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SHADOWING IN MULTI-DIMENSIONAL SHIFT SPACES

BY

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Abstract. We show that the class of expansive $\mathbb{Z}^d$ actions with P.O.T.P. is wider than the class of actions topologically hyperbolic in some direction $\nu \in \mathbb{Z}^d$. Our main tool is an extension of a result by Walters to the multi-dimensional symbolic dynamics case.

1. Introduction. In this paper we consider multi-dimensional shift spaces. The books [1, 11] give an introduction to one-dimensional shift spaces theory. Multi-dimensional shift spaces arise in a natural way when we generalize the standard shift map σ to a $\mathbb{Z}^d$-action $n \mapsto \sigma^n$. The vector $n \in \mathbb{Z}^d$ informs us how many cells we shift in each direction. The dynamics in higher dimensions is more complex than in the one-dimensional case (see, for example, [2, 9]). It turns out that there exist one-dimensional results which are not true in higher dimensions and also some higher dimensional properties have no analogue in dimension one [4].

Studying the pseudo-orbit tracing property (P.O.T.P.) of dynamical systems is an important part of stability theory (see [6, 7]). P.O.T.P. for group actions has recently been established by Pilyugin and Tikhomirov in [8]. In his fundamental paper [10] Peter Walters proved that a (one-dimensional) subshift has P.O.T.P. if and only if it is a shift of finite type. In this paper we prove an analogous result for multi-dimensional shift spaces. We also show a stronger property: every shift of finite type has Lipschitz P.O.T.P. and for $\varepsilon < 1$ any pseudo-orbit may be ε-traced by exactly one point. This result is used to study connections between P.O.T.P. of a $\mathbb{Z}^d$-action Φ and P.O.T.P. of the homeomorphisms Φ^ν where $\nu \in \mathbb{Z}^d$.

2. Preliminaries. Let $\mathcal{A}$ be a finite set, $d \in \mathbb{N}$, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \to \mathcal{A}$. For any $(j_1, \ldots, j_d) = j \in \mathbb{Z}^d$ we define $\|j\| = \max\{|j_i| : i = 1, \ldots, d\}$. The usual prefix metric on the one-dimensional full

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shift may be generalized to a metric ϱ on $\mathcal{A}^{\mathbb{Z}^d}$ given by $\varrho(x,y)=2^{-j}$ where
$$j=\sup(\{k\in\mathbb{N} : x_n=y_n,\ n\in\mathbb{Z}^d,\ \|n\|<k\}).$$
For each $n\in\mathbb{Z}^d$ we define a homeomorphism $\sigma^n : \mathcal{A}^{\mathbb{Z}^d}\to\mathcal{A}^{\mathbb{Z}^d}$ putting $(\sigma^n(x))_m=x_{m+n}$ for all $x\in\mathcal{A}^{\mathbb{Z}^d}$ and $m\in\mathbb{Z}^d$. The $\mathbb{Z}^d$-action $n\mapsto\sigma^n$ is called the *shift action* on $\mathbb{Z}^d$. The *d-dimensional full shift* is the space $\mathcal{A}^{\mathbb{Z}^d}$ with metric ϱ and the shift action. Any closed subset X of $\mathcal{A}^{\mathbb{Z}^d}$ invariant under σ (i.e. $\sigma^n(X)=X$ for all $n\in\mathbb{Z}^d$) is called a *d-dimensional shift space* (or simply a *shift space*). If X,Y are shift spaces and $X\subset Y$ then we say that X is a *subshift* of Y.

Given two d-dimensional shift spaces X, Y we may always assume that both are subshifts of some d-dimensional full shift. Namely, if $X\subset(\mathcal{A}_X)^{\mathbb{Z}^d}$ and $Y\subset(\mathcal{A}_Y)^{\mathbb{Z}^d}$ we may set $\mathcal{A}=\mathcal{A}_X\cup\mathcal{A}_Y$ and then $X,Y\subset\mathcal{A}^{\mathbb{Z}^d}$. Due to this observation, when we consider a finite number of shift spaces, we may always assume that they have the same alphabets.

