FINITARY ORBIT EQUIVALENCE AND MEASURED BRATTELI DIAGRAMS

by

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Abstract. We prove a strengthened version of Dye’s theorem on orbit equivalence, showing that if the transformation structures are represented as finite coordinate change equivalence relations of ergodic measured Bratteli diagrams, then there is a finitary orbit equivalence between these diagrams.

1. Introduction. Two ergodic measure preserving transformations of nonatomic Lebesgue probability spaces are, by a classical theorem of Dye [D], orbit equivalent: there exists a measure preserving isomorphism between the underlying spaces taking orbits to orbits on an invariant set of full measure. In this article, we prove a strengthened version of Dye’s theorem, showing that if the transformation orbit structures are represented by the procedure introduced initially by Vershik [V] as equivalence relations of ergodic measured Bratteli diagrams, then there is a finitary orbit equivalence between these diagrams. In other words, after a change of the transformation on invariant sets of measure zero, the transformation spaces acquire a Cantor-like topology from the Bratteli diagram representations, and the orbit equivalence is then given by a homeomorphism after removal of sets of measure zero in each space.

We remark that an interesting result concerning orbit equivalence has been obtained by N. Ormes [O]. He shows that any ergodic measure preserving transformation $S$ of a probability Lebesgue space $(X, \mu)$ is measure conjugate to a homeomorphism $T'$ whose orbits are the same as the $T$-orbits for a given Cantor minimal system $(Y, T)$. However, in the case where $(X, S)$ is also a Cantor system with an ergodic invariant measure $\mu$, it is not clear that the measure conjugacy from $X$ to $Y$ in his construction is a homeomorphism after removal of sets of measure zero.

The structure of the article is as follows. First we recall the Vershik procedure in the simple setting we need in order to proceed—this seems to be an easier version than the original one in [V], but the ideas come from [V].
This part has also been known precisely in [O] as remarked above. Next, we describe our finitary construction leading to finitary orbit equivalence of ergodic measured Bratteli diagrams, which is similar to the isomorphism theorem for Bernoulli schemes as found in Keane–Smorodinsky ([KS1], [KS2]); earlier versions for odometers have been given in Hamachi–Keane [HK] and for irrational rotations in Roychowdhury [R]. We remark that this proof differs from other proofs of Dye’s theorem and seems conceptually and calculationally simpler. In a subsequent article, which will be published elsewhere [HKY], we solve a problem of [GW] by obtaining a dynamical proof of the topological orbit equivalence theorem of Giordano, Putnam, and Skau [GPS].

2. Vershik’s procedure. Let \((X, \mathcal{A}, \mu, S)\) be an ergodic measure preserving system with \(\mu(X) = 1\); we suppose that \((X, \mathcal{A}, \mu)\) is a nonatomic Lebesgue space, so that the transformation \(S\) is aperiodic. We will see that \(S\) admits a subset \(B \in \mathcal{A}\) with return times 2 or 3. By an easy argument, there exists a set \(A_0 \in \mathcal{A}\) with \(\mu(A_0) > 0\) such that

\[
A_0 \cap SA_0 = \emptyset \quad \text{and} \quad \bigcup_{i=0}^{\infty} S^i A_0 = X,
\]

since otherwise we would have \(Sx = x\) for some set \(E \in \mathcal{A}\) of positive measure and for every \(x \in E\). We put

\[
A_1 = SA_0, \quad A_n = S^n A_0 \setminus \bigcup_{i=0}^{n-1} S^i A_0, \quad n \geq 2,
\]

and

\[
B = S(A_1 \cup A_3 \cup A_5 \cup \cdots).
\]

Since the set \(S(A_1 \cup A_3 \cup A_5 \cup \cdots)\) is disjoint from \(A_1 \cup A_3 \cup A_5 \cup \cdots\),

\[
B \cap SB = \emptyset.
\]

Moreover,

\[
B \cup SB \cup S^2 B = X.
\]

Indeed, put

\[
A'_n = A_n \setminus S^{-1} A_{n+1}, \quad n \geq 1.
\]

Then \(B = A_2 \cup A_4 \cup A_6 \cup \cdots \cup SA'_1 \cup SA'_3 \cup SA'_5 \cup \cdots\), and \(SB \supset A_3 \cup A_5 \cup A_7 \cup \cdots \cup SA'_2 \cup SA'_4 \cup SA'_6 \cup \cdots\). So \(A_0 = \bigcup_{i=1}^{\infty} SA'_i \subset B \cup SB\) and \(A_1 = SA_0 \subset SB \cup S^2 B\). Therefore, \(B \cup SB \cup S^2 B \supset A_0 \cup A_1 \cup A_2 \cup A_3 \cup \cdots = X\). Consequently, we see that if

\[
\tau(x) = \min\{n \geq 1 : S^n(x) \in B\},
\]
then for each \( x \in B \), \( \tau(x) \) is equal to either 2 or 3.

Next, consider the induced transformation \( S_B \) defined by
\[
S_B(x) := S^{\tau(x)}(x) \quad (x \in B)
\]
on \( B \), and apply the same argument. Using induction, this yields a cutting-and-stacking procedure whose stacks at the \( n \)th stage are of height at least \( 2^n \); taking further a sequence \( B_1, B_2, \ldots \) of sets in \( A \) which generate \( A \) and refining the stacks at each stage \( n \) so that the sets of this stage include \( B_n \), we obtain a cutting-and-stacking procedure which fully represents the original transformation \((X, A, \mu, S)\), up to a set of measure zero in the way that each interval of all of the stacks has length equal to the measure of the corresponding measurable subset of \( X \). Now define the measured Bratteli diagram (see next section for definitions) associated with this procedure by letting the vertex set \( V_n \) at stage \( n \) be the set of stacks at level \( n \), and by drawing an edge in \( E_n \) from a stack \( u \in V_{n-1} \) to a stack \( v \in V_n \) for each piece of the stack \( u \) which is cut by the procedure and placed in the stack \( v \). Each finite path \( a = a_1 \ldots a_n \) of the diagram starting from the root and consisting of \( n \) edges \( a_i \) corresponds to an interval appearing in the cutting-and-stacking procedure. \( a \) is equipped with the same measure as the length of the interval, and all the other finite paths \( b = b_1 \ldots b_n \) starting from the root and of the same length \( n \) and the same range vertex, \( r(b_n) = r(a_n) \), have the same measure as \( a \) does. Thus we have given a brief proof sketch of:

2.1. THEOREM (Vershik [V]). Let \((X, A, \mu, S)\) be an ergodic measure preserving transformation of a nonatomic Lebesgue probability space. Then there exists a measured Bratteli diagram and an isomorphism of \((X, A, \mu)\) with the path space of this diagram which takes \( S \)-orbits in \( X \) to full equivalence classes of the diagram.

3. Bratteli diagram and finite coordinate change relation. As is well known (see [B], [HPS]), a Bratteli diagram is a countable graph \( G = (V, E) \), whose vertex set \( V \) and edge set \( E \) satisfy the following conditions:

1. \( V \) and \( E \) are countable unions of pairwise disjoint finite sets \( V_n, n \geq 0 \), and \( E_n, n \geq 1 \), respectively, where \( V_0 \) is a singleton set.
(2) $E$ is equipped with source maps $s = s_n : E_n \to V_{n-1}$, and range maps $r = r_n : E_n \to V_n$. Call $s(a)$ and $r(a)$ for $a \in E$ the source vertex and the range vertex of $a$.

