\hat{\mathcal{C}}}{\sim} \hat{c}', \quad \hat{c}'_i \overset{\hat{\mathcal{C}}}{\sim} \hat{c}_i, \quad 1 < i < k. \end{aligned}$$

Then $\mathcal{C}'$ is $(1-\varepsilon)$-$\mathcal{S}_n$-invariant. ∎

8. Orbital extension and tower map. As seen in [HK] and [R], a tower map is a central ingredient of finitary orbit equivalence. We will introduce a more general notion of a tower map for finite coordinate change relations.

8.1. LEMMA. *Let $C \subset X$ be an open set, and $\{r_i\}_{i\geq 1}$ be a finite or countable sequence of positive numbers such that*

$$\sum_{i\geq 1} r_i = \mu(C).$$

Then there exist pairwise disjoint cylinders $C_{i,j} \subset C$, $i, j \geq 1$, satisfying, for each $i \geq 1$,

$$\sum_{j\geq 1} \mu(C_{i,j}) = r_i.$$

Proof. We inductively show that there exist positive numbers $\varepsilon_{i,j}$ with $\varepsilon_{i,j} < \min\{r_i, 1/(i+j)\}$ and pairwise disjoint clopen subsets $C_{i,j}$ of C for

$1 \le j \le n-i+1,\ 1 \le i \le n$ and $n \ge 1$ satisfying

$$(*) \qquad r_i - \sum_{j=1}^{n-i} \mu(C_{i,j}) > \mu(C_{i,n-i+1}) > r_i - \sum_{j=1}^{n-1} \mu(C_{i,j}) - \varepsilon_{i,n-i+1} \qquad (1 \le i \le n,\ n \ge 1).$$

For $n = 1$, it is enough to choose a positive number $\varepsilon_{1,1} < r_1$ and a clopen subset $C_{1,1}$ of C such that

$$r_1 - \varepsilon_{1,1} < \mu(C_{1,1}) < r_1.$$

Suppose $(*)$ holds for n. Put

$$D_i = \bigcup_{j=1}^{n-i+1} C_{i,j} \quad \text{for } 1 \le i \le n, \quad D = \bigcup_{i=1}^{n} D_i.$$

The induction hypothesis makes it possible to choose small positive numbers $\varepsilon_{i,n+i-2} < 1/(n+2)$ for $1 \le i \le n$ such that

$$r_i - \mu(D_i) - \varepsilon_{i,n+i-2} > 0.$$

We also choose a small positive number $\varepsilon_{n+1,1} < r_{n+1}$. Since

$$\mu(C \setminus D) > r_{n+1} + \sum_{i=1}^{n} (r_i - \mu(D_i)),$$

one then has pairwise disjoint clopen subsets $C_{i,n+1-i}$, $1 \le i \le n+1$, of $C \setminus D$ such that

$$r_i - \mu(D_i) > \mu(C_{i,n+2-i}) > r_i - \mu(D_i) - \varepsilon_{i,n+2-i} \quad (1 \le i \le n)$$

and

$$r_{n+1} > \mu(C_{n+1,1}) > r_{n+1} - \varepsilon_{n+1,1}.$$

This means that $(*)$ holds for $n+1$.

Letting n tend to infinity for $i \ge 1$ in the inequality

$$r_i > \mu\Big(\bigcup_{j=1}^{n-i+1} C_{i,j}\Big) > r_i - \varepsilon_{n-i+1},$$

one then shows

$$\sum_{j=1}^{\infty} \mu(C_{i,j}) = r_i.$$

The proof is completed by taking a finite partition of $C_{i,j}$ into cylinders for each $i \ge 1$ and $j \ge 1$. ∎

The basic idea of the following lemma comes from [HK, Lemma 2, the argument in Section 3, and the proof of Proposition 4.3], where the cylinders of the binary odometer space X and the ternary odometer space Y are compared with each other as subintervals of $[0, 1]$ through some identification.

