SHADOWING IN MULTI-DIMENSIONAL SHIFT SPACES

BY

PIOTR OPROCHA (Kraków)

Abstract. We show that the class of expansive $\mathbb{Z}^d$ actions with P.O.T.P. is wider than the class of actions topologically hyperbolic in some direction $\nu \in \mathbb{Z}^d$. Our main tool is an extension of a result by Walters to the multi-dimensional symbolic dynamics case.

1. Introduction. In this paper we consider multi-dimensional shift spaces. The books [1, 11] give an introduction to one-dimensional shift spaces theory. Multi-dimensional shift spaces arise in a natural way when we generalize the standard shift map $\sigma$ to a $\mathbb{Z}^d$-action $n \mapsto \sigma^n$. The vector $n \in \mathbb{Z}^d$ informs us how many cells we shift in each direction. The dynamics in higher dimensions is more complex than in the one-dimensional case (see, for example, [2, 9]). It turns out that there exist one-dimensional results which are not true in higher dimensions and also some higher dimensional properties have no analogue in dimension one [4].

Studying the pseudo-orbit tracing property (P.O.T.P.) of dynamical systems is an important part of stability theory (see [6, 7]). P.O.T.P. for group actions has recently been established by Pilyugin and Tikhomirov in [8]. In his fundamental paper [10] Peter Walters proved that a (one-dimensional) subshift has P.O.T.P. if and only if it is a shift of finite type. In this paper we prove an analogous result for multi-dimensional shift spaces. We also show a stronger property: every shift of finite type has Lipschitz P.O.T.P. and for $\varepsilon < 1$ any pseudo-orbit may be $\varepsilon$-traced by exactly one point. This result is used to study connections between P.O.T.P. of a $\mathbb{Z}^d$-action $\Phi$ and P.O.T.P. of the homeomorphisms $\Phi^\nu$ where $\nu \in \mathbb{Z}^d$.

2. Preliminaries. Let $A$ be a finite set, $d \in \mathbb{N}$, and let $A^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \to A$. For any $(j_1, \ldots, j_d) = j \in \mathbb{Z}^d$ we define $\|j\| = \max\{|j_i| : i = 1, \ldots, d\}$. The usual prefix metric on the one-dimensional full

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shift may be generalized to a metric $g$ on $A^\mathbb{Z}^d$ given by $g(x, y) = 2^{-j}$ where 
$$j = \sup\{k \in \mathbb{N} : x_n = y_n, n \in \mathbb{Z}^d, \|n\| < k\}.$$ 
For each $n \in \mathbb{Z}^d$ we define a homeomorphism $\sigma^n : A^\mathbb{Z}^d \to A^\mathbb{Z}^d$ putting 
$(\sigma^n(x))_m = x_{m+n}$ for all $x \in A^\mathbb{Z}^d$ and $m \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-action $n \mapsto \sigma^n$ is called the shift action on $\mathbb{Z}^d$. The $d$-dimensional full shift is the space $A^\mathbb{Z}^d$ 
with metric $g$ and the shift action. Any closed subset $X$ of $A^\mathbb{Z}^d$ invariant under $\sigma$ (i.e. $\sigma^n(X) = X$ for all $n \in \mathbb{Z}^d$) is called a $d$-dimensional shift space 
(or simply a shift space). If $X, Y$ are shift spaces and $X \subset Y$ then we say 
that $X$ is a subshift of $Y$.

Given two $d$-dimensional shift spaces $X, Y$ we may always assume that both are subshifts of some $d$-dimensional full shift. Namely, if $X \subset (A_X)^\mathbb{Z}^d$ 
and $Y \subset (A_Y)^\mathbb{Z}^d$ we may set $A = A_X \cup A_Y$ and then $X, Y \subset A^\mathbb{Z}^d$. Due to 
this observation, when we consider a finite number of shift spaces, we may 
always assume that they have the same alphabets.

For $x \in A^\mathbb{Z}^d$ and $F \subset \mathbb{Z}^d$ let $x_F$ denote the restriction of $x$ to $F$. If 
$F = \{a\}$ for some $a \in \mathbb{Z}^d$ we simply write $x_a$. A shape is a finite subset 
of $\mathbb{Z}^d$. A pattern on the shape $F$ is a function $f : F \to A$.