For $x\in\mathcal{A}^{\mathbb{Z}^d}$ and $F\subset\mathbb{Z}^d$ let x_F denote the restriction of x to F. If $F=\{a\}$ for some $a\in\mathbb{Z}^d$ we simply write x_a. A *shape* is a finite subset of $\mathbb{Z}^d$. A *pattern* on the shape F is a function $f:F\to\mathcal{A}$.

A pattern $f:F\to\mathcal{A}$ is said to be *allowed for the shift* X if there exists $x\in X$ such that $x_F=f$. If $f:F\to\mathcal{A}$ is a pattern then we write $[f]=\{x : x_F=f\}$. This generalizes the notion of one-dimensional *cylinder set* to d dimensions.

The *k-cube with lowest corner at the origin* is the set
$$\Lambda(k)=\{0,\dots,k-1\}^d.$$
For $n\in\mathbb{Z}^d$ the set $n+\Lambda(k)=\{n+m : m\in\Lambda(k)\}$ is called the *k-cube with the lowest corner at n*. The *k-cube centered at the origin* is the set $\overline{\Lambda}(k)=\{-k+1,\dots,k-1\}^d$. Observe that if $\varrho(x,y)\le 2^{-k}$ then $x_{\overline{\Lambda}(k)}=y_{\overline{\Lambda}(k)}$.

By a *k-block* we mean a pattern $f:\Lambda(k)\to\mathcal{A}$. A pattern f is called a *block* if it is a k-block for some k. We write $B_k(\mathcal{A})$ for the set of all k-blocks and $B(\mathcal{A})$ for the set of all possible blocks (i.e. $B(\mathcal{A})=\bigcup_{k=1}^{\infty}B_k(\mathcal{A})$). For any shift space X and $k\in\mathbb{N}$ we denote by $B_k(X)$ the set of all k-blocks allowed for X and by $B(X)$ the set of all blocks allowed for X.

If $f\in B_k(X)$ and $x\in X$ then we say that f *occurs in x with lowest corner at* $n\in\mathbb{Z}^d$ whenever $f(m)=x(m+n)$ for all $m\in\Lambda(k)$. We then write $f=x_{n+\Lambda(k)}$. Given $l\ge k$, we say that $f\in B_k(X)$ *occurs* in $x_{b+\Lambda(l)}$ if there exists $a\in\mathbb{Z}^d$ such that $a+\Lambda(k)\subset b+\Lambda(l)$ and $f=x_{a+\Lambda(k)}$.

3. Shifts of finite type. Let $\mathcal{F}$ be a set of patterns. We denote by $X_{\mathcal{F}}$ the set of all points of $\mathcal{A}^{\mathbb{Z}^d}$ which do not contain any pattern from $\mathcal{F}$, i.e.
$$x\in X_{\mathcal{F}}\ \Leftrightarrow\ \forall(f:E\to\mathcal{A})\in\mathcal{F}\ \forall n\in\mathbb{Z}^d\ \ x_{n+E}\ne f.$$
Elements of $\mathcal{F}$ are called *forbidden patterns*.

LEMMA 3.1. *A set $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift space if and only if there exists a set $\mathcal{F}$ of patterns such that $X = X_{\mathcal{F}}$.*

The proof is analogous to that in [1, Thm. 6.1.21] for the one-dimensional case, and therefore is omitted.

COROLLARY 3.2. *A set $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift space if and only if there exists $\mathcal{F} \subset B(\mathcal{A})$ such that $X = X_{\mathcal{F}}$.*

Proof. Any set $\mathcal{F} \subset B(X)$ is a set of patterns and so $X_{\mathcal{F}}$ is a shift space.