8.2. Lemma. *Assume that* $\mathcal{S}_Y$ *is ergodic. Let* $n \geq 1$ *and* $C_{i,j}$, $1 \leq j \leq n$, $i \geq 1$, *be pairwise disjoint cylinders of* X, *and* D_j, $1 \leq j \leq n$, *be pairwise disjoint cylinders of* Y. *Suppose that for each* $i \geq 1$, *the cylinders* $C_{i,j}$, $1 \leq j \leq n$, *have the same length*, *and*

$$C_{i,1} \sim \cdots \sim C_{i,n} \quad \textit{and} \quad \sum_{i \geq 1} \mu(C_{i,j}) = \nu(D_j) \quad (1 \leq j \leq n).$$

Then there are pairwise disjoint cylinders $D_{i,j,k} \subset D_j$, $i, k \geq 1$ $(1 \leq j \leq n)$, *satisfying the following conditions*:

(1) *For* $i, k \geq 1$, *the cylinders* $D_{i,j,k}$, $1 \leq j \leq n$, *have the same length*, *and*

$$D_{i,1,k} \sim D_{i,2,k} \sim \cdots \sim D_{i,n,k}.$$

(2) $\sum_{k \geq 1} \nu(D_{i,j,k}) = \mu(C_{i,j})$ $(i \geq 1,\ 1 \leq j \leq n)$.

Proof. First apply Proposition 5.5 for the cylinders $D_1, \ldots, D_n$ to get pairwise disjoint cylinders $D_{j,l} \subset D_j$, $l \geq 1, 1 \leq j \leq n$, such that for each $l \geq 1$ the cylinders $D_{j,l}$, $1 \leq j \leq n$, have the same length, and

$$D_{1,l} \sim D_{2,l} \sim \cdots \sim D_{n,l} \quad \text{and} \quad \sum_{l \geq 1} \nu(D_{j,l}) = \nu(D_j) \quad (1 \leq j \leq n).$$

Next apply Lemma 8.1 for the cylinder D_1 and the numbers $\mu(C_{i,1})$, $i \geq 1$, to get pairwise disjoint cylinders $D'_{i,k} \subset D_1$, $i, k \geq 1$, such that

$$\sum_{k \geq 1} \nu(D'_{i,k}) = \mu(C_{i,1}), \quad i \geq 1.$$

One then has a common refinement of the partitions $\{D'_{i,k} : i, k \geq 1\}$ and $\{D_{1,l} : l \geq 1\}$ of D_1. Write

$$D_{i,1,k,l} = D'_{i,k} \cap D_{1,l}, \quad i, k \geq 1, l \geq 1.$$

For $l \geq 1$, the equivalences $D_{1,l} \sim D_{2,l} \sim \cdots \sim D_{n,l}$ induce pairwise disjoint cylinders $D_{i,j,k,l} \subset D_{j,l}$ such that for $l \geq 1$,

$$D_{i,1,k,l} \sim D_{i,j,k,l} \quad (i, k \geq 1) \quad \text{and} \quad \sum_{i \geq 1} \sum_{k \geq 1} \nu(D_{i,j,k,l}) = \nu(D_{j,l}).$$

These cylinders $D_{i,j,k,l}$, with subscripts i, j, and (k, l), satisfy conditions (1)–(3) of the lemma. ∎

8.3. Definition. Let $C \subset X$ and $D \subset Y$ be open sets such that $\mu(C) = \nu(D)$, and $\mathcal{C} = \{C_{h,i,j} : h, i \geq 1,\ 1 \leq j \leq n(h)\}$ be a tower partition of C, where $n(h) \geq 1$ for $h \geq 1$ and

$$\{C_{h,i,1}, C_{h,i,2}, \ldots, C_{h,i,n(h)}\}, \quad h, i \geq 1,$$

are the $\mathcal{C}$-equivalence classes, where $C_{h,i,1}$'s are the bottom cylinders. Let $\mathcal{D}^{(0)} = \{D_{h,j} : h \geq 1,\ 1 \leq j \leq n(h)\}$ be a cylinder partition of D such that

$$(*) \qquad \nu(D_{h,j}) = \sum_{i \geq 1} \mu(C_{h,i,j}).$$

We also let $\mathcal{D}$ be a tower partition of D whose equivalence classes are

$$\{D_{h,i,1,k}, D_{h,i,2,k}, \ldots, D_{h,i,n(h),k}\}, \qquad i, h, k \geq 1,$$

where $D_{h,i,1,k}$'s are the bottom cylinders such that

$$\bigcup_{i \geq 1} \bigcup_{k \geq 1} D_{h,i,j,k} = D_{h,j} \pmod{\nu}$$

and

$$\sum_{k \geq 1} \nu(D_{h,i,j,k}) = \mu(C_{h,i,j}) \qquad (1 \leq j \leq n(h)).$$

We call the tower partition $\mathcal{D}$ an *orbital extension* of the tower partition $\mathcal{C}$ to the cylinder partition $\mathcal{D}^{(0)}$. We also call the map $\psi : \mathcal{D} \to \mathcal{C}$ defined by

$$\psi(D_{h,i,j,k}) = C_{h,i,j}$$

a *tower map.*

Lemma 8.2 implies the existence of an orbital extension $\mathcal{D}$ for any tower partition $\mathcal{C}$ and a cylinder partition $\mathcal{D}^{(0)}$ satisfying $(*)$.

