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FULL GROUPS, FLIP CONJUGACY, AND ORBIT EQUIVALENCE OF CANTOR MINIMAL SYSTEMS

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Abstract. We consider the full group $[\varphi]$ and topological full group $[[\varphi]]$ of a Cantor minimal system (X,φ). We prove that the commutator subgroups $D([\varphi])$ and $D([[\varphi]])$ are simple and show that the groups $D([\varphi])$ and $D([[\varphi]])$ completely determine the class of orbit equivalence and flip conjugacy of φ, respectively. These results improve the classification found in [GPS]. As a corollary of the technique used, we establish the fact that φ can be written as a product of three involutions from $[\varphi]$.

1. Introduction. One of the most remarkable results of ergodic theory is Dye's theorem which states that any two ergodic finite measure-preserving automorphisms of a Lebesgue space are orbit equivalent and, as a corollary, their full groups are isomorphic [D1]. Dye also proved that the full group is a complete invariant of orbit equivalence for ergodic finite measure-preserving actions of countable groups [D2]. Later, full groups have been studied in numerous papers from different points of view. In particular, the analogues of the theorems of Dye were established for infinite measure-preserving and non-singular automorphisms of a standard measure space.

The ideas developed in ergodic theory have been successfully applied to the study of orbit equivalence in Cantor and Borel dynamics. Giordano, Putnam, and Skau considered the notions of the full group $[\varphi]$ and topological full group $[[\varphi]]$ of a Cantor minimal system (X,φ) and showed that these groups completely determine the classes of orbit equivalence and flip conjugacy of φ [GPS]. In other words, they proved that, for minimal homeomorphisms φ_1 and φ_2, any algebraic isomorphism of the full groups $[\varphi_1]$ and $[\varphi_2]$ is spatially generated. Recently, Miller and Rosendal have shown that Dye's theorem holds in the context of Borel dynamics: two Borel aperiodic actions of countable groups are orbit equivalent if and only if their full groups are isomorphic [MilRos]. We should also mention that there are several papers where the algebraic and topological structures of the full groups

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of ergodic automorphisms have been studied. In particular, Eigen proved that the full group of an ergodic finite measure-preserving automorphism is simple, i.e. it has no proper normal subgroups [E1]. Notice that it is still an open problem to show the simplicity of the full group $[\varphi]$ generated by a minimal homeomorphism of a Cantor set. On the other hand, it is known that the topological full group $[[\varphi]]$ has a proper normal subgroup and so is not simple.

The goal of this paper is to prove that the class of orbit equivalence and flip conjugacy of a Cantor minimal system can be determined by proper simple subgroups of the full group and topological full group. To do this, we focus our study on the commutator subgroups $D([\varphi])$ and $D([[\varphi]])$ of $[\varphi]$ and $[[\varphi]]$, respectively. First of all, we prove in Section 3 that the groups $D([\varphi])$ and $D([[\varphi]])$ are simple (Theorem 3.4). Then we show that the commutator subgroup $D([[\varphi]])$ is a complete invariant of flip conjugacy (Theorem 5.13) and the group $D([\varphi])$ is a complete invariant of orbit equivalence (Theorem 5.2). These results make more precise the characterization of orbit equivalence and flip conjugacy found in [GPS]. In particular, Theorem 5.13 contains a new proof of the fact that $[[\varphi]]$ is a complete invariant of flip conjugacy. To show that the group $D([\varphi])$ determines the class of orbit equivalence, we follow the idea of the proof of Corollary 4.6 from [GPS]. The key point of our approach is the fact that every involution with clopen support belongs to $D([\varphi])$ (Corollary 4.8). In its turn, this result is based on the fact that every homeomorphism from $[\varphi]$, which is minimal on its support, can be written as a product of five commutators from $[\varphi]$ (Theorem 4.6).

In Section 4, we also answer the question of representation of a minimal homeomorphism φ as a product of involutions from $[\varphi]$. We note that this problem would be trivial if we knew that $[\varphi]$ is a simple group. The problem of writing every element of a transformation group as a product of involutions has been considered in measurable, Borel and Boolean dynamics. Apparently, the first results appeared in the paper of Anderson [A], where it was shown that every homeomorphism of a Cantor set is a product of six involutions from $H(X)$, the group of all homeomorphisms of a Cantor set X. The technique used by Anderson also works for the group of non-singular transformations of a Lebesgue space (this fact was mentioned in [E1]). In [F], Fathi suggested a new approach to this problem which allowed him to show that the group $\mathrm{Aut}(Y,\mu)$ of finite measure-preserving automorphisms of a Lebesgue space (Y,μ) is simple, and that every $f \in \mathrm{Aut}(Y,\mu)$ is a product of 10 involutions. His ideas were then used in [E1] to obtain a similar result for the full group of an ergodic automorphism of a Lebesgue space. Later, Ryzhikov improved Fathi's theorem and showed that every ergodic automorphism $f \in \mathrm{Aut}(X,\mu)$ is, in fact, a product of three involutions from its full group [R1]. We notice that here three is the least number possible.

Miller showed that an automorphism f is the product of two involutions from its full group if and only if f is dissipative [Mil]. We also mention [Fr], [R2] and [ChPr] where these questions were studied for automorphisms of complete Boolean algebras and homogeneous measure algebras. Developing ideas of Fathi and Ryzhikov, we prove that every minimal homeomorphism φ of a Cantor set is the product of three involutions from $[\varphi]$ (Theorem 4.14).

2. Preliminaries. Let X denote a Cantor set, i.e. a 0-dimensional compact metric space without isolated points. Denote by $H(X)$ the group of all homeomorphisms of X. For a homeomorphism $\varphi \in H(X)$ and a point $x \in X$, let $\mathrm{Orb}_\varphi(x) = \{\varphi^n(x) : n \in \mathbb{Z}\}$ denote the φ-orbit of x. The open set $\mathrm{supp}(\varphi) = \{x \in X : \varphi(x) \neq x\}$ is called the *support* of φ.

A homeomorphism $\varphi \in H(X)$ is called *periodic* if for each $x \in X$ there exists $n > 0$ such that $\varphi^n(x) = x$, and if the φ-orbit of x is infinite for all x, then φ is called *aperiodic*. A homeomorphism φ is called *minimal* if the φ-orbit of every point is dense in X. Let $[\varphi]$ denote the set of all homeomorphisms $f \in H(X)$ such that $f(x) \in \mathrm{Orb}_\varphi(x)$ for all $x \in X$. The set $[\varphi]$ is called the *full group* of φ. If $f \in [\varphi]$, then there is a function $n_f : X \to \mathbb{Z}$ such that $f(x) = \varphi^{n_f(x)}(x)$ for all $x \in X$, which is called the *cocycle associated to* f. The *topological full group* $[[\varphi]]$ of φ is defined as the set of all $f \in [\varphi]$ such that n_f is continuous. We refer the reader to [GPS] and [GW] for an in-depth study of various properties of full groups associated to a minimal homeomorphism of a Cantor set.

Given a group G, denote by $D(G)$ the subgroup generated by all elements of the form $[f, g] := fgf^{-1}g^{-1}$, $f, g \in G$. The group $D(G)$ is called the *commutator subgroup* of G.

Let $\varphi \in H(X)$ and let A be a clopen set. A point $x \in A$ is called *recurrent* for φ if there is $n > 0$ such that $\varphi^n(x) \in A$. Observe that if a clopen set A consists of recurrent points, then the function $n_A : A \to \mathbb{N}$ given by $n_A(x) = \min\{n > 0 : \varphi^n(x) \in A\}$ is well-defined and continuous. Then n_A is called the the *first return function*. Suppose that a clopen set A meets every φ-orbit and consists of recurrent points. Set $A_k = \{x \in A : n_A(x) = k\}$. Hence, X can be decomposed into clopen sets

$$X = \bigcup_{k\geq 1} \bigcup_{i=0}^{k-1} \varphi^i(A_k).$$

This decomposition is called a *Kakutani–Rokhlin* (*K-R*) *partition* built over the set A. By a *φ-tower*, we mean a non-empty disjoint family $\xi = (A_k, \varphi A_k, \ldots, \varphi^{k-1} A_k)$. The set A_k is called the *base* of ξ and k is called the *height* of ξ.

The following simple proposition explains the construction of the induced map.

