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PARTIAL VARIATIONAL PRINCIPLE FOR FINITELY GENERATED GROUPS OF POLYNOMIAL GROWTH AND SOME FOLIATED SPACES

BY

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Abstract. We generalize the notion of topological pressure to the case of a finitely generated group of continuous maps and introduce group measure entropy. Also, we provide an elementary proof that any finitely generated group of polynomial growth admits a group invariant measure and show that for a group of polynomial growth its measure entropy is less than or equal to its topological entropy. The dynamical properties of groups of polynomial growth are reflected in the dynamics of some foliated spaces.

1. Introduction. The concept of entropy of a transformation plays a crucial role in topological dynamics. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in [1] as an invariant of topological conjugacy. Later, Bowen [8] and Dinaburg [14] presented an equivalent approach to the notion of entropy in the case when the domain of the transformation is a metrizable space. The topological entropy $h(f)$ of a homeomorphism f measures the complexity of the transformation acting on a compact topological space in the sense that it shows the rate at which the action of the transformation disperses points.

Since the entropy appeared to be a very useful invariant in ergodic theory and dynamical systems, there were several attemps to find suitable generalizations of it to other systems, like groups, pseudogroups, graphs, foliations. Among others, Ghys, Langevin and Walczak [20] proposed a definition of topological entropy for finitely generated groups and pseudogroups of continuous transformations. Biś and Walczak [7] applied the notion of entropy of a group to hyperbolic groups in the sense of Gromov to study their geometry and dynamics. Friedland [19] used the notion of entropy to study some aspects of dynamics of graphs and semigroups.

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Adler, Konheim and McAndrew [1] stated the hypothesis, called the *variational principle*, that the topological entropy of a dynamical system, determined by a single transformation, is the supremum of all measure entropies taken with respect to all invariant Borel probability measures.

Dinaburg described the relation between topological entropy and measure entropy, two characteristics of a dynamical system determined by a single transformation, in the case of a space of finite dimension and a homeomorphism. Goodwyn [22] proved that the topological entropy is not less than the measure entropy of a dynamical system. Finally, Goodman [21] proved the hypothesis stated by Adler, Konheim and McAndrew in [1].

The notion of pressure, which is a generalization of topological entropy for an action of the group $\mathbb{Z}^N$ on a compact metric space, was introduced by Ruelle in [32]. Given a continuous real function ϕ on a compact metric space X one tries to maximize the functional $\Phi_f(\mu) = h_\mu(f) + \int_X \phi\, d\mu$, where $f : X \to X$ is a continuous map and $h_\mu(f)$ is the measure entropy of f with respect to an f-invariant measure μ. The supremum of $\Phi_f(\mu)$ over all f-invariant probability measures μ on the Borel σ-algebra is the topological pressure $P(f, \phi)$. Then the variational principle can be rewritten in the form

$$P(f, \phi) = \sup \Big\{ h_\mu(f) + \int_X \phi\, d\mu : \mu \in M(f) \Big\}$$

where $M(f)$ denotes the set of all f-invariant Borel probability measures defined on X.

A general proof of the variational principle for an action of $\mathbb{Z}_+$ was given by Walters [36] and by Denker [13]. Some generalization of the variational principle to actions of $\mathbb{Z}_+^N$ was found by Elsanousi [17]. A very short and elegant proof of the variational principle for an action of $\mathbb{Z}_+^N$ on a compact space was given by Misiurewicz [28]. A generalization to $\mathbb{R}^n$ actions was provided by Tagi-Zade [34].

In this paper we show that for arbitrary finitely generated groups of continuous maps, of polynomial growth, there exists a group invariant measure. The main result of the paper states that the group measure entropy of a finitely generated group of polynomial growth is less than or equal to its topological entropy. The dynamical properties of finitely generated groups of polynomial growth are reflected in the dynamics of some foliated spaces. The notion of foliation (or more generally of foliated space) generally corresponds to a decomposition of a manifold into the union of connected submanifolds of the same dimension, called leaves, which are piled up locally like pages of a book; for a detailed introduction see [9], [10].

For a foliated space (M_G, F_G) determined by the suspension of a group (G, G_1) of polynomial growth we find that the measure entropy of the foliation F_G is upper bounded by the geometric entropy of F_G multiplied by a constant dependent on the geometry of (M_G, F_G).

Therefore, we get some partial variational principle for groups of polynomial growth and its analogue for some foliated spaces.

The paper is organized as follows.

In Section 2 we recall different approaches to the problem of the existence of a group invariant measure, we construct an example of a group without any group invariant measure and we recall the known fact that a finitely generated abelian group admits a group invariant measure. Also, we provide an example of a non-abelian finitely generated group which has a group invariant measure. In Section 3, we recall the notion of the growth of a group and cite a few results which motivate our restricting attention to finitely generated groups of exponential or of polynomial growth. We study the algebraic structure of those groups and introduce the notions of "nice groups" which will be used later. The nice groups form a large class of groups which embraces abelian groups, hyperbolic groups, groups of polynomial growth, groups of exponential growth and others. In Section 4, we define and discuss the notion of topological pressure $P((G, G_1), f)$ for a finitely generated group (G, G_1). In Section 5, we define the measure entropy for a finitely generated group and prove the main result of the paper:

THEOREM 1. *For a nice group (G, G_1), measure $\mu \in M(X, (G, G_1))$ and $f \in C(X)$ we have the inequality*

$$h_\mu(G, G_1) + \int_X f\, d\mu \leq P((G, G_1), f)$$

where $M(X, (G, G_1))$ denotes the set of G-invariant measures.