(3) $s^{-1}(v)$ is nonempty for all $v \in V$ and $r^{-1}(v)$ is nonempty for all $v \in V \setminus V_0$.

The graph $G$ is associated with a one-sided infinite edge space $X = X_G$:

$$X = \{ (a_n)_{n \geq 1} : a_n \in E_n, r(a_n) = s(a_{n+1}) \ (n \geq 1) \}.$$ 

It is a compact space with the topology induced from the product topology of the infinite product space of the sets $E_n$ for $n \geq 1$, each equipped with the discrete topology. We let $\mathcal{A}$ be the $\sigma$-algebra consisting of all Borel subsets of $X$.

On $(X, \mathcal{A})$ one defines a finite coordinate change equivalence relation $S = S_X$ by letting, for $x$ and $y$ in $X$, $x \sim y$ if there is an integer $n \geq 1$ such that $x_i = y_i$ for $i \geq n$. For $x \in X$ we denote by $S(x)$ the equivalence class $\{ y \in X : x \sim y \}$, and call these classes $S$-orbits.

For $n \geq 1$, we denote by $S_n$ the subrelation of $S$, $x \overset{n}{\sim} y$, defined by $x_i = y_i$ for all $i > n$. We denote by $S_n(x)$ the $S_n$-equivalence class $\{ z \in X : x \overset{n}{\sim} z \}$.

For simplicity we identify the cylinder

$$[a_1 \ldots a_n] = \{ x \in X : x_i = a_i, 1 \leq i \leq n \}$$

of $X$ with the word $a_1 \ldots a_n$. If the cylinder is nonempty, the word is said to be admissible. For $m \leq n$, $S_m$ is also considered to be an equivalence relation on the set of all admissible words or cylinders $a_1 \ldots a_n$ of length $n$, namely

$$a_1 \ldots a_n \overset{m}{\sim} b_1 \ldots b_n \quad \text{if } a_i = b_i \text{ for all } m < i \leq n.$$ 

That is, a word $a_1 \ldots a_n$ corresponds to a path in the graph $G$ from the root to level $n$, and two words of length $n$ are $m$-equivalent ($\overset{m}{\sim}$) if the corresponding paths terminate at level $m$ at the same vertex in $V_m$ and coincide from level $m+1$ to $n$. When $m = n$, we also write $a_1 \ldots a_n \sim b_1 \ldots b_n$ if $a_1 \ldots a_n \overset{n}{\sim} b_1 \ldots b_n$.

4. Measured Bratteli diagram. Given a Bratteli diagram $G = (V, E)$, let $X = X_G$ and $S$ be the finite coordinate change relation on $(X, \mathcal{A})$. In what follows we will just write $X$ instead of $(X, \mathcal{A})$. A probability measure $\mu$ on $X$ is said to be $S$-invariant if for all $n \geq 1$ and all admissible cylinders $a_1 \ldots a_n$ and $b_1 \ldots b_n$ with $a_1 \ldots a_n \overset{n}{\sim} b_1 \ldots b_n$,

$$\mu(a_1 \ldots a_n) = \mu(b_1 \ldots b_n).$$

$\mu$ (or $S$) is said to be ergodic if any $S$-invariant set $E \in \mathcal{A}$ (i.e. for a.e. $x \in E$, $x' \sim x$ implies $x' \in E$) has measure 0 or 1.

Proof. Fix $x \in X$, and for $n \geq 1$ let $\mu_n^x$ be the probability measure on $X$ defined by

$$\mu_n^x = \frac{1}{\# \mathcal{S}_n(x)} \sum_{y \in \mathcal{S}_n(x)} \delta_y,$$

where $\delta_y$ means a Dirac measure. Then $\mu_n^x$ is $\mathcal{S}_n$-invariant. Actually for $a = a_1 \cdots a_n$,

$$\mu_n^x(a) = \frac{1}{A}$$

where $A$ is the number of words of length $n$ ending at the vertex $r(a_n)$. Since the set of all probability measures on a compact metric space is weak*-compact, one obtains a sequence of integers $n_1 < n_2 < \cdots$ such that

$$\mu(x) = w^* \lim_{k \to \infty} \frac{1}{\# \mathcal{S}_{n_k}(x)} \sum_{y \in \mathcal{S}_{n_k}(x)} \delta_y$$

exists. Then $\mu(x)$ is an $S$-invariant probability measure.

We remark two things on $S$-invariant probability measures $\mu$. The one is that the measure $\mu(a_1 \cdots a_n)$ of $a_1 \cdots a_n$ only depends on $r(a_n)$. The other is that there is a subgraph $G' = (V', E')$ of $G$ such that $\mu$ has support on the closed subset $X' = X_{G'}$ of $X$ on which $\mu$ is regular. The Bratteli diagram $G'$ is obtained by removing all the edges $e \in E$ together with the range vertex $r(e)$ if $e \in E_n$ and there is a path $e_1 \cdots e_{n-1}e$ in $G$ such that

$$\mu[e_1 \cdots e_{n-1}e] = 0.$$

We note that then all the other edges $f$ in $E_n$ ending at $r(e)$ are also removed, because any path $f_1 \cdots f_{n-1}f$ in $G$ also satisfies

$$\mu[f_1 \cdots f_{n-1}f] = \mu[e_1 \cdots e_{n-1}e] = 0.$$

Thus the rest $G'$ of $G$ also satisfies the conditions (1)–(3) of the previous section, and the measure $\mu$ is an $S'$-invariant probability measure on $X' = X_{G'}$, where $S'$ is the finite coordinate change equivalence relation, and takes positive measure for all cylinders of $X'$. So whenever a Bratteli diagram $G = (V, E)$ and an $S$-invariant probability measure $\mu$ on $X = X_G$ are given, we may and will assume that $\mu$ is regular.

Here are examples of measured Bratteli diagrams:

4.2. Example (Binary odometer).
The finite coordinate change relation $S$ arising from the measured Bratteli diagram is called the *binary odometer*.

4.3. Example (Irrational rotation).

The finite change relation arising from the diagram is called the *irrational rotation* by $\alpha = (\sqrt{5} - 1)/2$.

4.4. Example (Binomial relation).

The finite coordinate change relation arising from the diagram is called the *binomial relation*.

