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A MATRIX FORMALISM FOR CONJUGACIES OF HIGHER-DIMENSIONAL SHIFTS OF FINITE TYPE

BY

MICHAEL SCHRAUDNER (Santiago de Chile)

Abstract. We develop a natural matrix formalism for state splittings and amalgamations of higher-dimensional subshifts of finite type which extends the common notion of strong shift equivalence of $\mathbb{Z}^+$-matrices. Using the decomposition theorem every topological conjugacy between two $\mathbb{Z}^d$-shifts of finite type can thus be factorized into a finite chain of matrix transformations acting on the transition matrices of the two subshifts. Our results may be used algorithmically in computer explorations on topological conjugacies and in the search for new conjugacy invariants.

1. Preliminaries. In the classification theory of one-dimensional shifts of finite type (SFTs) the notion of strong shift equivalence of non-negative integer matrices as defined by Williams [15] is crucial: Two SFTs presented on directed graphs are topologically conjugate and thus exhibit identical dynamical properties if and only if the corresponding adjacency matrices are strong shift equivalent over $\mathbb{Z}^+$ (William's Classification Theorem 7.2.7 in [8]). In fact every strong shift equivalence is built up from elementary equivalences called splitting and amalgamation (= inverse splitting) codes. Those directly translate into matrix equations of the well known form (see Section 2.4 and the Decomposition-Theorem in Section 7.1 of [8]):

$$A \overset{\text{ESSE}}{\sim_{\mathbb{Z}^+}} B \;\Leftrightarrow\; \exists\, R, S\ \mathbb{Z}^+\text{-matrices} : A = R \cdot S \wedge B = S \cdot R.$$

This matrix formulation has been very useful in checking the invariance of certain properties under topological conjugacy, as it is enough to prove invariance only for one elementary step (see Sections 7.4 and 7.5 in [8]). Moreover this formalism has been used in searching for certain morphisms (factor codes, conjugacies) between given subshifts, e.g. finding chains of ESSEs to build up topological conjugacies between a given pair of matrices by computer (Example 7.3.12 in [8]).

In this note we introduce a similar matrix formalism capturing topological conjugacy for higher-dimensional SFTs. This addresses a question of

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Johnson and Madden posed at the end of [6]: Can one classify topological conjugacies between two-/higher-dimensional SFTs using a matrix condition?

In $d > 1$ dimensions there are d matrices each describing the allowed transitions in one of the d directions. Again splitting and amalgamation codes can be defined, acting in one direction. This immediately yields an equation for the transition matrix corresponding to that direction. Now this operation has to be extended properly to all other directions, giving the remaining equations for the other matrices. Our formalism should on the one hand be helpful in setting up computer searches for conjugacies between given matrix presentations of SFTs and on the other hand in identifying new invariants, possibly by computer assisted investigations of matrix properties that stay unchanged under splittings. Thus one can look for higher-dimensional generalizations of the Jordan form away from zero, the dimension group or the Bowen–Franks groups.

The outline of the paper is as follows: In Section 2 we give the necessary definitions from symbolic dynamics and introduce the notion of strict essentiality. Section 3 briefly reviews the one-dimensional setting and Section 4 contains the definition of our new matrix formalism (Definition 4.3) together with proofs that it is compatible with the one-dimensional theory (Proposition 4.6), that our matrix equations give rise to topological conjugacies (Lemma 4.8) and that every topological conjugacy between higher-dimensional SFTs is composed of a finite chain of such equations (Theorem 4.12). Moreover we show that the formalism converges to and preserves strictly essential presentations (Lemma 4.4, Example 4.5) and we investigate the possibilities and limitations of rearranging the order of splittings and amalgamations (Propositions 4.13 to 4.16). We end this paper by listing the strictly essential presentations of the two-dimensional full 2-shift and the two-dimensional golden-mean shift found by an extensive computer search.