In Section 6, we restrict our attention to finitely generated groups of polynomial growth. We prove (Proposition 8) that any finitely generated group of homomorphisms of a compact metric space, of polynomial growth, admits a group invariant measure. Finally, in Section 7 we show that the dynamical properties of groups of polynomial growth are reflected in the dynamics of some foliated spaces. Given a finitely generated group (G, G_1) of polynomial growth we construct a compact foliated space (M_G, F_G) modeled transversally on a compact metric space Γ, with analogous dynamical properties. Moreover, we get:

COROLLARY 3. *For a compact foliated space (M_G, F_G), determined by the suspension of a finitely generated group (G, G_1) of polynomial growth, with a continuous family g_{M_G} of Riemannian structures on the leaves, and for any measure $\mu \in M(X, (G, G_1))$ we get*

$$\sup\{h_\mu(G, G_1); \mu \in M(\Gamma, (G, G_1))\} \leq a h_{\mathrm{geom}}(F_G, g_{M_G}),$$

where $h_{\mathrm{geom}}(F_G, g_{M_G})$ is the geometric entropy of F_G with respect to the Riemannian structure g_{M_G}, and a denotes the maximum of the lengths of the free homotopy classes of curves homotopic to elements of G_1.

2. Existence of a group invariant measure. Let X be a compact metric space with distance function d. Consider a group G of homeomorphisms of X. The group G is assumed to be finitely generated, i.e. there exists a finite set $G_1 = \{\mathrm{id}_X, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ such that

$$G = \bigcup_{n \in \mathbb{N}} G_n,$$

where

$$G_n = \{g_1 \circ \cdots \circ g_n : X \to X\}_{g_1,\ldots,g_n \in G_1}.$$

We always assume that $\mathrm{id}_X \in G_1$. This implies that $G_m \subset G_n$ for all $m \le n$. To emphasize the generating set we shall write (G, G_1) instead of G.

DEFINITION 1. A Borel probability measure μ on X is said to be *G-invariant* if $\mu \circ g = \mu$ for any $g \in G$.

It is well known that if G is abelian then a G-invariant measure exists (see [17]). But in the case of an arbitrary finitely generated group (G, G_1) a G-invariant measure may not exist.

EXAMPLE 1. Let $f_i : S^1 \to S^1$ be diffeomorphisms of a circle with a source A_i and a sink B_i, $i = 1, 2$, such that $\{A_1, B_1\} \cap \{A_2, B_2\} = \emptyset$. Then the group (G, G_1) generated by $G_1 = \{\mathrm{id}_{S^1}, f_1, f_1^{-1}, f_2, f_2^{-1}\}$ has no G-invariant measure. Indeed, if μ were a G-invariant measure then $\operatorname{supp} \mu$ (the complement of the set of all $x \in S^1$ which admit an open neighbourhood V such that $\mu(V) = 0$) would be a subset of a nonwandering set. But in this case, the nonwandering set is empty.

EXAMPLE 2. The orthogonal group $O(n)$ acting on S^n is a non-abelian group admitting an $O(n)$-invariant Haar measure. Thus, a free subgroup F_2 of $O(n)$ admits an F_2-invariant measure.

Bounded groups and a group invariant measure. Ramachandran and Misiurewicz [31] considered a probability space $(X, \mathcal{A}, P)$ and a group G of measurable and nonsingular transformations defined on $(X, \mathcal{A}, P)$. They proved a necessary and sufficient condition for the existence of a finite G-invariant measure.

We say that a finite additive measure μ on $\mathcal{A}$ is *equivalent* to the measure P provided for any set $E \in \mathcal{A}$, $\mu(E) = 0$ iff $P(E) = 0$. A measurable transformation $f : X \to X$ is called *nonsingular* if for any $E \in \mathcal{A}$ the condition $P(E) > 0$ implies $P(f^{-1}(E)) > 0$. Two measurable sets E and F are said to be *equivalent* if

1) there exist sets E' and F' such that $P((E \setminus E') \cup (E' \setminus E)) = 0$ and $P((F \setminus F') \cup (F' \setminus F)) = 0$,
2) there exists a sequence (E_j) such that $E' = \bigcup_{j=1}^{\infty} E_j$,
3) there exists a sequence (F_j) such that $F' = \bigcup_{j=1}^{\infty} F_j$,
4) there exists a sequence $(g_j) \subset G$ such that for every j,

$$F_j = g_j(E_j).$$

Following [31], we say that a set $E \in \mathcal{A}$ is *bounded* if it is not equivalent to a measure-theoretically proper subset of itself. Finally, we say that a group G is bounded if X is bounded.

PROPOSITION 1 (Theorem 1 in [31]). *A finite G-invariant measure equivalent to P exists if and only if the group G is bounded.*

Measure preserving groups of transformations were studied by Alpern and Prasad [2] and Oxtoby and Ulam [29]. In the compact case, the topological and algebraic properties of those groups were investigated by Fathi [18].

3. Growth rate of a group. We recommend [25] as a survey of results on the growth rate of groups. In this section we shall consider only finitely generated groups. More precisely, a group G is said to be *finitely generated* if there exists a finite set $G_1 = \{g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$ such that

$$G = \bigcup_{n \in \mathbb{N}} G_n,$$

where

$$G_n = \{g_1 \circ \cdots \circ g_n : g_1, \ldots, g_n \in G_1\}.$$

We always assume that e, the neutral element of G, belongs to the generating set G_1. This implies that $G_m \subset G_n$ for all $m \le n$. Let $|G_n|$ denote the cardinality of G_n.