5. Hopf equivalence. Let $G = (V, E)$ be a Bratteli diagram and $\mu$ an ergodic, nonatomic, $S$-invariant probability measure on $X = X_G$. We assign in arbitrary manner the linear order from 0 to $k - 1$ to the set of edges $e \in E_n$ ending at the vertex $v \in V_n$, for each $n \geq 1$ and $e \in E_n$, where $k$ is the number of these edges. We denote by $\lambda(e)$ the order of the edge $e$. Call the edges $e$ and $f$ the minimal and maximal edges if $\lambda(e) = 0$ and $\lambda(f) = k - 1$. Write

\[
X_{\min} = \{ x \in X : x_n \text{ is minimal for all } n \geq 1 \},
\]
\[
X_{\max} = \{ x \in X : x_n \text{ is maximal for all } n \geq 1 \},
\]
\[
X^{(n)}_{\min} = \{ x \in X : x_i \text{ is minimal for all } 1 \leq i \leq n \},
\]
\[
X^{(n)}_{\max} = \{ x \in X : x_i \text{ is maximal for all } 1 \leq i \leq n \}.
\]
The clopen sets $X^{(n)}_{\min}$ and $X^{(n)}_{\max}$ are decreasing and satisfy
$$\bigcap_{n \geq 1} X^{(n)}_{\min} = X_{\min} \quad \text{and} \quad \bigcap_{n \geq 1} X^{(n)}_{\max} = X_{\max}.$$ Note that $X_{\min}$ and $X_{\max}$ are nonempty and $\mu(X^{(n)}_{\min}) = \mu(X^{(n)}_{\max})$, and hence $\mu(X_{\min}) = \mu(X_{\max})$. If $z, z' \in X_{\min}$ and $z \sim z'$ then $z = z'$. Since $\mu$ is ergodic and nonatomic, this means that $\mu(X_{\min}) = \mu(X_{\max}) = 0$.

Now we define the one-to-one map $S : X^c_{\max} \to X^c_{\min}$ by setting
$$Sx = x', \quad x \in X^c_{\max},$$
where $x'$ is next to $x$ in the sense that if we let $n = \min \{i \geq 1 : x_i \text{ is not maximal} \}$ then
$$\lambda(x'_n) = \lambda(x_n) + 1, \quad i < n,$$
$$\lambda(x'_i) = 0, \quad i < n,$$
$$x'_i = x_i, \quad i > n.$$ $S$ is continuous on the open set $X^c_{\max}$ and so also is $S^{-1}$ on $X^c_{\min}$, and
$$S(x) = \{S^n x : n \in \mathbb{Z} \}, \quad x \notin S(X_{\max}) \cup S(X_{\min}).$$
So, $S$ is a measure preserving automorphism of $X$, and is called an adic transformation ([V]). We define the $S$-invariant measurable subset
$$X_0 = \{ x \notin S(X_{\max}) \cup S(X_{\min}) : x \text{ is not periodic under } S \}$$
of full measure.

5.1. DEFINITION. Let $C$ be an open subset of $X$. We call a countable set $C = \{ C_\xi : \xi \in \Lambda \}$ of cylinders a cylinder partition of $C$ if the cylinders are pairwise disjoint subsets of $C$ and
$$\sum_{\xi \in \Lambda} \mu(C_\xi) = \mu(C).$$

5.2. DEFINITION. Let $O_1$ and $O_2$ be open subsets of $X$. We say that they are Hopf equivalent to each other if there are cylinder partitions $\{ C_{1,i} : i \geq 1 \}$ of $C_1$ and $\{ C_{2,i} : i \geq 1 \}$ of $C_2$ such that $C_{1,i} \sim C_{2,i}$, $i \geq 1$.

For any open set $O \subset X$ the return time $\tau$ under $S$ to $O$ is defined, because $\mu(O) > 0$ and $S$ is conservative. For any point $x \in X_0$ and any integer $k \geq 1$, the points $S^i x$, $0 \leq i < k$, are all different. Moreover, one then obtains an integer $n \geq 1$ such that
$$(S^i x)_j = x_j, \quad 0 \leq i < k, \quad j \geq n.$$
One also sees that for any \( x' \in X \) with \( x_j = x'_j, \ 1 \leq j < n \),
\[
(S^i x')_j = x'_j, \quad 0 \leq i < k, \quad j \geq n,
\]
and
\[
(S^i x)_j = (S^i x')_j, \quad 0 \leq i < k, \quad 1 \leq j < n.
\]
This means that for the cylinder \( C = [x_1 \ldots x_{n-1}] \), the return time \( \tau(x) \) to \( C \) satisfies
\[
\tau(x) \geq k, \quad x \in C.
\]

5.3. Lemma. Let \( C \subset X \) be a clopen set. Then the return time \( \tau \) to \( C \) satisfies, for each integer \( k \geq 1 \),
\[
\mu(C^k) = \mu(\text{Int } C^k),
\]
where \( C^k = \{ x \in C : \tau(x) = k \} \) and \( \text{Int} \) denotes interior, and hence \( \tau \) is a continuous function on \( C \) up to a null set.

Proof. \( C \) is the union of a finite number of pairwise disjoint cylinders \( E^r = c_{r,1} \ldots c_{r,q}, 1 \leq r \leq p, \) of the same length \( q \) for some \( q \geq 1 \). It is enough to show that for any \( 1 \leq r \leq p \) and any \( k \geq 1 \) with \( \mu(C^k) > 0 \), any point \( x \in X_0 \cap E^r \cap C^k \) is an interior point of \( C^k \). One has an integer \( n \geq q \) satisfying the following conditions:

(i) The words \( (S^i x)_1 \ldots (S^i x)_n, 0 \leq i < k \), are all different.
(ii) Any word in (i) except \( x_1 \ldots x_n \) has no prefix in \( \{ c_{r,1} \ldots c_{r,q} : 1 \leq r \leq p \} \).
(iii) \( (S^i x)_j = x_j, j > n, \ 1 \leq i \leq k \).
(iv) \( (S^k x)_1 \ldots (S^k x)_q \in \{ c_{r,1} \ldots c_{r,q} : 1 \leq r \leq p \} \).

Let \( x' \) be any point in \( X \) such that
\[
x'_j = x_j, \quad 1 \leq j \leq n.
\]
Then
\[
(S^i x')_1 \ldots (S^i x')_n = (S^i x)_1 \ldots (S^i x)_n, \quad 1 \leq i \leq k,
\]
\[
(S^i x')_j = x'_j, \quad j > n, \quad 1 \leq i \leq k,
\]
\[
(S^k x')_j = (S^k x)_j, \quad 1 \leq j \leq q,
\]
that is, \( x' \in C^k \). \( \blacksquare \)

5.4. Remark. The proof of Lemma 5.3 shows that there is a cylinder partition \( \{ C_{k, i,j} : i, k \geq 1, 0 \leq j < k \} \) of \( X \) satisfying:

1. \( C_{k, i,0} \subset C^k, i, k \geq 1 \).
2. \( \sum_{i \geq 1} \mu(C_{k, i,0}) = \mu(C^k), k \geq 1 \).
3. \( C_{k, i,j} = S^j(C_{k, i,0}), 0 < j < k, i, k \geq 1 \).
4. \( C_{k, i,0} \sim C_{k, i,1} \sim \cdots \sim C_{k, i, k-1}, i, k \geq 1 \).
5.5. Proposition. Let $A, B \subset X$ be clopen sets. If $\mu(A) = \mu(B)$, then $A$ and $B$ are Hopf equivalent to each other.