Although the following proposition was observed in [HK], for completeness we will include the proof.

8.4. Proposition. *Let $\phi : \mathcal{C} \to \mathcal{D}$ be a tower map, and $\mathcal{C}'$ a tower refinement of $\mathcal{C}$. Then there exist a tower refinement $\mathcal{D}'$ of $\mathcal{D}$ and a tower map $\psi : \mathcal{D}' \to \mathcal{C}'$ such that*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \phi\ } & \mathcal{D} \\ {\scriptstyle\pi}\big\uparrow & & \big\uparrow{\scriptstyle\pi} \\ \mathcal{C}' & \xleftarrow{\ \psi\ } & \mathcal{D}' \end{array}$$

commutes, i.e. $\phi \circ \pi \circ \psi = \pi$, and ψ is also a tower map from $\hat{\mathcal{D}}$ to $\hat{\mathcal{C}}$.

Proof. It follows from Lemma 8.2 that there exists an orbital extension $\mathcal{E}$ of the restriction $\mathcal{C}'|_{B(\mathcal{C})}$ to the partition $\mathcal{D}|_{B(\mathcal{D})}$, whose tower map $\psi : \mathcal{E} \to \mathcal{C}'|_{B(\mathcal{C})}$ is such that if $\psi(D') = C'$ then the cylinders $C \in \mathcal{C}$ and $\mathcal{D} \in \mathcal{D}$ determined by $C' \subset C$ and $D' \subset D$ satisfy $\phi(C) = D$. We extend the tower partition $\mathcal{E}$ of $B(\mathcal{D})$ to a tower partition of the whole space Y by $\mathcal{D}$. The resulting tower refinement $\mathcal{D}'$ admits a tower map $\psi : \mathcal{D}' \to \mathcal{C}'$ closing the above diagram, and which preserves the $\approx$ equivalence relations. ∎

9. The finitary construction. Let

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\phi_1} & \mathcal{D}_1 \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle\pi} \\ \mathcal{C}_2 & \xleftarrow{\psi_2} & \mathcal{D}_2 \end{array}$$

be a commuting diagram of tower refinements and tower maps. Assume also that ψ_2 preserves the $\approx$ equivalence relations.

9.1. LEMMA. *Let $y_1 \dots y_{n_2}, \tilde{y}_1 \dots \tilde{y}_{n_2} \in \mathcal{D}_2$, and $n_1 < n_2$, $\tilde{n}_1 < n_2$ be such that*

$$\pi(y_1 \dots y_{n_2}) = y_1 \dots y_{n_1}, \qquad \pi(\tilde{y}_1 \dots \tilde{y}_{n_2}) = \tilde{y}_1 \dots \tilde{y}_{\tilde{n}_1}.$$

If

$$\psi_2(y_1 \dots y_{n_2}) \approx \psi_2(\tilde{y}_1 \dots \tilde{y}_{n_2}) \quad \text{and} \quad y_1 \dots y_{n_2} \sim \tilde{y}_1 \dots \tilde{y}_{n_2}$$

then $n_1 = \tilde{n}_1$ and $y_k = \tilde{y}_k$ $(n_1 < k \le n_2)$.

Proof. First we show that $n_1 = \tilde{n}_1$ and $y_1 \dots y_{n_1} \sim \tilde{y}_1 \dots \tilde{y}_{n_1}$. Put

$$x_1 \dots x_{m_2} = \psi_2(y_1 \dots y_{n_2})$$

and let $m_1 < m_2$ be such that

$$x_1 \dots x_{m_1} = \pi(x_1 \dots x_{m_2}) \in \mathcal{C}_1.$$

By the commuting diagram,

$$\begin{aligned} \phi_1(x_1 \dots x_{m_1}) &= \phi_1(\pi(x_1 \dots x_{m_2})) = \phi_1 \pi \psi_2(y_1 \dots y_{n_2}) \\ &= \pi(y_1 \dots y_{n_2}) = y_1 \dots y_{n_1}. \end{aligned}$$