A pattern $f : F \to A$ is said to be allowed for the shift $X$ if there exists 
$x \in X$ such that $x_F = f$. If $f : F \to A$ is a pattern then we write $[f] = 
\{x : x_F = f\}$. This generalizes the notion of one-dimensional cylinder set to 
d dimensions.

The $k$-cube with lowest corner at the origin is the set 
$$A(k) = \{0, \ldots, k-1\}^d.$$ 
For $n \in \mathbb{Z}^d$ the set $n + A(k) = \{n + m : m \in A(k)\}$ is called the $k$-cube 
with the lowest corner at $n$. The $k$-cube centered at the origin is the set 
$$A(k) = \{-k+1, \ldots, k-1\}^d.$$ Observe that if $g(x, y) \leq 2^{-k}$ then $x_{\Lambda(k)} = y_{\Lambda(k)}$.

By a $k$-block we mean a pattern $f : A(k) \to A$. A pattern $f$ is called a 
block if it is a $k$-block for some $k$. We write $B_k(A)$ for the set of all $k$-blocks 
and $B(A)$ for the set of all possible blocks (i.e. $B(A) = \bigcup_{k=1}^{\infty} B_k(A)$). For 
any shift space $X$ and $k \in \mathbb{N}$ we denote by $B_k(X)$ the set of all $k$-blocks 
allowed for $X$ and by $B(X)$ the set of all blocks allowed for $X$.

If $f \in B_k(X)$ and $x \in X$ then we say that $f$ occurs in $x$ with lowest 
corner at $n \in \mathbb{Z}^d$ whenever $f(m) = x(m+n)$ for all $m \in A(k)$. We then 
write $f = x_{n+\Lambda(k)}$. Given $l \geq k$, we say that $f \in B_k(X)$ occurs in $x_{b+\Lambda(l)}$ if 
there exists $a \in \mathbb{Z}^d$ such that $a + \Lambda(k) \subset b + \Lambda(l)$ and $f = x_{a+\Lambda(k)}$.

3. Shifts of finite type. Let $\mathcal{F}$ be a set of patterns. We denote by $X_{\mathcal{F}}$ 
the set of all points of $A^\mathbb{Z}^d$ which do not contain any pattern from $\mathcal{F}$, i.e. 
$$x \in X_{\mathcal{F}} \iff \forall (f : E \to A) \in \mathcal{F} \forall n \in \mathbb{Z}^d \ x_{n+E} \neq f.$$ 
Elements of $\mathcal{F}$ are called forbidden patterns.
Lemma 3.1. A set $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift space if and only if there exists a set $\mathcal{F}$ of patterns such that $X = X_{\mathcal{F}}$.

The proof is analogous to that in [1, Thm. 6.1.21] for the one-dimensional case, and therefore is omitted.

Corollary 3.2. A set $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift space if and only if there exists $\mathcal{F} \subset B(\mathcal{A})$ such that $X = X_{\mathcal{F}}$.

Proof. Any set $\mathcal{F} \subset B(X)$ is a set of patterns and so $X_{\mathcal{F}}$ is a shift space.

Conversely, let $\mathcal{F}$ be any fixed set of patterns such that $X = X_{\mathcal{F}}$. For any $f : E \to \mathcal{A}$ is a pattern then there exists $k \in \mathbb{N}$ such that $E \subset \Lambda(k)$, and so $w + E \subset \Lambda(2k + 1)$ where $w = (k, \ldots, k) \in \mathbb{Z}^d$. We define $A_f \subset B(X)$ by

$$A_f = \{ g \in B_{2k+1}(\mathcal{A}) : g_{w+E} = f \}.$$ 

Set $\tilde{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} A_f$. Observe that $\tilde{\mathcal{F}} \subset B(\mathcal{A})$ and $X_{\mathcal{F}} = X_{\tilde{\mathcal{F}}}$.

Definition 3.3. Let $X$ be a shift space. We say that $X$ is a shift of finite type if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. A shift of finite type $X$ is $M$-step if $X = X_{\mathcal{F}}$ for some $\mathcal{F} \subset B_{M+1}(\mathcal{A})$.

