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PERIODS OF MORSE–SMALE DIFFEOMORPHISMS OF $\mathbb{S}^2$

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Abstract. The aim of this paper is to describe the set of periods of a Morse–Smale diffeomorphism of the two-dimensional sphere according to its homotopy class. The main tool for proving this is the Lefschetz fixed point theory.

1. Introduction and statement of the main results. An important class of dynamical systems on smooth compact manifolds consists of the Morse–Smale diffeomorphisms. These have a relatively simple orbit structure and this structure is preserved under small C^1 perturbations.

During the last quarter of the XXth century several papers were published analyzing the relationships between the dynamics of Morse–Smale diffeomorphisms and the topology of the manifold where they are defined; see for instance [5, 7, 11, 13].

Each Morse–Smale diffeomorphism has a finite set of periodic orbits. Franks [5] linked the periodic behavior of a Morse–Smale diffeomorphism to its action on homology. For a given manifold and a homotopy (or isotopy) class of maps on that manifold, this result provides a necessary condition for the Morse–Smale dynamics to occur in that homotopy (or isotopy) class. Narasimhan [11] showed for a compact surface that a diffeomorphism homotopic to the identity can exhibit a given Morse–Smale dynamics provided it satisfies Franks's condition [5] and two other necessary properties. Essentially, the homotopy class of the identity on a compact surface can admit any periodic behavior consistent with the Lefschetz zeta function. In Section 2 we provide precise definitions of all these notions.

We focus on Morse–Smale diffeomorphisms of the two-dimensional sphere $\mathbb{S}^2$. For orientation-reversing diffeomorphisms there are additional obstructions to the ones given by the Lefschetz zeta function; they were obtained by Batterson, Handel and Narasimhan [2]. Blanchard and Franks [3] have shown that if an orientation-reversing homeomorphism of $\mathbb{S}^2$ has periodic orbits with two distinct odd periods, then the topological entropy

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of this homeomorphism is positive. This implies that orientation-reversing Morse–Smale diffeomorphisms of $\mathbb{S}^2$ cannot have more than one odd period, because they have zero topological entropy.

Our aim in this work is to characterize the set $\operatorname{Per}(f)$ of periods for an orientation-preserving and for an orientation-reversing Morse–Smale diffeomorphism of $\mathbb{S}^2$. The three key tools for doing that are the results obtained by Franks [6] on the Lefschetz zeta function for C^1 maps having only hyperbolic periodic points, by Narasimhan [11] for orientation-preserving Morse–Smale diffeomorphisms of $\mathbb{S}^2$, and by Batterson, Handel and Narasimhan [2] for orientation-reversing ones.

Of our main results are stated in the following two theorems:

THEOREM 1. *Let f be an orientation-preserving Morse–Smale diffeomorphism of $\mathbb{S}^2$. Then* $\operatorname{Per}(f)$ *is a finite set containing* 1. *Conversely*, *any finite set of positive integers containing* 1 *is realizable as the set of periods for some orientation-preserving Morse–Smale diffeomorphism on* $\mathbb{S}^2$.

THEOREM 2. *Let f be an orientation-reversing Morse–Smale diffeomorphism of $\mathbb{S}^2$. Then* $\operatorname{Per}(f)$ *is either* $\{1\}$, *or* $S \cup \{2\}$, *where S is a finite set of positive integers with at most one element odd. Conversely*, $\{1\}$ *and any set of the form* $S \cup \{2\}$ *with S as above are each realizable as the set of periods of some orientation-reversing Morse–Smale diffeomorphism of* $\mathbb{S}^2$.

The paper is structured as follows. In Section 2 we provide all the definitions and basic tools and results necessary for this work. In Section 3 we prove our two theorems.

2. Preliminary definitions and basic results. We begin by recalling several definitions. Let $\operatorname{Diff}(\mathbb{M})$ be the space of C^1 diffeomorphisms of a compact manifold $\mathbb{M}$. The set $\operatorname{Diff}(\mathbb{M})$ is a topological space endowed with the topology of the supremum norm for the map and its differential. All the diffeomorphisms in this paper will be C^1 diffeomorphisms.

We denote by f^m the mth iterate of $f \in \operatorname{Diff}(\mathbb{M})$. A point $x \in \mathbb{M}$ is a *nonwandering point* of f provided that for any neighborhood $\mathcal{U}$ of x there exists a nonzero integer m such that $f^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The set of nonwandering points of f is denoted by $\Omega(f)$.