Proof. Let $0 < \varepsilon < \mu(A)/2$. By choosing a clopen subset $A' \subset A$ such that $\mu(A') > \mu(A) - \varepsilon$, it is enough to construct finitely many pairwise disjoint, and the same number of cylinders $[a_i]$ and $[b_i]$ satisfying the following conditions:

(i) $[a_i] \subset A$, $[b_i] \subset B$,
(ii) $[a_i] \sim [b_i]$, $i \geq 1$,
(iii) $\sum_{i} \mu([a_i]) > \mu(A') - \varepsilon$.

It follows from the ergodic theorem that there exists a measurable subset $E \subset X$ with $\mu(E) < \varepsilon$ and an integer $N \geq 1$ such that if $n \geq N$ and $x \notin E$ then

$$\#\{0 \leq k < n : S^k x \in A'\} < \#\{0 \leq k < n : S^k x \in B\},$$
$$\#\{0 < k \leq n : S^{-k} x \in A'\} < \#\{0 < k \leq n : S^{-k} x \in B\}.$$  

Take an integer $M \geq 2/\varepsilon$ and a cylinder $C \subset X$ such that the return time $\tau$ to $C$ satisfies

$$\tau(x) \geq MN, \quad x \in C.$$  

Then by Remark 5.4 one has a cylinder partition $\{C_{k,i,j} : 0 \leq j < k, i \geq 1, k \geq MN\}$ of $X$ satisfying:

1. $C_{k,i,0} \subset \{x \in C : \tau(x) = k\}, i \geq 1, k \geq MN$.
2. $\sum_{i \geq 1} \mu(C_{k,i,0}) = \mu(\{x \in C : \tau(x) = k\}), k \geq MN$.
3. $C_{k,i,j} = S^j(C_{k,i,0}), 0 \leq j < k, i \geq 1, k \geq MN$.
4. $C_{k,i,0} \sim C_{k,i,1} \sim \cdots \sim C_{k,i,k-1}, i \geq 1, k \geq MN$.

By decomposing the cylinder $C_{k,i,0}$ into finitely many cylinders for each $k$ and each $i$ if necessary, we may assume that the set $A'$ (resp. $B$) is the union of a countable number of the cylinders $C_{k,i,j}$ up to a null set.

Set

$$\Gamma = \{(k, i) \in [MN, \infty) \times [1, \infty) : \exists j \text{ with } N \leq j < k - N \text{ such that } C_{k,i,j} \cap E^c \neq \emptyset\},$$

and

$$H = \bigcup_{(k,i) \in \Gamma} \bigcup_{j=N}^{k-N-1} C_{k,i,j}.$$  

Then $\mu(H) > 1 - 2\varepsilon$ and for each $(k,i) \in \Gamma$, the number of cylinders in the collection $\{C_{k,i,j} : 0 \leq j < k\}$ that are contained in $A'$ is smaller than the number of those contained in $B$. Making a maximal number of one-to-one matchings between these cylinders completes the proof. \hfill \blacksquare
6. **Tower partition.** Let us use some definitions similar to those in [HK]. Let $G = (V, E)$ be a Bratteli diagram, $X = X_G$, and $\mu$ an $S$-invariant probability measure on $X$.

6.1. **Definition.** Let $C$ be an open set of $X$. A *tower partition* of $C$ is a cylinder partition $\mathcal{C}$ of $C$ endowed with an equivalence relation such that each equivalence class consists of cylinders having the same length and the same range vertex, and in each equivalence class, a specified cylinder called a *bottom cylinder* is designated. The union of the bottom cylinders is called the *bottom of the tower partition*, and is denoted by $B(C)$.

Given a tower partition $\mathcal{C}$, for each equivalence class $\{C_1, \ldots, C_d\}$ of $\mathcal{C}$ and each $1 \leq i, j \leq d$ we introduce partially defined homeomorphisms $C_{j,i} : C_i \rightarrow C_j$ by setting

$$C_{j,i}(x) = x', \quad x \in C_i,$$

where $x' \in C_j$, $x_k = x'_k$, $k > n$, and $n$ is the common length of the cylinders $C_i$ and $C_j$. Then

$$C_{i,i} = \text{Id}|C_i \quad \text{and} \quad C_{k,j}C_{j,i} = C_{k,i} \quad (1 \leq i, j, k \leq d).$$

We call $C_{j,i}$’s the *associated homeomorphisms* of $\mathcal{C}$. For $x \in C_i$ and $x' \in C_j$, we write

$$x \overset{\mathcal{C}}{\sim} x' \quad \text{if} \quad x' = C_{j,i}x,$$

and $\mathcal{C}(x) = \{x' : x \overset{\mathcal{C}}{\sim} x'\}$.

For a subset $E \subseteq C_i$ and $j$, we let $E' = C_{j,i}(E)$, and say that $E' \subseteq C_j$ is induced from $E$ by the $\mathcal{C}$-equivalence of $C_i$, or briefly that $E'$ is *induced from $E$ by $\mathcal{C}$*, and write

$$E \overset{\mathcal{C}}{\sim} E'.$$

When we discuss an equivalence $A = [a_1 \ldots a_m] \overset{m}{\sim} B = [b_1 \ldots b_m]$ of cylinders, we also say that for a subset $E \subseteq A$ the subset $E' \subseteq B$ defined below is induced from $E$ by the equivalence:

$$E' = \{y : y_i = b_i \quad (1 \leq i \leq m), \quad y_i = x_i \quad (i > m) \quad \text{for some} \quad x \in E\}.$$

We then write

$$E \overset{m}{\sim} E'.$$

Now we define tower extension and tower refinement, similarly to [HK].

6.2. **Definition.** An *extension* of an equivalence class of a tower partition is defined to be the set of cylinders constructed by juxtaposing the same word to the cylinders belonging to the equivalence class. That is, for a word $w$, it is the set of cylinders $cw$, where $c$ ranges over the equivalence class, and where $w$ can be an empty word.
6.3. Definition. Let \( C \) be a tower partition of an open subset \( C \) of \( X \), and \( E \) be a tower partition of the bottom \( B(C) \) such that \( E \) is, as a partition, finer than or equal to the partition of \( B(C) \) consisting of the bottom cylinders of the equivalence classes of \( C \). We let \( C' \) be the tower partition of \( C \) constructed in the way that for every equivalence class of \( E \) the equivalence class of \( C' \) is the collection of cylinders induced from every cylinder in the \( E \)-equivalence class by \( C \), where the bottom cylinder of the \( E \)-equivalence class is also the bottom cylinder of the \( C' \)-equivalence class. \( C' \) is called a \textit{tower refinement of} \( C \), and \( E \) is also said to be \textit{extended} to \( C' \) by \( C \). In particular, if \( E \) is trivial as an equivalence relation, we say that the tower refinement \( C' \) is a \textit{tower extension} of \( C \). In other words, each equivalence class of the tower extension is an extension of an equivalence class of \( C \).