Conversely, let $\mathcal{F}$ be any fixed set of patterns such that $X = X_{\mathcal{F}}$. If $f : E \to \mathcal{A}$ is a pattern then there exists $k \in \mathbb{N}$ such that $E \subset \overline{\Lambda}(k)$, and so $w + E \subset \Lambda(2k+1)$ where $w = (k, \dots, k) \in \mathbb{Z}^d$. We define $A_f \subset B(X)$ by

$$A_f = \{g \in B_{2k+1}(\mathcal{A}) : g_{w+E} = f\}.$$

Set $\widetilde{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} A_f$. Observe that $\widetilde{\mathcal{F}} \subset B(\mathcal{A})$ and $X_{\mathcal{F}} = X_{\widetilde{\mathcal{F}}}$. ∎

DEFINITION 3.3. Let X be a shift space. We say that X is a *shift of finite type* if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. A shift of finite type X is *M-step* if $X = X_{\mathcal{F}}$ for some $\mathcal{F} \subset B_{M+1}(\mathcal{A})$.

EXAMPLE 3.4 (Chessboard). Let $\mathcal{A}^{(n)} = \{0, 1, \dots, n-1\}$ be an alphabet interpreted as a set of n colors. We construct a shift $X^{(n)}$ of finite type such that adjacent cells of any point have different colors. Such a shift space may be obtained as $X^{(n)} = X_{\mathcal{F}^{(n)}}$ where the set of forbidden patterns $\mathcal{F}^{(n)}$ consists of:

$$\begin{array}{|c|c|}\hline a & a \\ \hline \end{array} \qquad \begin{array}{|c|}\hline a \\ \hline a \\ \hline \end{array}$$

where a is any color from $\mathcal{A}^{(n)}$. Observe that if we denote by $\mathcal{H}^{(n)}$ the set containing all possible patterns of the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline a & c \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline a & a \\ \hline b & c \\ \hline \end{array}$$

where $a, b, c \in \mathcal{A}^{(n)}$, then $X^{(n)} = X_{\mathcal{H}^{(n)}}$. However, $\mathcal{H}^{(n)} \subset B_2(\mathcal{A}^{(n)})$ and then $X^{(n)}$ is a 1-step shift of finite type.

In view of Corollary 3.2 every shift X of finite type may be defined by a finite set $\mathcal{F} \subset B(\mathcal{A})$. By the same arguments there always exists a positive integer M such that X is an M-step shift of finite type.

DEFINITION 3.5. Let X be a subshift of $\mathcal{A}^{\mathbb{Z}^d}$. A map $\phi : X \to \mathcal{A}^{\mathbb{Z}^d}$ is called *k-local* if there exists $\Phi : B_{2k+1}(X) \to \mathcal{A}$ such that for every $x \in X$ and $n \in \mathbb{Z}^d$,

$$\phi(x)_n = \Phi((\sigma^n(x))_{\overline{\Lambda}(k)}).$$

A map ϕ is called *local* if it is k-local for some $k \in \mathbb{N}$.

This definition generalizes the definition of a sliding block code (one-dimensional case). In fact, the well known Curtis–Lyndon–Hedlund theorem (see [1, Thm. 6.2.9]) may be extended to the d-dimensional case and k-local maps. This implies that the k-local maps are exactly the functions which are continuous and shift commuting (i.e. $\sigma^n(\phi(x)) = \phi(\sigma^n(x))$ for any $x \in X$ and $n \in \mathbb{Z}$). Furthermore, if X, Y are shift spaces and $\phi : X \to Y$ is a local map which is one-to-one and onto, then ϕ is a shift commuting homeomorphism (see [1, Thm. 1.5.14] for the one-dimensional case). This leads to the following definition:

Definition 3.6. Two shift spaces X, Y are *conjugate* if there exists a bijective local map $\phi : X \to Y$. Every such ϕ is called a *conjugacy* between X and Y.