2. Notations and basic definitions. Let $\mathcal{A}$ be a finite set of symbols and let $d \in \mathbb{N}$. The *d-dimensional full shift* on $\mathcal{A}$ is the set $\mathcal{A}^{\mathbb{Z}^d} = \{x = (x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mid \forall \vec{\jmath} \in \mathbb{Z}^d : x_{\vec{\jmath}} \in \mathcal{A}\}$ together with the shift maps $\sigma_{\vec{\jmath}} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ ($\vec{\jmath} \in \mathbb{Z}^d$) defined at each coordinate $\vec{\imath} \in \mathbb{Z}^d$ by $(\sigma_{\vec{\jmath}}(x))_{\vec{\imath}} := x_{\vec{\imath}+\vec{\jmath}}$. Equipped with the standard metric $d(x,y) := 2^{-k}$ where k is the largest integer such that $x_{\vec{\jmath}} = y_{\vec{\jmath}}$ at all coordinates $\vec{\jmath} \in \mathbb{Z}^d$ with $\|\vec{\jmath}\|_\infty \le k$, $\mathcal{A}^{\mathbb{Z}^d}$ becomes a compact, perfect, totally disconnected metric space on which the shift maps act as homeomorphisms.

Every closed, shift-invariant subset X of $\mathcal{A}^{\mathbb{Z}^d}$ together with the restricted shift maps $\sigma = (\sigma_{\vec{\jmath}}|_X)_{\vec{\jmath}\in\mathbb{Z}^d}$ is called a (d-dimensional) *subshift*. Subshifts (X, σ) *of finite type* (SFT) are defined using finite sets $P \subseteq \mathcal{A}^S$ of configurations on a finite set $S \subset \mathbb{Z}^d$ of coordinates, such that

$$X := \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall \vec{\jmath} \in \mathbb{Z}^d : x_{\vec{\jmath}+S} \in P\}.$$

A folklore result generalizing Proposition 2.3.9 in [8] and proved explicitly in Section 2 of [9] tells us that by using a higher block presentation every d-dimensional SFT (X, σ) can be presented as a matrix shift given by a set of square 0/1-matrices A_i $(1 \leq i \leq d)$ indexed by the symbols in $\mathcal{A}$ that describe the allowed transitions in each of the d directions such that

$$X := \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall \vec{\jmath} \in \mathbb{Z}^d,\, 1 \leq i \leq d : (A_i)_{x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_i}} = 1\}.$$

Another equivalent way of representing d-dimensional SFTs uses a straightforward d-dimensional generalization of Nasu's notion of a textile system [10]. This approach is used in [6] to prove the decomposition theorem in two dimensions.

Difficulties in the d-dimensional $(d > 1)$ setting not present in one dimension arise from the interplay of different matrices. For instance checking for a given set of matrices $A_1, \ldots, A_d$ whether the corresponding shift space X is non-empty is undecidable for $d > 1$ in general [2, 12], whereas the same question can be easily answered for $d = 1$. Moreover one cannot use $\mathbb{Z}^+$-matrices but has to stick to 0/1-matrices as edge labellings are crucial in higher dimensions. As we will see, these facts also influence and limit our matrix formalism.

For one-dimensional SFTs it is common to consider only essential adjacency matrices, i.e. matrices containing at least one non-zero entry in every row and every column. The appropriate generalization of this standing assumption to $d > 1$ dimensions is to require all transition matrices to be essential and in addition to demand that every non-zero entry corresponds to a transition used in some point of the shift space. We define this strengthened condition of essentiality explicitly:

Definition 2.1. A set of 0/1-matrices A_i $(1 \leq i \leq d)$ of size $n \times n$ defining a d-dimensional matrix shift X is called *strictly essential* if for every $1 \leq i \leq d$,

$$\forall 1 \leq c \leq n\ \exists 1 \leq a, b \leq n : \quad (A_i)_{a,c} = 1 \wedge (A_i)_{c,b} = 1, \tag{2.1}$$

$$\forall 1 \leq a, b \leq n : \quad (A_i)_{a,b} = 1 \ \Rightarrow\ \exists x \in X : x_{\vec{0}} = a \wedge x_{\vec{e}_i} = b \tag{2.2}$$

Those presentations appear to be the natural ones—containing no unused "extra" entries. In fact the emptiness problem stated above can be reformulated in this framework: A d-dimensional $(d > 1)$ SFT is non-empty if and only if it can be presented by a strictly essential set of transition matrices.

Remark 2.2. Obviously every non-trivial higher-dimensional SFT (X, σ) has a strictly essential presentation which can be obtained from a given set of transition matrices by deleting all those positive entries that do not affect the shift space X. The matrix formalism developed in Section 4 also gives an algorithm that successively identifies unused positive entries in the transition matrices by looking at sufficiently large higher block presentations (see Example 4.5).