A tower refinement \( C' \) of a tower partition \( C \) is, in other words, a refinement of \( C \) as a partition, and as an equivalence relation, \( C \) is a subrelation of \( C' \). Each cylinder \( c' \in C' \) is contained in a unique \( c \in C \), yielding a map \( \pi: C' \rightarrow C \) which preserves the measure \( \mu \). Let \( E^{(0)} \) be the cylinder partition of \( B(C) \), which is the restriction of \( C' \) to \( B(C) \) as a partition, and endow \( E^{(0)} \) with the trivial equivalence relation. The tower refinement of \( C \) extended from the tower partition \( E^{(0)} \) to \( C \) is a tower extension of \( C \) and is denoted by \( \hat{C} \). It is a subrelation of \( C' \) and we call the equivalence relation of \( \hat{C} \) the \textit{subrelation of} \( C' \) \textit{coming from} \( C \). If \( c' \) and \( \tilde{c}' \) are equivalent in \( \hat{C} \), then we write \( c' \approx \tilde{c}' \), namely \( c' = cu \), \( \tilde{c}' = \tilde{c}u \) for some word \( u \), where \( \pi(c') = c \), \( \pi(\tilde{c}') = \tilde{c} \), \( c \sim \tilde{c} \).

Clearly, if \( C' \) is a tower refinement of \( C \) and \( C'' \) is a tower refinement of \( C' \), then \( C'' \) is also a tower refinement of \( C \)—in this case, we write \( c'' \approx \tilde{c}'' \) for the subrelation of \( C'' \) coming from \( C \), namely, \( c'' = cuv \) and \( \tilde{c}'' = \tilde{c}uv \) for some word \( v \), where \( c = \pi^2(c'') \), \( \tilde{c} = \pi^2(\tilde{c}'') \), \( c \sim \tilde{c} \), and where \( \pi(c'') = cu \), \( \pi(\tilde{c}'') = \tilde{c}u \).

7. \((1-\varepsilon)\)-\(S_n\)-invariant tower partition. We will introduce a notion of \((1-\varepsilon)\)-\(S_n\)-invariant tower partition, which is quite similar to \((1-\varepsilon)\)-cyclic tower partition of odometers [HK]. Let \( X \) be the path space associated with a Bratteli diagram \( G = (V, E) \) and \( \mu \) be an \( S \)-invariant probability measure.

7.1. Definition. Let \( 0 < \varepsilon < 1 \) and \( n \geq 1 \). A tower partition \( C \) is said to be \((1-\varepsilon)\)-\(S_n\)-invariant if there exists a union \( E \) of finitely many \( C \)-equivalence classes of cylinders of the same length \( \geq n \) such that
\[
\sum_{c \in E} \mu(c) > 1 - \varepsilon,
\]
and for \( c, c' \in E \),
\[
c \sim^0 c' \Rightarrow c \sim c'.
\]
In the definition above it is easily seen that there exists a $C$-invariant set $E$ of positive measure $> 1 - \varepsilon$ such that for $x$ and $x'$ in $E$, if $x \sim x'$ then $x \sim x'$.

7.2. Proposition. Let $C$ be a tower partition, $n \geq 1$, and $0 < \varepsilon < 1$. There exists a tower refinement $C'$ of $C$ which is $(1 - \varepsilon)$-$S_n$-invariant.

Proof. Take a union $E$ of finitely many $C$-equivalence classes so that
\[
\sum_{c \in E} \mu(c) > 1 - \varepsilon.
\]
Also take an extension $\hat{C}$ of $C$ so that every cylinder $\hat{c} \in \hat{C}$ either is in $C$, or has constant length $\geq n$ and is contained in a cylinder $c \in E$. Put
\[
\hat{E} = \{ \hat{c} \in \hat{C} : \hat{c} = cu \text{ for some } c \in E \text{ and } u \}.
\]
We define the tower partition $C'$ by letting, for $\hat{c}, \hat{c}' \in \hat{C}$,
\[
\hat{c} \sim C' \hat{c}'
\]
if either $\hat{c}, \hat{c}' \notin \hat{E}$, $\hat{c}, \hat{c}' \in C$ and $\hat{c} \sim \hat{c}'$, or $\hat{c}, \hat{c}' \in \hat{E}$ and there exist a finite number of cylinders $\hat{c}_i \in \hat{E}$, $1 \leq i < k$ and $\hat{c}_i' \in \hat{E}$, $2 \leq i \leq k$, such that
\[
\hat{c}_i \sim \hat{c}_{i+1}, \quad 1 \leq i < k,
\]
\[
\hat{c}_1 \sim \hat{c}, \quad \hat{c}_k' \sim \hat{c}', \quad \hat{c}_i' \sim \hat{c}_i, \quad 1 < i < k.
\]
Then $C'$ is $(1 - \varepsilon)$-$S_n$-invariant.

8. Orbital extension and tower map. As seen in [HK] and [R], a tower map is a central ingredient of finitary orbit equivalence. We will introduce a more general notion of a tower map for finite coordinate change relations.

8.1. Lemma. Let $C \subset X$ be an open set, and $\{ r_i \}_{i \geq 1}$ be a finite or countable sequence of positive numbers such that
\[
\sum_{i \geq 1} r_i = \mu(C).
\]
Then there exist pairwise disjoint cylinders $C_{i,j} \subset C$, $i, j \geq 1$, satisfying, for each $i \geq 1$,
\[
\sum_{j \geq 1} \mu(C_{i,j}) = r_i.
\]

Proof. We inductively show that there exist positive numbers $\varepsilon_{i,j}$ with $\varepsilon_{i,j} < \min\{ r_i, 1/(i + j) \}$ and pairwise disjoint clopen subsets $C_{i,j}$ of $C$ for
For $n = 1$, it is enough to choose a positive number $\varepsilon_{1,1} < r_1$ and a clopen subset $C_{1,1}$ of $C$ such that
\[ r_1 - \varepsilon_{1,1} < \mu(C_{1,1}) < r_1. \]
Suppose $(\ast)$ holds for $n$. Put
\[ D_i = \bigcup_{j=1}^{n-i+1} C_{i,j} \quad \text{for } 1 \leq i \leq n, \quad D = \bigcup_{i=1}^{n} D_i. \]
The induction hypothesis makes it possible to choose small positive numbers $\varepsilon_{i,n+i-2} < 1/(n+2)$ for $1 \leq i \leq n$ such that
\[ r_i - \mu(D_i) - \varepsilon_{i,n+i-2} > 0. \]
We also choose a small positive number $\varepsilon_{n+1,1} < r_{n+1}$. Since
\[ \mu(C \setminus D) > r_{n+1} + \sum_{i=1}^{n} (r_i - \mu(D_i)), \]
one then has pairwise disjoint clopen subsets $C_{i,n+i-i}, 1 \leq i \leq n+1$, of $C \setminus D$ such that
\[ r_i - \mu(D_i) > \mu(C_{i,n+i-i}) > r_i - \mu(D_i) - \varepsilon_{i,n+i-i} \quad (1 \leq i \leq n) \]
and
\[ r_{n+1} > \mu(C_{n+1,1}) > r_{n+1} - \varepsilon_{n+1,1}. \]
This means that $(\ast)$ holds for $n + 1$. Letting $n$ tend to infinity for $i \geq 1$ in the inequality
\[ r_i > \mu\left(\bigcup_{j=1}^{n-i+1} C_{i,j}\right) > r_i - \varepsilon_{n-i+1}, \]
one then shows
\[ \sum_{j=1}^{\infty} \mu(C_{i,j}) = r_i. \]
The proof is completed by taking a finite partition of $C_{i,j}$ into cylinders for each $i \geq 1$ and $j \geq 1$. ■