PROPOSITION 2.1. *Let $\varphi \in H(X)$ and suppose a clopen set A meets every φ-orbit. Then*:

1. *A consists of recurrent points*;
2. *the homeomorphism $\varphi_A \in H(X)$ given by $\varphi_A(x) = \varphi^{n_A(x)}(x)$ for $x \in A$ and $\varphi_A(x) = x$ for $x \in X \setminus A$ belongs to $[[\varphi]]$*;
3. *the homeomorphism $g = \varphi_A^{-1}\varphi$ is periodic.*

Proof. (1) Since A meets every φ-orbit, we have $X = \bigcup_{i\in\mathbb{Z}} \varphi^i(A)$. By compactness of X, we get $X = \bigcup_{i=0}^{m} \varphi^i(A)$ for some m. This implies that A consists of recurrent points. The statement (2) is trivial and the proof of (3) is straightforward. ■

For a homeomorphism φ, denote by $\mathcal{M}(\varphi)$ the set of all Borel probability φ-invariant measures on X. Clearly, if $\gamma \in [\varphi]$, then $\gamma \circ \mu = \mu$ for every $\mu \in \mathcal{M}(\varphi)$. The following theorem answers the question when two clopen sets can be mapped onto each other by an element of the full group. This result will be one of our main tools in the study of full groups.

THEOREM 2.2 ([GW]). *Let (X, φ) be a Cantor minimal system.*

1. *If A, B are clopen subsets of X such that $\mu(B) < \mu(A)$ for every $\mu \in \mathcal{M}(\varphi)$, then there exists $\alpha \in [[\varphi]]$ with $\alpha(B) \subset A$. Moreover, α can be chosen such that $\alpha^2 = \mathrm{id}$ and $\alpha|(B \cup \alpha(B))^c = \mathrm{id}$ where $E^c := X \setminus E$.*
2. *If A, B are clopen sets with $\mu(A) = \mu(B)$ for all $\mu \in \mathcal{M}(\varphi)$, then there exists $\alpha \in [\varphi]$ such that $\alpha(B) = A$. Moreover, α can be chosen such that $\alpha^2 = \mathrm{id}$, α has clopen support, and $\alpha|(B \cup \alpha(B))^c = \mathrm{id}$.*

We will need a generalization of a theorem from [Ak].

PROPOSITION 2.3. *Let (X, φ) be a Cantor minimal system and let d be a metric on X compatible with the topology.*

1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if the d-diameter of a clopen set A is less than δ, then $\mu(A) < \varepsilon$ for every $\mu \in \mathcal{M}(\varphi)$.*
2. *If A is a clopen set, then $\inf\{\mu(A) : \mu \in \mathcal{M}(\varphi)\} > 0$.*

Proof. (1) Assume the converse: there exists $\varepsilon_0 > 0$ such that for every n there exist a clopen set A_n with $\mathrm{diam}(A_n) < 1/n$ and a measure $\mu_n \in M(\varphi)$ such that $\mu_n(A_n) \geq \varepsilon_0$. By compactness of $M(\varphi)$ and X, we may assume that $\mu_n \to \mu \in M(\varphi)$ and there exists $x_0 \in X$ such that every neighborhood of x_0 contains all but a finite number of the sets A_n.

Consider any clopen neighborhood O of x_0. Then $A_n \subseteq O$ and $\mu_n(O) \geq \mu_n(A_n) \geq \varepsilon_0$ for sufficiently large n. As $\mu_n \to \mu$, we get $\mu(O) \geq \varepsilon_0$. Hence $\mu(\{x_0\}) > 0$, a contradiction.

(2) This easily follows from the minimality of φ (see also [GW]). ■

From now on, the pair (X, φ) will always denote a Cantor minimal system.

3. Commutator subgroups. In this section, we show that, for a minimal homeomorphism φ, the commutator subgroups $D([\varphi])$ and $D([[\varphi]])$ are simple. In our proofs, we follow the arguments used in [F] and [E1].

The following simple statement describes the properties of periodic homeomorphisms. More detailed descriptions of periodic homeomorphisms from $[[\varphi]]$ and $[\varphi]$ can be found in [BDK].

LEMMA 3.1. *Let $f \in H(X)$ and $X_n := \{x \in X : |\mathrm{Orb}_f(x)| = n\}$ for $1 \le n < \infty$.*

(1) *If $f \in [[\varphi]]$ with φ a minimal homeomorphism, then X_n is clopen.*
(2) *If X_n is clopen, then there exists a clopen set X_n^0 such that $X_n = \bigcup_{i=0}^{n-1} f^i(X_n^0)$, a disjoint union.*

Proof. (1) Notice that $X_{\le n} := \{x \in X : |\mathrm{Orb}_f(x)| \le n\}$ is closed. Since $f \in [[\varphi]]$, the associated cocycle n_f is continuous. This implies that the set X_n is open for each $1 \le n < \infty$. Hence, X_n is clopen.

(2) See Lemma 3.2 in [BDK]. ∎

The following lemma states that every homeomorphism from the full group can be written as a product of homeomorphisms which have "small" supports.

LEMMA 3.2. *Let Γ denote either $[\varphi]$ or $[[\varphi]]$. Given $\delta > 0$ and $g \in \Gamma$, there exist $g_1, \ldots, g_m \in \Gamma$ and clopen sets $E_1, \ldots, E_m$ such that*

(1) $g = g_1 \ldots g_m$;
(2) $\mathrm{supp}(g_i) \subseteq E_i$ *and $\mu(E_i) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$ and $i = 1, \ldots, m$.*

Proof. Suppose first that $\Gamma = [\varphi]$. Using Proposition 2.3, take any decomposition of X into disjoint clopen sets $X = B_1 \sqcup \cdots \sqcup B_m$ such that $\mu(B_i) < \delta/2$ for any $\mu \in \mathcal{M}(\varphi)$ and $i = 1, \ldots, m$.

Given $f \in [\varphi]$, we have $\mu(B \setminus f(B)) = \mu(f(B) \setminus B)$ for every $\mu \in \mathcal{M}(\varphi)$ and every clopen set B. Therefore, by Theorem 2.2, there exists $g_1 \in [\varphi]$ such that $g_1|B_1 = g|B_1$, $g_1(g(B_1) \setminus B_1) = B_1 \setminus g(B_1)$, and $\mathrm{supp}(g_1) \subseteq B_1 \cup g(B_1)$. Setting $E_1 = B_1 \cup g(B_1)$, we obtain $\mu(\mathrm{supp}(g_1)) \le \mu(E_1) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$.

Let $g_1' = g_1^{-1} g$. Clearly, $\mathrm{supp}(g_1') \subseteq B_2 \sqcup \cdots \sqcup B_m$. Find $g_2 \in [\varphi]$ such that $g_2|B_2 = g_1'|B_2$ and $\mathrm{supp}(g_2) \subseteq B_2 \cup g_1'(B_2)$. Let $E_2 = B_2 \cup g_1'(B_2)$. Then $\mu(E_2) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$. Next, consider $g_2' := g_2^{-1} g_1'$. Clearly, $\mathrm{supp}(g_2') \subseteq B_3 \sqcup \cdots \sqcup B_m$ and $g = g_1 g_2 g_2'$. Repeating the above argument for each set B_i, we construct a family of homeomorphisms $g_i \in [\varphi]$, $i = 1, \ldots, m$, such that

$g = g_1 \dots g_m$ and $\mathrm{supp}(g_i) \subseteq B_i \cup g'_{i-1}(B_i)$. Here $g_m = g'_{m-1} = g_{m-1}^{-1} g'_{m-2}$. Setting $E_i = B_i \cup g'_{i-1}(B_i)$, we complete the proof for the case $\Gamma = [\varphi]$.

Suppose now that $\Gamma = [[\varphi]]$. If g is periodic, then by Lemma 3.1 and compactness of X, we can decompose X into a finite number of clopen sets

$$X = \bigcup_{i \in I} \bigcup_{j=0}^{n_i - 1} g^j(X_i^0),$$

where $g^{n_i}(x) = x$ for each $x \in X_i^0$. By Proposition 2.3, we can divide each set X_i^0 into clopen sets $\{A_{i,1}^0, \dots, A_{i,m_i}^0\}$ such that $\mu(\mathcal{A}_{i,j}) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$, where $\mathcal{A}_{i,j} = A_{i,j}^0 \cup \dots \cup g^{n_i - 1} A_{i,j}^0$. Set $g_{i,j} x = gx$ whenever $x \in \mathcal{A}_{i,j}$ and $g_{i,j} x = x$ elsewhere. Clearly, every $g_{i,j} \in [[\varphi]]$ and g is the product of the commuting elements $g_{i,j}$.