Every essential adjacency matrix of a one-dimensional SFT automatically meets the extra assumption (2.2). This fact is directly linked to the decidability or undecidability of the extension problem—asking whether a given admissible block occurs in a point of the subshift—for one-dimensional, respectively higher-dimensional SFTs (see [2, 12]).

The following definition and remarks are standard and appear already in [15]:

DEFINITION 2.3. A *division matrix* D is a 0/1-matrix of size $m \times n$ ($m \leq n \in \mathbb{N}$) such that every column in D has exactly one 1 and every row in D has at least one 1. To D we associate a surjective map $\Delta : \{1, \ldots, n\} \to \{1, \ldots, m\}$ satisfying

$$D_{i,j} = \delta_{i,\Delta(j)} \quad \text{for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

An *amalgamation matrix* C is a 0/1-matrix of size $m \times n$ ($n \leq m \in \mathbb{N}$) such that every row in C has exactly one 1 and every column in C has at least one 1. The corresponding surjective map $\Gamma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is given via

$$C_{i,j} = \delta_{\Gamma(i),j} \quad \text{for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

REMARK 2.4. The transpose of a division matrix is an amalgamation matrix and vice versa. In particular any permutation matrix is at the same time a division and an amalgamation matrix.

The product of two division (resp. amalgamation) matrices is again a division (resp. amalgamation) matrix.

Every $\mathbb{Z}^+$-matrix A can be decomposed into a product $A = D \cdot C$ of a division matrix D and an amalgamation matrix C. The 0/1-matrix $A^{\mathrm{cs}} = C \cdot D$ is called a *complete splitting* of A and is unique up to simultaneous row and column permutations (Factorization Lemma 5.3 in [15]).

3. Review: Splittings of one-dimensional shifts of finite type. We only briefly review the one-dimensional setting. For details on this have a look at Section 2.4 in [8] or Section 2.1 in [7].

Every transitive one-dimensional SFT (X, σ) can be presented as a vertex-shift on a finite strongly connected directed graph $G = (V, E)$ without parallel edges that is given by a single adjacency matrix A. A is an essential 0/1-matrix of size $|V| \times |V|$ with entries

$$A_{i,j} := \#\{e \in E \mid \mathfrak{i}(e) = i \wedge \mathfrak{t}(e) = j\} \quad (i, j \in V).$$

Elements of the shift space

$$X := \{(x_i)_{i\in\mathbb{Z}} \in V^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : A_{x_i,x_{i+1}} = 1\}$$

correspond to bi-infinite sequences of vertices in G and the shift map operates on X as the homeomorphism

$$\sigma : X \to X, \quad (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}.$$

Now state splitting is a procedure to construct new directed graphs from a given one. It comes in two flavors: Each vertex $v \in V$ in G is replaced by a finite number of vertices $v_1, \dots, v_{|P_v|}$ according to a partition P_v (into nonempty sets) of either the incoming or the outgoing edges at the particular vertex v. In the first case we distribute the incoming edges at v among the new vertices according to the partition P_v and we $|P_v|$-fold copy the outgoing edges at v and attach one copy to each of the new vertices $v_1, \dots, v_{|P_v|}$. In the second case the roles of incoming and outgoing edges at v is just reversed (the procedure just described takes place on the transposed graph). Partitioning the outgoing edges yields an *out-splitting*—the new graph is denoted $G^{\mathcal{P}}$ ($\mathcal{P} := \{P_v \mid v \in V\}$), whereas using partitions on the incoming edges gives an *in-splitting* $G_{\mathcal{P}}$ of G.

Investigating the adjacency matrices A, $A^{\mathcal{P}}$ of the graphs G and $G^{\mathcal{P}}$ an out-splitting (denoted as $A \xrightarrow{\text{os}} A^{\mathcal{P}}$) is given in terms of a matrix decomposition of the form $A = D \cdot E$ and $A^{\mathcal{P}} = E \cdot D$ where D is a division matrix and E is some rectangular 0/1-matrix. An in-splitting from A to $A_{\mathcal{P}}$ (denoted as $A \xrightarrow{\text{is}} A_{\mathcal{P}}$) is given as a decomposition $A = E \cdot C$, $A_{\mathcal{P}} = C \cdot E$ with C some amalgamation matrix and E again a rectangular 0/1-matrix.