Following de la Harpe [25] we introduce the following definitions:

DEFINITION 2. Let (G, G_1) be a finitely generated group. The *exponential growth rate* of (G, G_1) is the upper limit

$$w(G, G_1) = \limsup_{k \to \infty} \sqrt[k]{|G_k|}.$$

The limsup is in fact a limit because the inequality $|G_{k+n}| \le |G_k|\,|G_n|$ implies the existence of $\lim_{k \to \infty} \sqrt[k]{|G_k|}$.

DEFINITION 3. The group (G, G_1) is said to be of

(a) *exponential growth* if $w(G, G_1) > 1$,
(b) *subexponential growth* if $w(G, G_1) = 1$,
(c) *polynomial growth* of degree d if $|G_k| \le a k^d$ for some $a > 0$ and $d \ge 0$,

(e) *intermediate growth* if it is of subexponential growth and not of polynomial growth.

It is known that the property of being of exponential growth (resp., subexponential growth, polynomial growth, intermediate growth) depends only on the group G, and not on the choice of the generating set G_1. Also, a finitely generated group is necessarily of one (and only one) of three types: exponential growth, polynomial growth or intermediate growth (see [25]).

There are many results on finitely generated groups of exponential or polynomial growth; let us quote a few of them:

PROPOSITION 2 ([25, p. 187]). *A finitely generated group which contains a free semigroup on two generators is of exponential growth.*

PROPOSITION 3 ([23]). *A finitely generated group of polynomial growth has a nilpotent subgroup of finite index.*

PROPOSITION 4 ([15], [24], [4]). *If (G, G_1) is a finitely generated nilpotent group, then (G, G_1) is of polynomial growth and*

$$a_1 k^d \leq |G_k| \leq a_2 k^d,$$

where d is the homogeneous dimension of (G, G_1) and a_1, a_2 are some positive constants.

PROPOSITION 5 ([25, Proposition 22]). *The Heisenberg group (G, G_1) is of polynomial growth and there exist constants $c_1, c_2 > 0$ such that for all $k \in \mathbb{N}$,*

$$c_1 k^4 \leq |G_k| \leq c_2 k^4.$$

PROPOSITION 6 ([12]). *If (G, G_1) is a hyperbolic group, then there exist positive constants c_1, c_2 and $w > 1$ such that for all $k \in \mathbb{N}$,*

$$c_1 w^k \leq |G_k| \leq c_2 w^k.$$

LEMMA 1. *If (G, G_1) is a group of either exponential growth or polynomial growth, then there exists a constant $A \geq 1$ such that*

$$|G_m|\,|G_n| \leq A|G_{mn}|$$

for $m, n \in \mathbb{N}$ large enough.

Proof. (a) Let (G, G_1) be a group of exponential growth. Then

$$w = \lim_{k \to \infty} \sqrt[k]{|G_k|} > 1.$$

Therefore, for small $\varepsilon > 0$ and large $m, n \in \mathbb{N}$ we get

$$(w + \varepsilon)^{m+n} < (w - \varepsilon)^{mn}.$$

Moreover,

$$(w-\varepsilon)^m \le |G_m| \le (w+\varepsilon)^m,$$
$$(w-\varepsilon)^n \le |G_n| \le (w+\varepsilon)^n,$$
$$(w-\varepsilon)^{mn} \le |G_{mn}| \le (w+\varepsilon)^{mn}.$$

Thus,

$$|G_m|\,|G_n| \le (w+\varepsilon)^{m+n} \le (w-\varepsilon)^{mn} \le |G_{mn}|.$$

(b) Assume now that (G, G_1) is of polynomial growth. Then by Proposition 3, G has a nilpotent subgroup H of finite index. Bass [4] proved that for a finitely generated nilpotent group H there exist positive constants A_1, A_2, d such that

$$A_1 n^d \le |H_n| \le A_2 n^d.$$

Therefore, for the finitely generated group (G, G_1) of polynomial growth there exist positive constants A_3, A_4, d such that

$$A_3 n^d \le |G_n| \le A_4 n^d,$$

which implies that there exists a positive constant A such that

$$|G_m|\,|G_n| \le A|G_{mn}|. \ \blacksquare$$

The above mentioned result motivates the following definition:

Definition 4. A finitely generated group (G, G_1) is said to be *nice* if there exist constants $A \ge 1$ and k_0 such that for all $m, n > k_0$,

$$|G_n|\,|G_m| \le A|G_{mn}|.$$

Remark. Milnor [27] showed that the type of growth of the fundamental group of a compact Riemannian manifold M determines the geometry of M and is related to the growth type of the manifold. The growth type of the manifold is determined by the volumes of balls in the universal covering of M. One of the most important results relating both types of growth is a theorem due to Shvarts [33] and Milnor [27], which says that the fundamental group $\pi_1(M)$ of a compact manifold M and the universal covering of M have the same type of growth.

An approach to the growth of groups, originating from foliation theory, based on the paper of Egashira [16], was presented by Walczak in [35]. Badura [3] showed that any growth type can be realized by a leaf of a C^1-foliation of a compact manifold.

4. Topological pressure of a group. Let X be a compact metric space with distance function d. Consider a group G of homeomorphisms of X. The group G is assumed to be finitely generated, e.g. there exists a finite set

$G_1 = \{\mathrm{id}_X, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ such that

$$G = \bigcup_{n\in\mathbb{N}} G_n,$$

where

$$G_n = \{g_1 \circ \cdots \circ g_n : X \to X\}_{g_1,\ldots,g_n \in G_1}.$$

Denote by $C(X)$ the set of continuous functions defined on X, and let D denote the set of all neighbourhoods of the diagonal in $X \times X$.