Consider the case when g is not periodic. Choose an integer $k > 0$ such that $1/k < \delta$. As $g \in [[\varphi]]$, the set $X_{\geq k} := \{x \in X : |\mathrm{Orb}_g(x)| \geq k\}$ is clopen. Using the arguments of [BDM, Proposition 3] (see also [M2, Lemma 2.2]), we can show that there exists a clopen set $B \subset X_{\geq k}$ such that $g^i(B) \cap B = \emptyset$ for $i = 0, \dots, k-1$ and B meets every $g|X_{\geq k}$-orbit. It follows that $\mu(B) \leq 1/k < \delta$ for all $\mu \in \mathcal{M}(\varphi)$. By our choice of B, the induced homeomorphism g_B, defined as in Proposition 2.1, belongs to $[[\varphi]]$. Moreover, $\mu(\mathrm{supp}(g_B)) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$. Observe that $g_1 = g_B^{-1} g$ is periodic. We use the decomposition obtained for periodic homeomorphisms to complete the proof. ∎

LEMMA 3.3. *Suppose that G is a group and H is a normal subgroup of G. If $g_1, \dots, g_n, h_1, \dots, h_m \in G$ are such that $[g_i, h_j], [g_i, g_j]$, and $[h_i, h_j]$ belong to H for any i, j, then also $[g_1 \dots g_n, h_1 \dots h_m] \in H$.*

Proof. Note that

$$[g_1 g_2, h_i] = g_1 [g_2, h_i] g_1^{-1} [g_1, h_i], \qquad [g_j, h_1 h_2] = [g_j, h_1] h_1 [g_j, h_2] h_1^{-1}.$$

As H is a normal subgroup, $[g_1 g_2, h_i]$ and $[g_i, h_1 h_2]$ belong to H. Hence $[g_1 g_2, h_1 h_2] \in H$. The proof can be completed by induction. We leave the details to the reader. ∎

Now we are ready to show that the commutator subgroups $D([\varphi])$ and $D([[\varphi]])$ are simple. We should also mention that the simplicity of $D([[\varphi]])$ was first established by Matui [Ma, Theorem 4.9.], but with a completely different technique.

THEOREM 3.4. *Let (X, φ) be a Cantor minimal system.*

(1) *If H is a normal subgroup of $[\varphi]$ (or of $D([\varphi])$), then $D([\varphi]) \subseteq H$.*

(2) *If H is a normal subgroup of $[[\varphi]]$ (or of $D([[\varphi]])$), then $D([[\varphi]]) \subseteq H$.*

In particular, the groups $D([\varphi])$ and $D([[\varphi]])$ are simple.

Proof. First of all we notice that if $2\mu(B) < \mu(A)$ for any $\mu \in M(\varphi)$ with A and B clopen sets, then there exists $\alpha \in D([[\varphi]])$ such that $\alpha(B) \subset A$. Indeed, by setting $\alpha = \mathrm{id}$ on $A \cap B$, we may assume that $A \cap B = \emptyset$. Applying Theorem 2.2 twice, we find two involutions $\alpha_1, \alpha_2 \in [[\varphi]]$ such that $\alpha_1(B) \subset A$, $\alpha_2(\alpha_1(B)) \subset A \setminus \alpha_1(B)$, and $\mathrm{supp}(\alpha_1) = B \cup \alpha_1(B)$, $\mathrm{supp}(\alpha_2) = \alpha_1(B) \cup \alpha_2(\alpha_1(B))$. Set $\alpha = \alpha_1\alpha_2$. Then $\alpha(B) = \alpha_1(B) \subset A$. Since $\alpha_2 = \alpha\alpha_1^{-1}\alpha^{-1}$ we get $\alpha = \alpha_1\alpha_2 = [\alpha_1, \alpha_2]$.

We must show that $[g, h] \in H$ for any $g, h \in \Gamma$, where Γ denotes one of the groups $D([[\varphi]])$, $[[\varphi]]$, $D([\varphi])$, or $[\varphi]$. Take any non-trivial element $f \in H$. Then there exists a clopen set $E \subseteq X$ such that $f(E) \cap E = \emptyset$. Proposition 2.3 implies that $\delta = \frac{1}{2}\inf\{\mu(E) : \mu \in \mathcal{M}(\varphi)\} > 0$.

By Lemma 3.2, write g and h as $g = g_1 \ldots g_n$ and $h = h_1 \ldots h_m$, such that (1) $g_i, h_j \in \Gamma$ and (2) there exist clopen sets $E_i(g)$ and $E_j(h)$ with $\mathrm{supp}(g_i) \subseteq E_i(g)$, $\mathrm{supp}(h_j) \subseteq E_j(h)$ and $\mu(E_i(g) \cup E_j(h)) < \delta$ for every $\mu \in \mathcal{M}(\varphi)$ and every i, j. Due to Lemma 3.3, it is sufficient to prove that $[g_i, h_j] \in H$.

For convenience we omit the subscripts i and j. Consider any homeomorphisms $g, h \in \Gamma$ such that $\mathrm{supp}(g) \cup \mathrm{supp}(h) \subseteq F$, where F is a clopen set with $\mu(F) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$. As above find a homeomorphism $\alpha \in D([[\varphi]]) \subset \Gamma$ with $\alpha(F) \subseteq E$. Since H is a normal subgroup of Γ, the element $q = \alpha^{-1}f\alpha$ is in H.

Since H is a normal subgroup, we have $\widehat{h} = [h, q] = hqh^{-1}q^{-1} \in H$. Analogously, $[g, \widehat{h}] \in H$. Since $q(F) \cap F = \emptyset$, we see that g^{-1} and $qh^{-1}q^{-1}$ commute. Then

$$\begin{aligned}[g, \widehat{h}] &= g(hqh^{-1}q^{-1})g^{-1}(qhq^{-1}h^{-1}) \\ &= ghg^{-1}qh^{-1}q^{-1}qhq^{-1}h^{-1} = [g, h] \in H.\end{aligned}$$

This completes the proof. ∎

REMARK 3.5. For a measure $\mu \in \mathcal{M}(\varphi)$, set $[[\varphi]]_0 = \{\gamma \in [[\varphi]] : \int_X n_\gamma \, d\mu = 0\}$. The definition of $[[\varphi]]_0$ does not depend on the choice of μ [GPS, Section 5]. As proved in [GPS], the group $[[\varphi]]_0$ completely determines the class of flip conjugacy of φ. Clearly, $[[\varphi]]_0$ is a proper normal subgroup of $[[\varphi]]$. Therefore, by Theorem 3.4, $D([[\varphi]]) \subseteq [[\varphi]]_0$ and $D([[\varphi]]) = D([[\varphi]]_0)$. In [GPS] the authors asked if $[[\varphi]]_0$ is a simple group. This would show that the class of flip conjugacy is determined by a countable simple group. However, $[[\varphi]]_0$ is not simple, in general. Matui proved that the simplicity of $[[\varphi]]_0$ is equivalent to the 2-divisibility of the dimension group $K^0(X, \varphi)$ [Ma]. Nevertheless, in Section 5 we will show that $D([[\varphi]])$ is a complete invariant for flip conjugacy.

4. Product of involutions. In the section, we show that a minimal homeomorphism φ of a Cantor set X and involutions from $[\varphi]$ with clopen supports belong to $D([\varphi])$. This will allow us to prove that the simple group $D([\varphi])$ is a complete invariant for the class of orbit equivalence of φ. As a corollary of the technique used, we also find that φ can be written as a product of three involutions from $[\varphi]$. Our considerations are mainly based on the ideas of Fathi [F].

REMARK 4.1. Suppose that g is a periodic homeomorphism from $[\varphi]$ such that the space X can be decomposed into clopen sets

$$X = \bigcup_{i\in I} \bigcup_{j=0}^{n_i-1} g^j(X_i^0)$$

with $g^{n_i}(x) = x$ for all $x \in X_i^0$. We give two model situations when g can be easily written as a commutator in $[\varphi]$. Consider the following cases:

(1) Suppose that n_i is odd for $i \in I$. Then $g|X_i$ is a commutator in $[\varphi]$. Indeed, let $m = (n_i - 1)/2$ and define the homeomorphisms g_1 and g_2 as follows:

$$g_1(x) = \begin{cases} g(x) & \text{if } x \in \bigcup_{k=0}^{m-1} g^k(X_i^0), \\ g^{-m}(x) & \text{if } x \in g^m(X_i^0), \\ \mathrm{id} & \text{elsewhere}, \end{cases}$$

$$g_2(x) = \begin{cases} g(x) & \text{if } x \in \bigcup_{k=m}^{n_i-2} g^k(X_i^0), \\ g^{-m}(x) & \text{if } x \in g^{n_i-1}(X_i^0), \\ \mathrm{id} & \text{elsewhere}. \end{cases}$$

Then $g = g_1g_2$. Since $g_2 = \psi g_1^{-1}\psi^{-1}$ for some $\psi \in [\varphi]$, we see that $g = g_1g_2 = [g_1, \psi]$.

(2) Now suppose that n_i is even and $X_i^0 = X_i^0(l) \sqcup X_i^0(r)$ where $X_i^0(l)$ and $X_i^0(r)$ are $[\varphi]$-equivalent clopen sets, i.e. $\alpha(X_i^0(l)) = X_i^0(r)$ for some $\alpha \in [\varphi]$. Then $g|X_i$ can be written as a commutator in $[\varphi]$. Indeed, put $X_i(l) = \bigcup_{k=0}^{n_i-1} g^k(X_i^0(l))$ and $X_i(r) = \bigcup_{k=0}^{n_i-1} g^k(X_i^0(r))$. Define $l, r \in [\varphi]$ as follows: $l|X_i(l) = g|X_i(l)$ and $l = \mathrm{id}$ elsewhere; $r|X_i(r) = g|X_i(l)$ and $r = \mathrm{id}$ elsewhere. Note that $g = lr$. By Theorem 2.2, it is easy to see that $r = \psi l^{-1}\psi^{-1}$ for some $\psi \in [\varphi]$. Hence $g = lr = [l, \psi]$.