Every splitting from A to $A^{\mathcal{P}}$ (resp. $A_{\mathcal{P}}$) can equally well be seen as an amalgamation from $A^{\mathcal{P}}$ (resp. $A_{\mathcal{P}}$) back to A: There exists an out-amalgamation $A^{\mathcal{P}} = E \cdot D$, $A = D \cdot E$ from $A^{\mathcal{P}}$ to A (denoted as $A^{\mathcal{P}} \xrightarrow{\text{oa}} A$) if and only if $A \xrightarrow{\text{os}} A^{\mathcal{P}}$, and there is an in-amalgamation $A_{\mathcal{P}} = C \cdot E$, $A = E \cdot C$ from $A_{\mathcal{P}}$ to A (denoted as $A_{\mathcal{P}} \xrightarrow{\text{ia}} A$) if and only if $A \xrightarrow{\text{is}} A_{\mathcal{P}}$.

Whenever two 0/1 adjacency matrices are connected via a splitting the corresponding one-dimensional vertex-shifts are topologically conjugate and due to William's Decomposition Theorem [8, Theorem 7.1.2] every conjugacy between two one-dimensional SFTs can be broken down into a composition of finitely many state splittings and amalgamations. In fact, those elementary transformations can be rearranged either to start with a sequence of out-splittings followed by a sequence of in-amalgamations or to start with a sequence of in-splittings followed by a sequence of out-amalgamations (Theorem 3.4 of [5]).

The matrix formalism for splittings and amalgamations described above generates the algebraic relation of strong shift equivalence of $\mathbb{Z}^+$-matrices [15] which captures topological conjugacy between one-dimensional SFTs and has a large impact on all questions concerning invariants (see [13, 14] and [3, 4]).

4. Splittings of higher-dimensional shifts of finite type

DEFINITION 4.1. Let A, B be two non-negative matrices of size $m \times n$ ($m, n \in \mathbb{N}$) and let $A \ominus B$ denote their elementwise minimum. $\ominus$ defines a binary operation on the set of non-negative matrices of fixed size such that for $1 \le i \le m$ and $1 \le j \le n$, $(A \ominus B)_{i,j} = \min\{A_{i,j}; B_{i,j}\}$.

As long as we restrict multiplication from the right to division matrices D and from the left to amalgamation matrices C we get some kind of distribution law of the standard matrix product over $\ominus$:

LEMMA 4.2. *Let A, B be a pair of non-negative matrices of size $l \times m$.*

(1) *Every division matrix D of size $m \times n$ satisfies*
$$(A \ominus B) \cdot D = A \cdot D \ominus B \cdot D.$$
(2) *Every amalgamation matrix C of size $k \times l$ satisfies*
$$C \cdot (A \ominus B) = C \cdot A \ominus C \cdot B.$$

Proof. The first matrix equation can be checked entrywise (the calculations are left to the reader). The second equation follows by transposition. ∎

Now we can state the desired matrix conditions for state splittings of higher-dimensional SFTs presented by a collection of 0/1 transition matrices $A_1, \ldots, A_d$.

DEFINITION 4.3. An *out-splitting in direction i* ($1 \le i \le d$) is given as:

$$\begin{aligned} A_i &= D \cdot E \quad \xrightarrow{\mathrm{os}_i} \quad A'_i = E \cdot D \\ A_j &\quad (j \ne i) \qquad\quad A'_j = (D^{\mathsf{T}} \cdot A_j \cdot D) \ominus (E \cdot A_j \cdot E^{\mathsf{T}}). \end{aligned} \tag{4.1}$$

An *in-splitting in direction i* ($1 \le i \le d$) is given as:

$$\begin{aligned} A_i &= E \cdot C \quad \xrightarrow{\mathrm{is}_i} \quad A'_i = C \cdot E \\ A_j &\quad (j \ne i) \qquad\quad A'_j = (E^{\mathsf{T}} \cdot A_j \cdot E) \ominus (C \cdot A_j \cdot C^{\mathsf{T}}). \end{aligned} \tag{4.2}$$

Here D is a division matrix, C is an amalgamation matrix and E is a 0/1-matrix. Obviously the products $E \cdot D$ and $C \cdot E$ are also 0/1-matrices and so is A'_i. The same argument shows that the terms $D^{\mathsf{T}} \cdot A_j \cdot D$ and $C \cdot A_j \cdot C^{\mathsf{T}}$ give 0/1-matrices, forcing A'_j to also have this property.

Amalgamations are defined as inverse operations, i.e. going from the right-hand side of (4.1) or (4.2) back to the left.