Let $\delta > 0$ and let

$$N_\delta := \{(x,y) \in X \times X : d(x,y) < \delta\}$$

be the δ-neighbourhood of the diagonal in $X \times X$. For fixed $n \in \mathbb{N}$, $\delta > 0$ and a continuous function $f \in C(X)$ we put

$$N(\delta,n) := \bigcap_{g\in G_{n-1}} (g\times g)^{-1} N_\delta, \qquad f_n := \sum_{g\in G_{n-1}} f\circ g.$$

Modifying the definitions stated in [28, p. 1070], we give

Definition 5. A finite set $E \subset X$ is called

(a) (n,δ)-*separated* if $(x,y) \notin N(\delta,n)$ for any distinct $x,y \in E$,
(b) (n,δ)-*spanning* if for any $x \in X$ there exists $y \in E$ such that $(x,y) \in N(\delta,n)$.

Definition 6. Let

$$p(f,E) := \log \sum_{x\in E} \exp f(x),$$

$$P_{n,\delta}((G,G_1),f) := \sup\{p(f,E) : E \text{ is } (n,\delta)\text{-separated}\},$$

$$P_\delta((G,G_1),f) := \limsup_{n\to\infty} \frac{1}{|G_{n-1}|} P_{n,\delta}((G,G_1),f).$$

Lemma 2. *If* $\alpha < \beta$, *then* $P_{n,\alpha}((G,G_1),f) \ge P_{n,\beta}((G,G_1),f)$.

Definition 7. The quantity

$$P((G,G_1),f) := \lim_{\delta\to 0^+} P_\delta((G,G_1),f)$$

is called the *pressure* of the group (G,G_1) with respect to the function f. By Lemma 2, $P((G,G_1)f)$ is well defined.

Remark. It is easy to notice that the pressure of (G,G_1) depends on the generating set. However, if G_1 and G_1' are two generating sets of the same group G, then $P((G,G_1),f) > 0$ if and only if $P((G,G_1'),f) > 0$. Therefore we can speak about the group of positive pressure without referring to the generating set.

Moreover, the topological entropy $h_{\text{top}}((G, G_1))$ satisfies the equality

$$h_{\text{top}}((G, G_1)) = P((G, G_1), 0).$$

More information on the topological entropy of a group can be found in [5] and [6].

5. Partial variational principle. Denote by $M(X, (G, G_1))$ the set of G-invariant measures. Let $\mathcal{A}$ be a finite Borel partition of X. For a partition $\mathcal{A}$ we define the partition

$$\mathcal{A}_n := \bigvee_{g \in G_{n-1}} g^{-1}\mathcal{A}.$$

Modifying Conze's definition of measure entropy for abelian groups ([11]), we define measure entropy for an arbitrary finitely generated group in the following way:

DEFINITION 8. For a finite Borel partition $\mathcal{A}$ of X and measure $\mu \in M(X, (G, G_1))$ we define

$$h_\mu((G, G_1), \mathcal{A}) := \limsup_{n\to\infty} \frac{1}{|G_{n-1}|} H_\mu(\mathcal{A}_n),$$

where $H_\mu(\mathcal{A}_n)$ denotes the standard measure entropy of the partition $\mathcal{A}_n$. Finally,

$$h_\mu(G, G_1) := \sup\{h_\mu((G, G_1), \mathcal{A}) : \mathcal{A} \text{ a finite Borel partition of } X\}.$$

THEOREM 1. *For a nice group (G, G_1), measure $\mu \in M(X, (G, G_1))$ and $f \in C(X)$, we have the inequality*

$$h_\mu(G, G_1) + \int_X f\, d\mu \le P((G, G_1), f).$$

COROLLARY 1. *For a nice group (G, G_1) acting on a compact metric space X and any G-invariant measure $\mu \in M(X, (G, G_1))$,*

$$h_\mu(G, G_1) \le h(G, G_1).$$

To prove Theorem 1 we need a few technical lemmas.

LEMMA 3. *For a finite Borel partition $\mathcal{A}_n = \{a_1, \dots, a_s\}$ of X, any $\mu \in M(X, (G, G_1))$, and any positive ζ, there exists a finite Borel partition $\mathcal{B} = \{b_0, b_1, \dots, b_s\}$ of X such that*

(a) *b_i is a compact subset of a_i for any $i = 1, \dots, s$,*
(b) *$H_\mu(\mathcal{A}_n \mid \mathcal{B}) \le \zeta$.*

Proof. Choose $\zeta, \varepsilon > 0$ satisfying $\varepsilon|\mathcal{A}_n| \log |\mathcal{A}_n| < \zeta$. Since a Borel probability measure is regular, for each $a_i \in \mathcal{A}_n$ there exists a compact set $b_i \subset a_i$ such that $\mu(a_i \setminus b_i) < \varepsilon$, $i = 1, \dots, s$. Consider the partition

$\mathcal{B} = \{b_0, b_1, \ldots, b_s\}$, where $b_0 = X \setminus \bigcup_{i=1}^{s} b_i$. Following the proof of Theorem 8.6 in [37], we get

$$H_\mu(\mathcal{A}_n \,|\, \mathcal{B}) < \varepsilon |\mathcal{A}_n| \log |\mathcal{A}_n| < \zeta. \ \blacksquare$$

Let $\mathcal{B} = \{b_0, b_1, \ldots, b_s\}$ be the partition of X described in the above lemma. For distinct i and j we have

$$(b_i \times b_j) \cap \{(x,x) : x \in X\} = \emptyset.$$

Therefore

$$O_\varepsilon = (X \times X) \setminus \bigcup_{i \neq j;\, i,j=1}^{s} (b_i \times b_j)$$

is an open neighbourhood of the diagonal.