Example 3.4 (Chessboard). Let $\mathcal{A}^{(n)} = \{0, 1, \ldots, n-1\}$ be an alphabet interpreted as a set of $n$ colors. We construct a shift $X^{(n)}$ of finite type such that adjacent cells of any point have different colors. Such a shift space may be obtained as $X^{(n)} = X_{\mathcal{F}^{(n)}}$ where the set of forbidden patterns $\mathcal{F}^{(n)}$ consists of:

$$\begin{array}{|c|c|} \hline a & a \\ \hline \end{array}$$

where $a$ is any color from $\mathcal{A}^{(n)}$. Observe that if we denote by $\mathcal{H}^{(n)}$ the set containing all possible patterns of the form

$$\begin{array}{|c|c|} \hline a & b \\ \hline a & c \\ \end{array}; \quad \begin{array}{|c|c|} \hline a & a \\ b & c \\ \end{array}$$

where $a, b, c \in \mathcal{A}^{(n)}$, then $X^{(n)} = X_{\mathcal{F}^{(n)}}$. However, $\mathcal{H}^{(n)} \subset B_2(\mathcal{A}^{(n)})$ and then $X^{(n)}$ is a 1-step shift of finite type.

In view of Corollary 3.2 every shift $X$ of finite type may be defined by a finite set $\mathcal{F} \subset B(\mathcal{A})$. By the same arguments there always exists a positive integer $M$ such that $X$ is an $M$-step shift of finite type.

Definition 3.5. Let $X$ be a subshift of $\mathcal{A}^{\mathbb{Z}^d}$. A map $\phi : X \to \mathcal{A}^{\mathbb{Z}^d}$ is called $k$-local if there exists $\Phi : B_{2k+1}(X) \to \mathcal{A}$ such that for every $x \in X$ and $n \in \mathbb{Z}^d$,

$$\phi(x)_n = \Phi((\sigma^n(x))_{\Lambda(k)}).$$

A map $\phi$ is called local if it is $k$-local for some $k \in \mathbb{N}$. 
This definition generalizes the definition of a sliding block code (one-dimensional case). In fact, the well known Curtis–Lyndon–Hedlund theorem (see [1, Thm. 6.2.9]) may be extended to the $d$-dimensional case and $k$-local maps. This implies that the $k$-local maps are exactly the functions which are continuous and shift commuting (i.e. $\sigma^n(\phi(x)) = \phi(\sigma^n(x))$ for any $x \in X$ and $n \in \mathbb{Z}$). Furthermore, if $X$, $Y$ are shift spaces and $\phi : X \to Y$ is a local map which is one-to-one and onto, then $\phi$ is a shift commuting homeomorphism (see [1, Thm. 1.5.14] for the one-dimensional case). This leads to the following definition:

**Definition 3.6.** Two shift spaces $X$, $Y$ are *conjugate* if there exists a bijective local map $\phi : X \to Y$. Every such $\phi$ is called a *conjugacy* between $X$ and $Y$.

In view of previous facts the above definition of conjugacy is equivalent to the definition of topological conjugacy of two $\mathbb{Z}^d$-actions.

Let $X$ be a shift space and let $N$ be a positive integer. Define a map $\beta_N : X \to B_N(X)^{\mathbb{Z}^d}$ by $(\beta_N(x))_n = x_{n+A(N)}$.

**Definition 3.7.** Let $X$ be a shift space. Then the $N$th higher block shift $X^{[N]}$ is the image $X^{[N]} = \beta_N(X)$.

Observe that $\beta_N$ is an $N$-local invertible mapping, so the shift spaces $X$ and $X^{[N]}$ are conjugate.

**Proposition 3.8.** Let $X$ be an $M$-step shift of finite type. Then it is conjugate to a 1-step shift of finite type.

**Proof.** By previous remarks, $X$ and $X^{[M]}$ are conjugate. Because any $M+1$ block in $X$ may be regarded as a 2-block in $X^{[M]}$, this space is a 1-step shift of finite type. This is an immediate generalization of the one-dimensional case [1, Prop. 2.3.9].