Compared to the matrix conditions proposed at the end of [6] we have introduced an additional term in the definition of A'_j. This extra term is necessary to suppress all those positive entries in $D^\mathsf{T} \cdot A_j \cdot D$, respectively $C \cdot A_j \cdot C^\mathsf{T}$, that do not give rise to allowed blocks due to the restrictions imposed by the transition rules in direction j one step ahead or back in direction i: In an out-splitting the term $D^\mathsf{T} \cdot A_j \cdot D$ takes care only of the transition rules for a symbol in the original alphabet $\mathcal{A}$, whereas the term $E \cdot A_j \cdot E^\mathsf{T}$ anticipates one step and takes into account those for the succeeding symbol in direction i. In an in-splitting the term $C \cdot A_j \cdot C^\mathsf{T}$ keeps track of allowed transitions in direction j of a symbol in $\mathcal{A}$, whereas the term $E^\mathsf{T} \cdot A_j \cdot E$ looks back one step to utilize the direction j transition restrictions of the symbol preceding in direction i.

Without the extra term one still gets a conjugate SFT, but in general the greater simplicity of the equations has to be paid by the fact that the resulting matrix presentations will contain additional, unused non-zero entries. Our approach, on the other hand, though looking more complicated and less algebraic, has the advantage of preserving strictly essential presentations:

LEMMA 4.4. *Starting with a set of strictly essential* 0/1 *transition matrices*, *an in-/out-splitting as defined in* (4.2) *and* (4.1) *produces again a set of strictly essential* 0/1 *transition matrices. The same is obviously true for amalgamations.*

Moreover starting with a non-strictly essential presentation, unused non-zero entries can be identified (and eliminated) using complete splittings, i.e. investigating sufficient higher block presentations.

EXAMPLE 4.5. We start with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and move to the $(2,1)$-higher block presentation by performing a complete splitting in direction 1:

$$A = D \cdot C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \xrightarrow{\mathrm{cs}_1} \quad A' = A^{\mathrm{cs}} = C \cdot D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$B \qquad\qquad B' = (D^\mathsf{T} \cdot B \cdot D) \ominus (C \cdot B \cdot C^\mathsf{T}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Undoing this operation by an amalgamation we have to calculate the entries of B'':

$$A' = C \cdot D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad \xrightarrow{\mathrm{ca}_1} \quad A'' = D \cdot C = A$$

$$B' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \overset{(*)}{=} (D^{\mathsf{T}} \cdot B'' \cdot D) \ominus (C \cdot B'' \cdot C^{\mathsf{T}}) \qquad B'' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & b''_{3,2} & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} b''_{1,1} & b''_{1,1} & b''_{1,2} & b''_{1,3} \\ b''_{1,1} & b''_{1,1} & b''_{1,2} & b''_{1,3} \\ b''_{2,1} & b''_{2,1} & b''_{2,2} & b''_{2,3} \\ b''_{3,1} & b''_{3,1} & b''_{3,2} & b''_{3,3} \end{pmatrix} \ominus \begin{pmatrix} b''_{1,1} & b''_{1,2} & b''_{1,3} & b''_{1,1} \\ b''_{2,1} & b''_{2,2} & b''_{2,3} & b''_{2,1} \\ b''_{3,1} & b''_{3,2} & b''_{3,3} & b''_{3,1} \\ b''_{1,1} & b''_{1,2} & b''_{1,3} & b''_{1,1} \end{pmatrix}$$

Equation $(*)$ determines all entries of B'' except $b''_{3,2}$. Thus we may set $b''_{3,2} = 0$ without affecting the shift space. This freedom shows that the corresponding transition of a symbol γ followed by a symbol β in the second direction is never used in the original shift space $X_{A,B} \subset \{\alpha, \beta, \gamma\}^{\mathbb{Z}^d}$. So the pair A, B is not strictly essential.

By contrast, one can easily show that the matrices A'', B'' (with $b''_{3,2} = 0$) are strictly essential: $X_{A'',B''}$ contains a fixed point defined by the symbol α and a periodic point of orbit length 3 defined by the block $\begin{smallmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{smallmatrix}$. These two points already contain all allowed transitions from A'' and B''.

In a similar way, performing two complete splittings in the first direction, i.e. going to the $(3,1)$-higher block presentation one proves that the pair

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

actually defines a shift space which is empty.