The proof of the fact that $\varphi \in D([\varphi])$ consists of a series of lemmas.

LEMMA 4.2. *Suppose that $f \in [\varphi]$ has clopen support and $f|\mathrm{supp}(f)$ is minimal. Then for given $\delta > 0$ there exist $f_1, s, t \in [\varphi]$ such that*

(1) $f = f_1[s, t]$;
(2) $\mathrm{supp}(f_1)$ *is clopen*, $f_1|\mathrm{supp}(f_1)$ *is minimal*, *and* $\mu(\mathrm{supp}(f_1)) < \delta$ *for all* $\mu \in \mathcal{M}(\varphi)$;
(3) $\mathrm{supp}(s) \cup \mathrm{supp}(t) \cup \mathrm{supp}(f_1) \subseteq \mathrm{supp}(f)$.

Proof. Take any clopen set $A \subset \mathrm{supp}(f)$ such that $\mu(A) < \delta/2$ for all $\mu \in \mathcal{M}(\varphi)$ and $f^i(A) \cap A = \emptyset$ for $i = 1, 2, 3$.

Applying the first return function, construct the K-R partition Ξ over A (see Section 2 for the definition). Suppose that $\Xi = \{\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_m\}$ where the ξ_i are the f-towers with even heights and the ξ'_j are f-towers with odd heights. Let $h(\xi)$ denote the height and $B(\xi)$ denote the base of an f-tower ξ. Set

$$B = A \cup \bigcup_{i=1}^{n} f^{h(\xi_i)/2} B(\xi_i).$$

Clearly, $\mu(B) < \delta$ for all $\mu \in \mathcal{M}(\varphi)$.

Define f_1 as the induced homeomorphism f_B. Again using the first return function, consider the K-R partition $\mathcal{P}$ over B. Note that $\mathcal{P} = \{\xi'_1, \ldots, \xi'_m,\ \xi^1_1,\ \xi^2_1, \ldots, \xi^1_n, \xi^2_n\}$, where the ξ'_i's are f-towers with odd heights as above, ξ^1_i is the lower half of ξ_i, and ξ^2_i is the upper half.

Define the periodic homeomorphism g as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{\xi \in \mathcal{P}} f^{h(\xi)-1}(B(\xi)), \\ f^{-h(\xi)+1}(x) & \text{if } x \in f^{h(\xi)-1}(B(\xi)) \text{ for some } \xi \in \mathcal{P}. \end{cases}$$

Then $f = f_1 g$. Since ξ'_i has odd height, Remark 4.1 implies that $g|\xi'_i$ is a commutator in $[\varphi]$. Consider $g|(\xi^1_i \cup \xi^2_i)$, $i = 1, \ldots, n$. By construction, the bases $B(\xi^1_i)$ and $B(\xi^2_i)$ are $[\varphi]$-equivalent. Therefore, the application of Remark 4.1(2) ensures that $g = [s, t]$ for some $s, t \in [\varphi]$. Statements (2) and (3) are obvious. ∎

Remark 4.3. Clearly, we can construct f_1 in Lemma 4.2 such that $\mathrm{supp}(f_1)$ is always a proper subset of $\mathrm{supp}(f)$. Let now x_0 be a point from $\mathrm{supp}(f) \setminus \mathrm{supp}(f_1)$. Then one can find a sequence $\{C_n\}_{n\geq 1}$ of mutually disjoint clopen sets such that (1) $C_i \subseteq \mathrm{supp}(f)$; (2) $C_1 = \mathrm{supp}(f_1)$; (3) $x_0 \notin C_n$ for $n \geq 1$; (4) $\mathrm{diam}(C_n \cup \{x_0\}) \to 0$ as $n \to \infty$. Put $\delta_n = \inf\{\mu(C_n) : \mu \in \mathcal{M}(\varphi)\}$. By Proposition 2.3, every $\delta_n > 0$.

Lemma 4.4. *Suppose that f_1 and a sequence $\{C_n\}$ are as in Remark 4.3. Then there exists two sequences $\{f_i\}_{i\geq 1}$ and $\{g_i\}_{i\geq 1}$ of homeomorphisms from $[\varphi]$ such that for every $i \geq 1$,*

(1) $\mathrm{supp}(f_i)$ *is clopen and f_i is minimal on it*;
(2) $f_i = f_{i+1} g_i$;
(3) $\mathrm{supp}(f_i) \subseteq C_i$;
(4) $g_i = [s'_i, t'_i][s_i, t_i]$ *for some $s_i, t_i, s'_i, t'_i \in [\varphi]$ with* $\mathrm{supp}(s_i) \cup \mathrm{supp}(t_i) \cup \mathrm{supp}(s'_i) \cup \mathrm{supp}(t'_i) \subseteq C_i \cup C_{i+1}$.

Proof. We will explain only the first step of the construction: given f_1, C_1, and C_2, we will find f_2, g_1.

By Lemma 4.2, there exist homeomorphisms $\bar{f}, s, t \in [\varphi]$ with clopen supports in C_1 such that $f_1 = \bar{f}[s,t]$ and $\mu(\mathrm{supp}(\bar{f})) < \delta_2$ for all $\mu \in \mathcal{M}(\varphi)$. It follows from Theorem 2.2 that there exists a homeomorphism $t' \in [\varphi]$ with clopen support in $C_1 \cup C_2$ such that $t'(\mathrm{supp}(\bar{f})) \subset C_2$. Set $f_2 = t'\bar{f}t'^{-1}$. Then the support of f_2 is a clopen subset of C_2 and $f_2 \in [\varphi]$ is minimal on $\mathrm{supp}(f_2)$. Thus,

$$f = \bar{f}[s,t] = f_2[t', \bar{f}^{-1}][s,t].$$

Setting $g_1 = [t', \bar{f}^{-1}][s,t]$, we complete the proof. ∎

LEMMA 4.5. *Let f_1 be as in Lemma 4.2. Then $f_1 \in D([\varphi])$ and f_1 is a product of four commutators from $[\varphi]$.*

Proof. Let $\{f_i\}_{i\geq 1}$ and $\{g_i\}_{i\geq 1}$ be the sequences of homeomorphisms constructed in Lemma 4.4. Recall that $g_i = [s_i', t_i'][s_i, t_i]$ and $\mathrm{supp}(s_i') \cup \mathrm{supp}(t_i') \cup \mathrm{supp}(s_i) \cup \mathrm{supp}(t_i) \subseteq C_i \cup C_{i+1}$. Note that the homeomorphisms $\{g_{2k+1}\}_{k\geq 0}$ have mutually disjoint supports. So do $\{g_{2k}\}_{k\geq 1}$. Define the maps g_{odd} and g_{even} as follows:

$$g_{\mathrm{odd}}(x) = \begin{cases} g_i(x) & \text{whenever } x \in \mathrm{supp}(g_i) \text{ for odd } i', \\ x & \text{elsewhere}, \end{cases}$$

$$g_{\mathrm{even}}(x) = \begin{cases} g_i(x) & \text{whenever } x \in \mathrm{supp}(g_i) \text{ for even } i, \\ x & \text{elsewhere}. \end{cases}$$

It follows from the choice of the sets C_i and the property $\mathrm{diam}(C_n \cup \{x_0\}) \to 0$ that g_{odd} and g_{even} are homeomorphisms.

We see $g_{\mathrm{odd}} = [s'_{\mathrm{odd}}, t'_{\mathrm{odd}}][s_{\mathrm{odd}}, t_{\mathrm{odd}}]$ and $g_{\mathrm{even}} = [s'_{\mathrm{even}}, t'_{\mathrm{even}}][s_{\mathrm{even}}, t_{\mathrm{even}}]$ where the homeomorphisms s'_{odd}, s'_{even}, t'_{odd}, t'_{even} s_{odd}, s_{even}, and t_{odd}, t_{even} are defined similarly to g_{odd} and g_{even}.