4. **Shift spaces and shadowing.** Fix a positive number $\delta$. We say that a set $\xi = \{x^{(n)} \in A^{\mathbb{Z}^d} : n \in \mathbb{Z}^d\}$ is a $\delta$ *pseudo-orbit* if

$$\rho(x^{(n \pm e_i)}, \sigma^{\pm e_i}(x^{(n)})) < \delta$$

for any $n \in \mathbb{Z}^d$ and $i = 1, \ldots, d$, where $e_i \in \mathbb{Z}^d$ is the $i$th standard basis vector.

**Definition 4.1.** Let $X$ be a shift space. A $\delta$ *pseudo-orbit* $\xi = \{x^{(n)} \in A^{\mathbb{Z}^d} : n \in \mathbb{Z}^d\}$ is *$\varepsilon$-traced* by $x \in X$ if $\rho(x_n, \sigma^n(x)) < \varepsilon$ for any $n \in \mathbb{Z}^d$.

The definition below is a particular case of the general definition of P.O.T.P. (see [8]). Similarly to the one-dimensional case it is easy to see that P.O.T.P. is a topological conjugacy invariant.
Definition 4.2. A shift space \( X \) has the \textit{pseudo-orbit tracing property} (P.O.T.P., shadowing) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that each \( \delta \) pseudo-orbit \( \xi \subset X \) is \( \varepsilon \)-traced by some point \( y \in X \).

Definition 4.3. A shift space \( X \) has the \textit{Lipschitz pseudo-orbit tracing property} (Lipschitz P.O.T.P., Lipschitz shadowing) if there exists a constant \( L > 0 \) such that for any \( \delta \) pseudo-orbit \( \xi = \{ x^{(n)} \in A^{Z^d} : n \in Z^d \} \) there is a point \( x \in X \) satisfying
\[
\varrho(x^{(n)}, \sigma^n(x)) < L \delta, \quad n \in Z^d.
\]

The following definition generalizes the well known concept of expansiveness.

Definition 4.4. We say that a shift space \( X \) is \textit{expansive} if there exists a constant \( b > 0 \) (expansive constant) such that whenever for any \( x, y \in X \),
\[
\varrho(\sigma^n(x), \sigma^n(y)) < b \quad \text{for all } n \in Z^d,
\]
then \( x = y \).

The main tool we will use is the following:

Theorem 4.5. Let \( X \) be a shift space. Then the following conditions are equivalent:

1. \( X \) is a shift of finite type.
2. \( X \) has the pseudo-orbit tracing property.
3. \( X \) has the Lipschitz pseudo-orbit tracing property.

In particular, if \( X \) is an \( M \)-step shift of finite type then it has the Lipschitz pseudo-orbit tracing property with constant \( L = 2^{M+1} \).

Proof. The implication (3) \( \Rightarrow \) (2) is always true. We will show that (1) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (1) hold.

(1) \( \Rightarrow \) (3). Suppose that \( X \) is a shift of finite type. We may assume that \( X \) is an \( M \)-step shift, that is, there exists \( \mathcal{F} \subset B_{M+1}(A) \) such that \( X = X_{\mathcal{F}} \). This means that \( x \in X \) if and only if \( x_{n+\Lambda(M+1)} \notin \mathcal{F} \) for any \( n \in Z^d \). First, let us make an observation which is crucial for this part of the proof.

Let \( m > M \), \( \delta = 2^{-m} \) and let \( \xi = \{ x^{(n)} \in X : n \in Z^d \} \) be a \( \delta \) pseudo-orbit. By definition, \( \varrho(x^{(n)\pm e_i}, x^{(n)}) < \delta \) for any \( n \in Z^d \). This implies that
\[
(4.1) \quad x^{(n)\pm e_i}_{\Lambda(m)} = (x^{(n)})_{\Lambda(m)}.
\]
Let \( y \in A^{Z^d} \) with \( y(n) = x^{(n)}(0) \) for any \( n \in Z^d \). We will show that \( y \in X \).