Now we show that our matrix formalism for higher-dimensional splittings is compatible with strong shift equivalence of 0/1-matrices in one dimension:

Proposition 4.6. *Two essential square* 0/1*-matrices* A, B *are related via a one-dimensional in-splitting* (*resp. out-splitting*) *if and only if the d-tuples* $(A, \mathrm{Id}, \ldots, \mathrm{Id})$ *and* $(B, \mathrm{Id}, \ldots, \mathrm{Id})$ *are related via a higher-dimensional in-splitting* (*resp. out-splitting*).

Proof. A one-dimensional out-splitting $A \xrightarrow{\mathrm{os}} B$ given as $A = D \cdot E$, $B = E \cdot D$ induces a higher-dimensional out-splitting (in the first component) of the strictly essential d-tuple $(A, \mathrm{Id}, \ldots, \mathrm{Id})$ as follows:

$$\begin{array}{lcl} A = D \cdot E & \xrightarrow{\mathrm{os}_1} & A' = E \cdot D = B \\ \mathrm{Id} & & (D^{\mathsf{T}} \cdot \mathrm{Id} \cdot D) \ominus (E \cdot \mathrm{Id} \cdot E^{\mathsf{T}}) = (D^{\mathsf{T}} \cdot D) \ominus (E \cdot E^{\mathsf{T}}). \end{array}$$

Calculating the entries of $(D^\intercal \cdot D) \ominus (E \cdot E^\intercal)$ yields

$$((D^\intercal \cdot D) \ominus (E \cdot E^\intercal))_{i,j} = \min\{\delta_{\Delta(i),\Delta(j)}; \textstyle\sum_k E_{i,k}\, E_{j,k}\}$$
$$= \left.\begin{cases} 0, & \Delta(i) \neq \Delta(j) \text{ because } (D^\intercal \cdot D)_{i,j} = 0 \\ 1, & i = j \text{ because } \sum_k E_{i,k}E_{j,k} = \sum_k (E_{i,k})^2 \geq 1 \ (E \text{ is essential}) \\ 0, & \Delta(i) = \Delta(j) \wedge i \neq j \text{ because } \sum_k E_{i,k}E_{j,k} = 0 \end{cases}\right\} = \delta_{i,j}.$$

In the last case $\sum_k E_{i,k}E_{j,k} \geq 1$ would imply the existence of an index k^* such that $E_{i,k^*} = E_{j,k^*} = 1$ and thus $A_{\Delta(i),k^*} = \sum_l D_{\Delta(i),l}E_{l,k^*} \geq E_{i,k^*} + E_{j,k^*} > 1$ as $\Delta(i) = \Delta(j)$, which is an immediate contradiction to A being a 0/1-matrix.

Thus $(D^\intercal \cdot \mathrm{Id} \cdot D) \ominus (E \cdot \mathrm{Id} \cdot E^\intercal) = \mathrm{Id}$, which gives the desired result.

The proof for a one-dimensional in-splitting $A \xrightarrow{\text{is}} B$ is similar. ∎

COROLLARY 4.7. *Two one-dimensional vertex-shifts* (X_A, σ) *and* (X_B, σ) *are topologically conjugate if and only if the corresponding d-dimensional SFTs represented by the transition matrices* $A, \mathrm{Id}, \ldots, \mathrm{Id}$, *respectively* B, $\mathrm{Id}, \ldots, \mathrm{Id}$, *are related via a sequence of higher-dimensional splittings and amalgamations.*

Proof. Use Proposition 4.6 together with William's Decomposition Theorem [8]. ∎

Next we consider the case of general d-dimensional shifts of finite type:

LEMMA 4.8. *Whenever* $(A_1, \ldots, A_d)$ *and* $(A'_1, \ldots, A'_d)$ *are* (*not necessarily strictly essential*) *sets of* 0/1*-matrices that are connected via an in-/out-splitting*, *the corresponding* $\mathbb{Z}^d$*-SFTs* X, X' *are topologically conjugate.*

Proof. Suppose there is an out-splitting in direction 1 (the other cases are similar):

$$\begin{array}{lll} A_1 = D \cdot E & \xrightarrow{\text{os}_1} & A'_1 = E \cdot D \\ A_k \quad (k \neq 1) & & A'_k = (D^\intercal \cdot A_k \cdot D) \ominus (E \cdot A_k \cdot E^\intercal). \end{array}$$