One of the assumptions means that

$$\psi(\tilde{y}_1 \dots \tilde{y}_{n_2}) = \tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_2}$$

for some $\tilde{x}_1 \dots \tilde{x}_{m_1} \in \mathcal{C}_1$ with $\tilde{x}_1 \dots \tilde{x}_{m_1} \sim x_1 \dots x_{m_1}$. We then have

$$\tilde{y}_1 \dots \tilde{y}_{\tilde{n}_1} = \phi_1(\tilde{x}_1 \dots \tilde{x}_{m_1}) \sim \phi_1(x_1 \dots x_{m_1}) = y_1 \dots y_{n_1}.$$

Next we show that

$$\psi_2(\tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_2}) = \tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_2}.$$

As a matter of fact,

$$x_1 \dots x_{m_2} = \psi_2(y_1 \dots \dots \dots y_{n_2}) \approx \psi_2(\tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_2}),$$

and hence

$$\psi_2(\tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_2}) = \hat{x}_1 \dots \hat{x}_{m_1} x_{m_1+1} \dots x_{m_2}$$

for some $\hat{x}_1 \dots \hat{x}_{m_1} \in \mathcal{C}_1$ with $\hat{x}_1 \dots \hat{x}_{m_1} \sim x_1 \dots x_{m_1}$. Therefore

$$\phi_1(\hat{x}_1 \dots \hat{x}_{m_1}) = \tilde{y}_1 \dots \tilde{y}_{n_1} = \phi_1(\tilde{x}_1 \dots \tilde{x}_{m_1}).$$

Since ϕ_1 is injective on each equivalence class, $\hat{x}_1 \dots \hat{x}_{m_1} = \tilde{x}_1 \dots \tilde{x}_{m_1}$.

To complete the proof, we use the other assumption:

$$\tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_2} \sim y_1 \dots y_1 \dots y_{n_2} \sim \tilde{y}_1 \dots \tilde{y}_{n_2}.$$

This and the fact that $\psi_2(\tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_2}) = \psi_2(\tilde{y}_1 \dots \tilde{y}_{n_2})$, which we showed above, imply

$$y_{n_1+1} \dots y_{n_2} = \tilde{y}_{n_1+1} \dots \tilde{y}_{n_2},$$

because ψ_2 is injective on each equivalence class. ■

We next let

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "\phi_1"] & \mathcal{D}_1 \\ \mathcal{C}_2 \arrow[u, "\pi"] & \mathcal{D}_2 \arrow[l, "\psi_2"'] \arrow[u, "\pi"'] \\ \mathcal{C}_3 \arrow[u, "\pi"] \arrow[r, "\phi_3"] & \mathcal{D}_3 \arrow[u, "\pi"'] \end{tikzcd}$$

be a commuting diagram of tower refinements and tower maps, and assume that both ψ_2 and ϕ_3 preserve the corresponding $\approx$ equivalence relations. Then the following proposition is known from [HK]. Here we give a slightly different and shorter proof.

9.2. Proposition. *The tower map ϕ_3 preserves the $\cong$ equivalence relations. That is, for $x_1 \dots x_{m_3} \in \mathcal{C}_3$, let $y_1 \dots y_{n_3} = \phi_3(x_1 \dots x_{m_3}) \in \mathcal{D}_3$, and let $m_1 < m_3$, $n_1 < n_3$ be such that*

$$x_1 \dots x_{m_1} = \pi^2(x_1 \dots x_{m_3}) \in \mathcal{C}_1 \quad \text{and} \quad y_1 \dots y_{n_1} = \pi^2(y_1 \dots y_{n_3}) \in \mathcal{D}_1.$$

If $\tilde{x}_1 \dots \tilde{x}_{m_1} \in \mathcal{C}_1$ satisfies $x_1 \dots x_{m_1} \sim \tilde{x}_1 \dots \tilde{x}_{m_1}$ then

$$\phi_3(\tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_3}) \cong \phi_3(x_1 \dots x_{m_1} x_{m_1+1} \dots x_{m_3}),$$

i.e.

$$\phi_3(\tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_3}) = \tilde{y}_1 \dots \tilde{y}_{n_1} y_{n_1+1} \dots y_{n_3}$$

for some $\tilde{y}_1 \dots \tilde{y}_{n_1} \in \mathcal{D}_1$ with $y_1 \dots y_{n_1} \sim \tilde{y}_1 \dots \tilde{y}_{n_1}$.