Fix any \( a = (a_1, \ldots, a_d) \in \Lambda(m) \). Applying (4.1) we find that \( x^{(n)\pm e_i}(j) = x^{(n)}(j \pm e_i) \) for all \( j \in \Lambda(m), n \in Z^d \). We will use (4.1) recursively. For simplicity, we assume that \( a_i \geq 0 \). When \( a_i < 0 \) it is enough to replace \(-1
by 1 in the following equalities (i.e. increase values at the \(i\)th coordinate instead of decreasing them):

\[
(4.2) \quad x^{(n+a)}(0) = x^{(n+(a_1,\ldots,a_d))}(0) = (\sigma^{e_1}(x^{(n+(a_1,\ldots,a_d)-e_1)}))(0) = x^{(n+(a_1-1,a_2,\ldots,a_d))}(e_1) = \ldots = x^{(n+(a_2,\ldots,a_d))}(e_1) = \ldots = x^{(n+(0,a_3,\ldots,a_d))}(a_1 e_1 + a_2 e_2) = \ldots = x^{(n+(\ldots,0))}(a_1 e_1 + \ldots + a_d e_d) = x^{(n)}(a).
\]

We have just shown that \(y(n+a) = x^{(n+a)}(0) = x^{(n)}(a)\) for any \(a \in \Lambda(m)\). Observe that \(\Lambda(M+1) \subset \Lambda(m)\), so \(y_{n+\Lambda(M+1)} = x^{(n)}(a)\) for any \(n \in \mathbb{Z}^d\) and hence \(y \in X\).

The point \(y\) defined above is a good candidate to trace the pseudo-orbit \(\xi\) and, as we will see, it really does. The set \(\xi\) is a \(2^{-m}\) pseudo-orbit, so by (4.2) we obtain \(x^{(n+a)}(0) = x^{(n)}(a)\) for all \(a \in \Lambda(m)\). This implies that \(\varrho(\sigma^n(y), x^{(n)}) < 2^{-m}\) and so \(\xi\) is \(\delta\) traced by \(y\).

Let \(L = 2^{M+1}\). Take any \(\delta > 0\). If \(\delta > 2^{-M}\) then \(L\delta > 1\) and there is nothing to prove. Suppose that \(K \geq M\) is an integer such that \(2^{-(K+1)} < \delta \leq 2^{-K}\). Observe that any \(\delta\) pseudo-orbit is also a \(2^{-K}\) pseudo-orbit, thus by previous observations it is \(2^{-K}\)-traced. Additionally, \(2^{-K} \leq 2^{-(K+1)} L \leq L\delta\), which finishes the proof of (1) \(\Rightarrow\) (3).

(2) \(\Rightarrow\) (1). Suppose that \(X\) has P.O.T.P., fix \(\varepsilon = 1/2\) and take \(\delta > 0\) such that every \(\delta\) pseudo-orbit is \(\varepsilon\)-traced. Choose \(N\) large enough to have \(2^{-N} < \delta\).

We will show that \(X\) is an \(M\)-step shift of finite type where \(M = 2N+1\). Let \(\mathcal{F} = B_{M+1}(A) \setminus B_{M+1}(X)\). Obviously \(X \subset X_\mathcal{F}\). We have to show that \(X_\mathcal{F} \subset X\).

Fix \(y \in X_\mathcal{F}\). By the definition of \(X_\mathcal{F}\) for every \(n \in \mathbb{Z}^d\) we have \(y_{n+\Lambda(N+1)} \in B_{M+1}(X)\), thus for every \(n \in \mathbb{Z}^d\) there exists \(x^{(n)} \in X\) such that \(x^{(n)}(\Lambda(N+1)) = y_{n+\Lambda(N+1)}\). Set \(\xi = \{x^{(n)} \in X : n \in \mathbb{Z}^d\}\). Obviously \(\Lambda(N) \pm e_i \subset \Lambda(N+1), \) so

\[
(\sigma^{\pm e_i}(x^{(n)}))_{\Lambda(N)} = x^{(n)}_{\Lambda(N) \pm e_i} = y_{n+\Lambda(N) \pm e_i} = y_{(n \pm e_i) + \Lambda(N)} = x^{(n \pm e_i)}_{\Lambda(N)}.
\]

This implies that \(\varrho(\sigma^{\pm e_i}(x^{(n)}), x^{(n \pm e_i)}) \leq 2^{-N} < \delta\) and so \(\xi\) is a \(\delta\) pseudo-orbit. Thus there exists \(x \in X\) such that \(\xi\) is \(\varepsilon\)-traced by \(x\). Observe that \(\varrho(\sigma^n(x), x^{(n)}) < 1/2\), which implies that \(x_{n+\Lambda(1)} = x_{\Lambda(1)}^{(n)} = y_{n+\Lambda(1)}\). We have just shown that \(x(n) = y(n)\) for any \(n \in \mathbb{Z}^d\), so \(y = x\) and hence \(y \in X\).
Theorem 4.6. Let $X$ be a shift space. If $0 < \varepsilon < 1$ then for any $\delta$ pseudo-orbit $\xi \subset X$ there exists at most one point $x \in X$ which $\varepsilon$-traces $\xi$.