By definition of g_i, we have $g_i = f_{i+1}^{-1}f_i$. Since all the f_i's have disjoint supports, we can formally write down the infinite products

$$g_{\mathrm{odd}} = (f_2^{-1}f_1)(f_4^{-1}f_3)(f_6^{-1}f_5)\ldots = (f_1f_3f_5\ldots)(f_2^{-1}f_4^{-1}f_6^{-1}\ldots),$$
$$g_{\mathrm{even}} = (f_3^{-1}f_2)(f_5^{-1}f_4)(f_7^{-1}f_6)\ldots = (f_3^{-1}f_5^{-1}f_7^{-1}\ldots)(f_2f_4f_6\ldots).$$

Therefore, $f_1 = g_{\mathrm{odd}}g_{\mathrm{even}} = [s'_{\mathrm{odd}}, t'_{\mathrm{odd}}][s_{\mathrm{odd}}, t_{\mathrm{odd}}][s'_{\mathrm{even}}, t'_{\mathrm{even}}][s_{\mathrm{even}}, t_{\mathrm{even}}]$. ∎

THEOREM 4.6. *Let (X,φ) be a Cantor minimal system. Suppose that a homeomorphism $f \in [\varphi]$ has clopen support and is minimal on it. Then $f \in D([\varphi])$ and f is a product of five commutators from $[\varphi]$. In particular, $\varphi \in D([\varphi])$.*

Proof. This follows immediately from Lemmas 4.2 and 4.5. ∎

REMARK 4.7. Let $\mathcal{P}_1(X)$ denote the set of all Borel probability measures on X. For any $g \in \mathrm{Homeo}(X)$, $\varepsilon > 0$, and any $\mu_1, \ldots, \mu_n \in \mathcal{P}_1(X)$, define $U(g; \mu_1, \ldots, \mu_n; \varepsilon) = \{h \in \mathrm{Homeo}(X) : \mu_i(\{x \in X : h(x) \neq g(x)\}) < \varepsilon$

for $i = 1, \dots, n\}$. Let τ denote the topology on Homeo(X) generated by the base sets of the form $U(g; \mu_1, \dots, \mu_n; \varepsilon)$. This topology was defined and studied in [BDK]. In particular, it was shown that the topological group $[\varphi]$ is closed in Homeo(X) with respect to τ. On the other hand, the topological full group $[[\varphi]]$ is τ-dense in $[\varphi]$ ([BK]; see also [M1] for another proof).

Developing the ideas used in this section, one can show that $[[\varphi]] \subset D([\varphi])$. Hence, Theorem 3.4 implies that the full group of a Cantor minimal system has no τ-closed normal subgroups.

Now, we present several immediate corollaries of Theorem 4.6.

COROLLARY 4.8. *Suppose φ is a minimal homeomorphism and $g \in [\varphi]$ is an involution with clopen support. Then g is a product of ten commutators in $[\varphi]$.*

Proof. By Lemma 3.1, there exists a clopen set $A \subset \operatorname{supp}(g)$ such that $\operatorname{supp}(g)$ is a disjoint union of A and $g(A)$. Define f_1 as the induced map φ_A and let $f_2 = f_1 g$. Clearly, f_1 and f_2 have clopen supports and are minimal on their supports. Then Theorem 4.6 asserts that $g = f_1^{-1} f_2$ is a product of ten commutators from $[\varphi]$. ∎

It follows from Theorems 4.6 and 3.4 that $\varphi \in H$ for any normal subgroup H of $[\varphi]$. In particular, this fact allows us to show that φ is a product of involutions. The following result gives the number of involutions needed to represent φ. The proof is based on Theorem 3.4.

COROLLARY 4.9. *A minimal homeomorphism φ is a product of* 18 *involutions from $[\varphi]$ which have clopen supports.*

Proof. Let w be any involution from $[\varphi]$ with clopen support. Choose any clopen set E such that $w(E) \cap E = \emptyset$. Set $\delta = \inf\{\mu(E) : \mu \in \mathcal{M}(\varphi)\} > 0$. Take any clopen set A' with $\mu(A') < \delta$ for all $\mu \in \mathcal{M}(\varphi)$. Let φ_A be the induced homeomorphism of φ where A is a proper clopen subset of A'. It follows from Proposition 2.1 that $g = \varphi_A^{-1}\varphi$ is a periodic homeomorphism from $[[\varphi]]$. It is not hard to see that $g = st$, where s and t are involutions from $[[\varphi]]$ (see, for example, [Mil, Proposition 4.1]). Applying Lemma 4.5 to φ_A $(= f_1)$, we find that $\varphi_A = [s_1, t_1] \dots [s_4, t_4]$, where $\operatorname{supp}(s_i) \cup \operatorname{supp}(t_i) \subseteq A'$ and $s_i, t_i \in [\varphi]$ for each $i = 1, \dots, 4$.

We claim that if some homeomorphisms h and g are supported on A', then $[h, g]$ is a product of four conjugates of w. Indeed, by Theorem 2.2, find $\alpha \in [\varphi]$ such that $\alpha(A') \subseteq E$. Set $q = \alpha^{-1} w \alpha$. Clearly, $q(A') \cap A' = \emptyset$. Observe that qhq^{-1} commutes with h and g. Hence

$$\begin{aligned}
[h, g] &= hgh^{-1}g^{-1} = h(qh^{-1}q^{-1})(qhq^{-1})gh^{-1}g^{-1} \\
&= h(qh^{-1}q^{-1})g(qhq^{-1})h^{-1}g^{-1} \\
&= (hqh^{-1})(q^{-1})(gqg^{-1})(ghq^{-1}h^{-1}g^{-1}).
\end{aligned}$$

This implies that $[h, g]$ is the product of four conjugates of w. Therefore, φ_A is a product of 16 conjugates of w and $\varphi = \varphi_A g$ is a product of 18 involutions. ∎

We can refine Corollary 4.9 and show that φ can be written, in fact, as a product of three involutions with clopen supports from $[\varphi]$. In our proof we follow the arguments from [R1]. Recall that everywhere below, φ denotes a minimal homeomorphism.

DEFINITION 4.10. We say that a homeomorphism $g \in [\varphi]$ is an *n-cycle* on disjoint clopen sets $E_0, E_1, \ldots, E_{n-1}$ if: (1) $g(E_i) = E_{i+1}$ for $i = 0, \ldots, n-2$ and $g(E_{n-1}) = E_0$; (2) $g(x) = x$ for all $x \in X \setminus (E_0 \cup \cdots \cup E_{n-1})$.

LEMMA 4.11. *Let $g \in [\varphi]$ be an 18-cycle on disjoint clopen sets $E_0, E_1, \ldots$ $\ldots, E_{17}$ such that $g^{18}|E_0$ is a minimal homeomorphism. Then there exists an involution $d \in [\varphi]$ with clopen support such that the homeomorphism gd is an 18-cycle on $E_0, \ldots, E_{17}$, and $(gd)^{18} = \mathrm{id}$.*

Proof. By Corollary 4.9, there are involutions $h_0, \ldots, h_{17}$ from $[\varphi]$ with clopen supports such that $g^{18}|E_0 = h_0 \ldots h_{17}$ and $\mathrm{supp}(h_i) \subseteq E_0$.

Set $d_k = g^k h_k^{-1} g^{-k}$ and $d = d_0 d_1 \ldots d_{17}$. Then d is an involution with clopen support and

$$\begin{aligned}(gd)^{18}|E_0 &= (gd_{17}) \ldots (gd_1)(gd_0)|E_0 \\ &= (gg^{17}h_{17}^{-1}g^{-17})(gg^{16}h_{16}^{-1}g^{17}) \ldots (gg^2h_2^{-1}g^{-2})(ggh_1^{-1}g^{-1})(gh_0^{-1})|E_0 \\ &= g^{18}h_{17}^{-1} \ldots h_0^{-1}|E_0 = \mathrm{id}\,.\end{aligned}$$

Since gd is an 18-cycle on $E_0, \ldots, E_{17}$, we conclude that $(gd)^{18} = \mathrm{id}$. ∎

REMARK 4.12. Since the homeomorphism gd has period 18 on its support, i.e. for all $x \in \mathrm{supp}(gd)$ one has $(gd)^i(x) = x$ iff $i = 18k$ for $k \in \mathbb{Z}$, there are two involutions $s, t \in [\varphi]$ such that $gd = st$. Furthermore, $\mathrm{supp}(s) \subseteq E_1 \cup \cdots \cup E_{17}$.

LEMMA 4.13. *Given $n > 1$, there exists a clopen set A such that*

(1) $A, \varphi(A), \ldots, \varphi^{n-1}(A)$ *are disjoint*;
(2) $\varphi(B) \subseteq A$, *where* $B = X \setminus \bigcup_{i=0}^{n-1} \varphi^i(A)$.

Proof. Let E be a clopen set such that $\varphi^i(E) \cap E = \emptyset$ for $i = 1, \ldots, n^2$. Construct a K-R partition over E, i.e. $E = \bigcup_k E_k$, where $E_k = \{x \in E : \varphi^k(x) \in E$ and $\varphi^j(x) \notin E$ for $0 < j < k\}$. Observe that $E_k = \emptyset$ for all $k \leq n^2$.

(1) If $k = nl + r$, $0 < r < n$, then we set

$$A_k = \bigcup_{i=0}^{r-1} \varphi^{(n+1)i}(E_k) \cup \bigcup_{i=r}^{l-1} \varphi^{ni+r-1}(E_k).$$

This means that we choose r times every $(n+1)$th set from the family $\{E_k, \varphi(E_k), \ldots, \varphi^{k-1}(E_k)\}$ starting from the first one and then we take every nth set.