LEMMA 4. *Given $f \in C(X)$. For any $\zeta > 0$ there exists $0 < \delta^* \leq \zeta$ such that*

(a) *if* $(x,y) \in N_{\delta^*}$, *then* $(y,x) \in N_{\delta^*}$,
(b) *if* $(x,y), (y,z) \in N_{\delta^*}$ *then* $(x,z) \in O_\varepsilon$,
(c) *if* $(x,y) \in N_{\delta^*}$, *then* $|f(x) - f(y)| \leq \zeta$.

Proof. Choose $\zeta > 0$. Denote by $B(x, r_x)$ the ball in $X \times X$ centered at (x,x) of radius r_x such that $B(x, r_x) \subset O_\varepsilon$. Since the diagonal is compact, it is covered by a finite subfamily $(B(x_i, r_{x_i}))_{i=1}^{k}$. Now, it is easy to notice that there exists a $\delta > 0$ such that

$$N_\delta \subset \bigcup_{i=1}^{k} B(x_i, r_{x_i}).$$

By the continuity of f we get δ_1 such that if $d(x,y) < \delta_1$, then $|f(x) - f(y)| < \zeta$. Taking $\delta^* \leq \min\{\delta, \delta_1, \zeta\}$ completes the proof. $\blacksquare$

DEFINITION 9. Given a group (G, G_1) and a positive integer m, denote by $(G^{(m)}, G_1^{(m)})$ the group generated by the set $G_1^{(m)} = \{G_m \setminus G_{m-1}\} \cup \{\mathrm{id}_X\}$.

LEMMA 5.

$$\mathcal{A}_{nm} = \bigvee_{k \in G_{n-1}^{(m)}} k^{-1}\Big(\bigvee_{g \in G_{m-1}} g^{-1}\mathcal{A}\Big).$$

Proof. Notice that any element a of the partition $\mathcal{A}_{nm}$ may be written in the form

$$a = \bigcap_{g \in G_{mn-1}} g^{-1} A_g, \qquad \text{where } A_g \in \mathcal{A}.$$

On the other hand, any element b of $\bigvee_{k \in G_{n-1}^{(m)}} k^{-1}(\bigvee_{g \in G_{m-1}} g^{-1}\mathcal{A})$ may be written, with $A_{g,k} \in \mathcal{A}$, in the form

$$b = \bigcap_{k \in G_{n-1}^{(m)}} k^{-1}\Big(\bigcap_{g \in G_{m-1}} g^{-1} A_{g,k}\Big) = \bigcap_{k \in G_{n-1}^{(m)},\, g \in G_{m-1}} (g \circ k)^{-1} A_{g,k}.$$

Thus, we obtain equality of the above mentioned partitions. ∎

Let $\mathcal{C} = \bigvee_{k \in G_{n-1}^{(m)}} k^{-1}\mathcal{B}$, where the partition $\mathcal{B}$ was described in Lemma 3. For any $c \in \mathcal{C}$ define

$$\alpha(c) := \sup\{f_{nm}(x) : x \in c\}, \qquad \beta := \sum_{c \in \mathcal{C}} \exp \alpha(c).$$

It is clear that

$$\int_c f_{nm}\, d\mu \le \alpha(c)\mu(c).$$

LEMMA 6.

$$H_\mu(\mathcal{C}) + \int_X f_{nm}\, d\mu \le \log \beta.$$

Proof. By the definitions of measure entropy and f_{nm},

$$\begin{aligned} H_\mu(\mathcal{C}) + \int_X f_{nm}\, d\mu &\le -\sum_{c \in \mathcal{C}} \exp \alpha(c) \Big(\frac{\mu(c)}{\exp \alpha(c)}\Big) \log\Big(\frac{\mu(c)}{\exp \alpha(c)}\Big) \\ &= \beta \sum_{c \in \mathcal{C}} \frac{\exp \alpha(c)}{\beta}\, L\Big(\frac{\mu(c)}{\exp \alpha(c)}\Big), \end{aligned}$$

where $L(x) = -x \log x$. The concavity of $L(x)$ yields

$$\begin{aligned} H_\mu(\mathcal{C}) + \int_X f_{nm}\, d\mu &\le \beta L\Big(\sum_{c \in \mathcal{C}} \frac{\exp \alpha(c)}{\beta} \frac{\mu(c)}{\exp \alpha(c)}\Big) = \beta L\Big(\sum_{c \in \mathcal{C}} \frac{\mu(c)}{\sum_{c \in \mathcal{C}} \exp \alpha(c)}\Big) \\ &= \beta L(\beta^{-1}) = \log \beta. \ ∎ \end{aligned}$$

LEMMA 7. *Let E be an (nm, δ)-spanning set. Then for any $c \in \mathcal{C}$ there exists a point $z_c \in E$ such that*

$$\alpha(c) = \sup\{f_{nm}(x) : x \in c \text{ and } (x, z_c) \in N(\delta, nm)\}.$$

Proof. Fix $c \in \mathcal{C}$ and let x_0 be a point of the closure of c such that $\alpha(c) = f_{nm}(x_0)$. Then there exists $y \in E$ such that

$$(x_0, y) \in N(\delta, mn) = \bigcap_{g \in G_{mn-1}} (g \times g)^{-1} N_\delta.$$

Therefore, $(g(x_0), g(y)) \in N_\delta$ for any $g \in G_{mn-1}$. If $x_0 \in c$, we are done. If $x_0 \in \partial c$, then by the continuity of all $g \in G_{mn-1}$ and the fact that N_δ is an open set, there exists a ball $B(x_0, r)$ in X such that for each $x_1 \in B(x_0, r)$ and each $g \in G_{mn-1}$,

$$(g(x_1), g(y)) \in N_\delta.$$

So, taking $x_0' \in B(x_0, r) \cap c$ we get the desired point. ∎

LEMMA 8. *Let ζ and δ^* be as in Lemma 4. Then for any element c of the partition $\mathcal{C}$ there exists a point z_c such that*

(a) $f_{nm}(z_c) \geq \alpha(c) - \zeta|G_{nm-1}|$,
(b) $\operatorname{card}\{c \in \mathcal{C} : z_c = y\} \leq 2^{|G_{n-1}|}$.