Using the map $\Delta : \mathcal{A}' \to \mathcal{A}$ associated to the division matrix D define a 1-block-code $\phi := \Delta_\infty : X' \to \mathcal{A}^{\mathbb{Z}^d}$, $(x'_{\vec{\jmath}})_{\vec{\jmath} \in \mathbb{Z}^d} \mapsto (\Delta(x'_{\vec{\jmath}}))_{\vec{\jmath} \in \mathbb{Z}^d}$. Whenever $x' = (x'_{\vec{\jmath}})_{\vec{\jmath} \in \mathbb{Z}^d} \in X'$ one has, for all $\vec{\jmath} \in \mathbb{Z}^d$ and $2 \leq k \leq d$,

$$\begin{aligned} 1 &= (A'_1)_{x'_{\vec{\jmath}}, x'_{\vec{\jmath}+\vec{e}_1}} = (E \cdot D)_{x'_{\vec{\jmath}}, x'_{\vec{\jmath}+\vec{e}_1}} = E_{x'_{\vec{\jmath}}, \Delta(x'_{\vec{\jmath}+\vec{e}_1})} \\ &\leq \sum_{a' \in \mathcal{A}'} \delta_{\Delta(x'_{\vec{\jmath}}), \Delta(a')} \cdot E_{a', \Delta(x'_{\vec{\jmath}+\vec{e}_1})} = (D \cdot E)_{\Delta(x'_{\vec{\jmath}}), \Delta(x'_{\vec{\jmath}+\vec{e}_1})} \\ &= (A_1)_{\Delta(x'_{\vec{\jmath}}), \Delta(x'_{\vec{\jmath}+\vec{e}_1})}, \\ 1 &= (A'_k)_{x'_{\vec{\jmath}}, x'_{\vec{\jmath}+\vec{e}_k}} \leq (D^\intercal \cdot A_k \cdot D)_{x'_{\vec{\jmath}}, x'_{\vec{\jmath}+\vec{e}_k}} = (A_k)_{\Delta(x'_{\vec{\jmath}}), \Delta(x'_{\vec{\jmath}+\vec{e}_k})}. \end{aligned}$$

Therefore $x := (\Delta(x'_{\vec{\jmath}}))_{\vec{\jmath} \in \mathbb{Z}^d} = \phi(x') \in X$ and $\phi(X') \subseteq X$.

Let $\mathcal{B}_{2\times 1^{d-1}}(X) := \{x_{\vec{\imath}}\, x_{\vec{\imath}+\vec{e}_1} \mid x \in X \wedge \vec{\imath} \in \mathbb{Z}^d\}$ denote the set of blocks of dimension $2\times 1^{d-1}$ that show up in X. We define a map $\Psi : \mathcal{B}_{2\times 1^{d-1}}(X) \to \mathcal{A}'$, $a\,b \mapsto c'$, with $c' \in \mathcal{A}'$ being uniquely determined by $D_{a,c'} = 1$ and $E_{c',b} = 1$. Since $a\,b \in \mathcal{B}_{2\times 1^{d-1}}(X)$ one has $1 = (A_1)_{a,b} = (D \cdot E)_{a,b} = D_{a,c'} \cdot E_{c',b}$. This induces a 2-block-code

$$\psi := \Psi_\infty^{[0,1]\times[0]^{d-1}} : X \to \mathcal{A}'^{\mathbb{Z}^d}, \qquad (x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mapsto (\Psi(x_{\vec{\jmath}}\, x_{\vec{\jmath}+\vec{e}_1}))_{\vec{\jmath}\in\mathbb{Z}^d}.$$

For every $x := (x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \in X$ the image $x' = (x'_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} := \psi(x)$ satisfies, for all $\vec{\jmath} \in \mathbb{Z}^d$, $2 \le k \le d$,