The basic idea of the following lemma comes from [HK, Lemma 2, the argument in Section 3, and the proof of Proposition 4.3], where the cylinders of the binary odometer space $X$ and the ternary odometer space $Y$ are compared with each other as subintervals of $[0, 1]$ through some identification.
8.2. Lemma. Assume that $S_Y$ is ergodic. Let $n \geq 1$ and $C_{i,j}, 1 \leq j \leq n, i \geq 1$, be pairwise disjoint cylinders of $X$, and $D_j, 1 \leq j \leq n,$ be pairwise disjoint cylinders of $Y$. Suppose that for each $i \geq 1$, the cylinders $C_{i,j}, 1 \leq j \leq n,$ have the same length, and

$$C_{i,1} \sim \cdots \sim C_{i,n} \quad \text{and} \quad \sum_{i \geq 1} \mu(C_{i,j}) = \nu(D_j) \quad (1 \leq j \leq n).$$

Then there are pairwise disjoint cylinders $D_{i,j,k} \subset D_j, i, k \geq 1 \ (1 \leq j \leq n),$ satisfying the following conditions:

(1) For $i, k \geq 1$, the cylinders $D_{i,j,k}, 1 \leq j \leq n,$ have the same length, and

$$D_{i,1,k} \sim D_{i,2,k} \sim \cdots \sim D_{i,n,k}.$$

(2) $\sum_{k \geq 1} \nu(D_{i,j,k}) = \mu(C_{i,j}) \ (i \geq 1, \ 1 \leq j \leq n).$

Proof. First apply Proposition 5.5 for the cylinders $D_1, \ldots, D_n$ to get pairwise disjoint cylinders $D_{1,l} \subset D_j, l \geq 1, 1 \leq j \leq n,$ such that for each $l \geq 1$ the cylinders $D_{j,l}, 1 \leq j \leq n,$ have the same length, and

$$D_{1,l} \sim D_{2,l} \sim \cdots \sim D_{n,l} \quad \text{and} \quad \sum_{l \geq 1} \nu(D_{j,l}) = \nu(D_j) \quad (1 \leq j \leq n).$$

Next apply Lemma 8.1 for the cylinder $D_1$ and the numbers $\mu(C_{i,1}), i \geq 1,$ to get pairwise disjoint cylinders $D'_{i,k} \subset D_1, i, k \geq 1,$ such that

$$\sum_{k \geq 1} \nu(D'_{i,k}) = \mu(C_{i,1}), \quad i \geq 1.$$

One then has a common refinement of the partitions $\{D'_{i,k} : i, k \geq 1\}$ and $\{D_{1,l} : l \geq 1\}$ of $D_1$. Write

$$D_{i,1,k,l} = D'_{i,k} \cap D_{1,l}, \quad i, k \geq 1, l \geq 1.$$

For $l \geq 1$, the equivalences $D_{1,l} \sim D_{2,l} \sim \cdots \sim D_{n,l}$ induce pairwise disjoint cylinders $D_{i,j,k,l} \subset D_{j,l}$ such that for $l \geq 1,$

$$D_{i,1,k,l} \sim D_{i,j,k,l} \quad (i, k \geq 1) \quad \text{and} \quad \sum_{i \geq 1} \sum_{k \geq 1} \nu(D_{i,j,k,l}) = \nu(D_{j,l}).$$

These cylinders $D_{i,j,k,l}$, with subscripts $i, j, \text{and} \ (k, l),$ satisfy conditions (1)–(3) of the lemma. ■

8.3. Definition. Let $C \subset X$ and $D \subset Y$ be open sets such that $\mu(C) = \nu(D),$ and $C = \{C_{h,i,j} : h, i \geq 1, 1 \leq j \leq n(h)\}$ be a tower partition of $C$, where $n(h) \geq 1$ for $h \geq 1$ and

$$\{C_{h,i,1}, C_{h,i,2}, \ldots, C_{h,i,n(h)}\}, \quad h, i \geq 1,$$
are the $\mathcal{C}$-equivalence classes, where $C_{h,i,1}$’s are the bottom cylinders. Let $D^{(0)} = \{D_{h,j} : h \geq 1, 1 \leq j \leq n(h)\}$ be a cylinder partition of $D$ such that
\begin{equation}
\nu(D_{h,j}) = \sum_{i \geq 1} \mu(C_{h,i,j}).
\end{equation}
We also let $D$ be a tower partition of $D$ whose equivalence classes are
$$\{D_{h,i,1,k}, D_{h,i,2,k}, \ldots, D_{h,i,n(h),k}\}, \quad i, h, k \geq 1,$$
where $D_{h,i,1,k}$’s are the bottom cylinders such that
$$\bigcup_{i \geq 1} \bigcup_{k \geq 1} D_{h,i,j,k} = D_{h,j} \pmod{\nu}$$
and
$$\sum_{k \geq 1} \nu(D_{h,i,j,k}) = \mu(C_{h,i,j}) \quad (1 \leq j \leq n(h)).$$
We call the tower partition $D$ an orbital extension of the tower partition $C$ to the cylinder partition $D^{(0)}$. We also call the map $\psi : D \to C$ defined by $\psi(D_{h,i,j,k}) = C_{h,i,j}$ a tower map.

Lemma 8.2 implies the existence of an orbital extension $D$ for any tower partition $C$ and a cylinder partition $D^{(0)}$ satisfying $(\ast)$.

Although the following proposition was observed in [HK], for completeness we will include the proof.

8.4. **Proposition.** Let $\phi : C \to D$ be a tower map, and $C'$ a tower refinement of $C$. Then there exist a tower refinement $D'$ of $D$ and a tower map $\psi : D' \to C'$ such that

\[
\begin{array}{ccc}
\mathcal{C} & \overset{\phi}{\longrightarrow} & \mathcal{D} \\
\uparrow & & \uparrow \\
\mathcal{C}' & \overset{\psi}{\longleftarrow} & \mathcal{D}'
\end{array}
\]

commutes, i.e. $\phi \circ \pi \circ \psi = \pi$, and $\psi$ is also a tower map from $\hat{D}$ to $\hat{C}$.

**Proof.** It follows from Lemma 8.2 that there exists an orbital extension $\mathcal{E}$ of the restriction $\mathcal{C}'|_{B(C)}$ to the partition $\mathcal{D}|_{B(D)}$, whose tower map $\psi : \mathcal{E} \to \mathcal{C}'|_{B(C)}$ is such that if $\psi(D') = C'$ then the cylinders $C \in C$ and $D \in D$ determined by $C' \subset C$ and $D' \subset D$ satisfy $\phi(C) = D$. We extend the tower partition $\mathcal{E}$ of $B(D)$ to a tower partition of the whole space $Y$ by $D$. The resulting tower refinement $D'$ admits a tower map $\psi : D' \to C'$ closing the above diagram, and which preserves the $\approx$ equivalence relations. $\blacksquare$
9. The finitary construction. Let
\[ C_1 \xrightarrow{\phi_1} D_1 \]
\[ \pi \uparrow \quad \uparrow \pi \]
\[ C_2 \xleftarrow{\psi_2} D_2 \]

be a commuting diagram of tower refinements and tower maps. Assume also that \( \psi_2 \) preserves the \( \approx \) equivalence relations.