Proof. (a) Let E be an (mn, δ)-spanning set. By Lemma 7 for $x \in c$ there exists $z_c \in E$ such that

$$(x, z_c) \in \bigcap_{g \in G_{nm-1}} (g \times g)^{-1} N_{\delta^*}.$$

Therefore, $(h(x), h(z_c)) \in N_{\delta^*}$ for any $h \in G_{mn-1}$. By Lemma 4 we get

$$|f(h(x)) - f(h(z_c))| < \zeta.$$

Thus

$$\begin{aligned} f_{nm}(z_c) &= \sum_{h \in G_{mn-1}} f \circ h(z_c) \geq \sum_{h \in G_{mn-1}} (f \circ h(x) - \zeta) \\ &\geq \sup\{f_{mn}(x) : x \in c\} - \zeta|G_{mn-1}| = \alpha(c) - \zeta|G_{mn-1}|. \end{aligned}$$

(b) The proof is similar to the proof of equation (8) in [28, p. 1072]. ∎

Proof of Theorem 1. Fix $\zeta > 0$ and choose large m such that

$$\frac{\log 2}{|G_{m-1}|} \leq \zeta.$$

Let E be an (mn, δ^*)-separated set. By Lemma 8,

$$\begin{aligned} 2^{|G_{n-1}|} \sum_{y \in E} \exp f_{mn}(y) &\geq \operatorname{card}\{c \in \mathcal{C} : z_c = y\} \sum_{y \in E} \exp f_{mn}(y) \\ &\geq \sum_{c \in \mathcal{C}} \exp(\alpha(c) - \zeta|G_{mn-1}|). \end{aligned}$$

Taking logarithms of both sides we arrive at

$$|G_{n-1}| \log 2 + \log\Big(\sum_{y \in E} \exp f_{mn}(y)\Big) \geq -\zeta|G_{mn-1}| + \log \sum_{c \in \mathcal{C}} \exp \alpha(c).$$

Thus,

$$|G_{n-1}| \log 2 + p(f_{mn}, E) \geq -\zeta|G_{mn-1}| + \log \beta. \tag{1}$$

On the other hand, by Lemma 6,

$$\begin{aligned} \frac{H_\mu(\mathcal{C})}{|G_{mn-1}|} + \int_X f\, d\mu &\leq \frac{H_\mu(\mathcal{C}) + |G_{mn-1}| \int_X f\, d\mu}{|G_{mn-1}|} = \frac{H_\mu(\mathcal{C}) + \int_X f_{mn}\, d\mu}{|G_{mn-1}|} \\ &\leq \frac{\log \beta}{|G_{mn-1}|}. \end{aligned}$$

Let A be as in Definition 4. By (1) we get

$$\frac{\log\beta}{|G_{mn-1}|} \le \frac{|G_{n-1}|\log 2 + p(f_{mn},E) + \zeta|G_{mn-1}|}{|G_{mn-1}|} \le \frac{P_{mn,\delta^*}((G,G_1),f)}{|G_{mn-1}|} + (A+1)\zeta$$

because $\log 2 \le \zeta|G_{m-1}|$ and

$$|G_{n-1}|\,|G_{m-1}| \le A|G_{(n-1)(m-1)}| \le A|G_{mn-1}|.$$

So, finally we obtain

$$\frac{H_\mu(\mathcal{C})}{|G_{mn-1}|} + \int_X f\,d\mu \le \frac{P_{mn,\delta^*}((G,G_1),f)}{|G_{mn-1}|} + (A+1)\zeta. \tag{2}$$

The construction of the partition $\mathcal{B}$ implies that for any $g \in G_n^{(m)}$,

$$H_\mu(g^{-1}\mathcal{A}_m \mid g^{-1}\mathcal{B}) \le \zeta.$$

So,

$$H_\mu(\mathcal{A}_{mn} \mid \mathcal{C}) = H_\mu\Big(\bigvee_{g\in G_{n-1}^{(m)}} g^{-1}\mathcal{A}_m \,\Big|\, \bigvee_{g\in G_{n-1}^{(m)}} g^{-1}\mathcal{B}\Big) \le |G_{n-1}^{(m)}|\zeta \le |G_{mn-1}|\zeta.$$

Using the basic properties of conditional entropy we arrive at

$$H_\mu(\mathcal{A}_{mn}) \le H_\mu(\mathcal{C}) + H_\mu(\mathcal{A}_{mn} \mid \mathcal{C}) \le H_\mu(\mathcal{C}) + |G_{mn-1}|\zeta.$$

Therefore,

$$\frac{H_\mu(\mathcal{A}_{mn})}{|G_{mn-1}|} + \int_X f\,d\mu \le \bigg(\frac{H_\mu(\mathcal{C})}{|G_{mn-1}|} + \int_X f\,d\mu\bigg) + \zeta\,\frac{|G_{mn-1}|}{|G_{mn-1}|} \le \frac{P_{mn,\delta^*}((G,G_1),f)}{|G_{mn-1}|} + (A+2)\zeta.$$

Passing to the limsup with respect to n we obtain

$$h_\mu((G,G_1),\mathcal{A}) + \int_X f\,d\mu \le P_{\delta^*}((G,G_1),f) + (A+2)\zeta.$$

Letting $\zeta \to 0^+$ (then also $\delta^* \to 0^+$) and taking into consideration that $\mathcal{A}$ is an arbitrary finite Borel partition, we arrive at

$$h_\mu(G,G_1) + \int_X f\,d\mu \le P((G,G_1),f). \ \blacksquare$$

6. Existence of a group invariant measure for a group of polynomial growth. In this section (G,G_1) is a finitely generated group of homeomorphisms of a compact metric space (X,d). Again, we assume that $G_1 = \{\mathrm{id}_X, g_1, g_1^{-1}, \dots, g_k, g_k^{-1}\}$.