Suppose that $x \in \mathbb{M}$. If $f(x) = x$ and the derivative of f at x, denoted by Df_x, has spectrum disjoint from the unit circle, then x is called a *hyperbolic fixed point*. If all the eigenvalues of Df_x lie inside the unit circle, then x is called a *sink*. When all the eigenvalues have modulus greater than one, x is called a *source*. Otherwise x is called a *saddle*.

Suppose that $y \in \mathbb{M}$. If $f^p(y) = y$, then y is a *periodic point* of f of period p if moreover $f^j(y) \neq y$ for all $0 \leq j < p$. This y is a *hyperbolic periodic point* if y is a hyperbolic fixed point of f^p. The set $\{y, f(y), \ldots, f^{p-1}(y)\}$ is called the *periodic orbit* of the periodic point y. By $\mathrm{Per}(f)$ we denote the set of periods of all periodic points of f.

Assume ϱ is a metric on $\mathbb{M}$, and x is a hyperbolic fixed point of f. The *stable manifold* of x is

$$W^{\mathrm{s}}(x) = \{y \in \mathbb{M} : \varrho(x, f^m(y)) \to 0 \text{ as } m \to \infty\},$$

and the *unstable manifold* of p is

$$W^{\mathrm{u}}(x) = \{y \in \mathbb{M} : \varrho(x, f^{-m}(y)) \to 0 \text{ as } m \to \infty\}.$$

For a hyperbolic periodic point x of period p, the stable and unstable manifolds are defined to be the stable and unstable manifolds of x under f^p.

A diffeomorphism $f : \mathbb{M} \to \mathbb{M}$ is *Morse–Smale* if

(i) $\Omega(f)$ is finite,
(ii) all periodic points are hyperbolic,
(iii) for each $x, y \in \Omega(f)$ if the manifolds $W^{\mathrm{s}}(x)$ and $W^{\mathrm{u}}(y)$ intersect then they intersect transversally.

The first condition implies that $\Omega(f)$ is the set of all periodic points of f.

Two diffeomorphisms $f, g \in \mathrm{Diff}(\mathbb{M})$ are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{M} \to \mathbb{M}$ such that $h \circ f = g \circ h$. A diffeomorphism f is *structurally stable* provided that there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(\mathbb{M})$ such that each $g \in \mathcal{U}$ is topologically equivalent to f.

The class of Morse–Smale diffeomorphisms is structurally stable inside the class of all diffeomorphisms (see [12]), so to understand the dynamics of this class of maps is a relevant problem.

2.1. *Lefschetz zeta function.* We will study the set of periods of Morse–Smale diffeomorphisms of $\mathbb{S}^2$ using the Lefschetz fixed point theory. The key 1920 work of Lefschetz was to relate the homology class of a given map with the earlier work of Brouwer on indices of self-maps of compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, one obtains information about the existence of fixed points.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ on a compact n-dimensional manifold, its *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{trace}(f_{*k}),$$

where $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ is the homomorphism induced by f on the kth rational homology group of $\mathbb{M}$. For $k = 0, \dots, n$, $H_k(\mathbb{M}, \mathbb{Q})$ is a finite-dimensional vector space over $\mathbb{Q}$, and f_{*k} is a linear map given by a matrix with integer entries. Thus, one of the most useful results for proving the existence of fixed points, or more generally of periodic points for a continuous self-map f of a compact manifold, is the Lefschetz fixed point theorem and its improvements (see for instance ([1, 4, 7–10]):

THEOREM 3. *Let $f : \mathbb{M} \to \mathbb{M}$ be a continuous map on a compact manifold, and let $L(f)$ be its Lefschetz number. If $L(f) \neq 0$ then f has a fixed point.*

For a proof of Theorem 3 see [4].

Our aim is to describe the set of periods of f. To this end, it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all the iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\biggl(\sum_{m=1}^{\infty} \frac{L(f^m)}{m}\, t^m\biggr).$$

This function generates the sequence of all Lefschetz numbers, and it may be computed independently through [6]

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim \mathbb{M}$, $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$, I_{n_k} is the $n_k \times n_k$ identity matrix, and we set $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$.