9.1. Lemma. Let \( y_1 \ldots y_{n_2}, \tilde{y}_1 \ldots \tilde{y}_{n_2} \in D_2 \), and \( n_1 < n_2, \tilde{n}_1 < n_2 \) be such that
\[ \pi(y_1 \ldots y_{n_2}) = y_1 \ldots y_{n_1}, \quad \pi(\tilde{y}_1 \ldots \tilde{y}_{n_2}) = \tilde{y}_1 \ldots \tilde{y}_{\tilde{n}_1}. \]

If
\[ \psi_2(y_1 \ldots y_{n_2}) \approx \psi_2(\tilde{y}_1 \ldots \tilde{y}_{n_2}) \quad \text{and} \quad y_1 \ldots y_{n_2} \sim \tilde{y}_1 \ldots \tilde{y}_{n_2} \]

then \( n_1 = \tilde{n}_1 \) and \( y_k = \tilde{y}_k \) (\( n_1 < k \leq n_2 \)).

Proof. First we show that \( n_1 = \tilde{n}_1 \) and \( y_1 \ldots y_{n_1} \sim \tilde{y}_1 \ldots \tilde{y}_{n_1} \). Put
\[ x_1 \ldots x_{m_2} = \psi_2(y_1 \ldots y_{n_2}) \]
and let \( m_1 < m_2 \) be such that
\[ x_1 \ldots x_{m_1} = \pi(x_1 \ldots x_{m_2}) \in \mathcal{C}_1. \]

By the commuting diagram,
\[ \phi_1(x_1 \ldots x_{m_1}) = \phi_1(\pi(x_1 \ldots x_{m_2})) = \phi_1 \pi \psi_2(y_1 \ldots y_{n_2}) \]
\[ \quad = \pi(y_1 \ldots y_{n_2}) = y_1 \ldots y_{n_1}. \]

One of the assumptions means that
\[ \psi(\tilde{y}_1 \ldots \tilde{y}_{n_2}) = \tilde{x}_1 \ldots \tilde{x}_{m_1} x_{m_1+1} \ldots x_{m_2} \]
for some \( \tilde{x}_1 \ldots \tilde{x}_{m_1} \in \mathcal{C}_1 \) with \( \tilde{x}_1 \ldots \tilde{x}_{m_1} \sim x_1 \ldots x_{m_1} \). We then have
\[ \tilde{y}_1 \ldots \tilde{y}_{n_1} = \phi_1(\tilde{x}_1 \ldots \tilde{x}_{m_1}) \sim \phi_1(x_1 \ldots x_{m_1}) = y_1 \ldots y_{n_1}. \]

Next we show that
\[ \psi_2(\tilde{y}_1 \ldots \tilde{y}_{n_1} y_{n_1+1} \ldots y_{n_2}) = \tilde{x}_1 \ldots \tilde{x}_{m_1} x_{m_1+1} \ldots x_{m_2} \]
As a matter of fact,
\[ x_1 \ldots x_{m_2} = \psi_2(y_1 \ldots y_{n_2}) \approx \psi_2(\tilde{y}_1 \ldots \tilde{y}_{n_1} y_{n_1+1} \ldots y_{n_2}), \]
and hence
\[ \psi_2(\tilde{y}_1 \ldots \tilde{y}_{n_1} y_{n_1+1} \ldots y_{n_2}) = \tilde{x}_1 \ldots \tilde{x}_{m_1} x_{m_1+1} \ldots x_{m_2} \]
for some \( \tilde{x}_1 \ldots \tilde{x}_{m_1} \in \mathcal{C}_1 \) with \( \tilde{x}_1 \ldots \tilde{x}_{m_1} \sim x_1 \ldots x_{m_1} \). Therefore
\[ \phi_1(\tilde{x}_1 \ldots \tilde{x}_{m_1}) = \tilde{y}_1 \ldots \tilde{y}_{n_1} = \phi_1(\tilde{x}_1 \ldots \tilde{x}_{m_1}). \]
Since \( \phi_1 \) is injective on each equivalence class, \( \tilde{x}_1 \ldots \tilde{x}_{m_1} = \tilde{x}_1 \ldots \tilde{x}_{m_1} \).
To complete the proof, we use the other assumption:
\[ \tilde{y}_1 \cdots \tilde{y}_{n_1} y_{n_1+1} \cdots y_{n_2} \sim y_1 \cdots y_1 \cdots y_{n_2} \sim \tilde{y}_1 \cdots \tilde{y}_{n_2}. \]
This and the fact that \( \psi_2(\tilde{y}_1 \cdots \tilde{y}_{n_1} y_{n_1+1} \cdots y_{n_2}) = \psi_2(\tilde{y}_1 \cdots \tilde{y}_{n_2}) \), which we showed above, imply
\[ y_{n_1+1} \cdots y_{n_2} = \tilde{y}_{n_1+1} \cdots \tilde{y}_{n_2}, \]
because \( \psi_2 \) is injective on each equivalence class.

We next let
\[
\begin{array}{ccc}
C_1 & \xrightarrow{\phi_1} & D_1 \\
\uparrow \pi & & \uparrow \pi \\
C_2 & \xleftarrow{\psi_2} & D_2 \\
\uparrow \pi & & \uparrow \pi \\
C_3 & \xrightarrow{\phi_3} & D_3
\end{array}
\]
be a commuting diagram of tower refinements and tower maps, and assume that both \( \psi_2 \) and \( \phi_3 \) preserve the corresponding \( \approx \) equivalence relations. Then the following proposition is known from [HK]. Here we give a slightly different and shorter proof.

9.2. Proposition. The tower map \( \phi_3 \) preserves the \( \cong \) equivalence relations. That is, for \( x_1 \cdots x_{m_3} \in C_3 \), let \( y_1 \cdots y_{n_3} = \phi_3(x_1 \cdots x_{m_3}) \in D_3 \), and let \( m_1 < m_3, n_1 < n_3 \) be such that
\[ x_1 \cdots x_{m_1} = \pi^2(x_1 \cdots x_{m_3}) \in C_1 \quad \text{and} \quad y_1 \cdots y_{n_1} = \pi^2(y_1 \cdots y_{n_3}) \in D_1. \]
If \( \tilde{x}_1 \cdots \tilde{x}_{m_1} \in C_1 \) satisfies \( x_1 \cdots x_{m_1} \sim \tilde{x}_1 \cdots \tilde{x}_{m_1} \) then
\[ \phi_3(\tilde{x}_1 \cdots \tilde{x}_{m_1} x_{m_1+1} \cdots x_{m_3}) \cong \phi_3(x_1 \cdots x_{m_1+1} \cdots x_{m_3}), \]
i.e.
\[ \phi_3(\tilde{x}_1 \cdots \tilde{x}_{m_1} x_{m_1+1} \cdots x_{m_3}) = \tilde{y}_1 \cdots \tilde{y}_{n_1} y_{n_1+1} \cdots y_{n_3} \]
for some \( \tilde{y}_1 \cdots \tilde{y}_{n_1} \in D_1 \) with \( y_1 \cdots y_{n_1} \sim \tilde{y}_1 \cdots \tilde{y}_{n_1} \).