Proof. Let $m_1 < m_2 < m_3$ and $n_1 < n_2 < n_3$ be such that

$$x_1 \dots x_{m_2} = \pi(x_1 \dots x_{m_3}), \qquad y_1 \dots y_{n_2} = \pi(y_1 \dots y_{n_3}).$$

By the commuting diagram,

$$\psi_2(y_1 \dots y_{n_2}) = x_1 \dots x_{m_2} \quad \text{and} \quad \phi_1(x_1 \dots x_{m_1}) = y_1 \dots y_{m_1}.$$

Since ϕ_3 respects the $\approx$ equivalence relation,

$$\tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_2} x_{m_2+1} \dots x_{m_3} \approx x_1 \dots x_{m_2} x_{m_2+1} \dots x_{m_3}$$

implies

$$\phi_3(\tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_2} x_{m_2+1} \dots x_{m_3}) = \tilde{y}_1 \dots \tilde{y}_{n_2} y_{n_2+1} \dots y_{n_3}$$

for some $\tilde{y}_1 \dots \tilde{y}_{n_2} \in \mathcal{D}_2$, where $y_1 \dots y_{n_2} \sim \tilde{y}_1 \dots \tilde{y}_{n_2}$. We also have

$$\psi_2(\tilde{y}_1 \dots \tilde{y}_{n_2}) = \tilde{x}_1 \dots \tilde{x}_{m_1} x_{m_1+1} \dots x_{m_2},$$

and

$$\psi_2(y_1 \dots y_{n_2}) = x_1 \dots x_{m_2}.$$

Apply Lemma 9.1 for $y_1 \dots y_{m_2}$ and $\tilde{y}_1 \dots \tilde{y}_{n_2}$ to conclude that $y_1 \dots y_{n_2} \approx \tilde{y}_1 \dots \tilde{y}_{n_2}$, so $y_k = \tilde{y}_k$ for all $n_1 < k \le n_2$. ■

Now we are ready to state and prove our main theorem.

9.3. THEOREM. *Finite coordinate change relations arising from measured Bratteli diagrams admitting nonatomic ergodic probability invariant measures are all finitary orbit equivalent.*

Proof. Choose a sequence $\{\varepsilon_n\}$ with $0 < \varepsilon_n < 1$ and

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

Using Propositions 7.2 and 8.4 inductively, we have a commuting diagram

$$\begin{array}{ccc}
\mathcal{C}^0 & \xrightarrow{\phi_0} & \mathcal{D}^0 \\
{\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle\pi} \\
\mathcal{C}^1 & \xleftarrow{\psi_1} & \mathcal{D}^1 \\
{\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle\pi} \\
\mathcal{C}^2 & \xrightarrow{\phi_2} & \mathcal{D}^2 \\
\uparrow & & \uparrow \\
 & \dots &
\end{array}$$

where $\mathcal{C}^0 = \{X\}$, $\mathcal{D}^0 = \{Y\}$, $\phi_0(X) = Y$, $\mathcal{C}^{2n+1}$ is $(1-\varepsilon_{2n+1})$-$\mathcal{S}_{2n+1}$-invariant $(n \ge 0)$, $\mathcal{D}^{2n}$ is $(1-\varepsilon_{2n})$-$\mathcal{S}_{2n}$-invariant $(n \ge 1)$, the maps commute, and ϕ_{2n} and ψ_{2n+1} preserve the respective $\approx$ equivalence relations at each stage of the induction. By the definition of cylinder partition, μ-almost every $x \in X$ belongs to a cylinder $c_n(x) \in \mathcal{C}^n$ for each $n \ge 0$. Similarly, ν-almost every $y \in Y$ belongs to a cylinder $d_n(y) \in \mathcal{D}^n$ for each $n \ge 0$. Moreover, if $d_0 \xleftarrow{\pi} d_1 \xleftarrow{\pi} d_2 \leftarrow \cdots$ is a chain, then $\bigcap_{n \ge 0} d_n$ is a singleton in Y. Therefore for μ-almost every $x \in X$ there is a $y = \Phi(x) \in Y$ such that for each $n \ge 0$,

$$\phi_{2n}(c_{2n}(x)) = d_{2n}(y).$$

Analogously (or by commutativity),

$$\psi_{2n+1}(d_{2n+1}(y)) = c_{2n+1}(x),$$

defining $\Psi(y) = x$ and proving that Φ and Ψ are finitary and inverses of each other.