In view of previous facts the above definition of conjugacy is equivalent to the definition of topological conjugacy of two $\mathbb{Z}^d$-actions.

Let X be a shift space and let N be a positive integer. Define a map $\beta_N : X \to B_N(X)^{\mathbb{Z}^d}$ by $(\beta_N(x))_n = x_{n+\Lambda(N)}$.

Definition 3.7. Let X be a shift space. Then the Nth *higher block shift* $X^{[N]}$ is the image $X^{[N]} = \beta_N(X)$.

Observe that β_N is an N-local invertible mapping, so the shift spaces X and $X^{[N]}$ are conjugate.

Proposition 3.8. *Let X be an M-step shift of finite type. Then it is conjugate to a* 1*-step shift of finite type.*

Proof. By previous remarks, X and $X^{[M]}$ are conjugate. Because any $M+1$ block in X may be regarded as a 2-block in $X^{[M]}$, this space is a 1-step shift of finite type. This is an immediate generalization of the one-dimensional case [1, Prop. 2.3.9]. ∎

4. Shift spaces and shadowing. Fix a positive number δ. We say that a set $\xi = \{x^{(n)} \in \mathcal{A}^{\mathbb{Z}^d} : n \in \mathbb{Z}^d\}$ is a δ *pseudo-orbit* if

$$\varrho(x^{(n\pm e_i)}, \sigma^{\pm e_i}(x^{(n)})) < \delta$$

for any $n \in \mathbb{Z}^d$ and $i = 1, \ldots, d$, where $e_i \in \mathbb{Z}^d$ is the ith standard basis vector.

Definition 4.1. Let X be a shift space. A δ pseudo-orbit $\xi = \{x^{(n)} \in \mathcal{A}^{\mathbb{Z}^d} : n \in \mathbb{Z}^d\}$ is ε*-traced* by $x \in X$ if $\varrho(x_n, \sigma^n(x)) < \varepsilon$ for any $n \in \mathbb{Z}^d$.

The definition below is a particular case of the general definition of P.O.T.P. (see [8]). Similarly to the one-dimensional case it is easy to see that P.O.T.P. is a topological conjugacy invariant.

DEFINITION 4.2. A shift space X has the *pseudo-orbit tracing property* (*P.O.T.P.*, *shadowing*) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that each δ pseudo-orbit $\xi \subset X$ is ε-traced by some point $y \in X$.

DEFINITION 4.3. A shift space X has the *Lipschitz pseudo-orbit tracing property* (*Lipschitz P.O.T.P.*, *Lipschitz shadowing*) if there exists a constant $L > 0$ such that for any δ pseudo-orbit $\xi = \{x^{(n)} \in \mathcal{A}^{\mathbb{Z}^d} : n \in \mathbb{Z}^d\}$ there is a point $x \in X$ satisfying

$$\varrho(x^{(n)}, \sigma^n(x)) < L\delta, \quad n \in \mathbb{Z}^d.$$

The following definition generalizes the well known concept of expansiveness.

DEFINITION 4.4. We say that a shift space X is *expansive* if there exists a constant $b > 0$ (*expansive constant*) such that whenever for any $x, y \in X$,

$$\varrho(\sigma^n(x), \sigma^n(y)) < b \quad \text{for all } n \in \mathbb{Z}^d,$$

then $x = y$.

The main tool we will use is the following:

THEOREM 4.5. *Let X be a shift space. Then the following conditions are equivalent*:

(1) *X is a shift of finite type.*
(2) *X has the pseudo-orbit tracing property.*
(3) *X has the Lipschitz pseudo-orbit tracing property.*

In particular, if X is an M-step shift of finite type then it has the Lipschitz pseudo-orbit tracing property with constant $L = 2^{M+1}$.

Proof. The implication (3)⇒(2) is always true. We will show that (1)⇒(3) and (2)⇒(1) hold.