Proof. Fix any $0 < \varepsilon < 1$ and let $\xi = \{x^{(n)} : n \in \mathbb{Z}^d\} \subset X$ be any fixed $\delta$ pseudo-orbit. Suppose that $\xi$ is $\varepsilon$-traced by some point $x$. For any $n \in \mathbb{Z}^d$ we have $g(x^{(n)}, \sigma^n(x)) = 2^{-j} < \varepsilon < 1$. Observe that $g(x^{(n)}, \sigma^n(x)) \leq 1/2$ and so $x^{(n)}_0 = (\sigma^n(x))_0 = x_n$ for any $n \in \mathbb{Z}^d$. This implies that there is at most one such $x$.

The orbit of any point $y \in X$ is a $\delta$ pseudo-orbit for any $\delta > 0$. This implies the following:

Corollary 4.7. Let $X$ be a shift space. Then $X$ is expansive with expansive constant $b = 1$.

We may also use Theorem 4.6 to define $\delta_0$ such that any $\delta$ pseudo-orbit is traced by exactly one point provided that $\delta < \delta_0$. Strictly speaking, we have the following:

Corollary 4.8. Let $X$ be an $M$-step shift of finite type, let $0 < \varepsilon < 1$ and let $\delta_0 = \varepsilon 2^{-(M+1)}$. If $\delta < \delta_0$ then every $\delta$ pseudo-orbit $\xi \subset X$ is $\varepsilon$-traced by exactly one point $y_\xi \in X$.

Proof. Let $\xi \subset X$ be a $\delta$ pseudo-orbit, where $\delta < \delta_0$. By Theorem 4.5 the pseudo-orbit $\xi$ is $L\delta$-traced by some point $y \in X$, where $L = 2^{M+1}$. Observe that $L\delta < \varepsilon < 1$, so by Theorem 4.6 there is exactly one such $y$.

5. Topologically Anosov homeomorphisms and shadowing. We recall that a homeomorphism $h$ is topologically Anosov (or equivalently topologically hyperbolic [3, 5]) if it is expansive and has P.O.T.P. The authors of [8] proved that if for a given $\mathbb{Z}^d$-action $\Phi$ there exists $\nu \in \mathbb{Z}^d$ such that the homeomorphism $f = \Phi^\nu$ is topologically Anosov then $\Phi$ has P.O.T.P. We will show that the assumptions about $f$ cannot be weakened (it is not enough to assume that $f$ has P.O.T.P. or $f$ is expansive alone). We will also show that there exist $\mathbb{Z}^d$-actions with P.O.T.P. which are not topologically Anosov for any $\nu \in \mathbb{Z}^d$, so [8, Thm. 1] is only a sufficient condition.

Example 5.1. Consider a one-dimensional shift space $X$ which is not of finite type (e.g. $X$ may be an “even shift” because it belongs to the class of strictly sofic shift spaces [1, Ex. 2.1.9]). Let $\mathcal{F}$ be the set of forbidden words for $X$, i.e. $X = X_\mathcal{F}$. We define a set $\mathcal{F}'$ of two-dimensional patterns as follows:

$$\mathcal{F}' = \{\begin{array}{cccc} u_1 & u_2 & \cdots & u_{|u|} \end{array} : u \in \mathcal{F}\} \cup \left\{ \begin{array}{cc} a & \\ b \end{array} : a, b \in \mathcal{A}, a \neq b \right\}.$$
Observe that the two-dimensional shift space $Y = X_{F'}$ contains points which consist of infinitely many copies of elements of $X$ and any point of $Y$ is determined by symbols on the $\mathbb{Z} \times \{0\}$ line. Strictly speaking, $y \in Y$ if:

1. $y(i, j) = y(i, j + m)$ for all $(i, j) \in \mathbb{Z}^2$ and $m \in \mathbb{Z}$.
2. $y(\cdot, j) \in X$.