(2) If $k = nl$, then we set

$$A_k = \bigcup_{i=0}^{l-1} \varphi^{ni}(E_k).$$

Denoting $A = \bigcup_k A_k$, we get the result. ∎

THEOREM 4.14. *Let (X, φ) be a Cantor minimal system. Then there exist involutions $i_1, i_2, i_3 \in [\varphi]$ with clopen supports such that $\varphi = i_1 i_2 i_3$.*

Proof. Find a clopen set A satisfying the conditions of Lemma 4.13 for $n = 18$. Let $B = X \setminus \bigcup_{i=0}^{17} \varphi^i(A)$. Clearly, $\varphi(B) \cap B = \emptyset$. Define an involution b as follows: $b|B = \varphi|B$ and $b|\varphi(B) = \varphi^{-1}|\varphi(B)$. It follows that $g = b\varphi$ is an 18-cycle on the clopen sets $A_0, \ldots, A_{17}$ where $A_i = \varphi^i(A)$.

By Lemma 4.11, we can find an involution d with clopen support such that $(gd)^{18} = \mathrm{id}$. It follows from Remark 4.12 that there exist involutions s and t such that $gd = st$. This implies that $\varphi = b^{-1} s t d^{-1}$ is the product of four involutions. As mentioned in Remark 4.12, $\mathrm{supp}(s) \subseteq A_1 \cup \cdots \cup A_{17}$. Hence $\mathrm{supp}(s) \cap \mathrm{supp}(b) = \emptyset$. It follows that $w = b^{-1}s$ is an involution. This proves that $\varphi = wtd^{-1}$ is the product of three involutions with clopen supports. ∎

5. Flip conjugacy and orbit equivalence. In the section, we show that the classes of orbit equivalence and flip conjugacy of a Cantor minimal system are completely determined by simple groups.

DEFINITION 5.1.

(1) Cantor minimal systems (X_1, φ_1) and (X_2, φ_2) are called *orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(\mathrm{Orb}_{\varphi_1}(x)) = \mathrm{Orb}_{\varphi_2}(F(x))$ for all $x \in X_1$.
(2) (X_1, φ_1) and (X_2, φ_2) are called *flip conjugate* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F \circ \varphi_1 \circ F^{-1}$ is equal to either φ_2 or φ_2^{-1}.

The following theorem can be deduced from [GPS].

THEOREM 5.2. *Let (X_1, φ_1) and (X_2, φ_2) be Cantor minimal systems. The homeomorphisms φ_1 and φ_2 are orbit equivalent if and only if $D([\varphi_1])$ and $D([\varphi_2])$ are isomorphic as algebraic groups.*

Proof. The theorem can be proved in the same way as Corollary 4.6 from [GPS]. We notice only that according to Corollary 4.8 every involution with clopen support belongs to the commutator subgroup. We also recall that $\varphi \in D([\varphi])$ (see Theorem 4.6). ∎

Now, we start studying the class of flip conjugacy of a Cantor minimal system in terms of topological full groups. In our arguments, we mainly follow the proof of Theorem 384D from [Fr]. This theorem was used to show that certain transformation groups of complete Boolean algebras have no outer automorphisms (see also [E2]).

Let Γ denote one of the following groups: (1) the topological full group $[[\varphi]]$; (2) the group $[[\varphi]]_0$ defined in Remark 3.5; (3) the commutator subgroup $D([[\varphi]])$ of $[[\varphi]]$. Notice that we have the following inclusions:

$$D([[\varphi]]) \subseteq [[\varphi]]_0 \subsetneq [[\varphi]]. \tag{5.1}$$

We will show that every group from (5.1) is a complete invariant of flip conjugacy of φ. The proposed proof works for any of these groups Γ, because the only group property we exploit is the existence of many "small involutions" in the following sense.

LEMMA 5.3. *Each group Γ has many involutions, in the sense that for any clopen set A and any $x_0 \in A$, there exists $h \in \Gamma$ such that $hx_0 \neq x_0$, $h^2 = 1$ and* $\mathrm{supp}(h) \subseteq A$. *Moreover, for every $n > 0$, there exists $h \in \Gamma$ such that h is supported by A, $x_0 \in \mathrm{supp}(h)$ and $h|\mathrm{supp}(h)$ has period n.*

Proof. By (5.1), it suffices to establish the result for $D([[\varphi]])$ only. Let us find integers $0 = m_0 < m_1 < \cdots < m_{2n-1}$ such that $\varphi^{m_i}(x_0) \in A$ for $i = 0, \ldots, 2n-1$. Take a clopen neighborhood V of x_0 such that $\varphi^{m_i}(V) \subset A$ and $\varphi^{m_i}(V) \cap \varphi^{m_j}(V) = \emptyset$ for all $i, j = 0, \ldots, 2n-1$, $i \neq j$.

Define a homeomorphism $l \in [[\varphi]]$ as follows: $l(x) = \varphi^{m_{i+1}-m_i}(x)$ if $x \in \varphi^{m_i}(V)$ for $i = 0, 1, \ldots, n-2$, $l(x) = \varphi^{-m_{n-1}}(x)$ if $x \in \varphi^{m_{n-1}}(V)$, and $l(x) = x$ elsewhere. Similarly, define a homeomorphism $r \in [[\varphi]]$: $r(x) = \varphi^{m_{i+1}-m_i}(x)$ if $x \in \varphi^{m_i}(V)$ for $i = n, n+1, \ldots, 2n-2$, $r(x) = \varphi^{-(m_{2n-1}-m_n)}(x)$ if $x \in \varphi^{m_{2n-1}}(V)$, and $r(x) = x$ elsewhere.

It is not hard to see that there exists $\alpha \in [[\varphi]]$ such that $l = \alpha r \alpha^{-1}$. Therefore, $h = lr^{-1} \in D([[\varphi]])$ and h has period n on its support $\varphi^{m_0}(V) \cup \cdots \cup \varphi^{m_{2n-1}}(V)$. ∎

As a corollary of the lemma, we obtain the following result.

COROLLARY 5.4. *The family $\{\mathrm{supp}(g) : g \in \Gamma$ and $h^2 = 1\}$ of clopen sets generates the clopen topology of X.*

We will need some notions of the theory of Boolean algebras. We refer the reader to the book [Fr] for a comprehensive coverage of the theory of Boolean algebras and their automorphisms.

Let X be a Cantor set. Recall that an open set A is called *regular open* if $A = \mathrm{int}(\overline{A})$. Denote the family of all regular open sets by $RO(X)$. Notice that the family $CO(X)$ of all clopen sets is contained in $RO(X)$.

Let $\mathcal{A}$ be a Boolean algebra and $\mathcal{H} \subset \mathcal{A}$. Define $\sup(\mathcal{H})$ to be the smallest element of $\mathcal{A}$ that contains all elements of $\mathcal{H}$. If $\sup(\mathcal{H})$ exists for any family $\mathcal{H} \subseteq \mathcal{A}$, then the Boolean algebra $\mathcal{A}$ is called *complete*.

THEOREM 5.5 (Theorem 314P of [Fr]). *$RO(X)$ is a complete Boolean algebra with Boolean operations given by*

$$A \vee B = \operatorname{int}(\overline{A \cup B}), \qquad A \wedge B = A \cap B, \qquad A \setminus_{RO(X)} B = A \setminus \overline{B}$$

and with suprema given by $\sup(\mathcal{H}) = \operatorname{int}(\overline{\bigcup \mathcal{H}})$.

REMARK 5.6. Notice that finite set-theoretical unions of clopen sets coincide with the Boolean ones.

LEMMA 5.7. *If $A, B \in RO(X)$ and $A \not\subset B$, then $A \setminus B$ contains a clopen set.*

Proof. Let $C = A \cap B$. If $C = \emptyset$, then the result is clear. Assume that $C \neq \emptyset$ and $A \setminus C = A \setminus B$ has no internal points. Then $A \subset \overline{C}$. This implies that $A = C$, a contradiction. ■

Now we are ready to start the proof of the main result of this section. Recall that for a Cantor minimal system (X, φ), Γ stands for one of the groups: $[[\varphi]]$, $[[\varphi]]_0$, or $D([[\varphi]])$.

THEOREM 5.8. *Let (X_1, φ_1) and (X_2, φ_2) be Cantor minimal systems. If $\alpha : \Gamma_1 \to \Gamma_2$ is a group isomorphism, then α is spatial, i.e. there exists a homeomorphism $\widehat{\alpha} : X_1 \to X_2$ such that $\alpha(g) = \widehat{\alpha} g \widehat{\alpha}^{-1}$ for any $g \in \Gamma_1$.*

Proof. The idea of the proof is the following: we study the local subgroup $\Gamma_A := \{g \in \Gamma : gx = x \text{ for all } x \in X \setminus A\}$, where a clopen set A is the support of an involution from Γ, and describe Γ_A in group terms. This description allows us to construct an automorphism $\widehat{\alpha} : RO(X_1) \to RO(X_2)$ of Boolean algebras of regular open sets, which also sends clopen sets onto clopen sets. This automorphism gives rise to the spatial realization of α. For convenience, we will omit the index i in the notation of Cantor minimal system (X_i, φ_i) and the group Γ_i. We split the proof of the theorem into two lemmas.