(1)⇒(3). Suppose that X is a shift of finite type. We may assume that X is an M-step shift, that is, there exists $\mathcal{F} \subset B_{M+1}(\mathcal{A})$ such that $X = X_{\mathcal{F}}$. This means that $x \in X$ if and only if $x_{n+\Lambda(M+1)} \notin \mathcal{F}$ for any $n \in \mathbb{Z}^d$. First, let us make an observation which is crucial for this part of the proof.

Let $m > M$, $\delta = 2^{-m}$ and let $\xi = \{x^{(n)} \in X : n \in \mathbb{Z}^d\}$ be a δ pseudo-orbit. By definition, $\varrho(x^{(n\pm e_i)}, \sigma^{\pm e_i}(x^{(n)})) < \delta$ for any $n \in \mathbb{Z}^d$. This implies that

$$x^{(n\pm e_i)}_{\overline{\Lambda}(m)} = (\sigma^{\pm e_i}(x^{(n)}))_{\overline{\Lambda}(m)}. \tag{4.1}$$

Let $y \in \mathcal{A}^{\mathbb{Z}^d}$ with $y(n) = x^{(n)}(0)$ for any $n \in \mathbb{Z}^d$. We will show that $y \in X$.

Fix any $a = (a_1, \ldots, a_d) \in \overline{\Lambda}(m)$. Applying (4.1) we find that $x^{(n\pm e_i)}(j) = x^{(n)}(j \pm e_i)$ for all $j \in \overline{\Lambda}(m)$, $n \in \mathbb{Z}^d$. We will use (4.1) recursively. For simplicity, we assume that $a_i \geq 0$. When $a_i < 0$ it is enough to replace -1

by 1 in the following equalities (i.e. increase values at the ith coordinate instead of decreasing them):

$$(4.2)\quad \begin{aligned} x^{(n+a)}(0) = x^{(n+(a_1,\dots,a_d))}(0) &\overset{(4.1)}{=} (\sigma^{e_1}(x^{(n+(a_1,\dots,a_d)-e_1)}))(0) \\ &= x^{(n+(a_1-1,a_2,\dots,a_d))}(e_1) \overset{(4.1)}{=} \cdots \\ &= x^{(n+(0,a_2,\dots,a_d))}(a_1e_1) \overset{(4.1)}{=} \cdots \\ &= x^{(n+(0,0,a_3,\dots,a_d))}(a_1e_1+a_2e_2) \overset{(4.1)}{=} \cdots \\ &= x^{(n+(0,\dots,0))}(a_1e_1+\cdots+a_de_d) \\ &= x^{(n)}(a). \end{aligned}$$

We have just shown that $y(n+a) = x^{(n+a)}(0) = x^{(n)}(a)$ for any $a \in \overline{\Lambda}(m)$. Observe that $\Lambda(M+1) \subset \overline{\Lambda}(m)$, so $y_{n+\Lambda(M+1)} = x^{(n)}_{0+\Lambda(M+1)} \notin \mathcal{F}$ for any $n \in \mathbb{Z}^d$ and hence $y \in X$.

The point y defined above is a good candidate to trace the pseudo-orbit ξ and, as we will see, it really does. The set ξ is a 2^{-m} pseudo-orbit, so by (4.2) we obtain $x^{(n+a)}(0) = x^{(n)}(a)$ for all $a \in \overline{\Lambda}(m)$. This implies that $\varrho(\sigma^n(y), x^{(n)}) < 2^{-m}$ and so ξ is δ traced by y.

Let $L = 2^{M+1}$. Take any $\delta > 0$. If $\delta > 2^{-M}$ then $L\delta > 1$ and there is nothing to prove. Suppose that $K \geq M$ is an integer such that $2^{-(K+1)} < \delta \leq 2^{-K}$. Observe that any δ pseudo-orbit is also a 2^{-K} pseudo-orbit, thus by previous observations it is 2^{-K}-traced. Additionally, $2^{-K} \leq 2^{-(K+1)}L \leq L\delta$, which finishes the proof of (1)⇒(3).