The map $\sigma^{(1,0)}$ is expansive with expansive constant $b = 1/2$; however, $Y$ does not have P.O.T.P. because it is not a shift of finite type.

Next, observe that if $\xi = \{x^{(n)} : n \in \mathbb{N}\}$ is a $2^{-k}$ pseudo-orbit for $\sigma^{(0,1)}$ then $x^{(n)}_{A(k)} = x^{(0)}_{A(k)}$ for all $n \in \mathbb{N}$ (every point of $Y$ consists of vertical lines of the same symbol). This implies that $\xi$ is $2^{-k}$-traced by $x_0$. Thus the map $\sigma^{(0,1)}$ has P.O.T.P. but $(Y, \sigma)$ does not.

Example 5.1 shows that even if we know that for some $\nu \in \mathbb{Z}^d$ the mapping $\Phi^{\nu}$ for a $\mathbb{Z}^d$-action $\Phi$ is expansive or has P.O.T.P. we may say nothing about P.O.T.P. of $\Phi$ unless we can find a $\nu$ such that $\Phi^{\nu}$ has both properties at the same time (is topologically Anosov).

Next, we will show that there exist $\mathbb{Z}^d$-actions with P.O.T.P. which are not topologically Anosov for any $\nu \in \mathbb{Z}^d$. In the following example we present a $\mathbb{Z}^2$-action $\Phi$ with P.O.T.P. but with $\Phi^{\nu}$ not expansive for any $\nu \in \mathbb{Z}^2$.

**Example 5.2.** Consider the full two-dimensional shift $X$ over the two-letter alphabet $A = \{0, 1\}$. Figure 1 shows that for any $n \in \mathbb{Z}^2$ the mapping $\sigma^n$ is not expansive. Given $b > 0$, fix $k$ large enough that $2^{-k} < b$. If we choose $\mu \notin \bigcup_{s \in \mathbb{N}} ns + A(k)$ and $x, y \in X$ such that $x(i, j) = y(i, j)$ for all $(i, j) \neq \mu$ and $x(\mu) \neq y(\mu)$ then $g(f^l(x), f^l(y)) < b$ for all $l \in \mathbb{Z}$ where $f = \sigma^n$.

In the following example we construct a $\mathbb{Z}^2$-action $T$ which has P.O.T.P. but $T^{\nu}$ does not have P.O.T.P. for any nonzero $\nu \in \mathbb{Z}^2$ (and $T^0$ is not expansive).
Example 5.3. We will construct a two-dimensional shift $X$ of finite type (Wang tiling) as follows. The alphabet $\mathcal{A}$ of $X$ consists of $1 \times 1$ closed squares (tiles) with colored edges as in Fig. 2.

Elements of $\mathcal{A}$ are divided into five groups. Two tiles are only allowed to touch along edges of the same color, so tiles from groups 1, 2 and 3 may not appear together at any point of $X$. Then we obtain three types of points in $X$ as presented in Figure 3. Observe that we may construct points with black regions (strip-like patterns) as wide as we want. Thus for any nonzero $\nu \in \mathbb{Z}^2$ and any $\delta > 0$ we can construct a $\delta$ pseudo-orbit $\xi = \{x^{(n)}\}_{n \in \mathbb{N}}$ for the mapping $\sigma^\nu$ with the property that for some $k, l \in \mathbb{Z}$ the points $x^{(k)}$ and $x^{(l)}$ are of different type. We may also choose $\xi$ so that any $x$ which $\frac{1}{2}$-traces it must contain symbols from two different groups 1, 2 or 3. This implies that $x \notin X$ and so $\sigma^\nu$ does not have P.O.T.P.

Remark 5.4. It is clear that in the case of shift spaces, $\sigma^0$ always has P.O.T.P. It would be nice to construct a $\mathbb{Z}^d$-action $T$ with P.O.T.P. such that $T^\nu$ does not have P.O.T.P. for any $\nu \in \mathbb{Z}^d$.

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Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30
30-059 Kraków, Poland
E-mail: oprocha@agh.edu.pl

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