DEFINITION 5.9. Let $\pi \in \Gamma$ be any involution. Set

$$C_\pi = \{g \in \Gamma : g\pi = \pi g\},$$

the centralizer of π in Γ;

$$U_\pi = \{g \in C_\pi : g^2 = 1 \text{ and } g(hgh^{-1}) = (hgh^{-1})g \text{ for all } h \in C_\pi\},$$

the involutions from C_π which commute with all their conjugates in C_π;

$$V_\pi = \{g \in \Gamma : gh = hg \text{ for all } h \in U_\pi\},$$

the centralizer of U_π in Γ;

$$S_\pi = \{g^2 : g \in V_\pi\};$$

and

$$W_\pi = \{g \in \Gamma : gh = hg \text{ for all } h \in S_\pi\},$$

the centralizer of S_π in Γ.

Clearly, for any involution π, $\mathrm{supp}(\pi)$ is a clopen set and $\alpha(W_\pi) = W_{\alpha(\pi)}$ where α is as in Theorem 5.8.

LEMMA 5.10. *Let $\pi \in \Gamma$ be an involution. Then $W_\pi = \Gamma_{\mathrm{supp}(\pi)}$.*

Proof. To prove the result, we will in turn study the properties of C_π, U_π, V_π, S_π, and W_π.

(1) $g(\mathrm{supp}(\pi)) = \mathrm{supp}(\pi)$ for all $g \in C_\pi$. $\lhd$ It is easy to see that $\mathrm{supp}(g\pi g^{-1}) = g(\mathrm{supp}(\pi))$. Since $g\pi g^{-1} = \pi$, one has $g(\mathrm{supp}(\pi)) = \mathrm{supp}(\pi)$. $\rhd$

(2-i) $\mathrm{supp}(g) \subseteq \mathrm{supp}(\pi)$ for all $g \in U_\pi$. $\lhd$ Assume the converse. Then there exists a clopen set $A \subset X \setminus \mathrm{supp}(\pi)$ such that $gA \cap A = \emptyset$. By Lemma 5.3, find a homeomorphism $h \in \Gamma$ with support in A such that for a clopen set $V \subset A$ one has $h^i(V) \cap V = \emptyset$, $i = 1, 2$. Note that $h \in C_\pi$. Then

$$g(hgh^{-1})(V) = g^2h^{-1}(V) = h^{-1}(V),$$
$$(hgh^{-1})g(V) = hg^2(V) = h(V).$$

The choice of h guarantees that $g(hgh^{-1}) \neq (hgh^{-1})g$. Hence $g \notin U_\pi$, which is a contradiction. $\rhd$

(2-ii) If a clopen set A is π-invariant, then $\pi_A \in U_\pi$ where the homeomorphism π_A coincides with π on A and is equal to id elsewhere.

(3-i) $V_\pi \subset C_\pi$, because $\pi \in U_\pi$.

(3-ii) If $g \in V_\pi$, then $g(B) \subseteq B \cup \pi(B)$ for all clopen sets $B \subseteq \mathrm{supp}(\pi)$. $\lhd$ Assume the converse, i.e. $g(B)$ is not in $B_0 = B \cup \pi(B)$ for some set B. Note that $\pi(B_0) = B_0$. As $B \subset B_0$, we have $C = g(B_0) \setminus B_0 \neq \emptyset$. Since $\pi g(B_0) = g\pi(B_0) = g(B_0)$, we obtain

$$\pi(C) = \pi(g(B_0) \setminus B_0) = \pi g(B_0) \setminus \pi(B_0) = g(B_0) \setminus B_0 = C.$$

Since $g \in V_\pi \subset C_\pi$, we see that $\mathrm{supp}(\pi)$ is g-invariant and $C \subset \mathrm{supp}(\pi)$. As π is an involution, $C = C' \sqcup C''$ for some clopen set C' with $C'' = \pi(C')$. Note that $g(C) \cap C = \emptyset$. Therefore

$$\pi_C g(C') = g(C') \neq g(C'') = g\pi_C(C').$$

Thus, g does not commute with $\pi_C \in U_\pi$, a contradiction. $\rhd$

(3-iii) If $g \in V_\pi$, then $g^2(B) = B$ for any clopen $B \subseteq \mathrm{supp}(\pi)$. $\lhd$ Assume the converse, i.e. there is a clopen set $B \subset \mathrm{supp}(\pi)$ such that $g^2(B) \cap B = \emptyset$. We can also assume that $g(B) \cap B = \emptyset$. We know that $g(B) \subset B \cup \pi(B)$. This implies that $g(B) \subset \pi(B)$. By the same argument applied to $g(B)$, we obtain $g^2(B) \subset \pi^2(B) = B$. If $g^2(B) \neq B$, then $\mu(B) = 0$ for any $\mu \in \mathcal{M}(\varphi)$, which contradicts the minimality of φ. Therefore, $g^2(B) = B$. $\triangleright$

(4-i) If $g \in S_\pi$, then $\mathrm{supp}(g) \subset X \setminus \mathrm{supp}(\pi)$. $\triangleleft$ This follows from (3-iii). $\triangleright$

(4-ii) For any clopen set $C \subset X \setminus \mathrm{supp}(\pi)$, there is an involution $h \in S_\pi$ supported on C. $\triangleleft$ By Lemma 5.3, there exists a periodic homeomorphism g of order 4 with support in C. Property (2-i) implies that $g \in V_\pi$. Therefore, $g^2 \in S_\pi$. $\triangleright$

(5) $W_\pi = \mathit{\Gamma}_{\mathrm{supp}(\pi)}$. $\triangleleft$ It follows from (4-i) that $\mathit{\Gamma}_{\mathrm{supp}(\pi)} \subset W_\pi$. To get the reverse inclusion, we consider any $g \in W_\pi$ and suppose that there exists a clopen set $B \subset X \setminus \mathrm{supp}(\pi)$ such that $g(B) \cap B = \emptyset$. By (4-ii), find an involution h from S_π with support in B. Take any clopen set $C \subset B$ with $h(C) \cap C = \emptyset$. Therefore,

$$hg(C) = g(C) \neq gh(C).$$

This implies that $gh \neq hg$, a contradiction. Thus, $W_\pi = \mathit{\Gamma}_{\mathrm{supp}(\pi)}$. $\triangleright$

This completes the proof of the lemma. ■

The following lemma gives the spatial realization of the group isomorphism α.

LEMMA 5.11. *Let (X_i, φ_i), $\mathit{\Gamma}_i$, α be as in Theorem 5.8. The map $\mathit{\Lambda} : RO(X_1) \to RO(X_2)$ given by*

$$\mathit{\Lambda}(A) = \bigvee \{\mathrm{supp}(\alpha(\pi)) : \pi \in \mathit{\Gamma}_1,\ \pi^2 = 1 \text{ and } \mathrm{supp}(\pi) \subseteq A\}$$

is a Boolean algebra isomorphism. Furthermore, $\mathit{\Lambda}(CO(X_1)) = CO(X_2)$.

Proof. (1) Note that if $g, \pi \in \mathit{\Gamma}_1$ and $\pi^2 = 1$, then

$$\mathrm{supp}(g) \subseteq \mathrm{supp}(\pi) \Leftrightarrow \mathrm{supp}(\alpha(g)) \subseteq \mathrm{supp}(\alpha(\pi)).$$

$\triangleleft$ By Lemma 5.10 (5), we see that $\mathrm{supp}(g) \subseteq \mathrm{supp}(\pi)$ iff $g \in W_\pi$ iff $\alpha(g) \in W_{\alpha(\pi)}$ iff $\mathrm{supp}(\alpha(g)) \subseteq \mathrm{supp}(\alpha(\pi))$. $\triangleright$

(2) It easily follows from the definition of $\mathit{\Lambda}$ that $\mathit{\Lambda}$ is order-preserving, i.e. if $A \subseteq B$ with $A, B \in RO(X_1)$, then $\mathit{\Lambda}(A) \subseteq \mathit{\Lambda}(B)$.