REMARK. It is clear that the group (G, G_1) generated by $G_1 = \{\mathrm{id}_{S^1}, f_1, f_1^{-1}, f_2, f_2^{-1}\}$, where f_i, $i = 1, 2$, were described in Example 1, has no G-invariant measure. It is an example of a group of exponential growth. That is why in the following we restrict our attention to the groups of polynomial growth.

DEFINITION 10. Let (G, G_1) be a finitely generated group and $A \subset G$. The G_1-*boundary* of a subset A of G is the set

$$\partial_{G_1} A := \{g \in G : g \notin A \text{ and } \exists_{s \in G_1} sg \in A\}.$$

PROPOSITION 7 ([30]). *Let (G, G_1) be a finitely generated group of polynomial growth. Then*

$$\lim_{n\to\infty} \frac{|G_n \cup \partial_{G_1} G_n|}{|G_n|} = 1.$$

COROLLARY 2. *Let (G, G_1) be a finitely generated group of polynomial growth, and let $g_0 \in G_1$. Define $A_1^{(n)} = G_{n-1} \setminus g_0 G_{n-1}$ and $A_2^{(n)} = g_0 G_{n-1} \setminus G_{n-1}$. Then*

$$\lim_{n\to\infty} \frac{|A_i^{(n)}|}{|G_{n-1}|} = 0, \qquad i = 1, 2.$$

Proof. By Proposition 7,

$$\lim_{n\to\infty} \frac{|\partial_{G_1} G_n|}{|G_n|} = 0. \tag{3}$$

It is easy to observe that $A_1^{(n)} = G_{n-1} \setminus g_0 G_{n-1} \subset \partial_{G_1} G_{n-2}$. So, by (3),

$$\lim_{n\to\infty} \frac{|A_1^{(n)}|}{|G_{n-1}|} = 0.$$

In a similar way we observe that

$$A_2^{(n)} = g_0 G_{n-1} \setminus G_{n-1} \subset \partial_{G_1} G_{n-1}$$

and using the same argument we conclude that

$$\lim_{n\to\infty} \frac{|A_2^{(n)}|}{|G_{n-1}|} = 0. \ \blacksquare$$

PROPOSITION 8. *If (G, G_1) is a finitely generated group of homeomorphisms of a compact metric space (X, d), of polynomial growth, then there exists a G-invariant measure.*

Proof. Let E_n be an (n, δ)-separated subset of X. Choose a continuous function $f \in C(X)$ and fix $g_0 \in G_1$. Define a measure σ_n concentrated on E_n by

$$\sigma(\{y\}) := \frac{\exp f_n(y)}{\sum_{y \in E_n} \exp f_n(y)}$$

for each $y \in E_n$, and let

$$\mu_n := \frac{1}{|G_{n-1}|} \sum_{g \in G_{n-1}} \sigma_n \circ g.$$

It is well known that the space $M(X)$ of all Borel probability measures defined on X is a compact metric space in the weak w^*-topology. Therefore, the sequence μ_n has a cluster point in $M(X)$.

Consider the mapping $\Phi_{f,g_0} : M(X) \to \mathbb{R}$ defined by

$$\Phi_{f,g_0}(m) := \int_X f\, dm - \int_X f \circ g_0\, dm$$

for any measure $m \in M(X)$. It is easy to check that Φ_{f,g_0} is a continuous map. Thus if μ is a cluster point of the sequence (μ_n) then $\Phi_{f,g_0}(\mu)$ is a cluster point of $(\Phi_{f,g_0}(\mu_n))$. To calculate the norm of the functional Φ_{f,g_0} note first that

$$\begin{aligned}
\Phi_{f,g_0}(\mu_n) &= \int_X (f - f \circ g_0)\, d\Big(\frac{1}{|G_{n-1}|} \sum_{g \in G_{n-1}} \sigma_n \circ g\Big) \\
&= \frac{1}{|G_{n-1}|} \sum_{g \in G_{n-1}} \int_X (f - f \circ g_0)\, d\sigma_n \circ g \\
&= \frac{1}{|G_{n-1}|} \sum_{g \in G_{n-1}} \int_X (f \circ g^{-1} - f \circ g_0 \circ g^{-1})\, d\sigma_n \\
&= \frac{1}{|G_{n-1}|} \sum_{g \in G_{n-1}} \sum_{y \in E_n} (f \circ g^{-1}(y) - f \circ g_0 \circ g^{-1}(y))\, \frac{\exp f_n(y)}{\sum_{y \in E_n} \exp f_n(y)} \\
&= \frac{1}{|G_{n-1}|} \sum_{y \in E_n} \frac{\exp f_n(y)}{\sum_{y \in E_n} \exp f_n(y)} \sum_{g \in A_1^{(n)} \cup A_2^{(n)}} (f \circ g^{-1}(y) - f \circ g_0 \circ g^{-1}(y)),
\end{aligned}$$

where $A_1^{(n)} = G_{n-1} \setminus g_0 G_{n-1}$ and $A_2^{(n)} = g_0 G_{n-1} \setminus G_{n-1}$.