2.2. *Results of Franks and Batterson et al.* If γ is a hyperbolic periodic orbit of period p, then for each $x \in \gamma$ let E_x^{u} denote the subspace of the tangent space $T_x\mathbb{M}$ generated by the eigenvectors of Df_x^p corresponding to eigenvalues whose moduli are greater than one. Let E_x^{s} be the subspace of $T_x\mathbb{M}$ generated by the remaining eigenvectors. Define the *orientation type* Δ of γ to be $+1$ if $Df_x^p : E_x^{\mathrm{u}} \to E_x^{\mathrm{u}}$ preserves orientation, and -1 if it reverses orientation. Note that Δ is well defined. The *index* u of γ is the dimension of E_x^{u} for some $x \in \gamma$. Finally, we define the triple (p, u, Δ) associated to γ.

For a diffeomorphism having all its periodic orbits hyperbolic, its *periodic data* Σ is defined to be the collection of all triples (p, u, Δ), where the same triple can occur more than once provided it corresponds to different periodic orbits. Franks [5] proved the following result.

THEOREM 4. *Let f be a C^1 map on a compact manifold having finitely many periodic orbits all of which are hyperbolic, and let Σ be the period data of f. Then the Lefschetz zeta function of f satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}.$$

Using this notion of periodic data Narasimhan [11] studied the Morse–Smale diffeomorphisms homotopic to the identity on compact connected surfaces. Her result can be stated as follows.

THEOREM 5. *Let $\mathbb{M}$ be a compact, connected 2-manifold. There exists a Morse–Smale diffeomorphism f of $\mathbb{M}$ homotopic to the identity with periodic data $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$ if and only if*

(a) *$u_i = 0$ and $u_j = 2$ for some i and j,*
(b) *$\mathcal{Z}_f(t) = (1-t)^{-\chi(\mathbb{M})}$, where $\chi(\mathbb{M})$ is the Euler characteristic of $\mathbb{M}$.*

In particular, taking into account that all orientation-preserving Morse–Smale diffeomorphisms on $\mathbb{S}^2$ are homotopic the identity and that $\chi(\mathbb{S}^2) = 2$, we have the following corollary.

COROLLARY 6. *There exists an orientation-preserving Morse–Smale diffeomorphism f on $\mathbb{S}^2$ with periodic data $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$ if and only if*

(a) *$u_i = 0$ and $u_j = 2$ for some i and j,*
(b) *$\mathcal{Z}_f(t) = (1-t)^{-2}$.*

For the existence of orientation-reversing Morse–Smale diffeomorphisms there are some extra requirements, as can be seen from the following result of Batterson, Handel and Narasimhan [2].

THEOREM 7. *There exists an orientation-reversing Morse–Smale diffeomorphism f on $\mathbb{S}^2$ with periodic data $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$ if and only if*

(a) *$u_i = 0$ and $u_j = 2$ for some i and j,*
(b) *$\Delta_i = +1$ if $u_i = 0$ and $\Delta_i = (-1)^{p_i}$ if $u_i = 2$,*
(c) *$\mathcal{Z}_f(t) = (1-t^2)^{-1} = \prod_{i=1}^n (1-\Delta_i t^{p_i})^{(-1)^{u_i+1}}$,*
(d) *if p_i and p_j are odd, then $p_i = p_j$,*
(e) *if $p_i > 2$ for some i, then the data contains one of the following triples: $(2,0,+1)$, $(2,2,+1)$ or $(2,1,-1)$.*

3. Proofs of Theorems 1 and 2. It is well known that the homology groups of the sphere with rational coefficients are $H_0(\mathbb{S}^2,\mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{S}^2,\mathbb{Q}) = 0$ and $H_2(\mathbb{S}^2,\mathbb{Q}) = \mathbb{Q}$. Let f be a continuous self-map of $\mathbb{S}^2$. Then the induced homomorphisms on homology are $f_{*0} = 1$, $f_{*1} = 0$ and $f_{*2} = d$ where d is the degree of the map f. From (1) we have

$$\mathcal{Z}_f(t) = [\det(I_1 - tf_{*0})\det(I_1 - tf_{*2})]^{-1} = [(1-t)(1-dt)]^{-1}.$$