Proof. Let \( m_1 < m_2 < m_3 \) and \( n_1 < n_2 < n_3 \) be such that
\[ x_1 \cdots x_{m_2} = \pi(x_1 \cdots x_{m_3}), \quad y_1 \cdots y_{n_2} = \pi(y_1 \cdots y_{n_3}). \]
By the commuting diagram,
\[ \psi_2(y_1 \cdots y_{n_2}) = x_1 \cdots x_{m_2} \quad \text{and} \quad \phi_1(x_1 \cdots x_{m_1}) = y_1 \cdots y_{m_1}. \]
Since \( \phi_3 \) respects the \( \approx \) equivalence relation,
\[ \tilde{x}_1 \cdots \tilde{x}_{m_1} x_{m_1+1} \cdots x_{m_2} x_{m_2+1} \cdots x_{m_3} \cong x_1 \cdots x_{m_2} x_{m_2+1} \cdots x_{m_3} \]
implies
\[ \phi_3(\tilde{x}_1 \cdots \tilde{x}_{m_1} x_{m_1+1} \cdots x_{m_2} x_{m_2+1} \cdots x_{m_3}) = \tilde{y}_1 \cdots \tilde{y}_{n_2} y_{n_2+1} \cdots y_{n_3}. \]
for some $\tilde{y}_1 \ldots \tilde{y}_{n_2} \in \mathcal{D}_2$, where $y_1 \ldots y_{n_2} \sim \tilde{y}_1 \ldots \tilde{y}_{n_2}$. We also have

$$\psi_2(y_1 \ldots y_{n_2}) = \tilde{x}_1 \ldots \tilde{x}_{m_1} x_{m_1+1} \ldots x_{m_2},$$

and

$$\psi_2(y_1 \ldots y_{n_2}) = x_1 \ldots x_{m_2}.$$  

Apply Lemma 9.1 for $y_1 \ldots y_{m_2}$ and $\tilde{y}_1 \ldots \tilde{y}_{n_2}$ to conclude that $y_1 \ldots y_{n_2} \approx \tilde{y}_1 \ldots \tilde{y}_{n_2}$, so $y_k = \tilde{y}_k$ for all $n_1 < k \leq n_2$.

Now we are ready to state and prove our main theorem.

**9.3. Theorem.** Finite coordinate change relations arising from measured Bratteli diagrams admitting nonatomic ergodic probability invariant measures are all finitary orbit equivalent.

**Proof.** Choose a sequence $\{\varepsilon_n\}$ with $0 < \varepsilon_n < 1$ and

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty.$$  

Using Propositions 7.2 and 8.4 inductively, we have a commuting diagram

$$
\begin{array}{ccc}
\mathcal{C}^0 & \xrightarrow{\phi_0} & \mathcal{D}^0 \\
\pi \uparrow & & \uparrow \pi \\
\mathcal{C}^1 & \xleftarrow{\psi_1} & \mathcal{D}^1 \\
\pi \uparrow & & \uparrow \pi \\
\mathcal{C}^2 & \xrightarrow{\phi_2} & \mathcal{D}^2 \\
\uparrow & & \uparrow \\
& \vdots & \\
\end{array}
$$

where $\mathcal{C}^0 = \{X\}$, $\mathcal{D}^0 = \{Y\}$, $\phi_0(X) = Y$, $\mathcal{C}^{2n+1}$ is $(1-\varepsilon_{2n+1})$-$\mathcal{S}_{2n+1}$-invariant ($n \geq 0$), $\mathcal{D}^{2n}$ is $(1-\varepsilon_{2n})$-$\mathcal{S}_{2n}$-invariant ($n \geq 1$), the maps commute, and $\phi_{2n}$ and $\psi_{2n+1}$ preserve the respective $\approx$ equivalence relations at each stage of the induction. By the definition of cylinder partition, $\mu$-almost every $x \in X$ belongs to a cylinder $c_n(x) \in \mathcal{C}^n$ for each $n \geq 0$. Similarly, $\nu$-almost every $y \in Y$ belongs to a cylinder $d_n(y) \in \mathcal{D}^n$ for each $n \geq 0$. Moreover, if $d_0 \leftarrow d_1 \leftarrow d_2 \leftarrow \cdots$ is a chain, then $\bigcap_{n \geq 0} d_n$ is a singleton in $Y$. Therefore for $\mu$-almost every $x \in X$ there is a $y = \Phi(x) \in Y$ such that for each $n \geq 0$,

$$\phi_{2n}(c_{2n}(x)) = d_{2n}(y).$$

Analogously (or by commutativity),

$$\psi_{2n+1}(d_{2n+1}(y)) = c_{2n+1}(x).$$
defining $\Psi(y) = x$ and proving that $\Phi$ and $\Psi$ are finitary and inverses of each other.

Finally, let $x$ and $\tilde{x}$ be two points of $X$ belonging to the same $S$-orbit. By $(1 - \varepsilon_n)\cdot S_n$-invariance and the Borel–Cantelli lemma, $\mu$-almost all such $x$ (and $\tilde{x}$) belong to the sets

$$E_{2n+1} = \bigcup_{c \in E_{2n+1}} c$$

for all sufficiently large $n$, where $E_{2n+1}$ is the collection of cylinders associated with the $(1 - \varepsilon_{n+1})\cdot S_{2n+1}$-invariant tower $\mathcal{C}^{2n+1}$, whose total measure is at least $1 - \varepsilon_{2n+1}$. This means that the cylinders $c_{2n+1}(x)$ and $c_{2n+1}(\tilde{x})$ are $\mathcal{C}_{2n+1}$-equivalent for all large $n$. In other words,

$$S(x) = \bigcup_{n \geq 0} \mathcal{C}^{2n+1}(x)$$

for $\mu$-almost every $x \in X$, and in particular the cylinders $c_{2n+1}(x)$ and $c_{2n+1}(\tilde{x})$ are obtained from the cylinders $c_{2n}(x)$ and $c_{2n}(\tilde{x})$ by juxtaposition of the same word $w$. Then by Proposition 9.2, $\phi_{2n+2}$ respects the $\cong$ equivalence relation. Therefore the corresponding cylinders $d_{2n+2}(\Phi(x)) = \phi_{2n+2}(c_{2n+2}(x))$ and $d_{2n+2}(\Phi(\tilde{x})) = \phi_{2n+2}(c_{2n+2}(\tilde{x}))$ are obtained from $d_{2n}(\Phi(x))$ and $d_{2n}(\Phi(\tilde{x}))$ by juxtaposition of the same word. This shows that $y = \Phi(x)$ and $\tilde{y} = \Phi(\tilde{x})$ differ in at most finitely many coordinates. Symmetrically we also see that $x = \Psi(y)$ and $\tilde{x} = \Psi(\tilde{y})$ differ in at most finitely many coordinates.

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