(3) Let $\pi \in \mathit{\Gamma}_1$ be such that $\pi^2 = 1$ and $A \in RO(X_1)$. If $\mathrm{supp}(\pi) \not\subset A$, then $\mathrm{supp}(\alpha(\pi)) \not\subset \mathit{\Lambda}(A)$. $\triangleleft$ Take a non-empty clopen set $V \subseteq \mathrm{supp}(\pi) \setminus A$ and a homeomorphism $h \in \mathit{\Gamma}_1$ of order 4 with support in V (see Lemma 5.7). As $\mathrm{supp}(h) \subset \mathrm{supp}(\pi)$, we have $\mathrm{supp}(\alpha(h)) \subset \mathrm{supp}(\alpha(\pi))$ (see (1)). On the other hand, if π' is an involution with support in A, then $h \in V_{\pi'}$ and $h^2 \in S_{\pi'}$ (see (2-i) of Lemma 5.10 and the definition of $V_{\pi'}$). Thus, $\alpha(h^2) \in S_{\alpha(\pi')}$ and $\mathrm{supp}(\alpha(h^2)) \cap \mathrm{supp}(\alpha(\pi')) = \emptyset$ (see (4-i) of Lemma 5.10). As π' is arbitrary, Corollary 5.4 implies that $\mathrm{supp}(\alpha(h^2)) \cap \mathit{\Lambda}(A) = \emptyset$. Hence, $\mathrm{supp}(\alpha(\pi)) \setminus \mathit{\Lambda}(A) \supset \mathrm{supp}(\alpha(h^2)) \neq \emptyset$. $\triangleright$

(4) Define the map $\Lambda^* : RO(X_2) \to RO(X_1)$ as follows:

$$\Lambda^*(B) = \bigvee\{\mathrm{supp}(\alpha^{-1}(\pi)) : \pi \in \Gamma_2,\ \pi^2 = 1 \text{ and } \mathrm{supp}(\pi) \subseteq B\}.$$

(5) $\Lambda^*\Lambda(A) = A$ for every $A \in RO(X_1)$ and $\Lambda\Lambda^*(B) = B$ for every $B \in RO(X_2)$. $\lhd$ By Lemma 5.4, the set A can be covered with clopen sets $\{C_n\}$ which are supports of involutions from Γ_1. Take an involution π whose support is C_n. As $C_n \subset A$, $\alpha(\pi)$ is an involution with support in $\Lambda(A)$. Hence $C_n = \mathrm{supp}(\alpha^{-1}\alpha(\pi))$ is contained in $\Lambda^*\Lambda(A)$. Since C_n is arbitrary, we get $A \subseteq \Lambda^*\Lambda(A)$.

If $\pi \in \Gamma_2$ is an involution with support in $\Lambda(A)$, then $\alpha^{-1}(\pi)$ is an involution whose support is in A (see (3)). Since π is arbitrary, $\Lambda^*\Lambda(A) \subseteq A$, which shows that $\Lambda^*\Lambda(A) = A$. Analogously, one can show that $\Lambda\Lambda^*(B) = B$ for all $B \in RO(X_2)$. $\rhd$

(6) Since $\Lambda^*\Lambda(A) = A$, it follows that for any $A, B \in RO(X_1)$, $\Lambda(A) \subseteq \Lambda(B)$ iff $A \subseteq B$. Moreover, Λ is a bijection of $RO(X_1)$ and $RO(X_2)$.

(7) $\Lambda : RO(X_1) \to RO(X_2)$ is a Boolean algebra isomorphism and Λ^* is its inverse. To see this, we refer the reader to Theorem 312L of [Fr], which asserts that property (6) implies that Λ is a Boolean algebra isomorphism.

It remains only to prove that Λ sends clopen sets onto clopen sets.

(8) If $\pi \in \Gamma_1$ is an involution, then $\Lambda(\mathrm{supp}(\pi)) = \mathrm{supp}(\alpha(\pi))$. $\lhd$ Indeed, by definition of Λ, $\mathrm{supp}(\alpha(\pi)) \subset \Lambda(\mathrm{supp}(\pi))$. On the other hand,

$$\mathrm{supp}(\alpha(\pi)) = \Lambda\Lambda^{-1}(\mathrm{supp}(\alpha(\pi))) \supseteq \Lambda(\mathrm{supp}(\alpha^{-1}\alpha(\pi))) = \Lambda(\mathrm{supp}(\pi)). \rhd$$

(9) $\Lambda(CO(X_1)) = CO(X_2)$. $\lhd$ We note that if A is the support of an involution from Γ_1, then we see from (8) that $\Lambda(A) \in CO(X_2)$. Now if A is an arbitrary clopen set, then, by Corollary 5.4, A is a finite union of supports of involutions. Since Λ is a Boolean algebra isomorphism and finite unions of clopen sets in $RO(X_2)$ coincide with the set-theoretical unions, we conclude that $\Lambda(A) \in CO(X_2)$. $\rhd$

(10) For any $B \in RO(X_2)$ and $g \in \Gamma_1$, we have $\alpha(g)(B) = \Lambda g\Lambda^{-1}(B)$. $\lhd$ Assume the converse, i.e. $h = \alpha(g)^{-1}\Lambda g\Lambda^*$ is not the identity automorphism of $RO(X_2)$. Notice that h also preserves $CO(X_2)$. Then there exists a clopen set V such that $h(V) \cap V = \emptyset$. Let π be an involution from Γ_2 with support in V. Then $\alpha^{-1}(\pi)$ is supported by $\Lambda^{-1}(V)$. Hence, $g\alpha(\pi)^{-1}g^{-1}$ is supported by $g(\Lambda^{-1}(V))$ and $\alpha(g\alpha^{-1}(\pi)g^{-1}) = \alpha(g)\pi\alpha^{-1}(g)$ is supported by $\Lambda g\Lambda^{-1}(V)$. On the other hand, $\alpha(g)\pi\alpha^{-1}(g)$ is supported by $\alpha(g)(V)$.

Furthermore, we deduce that $\alpha(g)(V) \cap \Lambda g\Lambda^{-1}(V) \neq \emptyset$. It follows that $V \cap \alpha^{-1}(g)\Lambda g\Lambda^{-1}(V) \neq \emptyset$, which is a contradiction. $\rhd$

Let us continue the proof of Theorem 5.8. Since Λ is an isomorphism of $CO(X_1)$ and $CO(X_2)$, it can be extended to a homeomorphism $\widehat{\alpha} : X_1 \to X_2$.

Moreover, it follows from (10) that $\alpha(g)(x) = \widehat{\alpha} g \widehat{\alpha}^{-1}(x)$ for any $g \in \Gamma_1$ and $x \in X_2$. This completes the proof of the theorem. ∎

To establish the fact that the group Γ is a complete invariant of flip conjugacy, we need to prove the following lemma. The proof we present here is analogous to Proposition 5.8 from [GPS].

LEMMA 5.12. *For a Cantor minimal system* (X, φ), *let* Γ *denote any of the groups from* (5.1). *Then the topological full group of* Γ *is equal to* $[[\varphi]]$.

Proof. It is sufficient to show that φ belongs to the topological full group of $D([[\varphi]])$. For every $x \in X$ find a clopen neighborhood V_x such that $\varphi^j(V_x) \cap V_x = \emptyset$ for $j = 1, 2$. Define a homeomorphism h_x as follows:

$$h_x(y) = \begin{cases} \varphi(y), & x \in V_x \cup \varphi(V_x), \\ \varphi^{-2}(y), & x \in \varphi^2(V_x). \end{cases}$$

It is not hard to see that $h_x \in D([[\varphi]])$ (this follows from the proof of Lemma 5.3).

By compactness of X, there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{j=1}^n V_{x_j}$. Set

$$U_1 = V_{x_1}, \quad U_2 = V_{x_2} \setminus U_1, \ \ldots, \ U_n = V_{x_n} \setminus (U_1 \cup \cdots \cup U_{n-1}).$$

Then $\{U_1, \ldots, U_n\}$ forms a clopen partition of X. Then $\varphi(x) = h_{x_i}(x)$ whenever $x \in U_i$. This completes the proof. ∎

THEOREM 5.13. *Let* (X_1, φ_1) *and* (X_2, φ_2) *be Cantor minimal systems. Then* φ_1 *and* φ_2 *are flip conjugate if and only if one of the following statements holds*:

(1) $D([[\varphi_1]]) \cong D([[\varphi_2]])$;
(2) $[[\varphi_1]]_0 \cong [[\varphi_2]]_0$;
(3) $[[\varphi_1]] \cong [[\varphi_2]]$.

Proof. Notice that the cases (2) and (3) are already proved in [GPS]. Here we present a unified proof for all three cases. Let Γ_i denote one of the following groups: $[[\varphi_i]]$, $[[\varphi_i]]_0$, $D([[\varphi_i]])$, $i = 1, 2$.

It is clear that the map implementing the flip conjugacy of φ_1 and φ_2 can be lifted to an isomorphism between Γ_1 and Γ_2.

Conversely, by Theorem 5.8 every isomorphism between Γ_1 and Γ_2 is spatial. By Lemma 5.12, this spatial isomorphism can be extended to an isomorphism of $[[\varphi_1]]$ and $[[\varphi_2]]$. Then it follows from Corollary 2.7 of [BoTo] that φ_1 and φ_2 are flip conjugate. ∎

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