Clearly $d = 1$ for orientation-preserving diffeomorphisms, and $d = -1$ for reversing ones. Thus,

$$\mathcal{Z}_f(t) = \begin{cases} (1-t)^{-2} & \text{if } f \text{ is orientation-preserving,} \\ (1-t^2)^{-1} & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof of Theorem 1. Let f be an orientation-preserving Morse–Smale diffeomorphism of $\mathbb{S}^2$. Clearly, since a Morse–Smale diffeomorphism has finitely many periodic orbits, $\mathrm{Per}(f)$ is a finite set of positive integers. Moreover, f always has fixed points. Indeed, by the previous paragraph, since f preserves orientation we know that $\mathcal{Z}_f(t) = (1-t)^{-2}$. Therefore, Theorem 4 implies that $1 \in \operatorname{Per}(f)$. So the first part of Theorem 1 is proved.

Now, we prove the converse, i.e. for a given finite set S of positive integers containing 1 we shall find an orientation-preserving Morse–Smale diffeomorphism f such that $\operatorname{Per}(f) = S$.

By Corollary 6 there exists an orientation-preserving Morse–Smale diffeomorphism f of $\mathbb{S}^2$ having a sink and a source corresponding to the triples $(1,0,+1)$ and $(1,2,+1)$ respectively, and $(p,1,-1)$ and $(p,0,-1)$ for every $p \in S$, so its Lefschetz zeta function is of the form

$$\mathcal{Z}_f(t) = \prod_{p\in S} \frac{1+t^p}{(1-t)(1-t)(1+t^p)} = \frac{1}{(1-t)^2}.$$

Clearly this orientation-preserving Morse–Smale diffeomorphism f of $\mathbb{S}^2$ satisfies $\operatorname{Per}(f) = S$. Thus the proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Let f be an orientation–reversing Morse–Smale diffeomorphism of $\mathbb{S}^2$. Then f^2 is an orientation-preserving Morse–Smale diffeomorphism. Hence, by Theorem 1, f^2 has a fixed point and therefore $\operatorname{Per}(f)$ contains 1 or 2.

Now by Theorem 7(d), $\operatorname{Per}(f)$ has at most one element odd, and statement (e) of this same result says that if $\operatorname{Per}(f) \neq \{1\}$ then it contains 2.

Thus, $\operatorname{Per}(f)$ is either $\{1\}$, or the union of $\{2\}$ and a finite set of positive integers (maybe empty) with no more than one element odd. This completes the proof of the first part of Theorem 2.

Now we prove the converse. First we want to find an orientation-reversing Morse–Smale diffeomorphism f of $\mathbb{S}^2$ such that $\operatorname{Per}(f) = \{1\}$. Consider here $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^2 : x^2+y^2+z^2 = 1\}$. Let f_1 be an orientation-preserving diffeomorphism of $\mathbb{S}^2$ for which the point $(0,0,1)$ is a global hyperbolic attractor in $\mathbb{S}^2 \setminus \{(0,0,-1)\}$ and $(0,0,-1)$ is a hyperbolic repellor, and let f_2 be the symmetry on $\mathbb{S}^2$ with respect to the $x=0$ plane. Then $f = f_2 \circ f_1$ is an orientation-reversing Morse–Smale diffeomorphism of $\mathbb{S}^2$ which has only two periodic points, the fixed points $(0,0,\pm 1)$.

Now let S be any finite set of positive integers with at most one element odd. We want to find an orientation-reversing Morse–Smale diffeomorphism f of $\mathbb{S}^2$ such that $\operatorname{Per}(f) = S \cup \{2\}$. By Theorem 7 there exists an orientation-reversing Morse–Smale diffeomorphism f which has a sink and a source corresponding to the triples $(p_1, 0, +1)$ and $(p_2, 2, (-1)^{p_2})$ with $p_1 = p_2 = 2$ respectively, and to the triples $(2,1,1)$ and $(p,1,-1)$ and $(p,0,-1)$ for every $p \in S$, so its Lefschetz zeta function is of the form

$$\mathcal{Z}_f(t) = \prod_{p\in S} \frac{(1-t^2)(1+t^p)}{(1-t^2)(1-t^2)(1+t^p)} = \frac{1}{1-t^2}.$$

Clearly this orientation-reversing Morse–Smale diffeomorphism f of $\mathbb{S}^2$ satisfies that $\mathrm{Per}(f) = S \cup \{2\}$. So the proof of Theorem 2 is complete. ∎

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