(2)⇒(1). Suppose that X has P.O.T.P., fix $\varepsilon = 1/2$ and take $\delta > 0$ such that every δ pseudo-orbit is ε-traced. Choose N large enough to have $2^{-N} < \delta$.

We will show that X is an M-step shift of finite type where $M = 2N+2$. Let $\mathcal{F} = B_{M+1}(\mathcal{A}) \setminus B_{M+1}(X)$. Obviously $X \subset X_{\mathcal{F}}$. We have to show that $X_{\mathcal{F}} \subset X$.

Fix $y \in X_{\mathcal{F}}$. By the definition of $X_{\mathcal{F}}$ for every $n \in \mathbb{Z}^d$ we have $y_{n+\overline{\Lambda}(N+1)} \in B_{M+1}(X)$, thus for every $n \in \mathbb{Z}^d$ there exists $x^{(n)} \in X$ such that $x^{(n)}_{\overline{\Lambda}(N+1)} = y_{n+\overline{\Lambda}(N+1)}$. Set $\xi = \{x^{(n)} \in X : n \in \mathbb{Z}^d\}$. Obviously $\overline{\Lambda}(N) \pm e_i \subset \overline{\Lambda}(N+1)$, so

$$(\sigma^{\pm e_i}(x^{(n)}))_{\overline{\Lambda}(N)} = x^{(n)}_{\overline{\Lambda}(N)\pm e_i} = y_{n+(\overline{\Lambda}(N)\pm e_i)} = y_{(n\pm e_i)+\overline{\Lambda}(N)} = x^{(n\pm e_i)}_{\overline{\Lambda}(N)}.$$

This implies that $\varrho(\sigma^{\pm e_i}(x^{(n)}), x^{(n\pm e_i)}) \leq 2^{-N} < \delta$ and so ξ is a δ pseudo-orbit. Thus there exists $x \in X$ such that ξ is ε-traced by x. Observe that $\varrho(\sigma^n(x), x^{(n)}) < 1/2$, which implies that $x_{n+\overline{\Lambda}(1)} = x^{(n)}_{\overline{\Lambda}(1)} = y_{n+\overline{\Lambda}(1)}$. We have just shown that $x(n) = y(n)$ for any $n \in \mathbb{Z}^d$, so $y = x$ and hence $y \in X$. ■

THEOREM 4.6. *Let X be a shift space. If $0 < \varepsilon < 1$ then for any δ pseudo-orbit $\xi \subset X$ there exists at most one point $x \in X$ which ε-traces ξ.*

Proof. Fix any $0 < \varepsilon < 1$ and let $\xi = \{x^{(n)} : n \in \mathbb{Z}^d\} \subset X$ be any fixed δ pseudo-orbit. Suppose that ξ is ε-traced by some point x. For any $n \in \mathbb{Z}^d$ we have $\varrho(x^{(n)}, \sigma^n(x)) = 2^{-j} < \varepsilon < 1$. Observe that $\varrho(x^{(n)}, \sigma^n(x)) \leq 1/2$ and so $x_0^{(n)} = (\sigma^n(x))_0 = x_n$ for any $n \in \mathbb{Z}^d$. This implies that there is at most one such x. ∎

The orbit of any point $y \in X$ is a δ pseudo-orbit for any $\delta > 0$. This implies the following:

COROLLARY 4.7. *Let X be a shift space. Then X is expansive with expansive constant $b = 1$.*

We may also use Theorem 4.6 to define δ_0 such that any δ pseudo-orbit is traced by exactly one point provided that $\delta < \delta_0$. Strictly speaking, we have the following:

COROLLARY 4.8. *Let X be an M-step shift of finite type, let $0 < \varepsilon < 1$ and let $\delta_0 = \varepsilon 2^{-(M+1)}$. If $\delta < \delta_0$ then every δ pseudo-orbit $\xi \subset X$ is ε-traced by exactly one point $y_\xi \in X$.*

Proof. Let $\xi \subset X$ be a δ pseudo-orbit, where $\delta < \delta_0$. By Theorem 4.5 the pseudo-orbit ξ is $L\delta$-traced by some point $y \in X$, where $L = 2^{M+1}$. Observe that $L\delta < \varepsilon < 1$, so by Theorem 4.6 there is exactly one such y. ∎

5. Topologically Anosov homeomorphisms and shadowing. We recall that a homeomorphism h is *topologically Anosov* (or equivalently topologically hyperbolic [3, 5]) if it is expansive and has P.O.T.P. The authors of [8] proved that if for a given $\mathbb{Z}^d$-action Φ there exists $\nu \in \mathbb{Z}^d$ such that the homeomorphism $f = \Phi^\nu$ is topologically Anosov then Φ has P.O.T.P. We will show that the assumptions about f cannot be weakened (it is not enough to assume that f has P.O.T.P. or f is expansive alone). We will also show that there exist $\mathbb{Z}^d$-actions with P.O.T.P. which are not topologically Anosov for any $\nu \in \mathbb{Z}^d$, so [8, Thm. 1] is only a sufficient condition.

EXAMPLE 5.1. Consider a one-dimensional shift space X which is not of finite type (e.g. X may be an "even shift" because it belongs to the class of strictly sofic shift spaces [1, Ex. 2.1.9]). Let $\mathcal{F}$ be the set of forbidden words for X, i.e. $X = X_{\mathcal{F}}$. We define a set $\mathcal{F}'$ of two-dimensional patterns as follows:

$$\mathcal{F}' = \left\{ \boxed{u_1 \mid u_2 \mid \cdots \mid u_{|u|}} : u \in \mathcal{F} \right\} \cup \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} : a, b \in \mathcal{A},\ a \neq b \right\}.$$

Observe that the two-dimensional shift space $Y = X_{\mathcal{F}'}$ contains points which consist of infinitely many copies of elements of X and any point of Y is determined by symbols on the $\mathbb{Z} \times \{0\}$ line. Strictly speaking, $y \in Y$ if:

(1) $y(i,j) = y(i, j+m)$ for all $(i,j) \in \mathbb{Z}^2$ and $m \in \mathbb{Z}$.
(2) $y(\cdot, j) \in X$.

The map $\sigma^{(1,0)}$ is expansive with expansive constant $b = 1/2$; however, Y does not have P.O.T.P. because it is not a shift of finite type.

Next, observe that if $\xi = \{x^{(n)} : n \in \mathbb{N}\}$ is a 2^{-k} pseudo-orbit for $\sigma^{(0,1)}$ then $x^{(n)}_{\overline{\Lambda}(k)} = x^{(0)}_{\overline{\Lambda}(k)}$ for all $n \in \mathbb{N}$ (every point of Y consists of vertical lines of the same symbol). This implies that ξ is 2^{-k}-traced by x_0. Thus the map $\sigma^{(0,1)}$ has P.O.T.P. but (Y, σ) does not.

Example 5.1 shows that even if we know that for some $\nu \in \mathbb{Z}^d$ the mapping Φ^ν for a $\mathbb{Z}^d$-action Φ is expansive or has P.O.T.P. we may say nothing about P.O.T.P. of Φ unless we can find a ν such that Φ^ν has both properties at the same time (is topologically Anosov).

Next, we will show that there exist $\mathbb{Z}^d$-actions with P.O.T.P. which are not topologically Anosov for any $\nu \in \mathbb{Z}^d$. In the following example we present a $\mathbb{Z}^2$-action Φ with P.O.T.P. but with Φ^ν not expansive for any $\nu \in \mathbb{Z}^2$.