Finally, let x and $\tilde{x}$ be two points of X belonging to the same $\mathcal{S}$-orbit. By $(1-\varepsilon_n)$-$\mathcal{S}_n$-invariance and the Borel–Cantelli lemma, μ-almost all such x (and $\tilde{x}$) belong to the sets

$$E_{2n+1} = \bigcup_{c\in\mathcal{E}_{2n+1}} c$$

for all sufficiently large n, where $\mathcal{E}_{2n+1}$ is the collection of cylinders associated with the $(1-\varepsilon_{2n+1})$-$\mathcal{S}_{2n+1}$-invariant tower $\mathcal{C}^{2n+1}$, whose total measure is at least $1-\varepsilon_{2n+1}$. This means that the cylinders $c_{2n+1}(x)$ and $c_{2n+1}(\tilde{x})$ are $\mathcal{C}_{2n+1}$-equivalent for all large n. In other words,

$$\mathcal{S}(x) = \bigcup_{n\geq 0} \mathcal{C}^{2n+1}(x)$$

for μ-almost every $x \in X$, and in particular the cylinders $c_{2n+1}(x)$ and $c_{2n+1}(\tilde{x})$ are obtained from the cylinders $c_{2n}(x)$ and $c_{2n}(\tilde{x})$ by juxtaposition of the same word w. Then by Proposition 9.2, ϕ_{2n+2} respects the $\cong$ equivalence relation. Therefore the corresponding cylinders $d_{2n+2}(\Phi(x)) = \phi_{2n+2}(c_{2n+2}(x))$ and $d_{2n+2}(\Phi(\tilde{x})) = \phi_{2n+2}(c_{2n+2}(\tilde{x}))$ are obtained from $d_{2n}(\Phi(x))$ and $d_{2n}(\Phi(\tilde{x}))$ by juxtaposition of the same word. This shows that $y = \Phi(x)$ and $\tilde{y} = \Phi(\tilde{x})$ differ in at most finitely many coordinates. Symmetrically we also see that $x = \Psi(y)$ and $\tilde{x} = \Psi(\tilde{y})$ differ in at most finitely many coordinates. ∎

We would like to thank the referee for the valuable comments. The first author would like to thank Wesleyan University for inviting him as a Van Vleck Visiting Researcher and for their warm hospitality during his stay when this work was done. The research of the first and the second authors was supported respectively by JSPS and the COE program at Keio University. We also thank Professors Hiroshi Fujisaki and Fumiaki Sugisaki for valuable discussions.

REFERENCES

[B] O. Bratteli, *Inductive limits of finite dimensional C^* algebras*, Trans. Amer. Math. Soc. 171 (1972), 195–234.

[D] H. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. 81 (1959), 119–159.

[GPS] T. Giordano, I. E. Putnam and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.

[GW] E. Glasner and B. Weiss, *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math. 6 (1995), 559–579.

[HK] T. Hamachi and M. S. Keane, *Finitary orbit equivalence of odometers*, Bull. London Math. Soc., to appear.

[HKY] T. Hamachi, M. Keane, and H. Yuasa, *Universally measure preserving homeomorphisms of Cantor minimal systems*, preprint.

[HPS] R. H. Herman, I. E. Putnam and C. F. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Internat. J. Math. 3 (1992), 827–864.

[KS1] M. S. Keane and M. Smorodinsky, *A class of finitary codes*, Israel J. Math. 26 (1977), 352–371.

[KS2] —, —, *Bernoulli schemes of the same entropy are finitary isomorphic*, Ann. of Math. 109 (1979), 397–406.

[O] N. S. Ormes, *Strong orbit realization for minimal homeomorphisms*, J. Anal. Math. 71 (1997), 103–133.

[R] M. K. Roychowdhury, *Irrational rotation of the circle and the binary odometer are finitary orbit equivalent*, Publ. RIMS Kyoto Univ. 43 (2007), 385–402.

[V] A. M. Vershik, *A theorem on periodical Markov approximation in ergodic theory*, in: Ergodic Theory and Related Topics (Vitte, 1981), Math. Res. 12, Akademie-Verlag, Berlin, 1982, 195–206.

Graduate School of Mathematics
Kyushu University
Chuo-Ku, Fukuoka 810-8560, Japan
E-mail: hamachi@math.kyushu-u.ac.jp

Department of Mathematics
Colorado State University
101 Weber Building
Fort Collins, CO 80523-1874, U.S.A.
E-mail: mrinal@math.colostate.edu

Department of Mathematics
and Computer Science
Wesleyan University
Middletown, CT 06459, U.S.A.
E-mail: mkeane@wesleyan.edu
and
Philips Research Laboratories
Eindhoven
and
Member of Center of Excellence
Department of Mathematics
Keio University

Received 6 January 2007 (4851)