Finally, in view of Corollary 2 we arrive at

$$\begin{aligned}
\|\Phi_{f,g_0}(\mu_n)\| &\le \frac{1}{|G_{n-1}|} \sum_{y \in E_n} \frac{\exp f_n(y)}{\sum_{y \in E_n} \exp f_n(y)}\, 2\|f\|\, |A_1^{(n)} \cup A_2^{(n)}| \\
&\le 4\|f\|\, \frac{\max\{|A_1^{(n)}|, |A_2^{(n)}|\}}{|G_{n-1}|}.
\end{aligned}$$

So, letting $n \to \infty$ we see that $\Phi_{f,g_0}(\mu) = 0$, and therefore the measure μ is g_0-invariant. But g_0 is an arbitrary element of G_1, thus μ is G-invariant. ∎

7. Suspension of a group of polynomial growth and the variational principle for the geometric entropy of foliations. The geometric entropy $h_{\text{geom}}(F, g_*)$ of a foliation F on a compact Riemannian manifold (M, g_*), defined by Ghys, Langevin and Walczak [20] for a regular foliation, measures the exponential rate of growth of separated leaves of F. An equivalent definition of $h_{\text{geom}}(F, g_*)$ was given in terms of points separated by elements of a holonomy pseudogroup. Given a foliation F on a compact Riemannian manifold (M, g) and a nice covering $\mathcal{U}$ which determines a holonomy pseudogroup $(H_{\mathcal{U}}, H_1)$ of the foliated manifold (M, F) (see [20]), we get:

PROPOSITION 9 (see [20]). *The geometric entropy $h_{\text{geom}}(F, g)$ of a foliated manifold (M, F) (with respect to a continuous family g of Riemannian structures on the leaves) is equal to*

$$h_{\text{geom}}(F, g) = \sup_{\mathcal{U}} \frac{h(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_1)}{\triangle(\mathcal{U})},$$

where $\mathcal{U}$ ranges over the family of all finite nice coverings of (M, F), and $\triangle(\mathcal{U})$ denotes the the maximum of the diameters of the plaques of $\mathcal{U}$ measured with respect to the Riemannian structures induced on the leaves.

The variational principle for the geometric entropy of foliations is an open problem. Walczak ([35, p. 141]) writes that it seems interesting and important to search for a good definition of a measure-theoretic entropy for foliations which could provide a kind of variational principle for geometric entropy. In this section we show that Theorem 1 provides a kind of partial variational principle for geometric entropy for some class of foliations.

We present a suspension construction which directly relates the dynamics of a group to the dynamics of the foliated space. To do this, take a compact metric space (Z, d), a compact Riemannian manifold B and its fundamental group $G = \pi_1(B, b)$ at a base point $b \in B$. The fundamental group $\pi_1(B)$ acts on the right in a natural way on $\widetilde{B}$, the universal covering of B. Assume that there exists a left action of G on Z. Let

$$M := (\widetilde{B} \times Z)/=_r$$

where the equivalence relation $=_r$ is defined in the following way: $(xg, z) =_r (x, gz)$ for any $g \in G$, $x \in \widetilde{B}$ and $z \in Z$. The space M fibres over B with fibre Z. Moreover, M can be equipped with a foliation F which consists of the leaves of the form $L = \pi(\widetilde{B} \times \{z\})$, where $z \in Z$ and $\pi : \widetilde{B} \times Z \to M$ is the canonical projection. The foliated space (M, F) is a fibre bundle with fibre Z. Then the holonomy group of (M, F) coincides with $G = \pi_1(B, b)$.

Given a finitely generated group (G, G_1) of polynomial growth there is a compact manifold M such that $G = \pi_1(M)$. Denote by Γ the one-point compactification of the graph of (G, G_1). Then G acts on the compact metric space Γ (see [26]), so we can consider the compact foliated space

$$M_G := (\widetilde{M} \times \Gamma)/=_r$$

with leaves $F_G = \{L = \pi(\widetilde{M} \times \{\gamma\}) : \gamma \in \Gamma\}$. The distance function d_Γ on Γ and the Riemannian metric g_M on M lifted via the canonical projection π to the leaves of F_G determine the natural metric g_{M_G} on M_G which coincides with d_Γ on Γ and with g_M along the leaves.

Following Example 4.3 in [20] or the last section in [7] we obtain

$$\frac{1}{a}\, h(G, G_1) \leq h_{\mathrm{geom}}(F_G),$$

where a denotes the maximum of the lengths of the free homotopy classes of curves homotopic to elements of G_1. Finally, we get a kind of partial variational principle for the geometric entropy of the foliation (M_G, F_G) modelled transversally on Γ.

Corollary 3. *For a compact foliated space (M_G, F_G), determined by the suspension construction of a finitely generated group (G, G_1) of polynomial growth, with a continuous family g_{M_G} of Riemannian structures on the leaves, and for any measure $\mu \in M(\Gamma, (G, G_1))$ we get*

$$\sup\{h_\mu(G, G_1) : \mu \in M(\Gamma, (G, G_1))\} \leq a h_{\mathrm{geom}}(F_G, g_{M_G}),$$

where $h_{\mathrm{geom}}(F_G, g_{M_G})$ is the geometric entropy of F_G with respect to the Riemannian structure g_{M_G}, and a denotes the maximum of the lengths of the free homotopy classes of curves homotopic to elements of G_1.

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