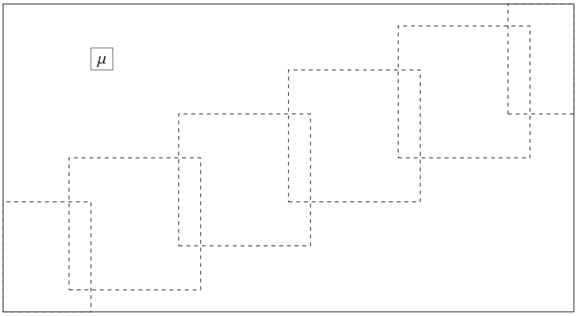

Fig. 1. Sketch of the set $\bigcup_{s\in\mathbb{Z}} ns + \overline{\Lambda}(k)$ from Example 5.2

Example 5.2. Consider the full two-dimensional shift X over the two-letter alphabet $\mathcal{A} = \{0, 1\}$. Figure 1 shows that for any $n \in \mathbb{Z}^2$ the mapping σ^n is not expansive. Given $b > 0$, fix k large enough that $2^{-k} < b$. If we choose $\mu \notin \bigcup_{s\in\mathbb{Z}} ns + \overline{\Lambda}(k)$ and $x, y \in X$ such that $x(i,j) = y(i,j)$ for all $(i,j) \neq \mu$ and $x(\mu) \neq y(\mu)$ then $\varrho(f^l(x), f^l(y)) < b$ for all $l \in \mathbb{Z}$ where $f = \sigma^n$.

In the following example we construct a $\mathbb{Z}^2$-action T which has P.O.T.P. but T^ν does not have P.O.T.P. for any nonzero $\nu \in \mathbb{Z}^2$ (and T^0 is not expansive).

Example 5.3. We will construct a two-dimensional shift X of finite type (Wang tiling) as follows. The alphabet $\mathcal{A}$ of X consists of 1×1 closed squares (tiles) with colored edges as in Fig. 2.

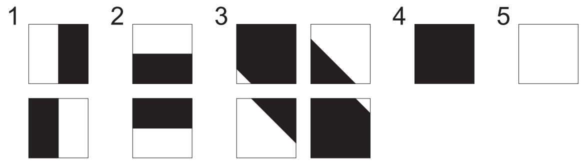

Fig. 2

Elements of $\mathcal{A}$ are divided into five groups. Two tiles are only allowed to touch along edges of the same color, so tiles from groups $1, 2$ and 3 may not appear together at any point of X. Then we obtain three types of points in X as presented in Figure 3. Observe that we may construct points with black regions (strip-like patterns) as wide as we want. Thus for any nonzero $\nu \in \mathbb{Z}^2$ and any $\delta > 0$ we can construct a δ pseudo-orbit $\xi = \{x^{(n)}\}_{n \in \mathbb{N}}$ for the mapping σ^ν with the property that for some $k, l \in \mathbb{Z}$ the points $x^{(k)}$ and $x^{(l)}$ are of different type. We may also choose ξ so that any x which $\frac{1}{2}$-traces it must contain symbols from two different groups $1, 2$ or 3. This implies that $x \notin X$ and so σ^ν does not have P.O.T.P.

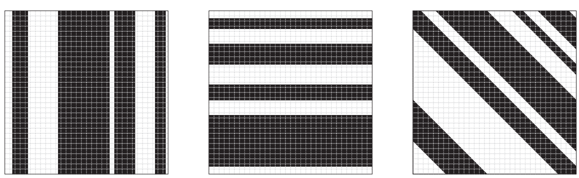

Fig. 3. Three types of points in X

Remark 5.4. It is clear that in the case of shift spaces, σ^0 always has P.O.T.P. It would be nice to construct a $\mathbb{Z}^d$-action T with P.O.T.P. such that T^ν does not have P.O.T.P. for any $\nu \in \mathbb{Z}^d$.

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