$$\begin{aligned}
(A'_1)_{x'_{\vec{\jmath}},x'_{\vec{\jmath}+\vec{e}_1}} &= (A'_1)_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),\Psi(x_{\vec{\jmath}+\vec{e}_1}\, x_{\vec{\jmath}+2\vec{e}_1})} \\
&= \sum_{a\in\mathcal{A}} E_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),a} \cdot D_{a,\Psi(x_{\vec{\jmath}+\vec{e}_1}\, x_{\vec{\jmath}+2\vec{e}_1})} \\
&= E_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),\Delta(\Psi(x_{\vec{\jmath}+\vec{e}_1}\, x_{\vec{\jmath}+2\vec{e}_1}))} = E_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),x_{\vec{\jmath}+\vec{e}_1}} = 1, \\
(A'_k)_{x'_{\vec{\jmath}},x'_{\vec{\jmath}+\vec{e}_k}} &= ((D^{\mathsf{T}} \cdot A_k \cdot D) \ominus (E \cdot A_k \cdot E^{\mathsf{T}}))_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),\Psi(x_{\vec{\jmath}+\vec{e}_k}\, x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})} \\
&= (A_k)_{\Delta(\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1})),\Delta(\Psi(x_{\vec{\jmath}+\vec{e}_k}\, x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k}))} \\
&\quad \ominus \sum_{a,b\in\mathcal{A}} E_{\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1}),a} \cdot (A_k)_{a,b} \cdot E_{\Psi(x_{\vec{\jmath}+\vec{e}_k}\, x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k}),b} \\
&\ge (A_k)_{x_{\vec{\jmath}},x_{\vec{\jmath}+\vec{e}_k}} \ominus (A_k)_{x_{\vec{\jmath}+\vec{e}_1},x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k}} = 1 \ominus 1 = 1,
\end{aligned}$$

so $x' = \psi(x) \in X'$ and $\psi(X) \subseteq X'$.

It is easy to see that $\phi(\psi(x)) = (\Delta(\Psi(x_{\vec{\jmath}} x_{\vec{\jmath}+\vec{e}_1})))_{\vec{\jmath}\in\mathbb{Z}^d} = (x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} = x$ for all $x \in X$ and $\psi(\phi(x')) = (\Psi(\Delta(x'_{\vec{\jmath}})\,\Delta(x'_{\vec{\jmath}+\vec{e}_1})))_{\vec{\jmath}\in\mathbb{Z}^d} = (x'_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} = x'$ for all $x' \in X'$. Thus the d-dimensional SFTs X and X' are topologically conjugate. ∎

In the following we generalize the well known notion of a bipartite code defined by Nasu in [11] to the d-dimensional setting:

Definition 4.9. Suppose X, X' are d-dimensional subshifts with alphabets $\mathcal{A}, \mathcal{A}'$. A map $\phi : X \to X'$ is a *direction-i* ($1 \le i \le d$) *bipartite code* induced by an injective map $f : \mathcal{A} \to \mathcal{C} \times \mathcal{D}$ ($\mathcal{C}, \mathcal{D}$ finite sets) if there exists an injective map $\widetilde{f} : \mathcal{A}' \to \mathcal{D} \times \mathcal{C}$ such that for all $(x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \in X$ and $(x'_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} := \phi((x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d}) \in X'$ there exist $(c_{\vec{\jmath}} \in \mathcal{C})_{\vec{\jmath}\in\mathbb{Z}^d}$ and $(d_{\vec{\jmath}} \in \mathcal{D})_{\vec{\jmath}\in\mathbb{Z}^d}$ and for all $\vec{\jmath} \in \mathbb{Z}^d$ one has $f(x_{\vec{\jmath}}) = [c_{\vec{\jmath}}, d_{\vec{\jmath}}]$ and either $\widetilde{f}(x'_{\vec{\jmath}}) = [d_{\vec{\jmath}}, c_{\vec{\jmath}+\vec{e}_i}]$ or $\widetilde{f}(x'_{\vec{\jmath}}) = [d_{\vec{\jmath}-\vec{e}_i}, c_{\vec{\jmath}}]$.

Every bipartite code ϕ is a topological conjugacy and whenever ϕ is a direction-i bipartite code then so is ϕ^{-1}.

Let $h : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a surjective map between two finite alphabets. We investigate the bipartite codes induced by the maps $f : \mathcal{A} \to \widetilde{\mathcal{A}} \times \mathcal{A}$, $a \mapsto [h(a), a]$, respectively $f : \mathcal{A} \to \mathcal{A} \times \widetilde{\mathcal{A}}$, $a \mapsto [a, h(a)]$.

LEMMA 4.10. *The bipartite codes*

$$l_i^+ : X \to X', \qquad (x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mapsto ([x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_i})])_{\vec{\jmath}\in\mathbb{Z}^d}$$

(*and* $l_i^- : X \to X'$, $(x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mapsto ([h(x_{\vec{\jmath}-\vec{e}_i}), x_{\vec{\jmath}}])_{\vec{\jmath}\in\mathbb{Z}^d}$) *induce out-splittings* (*resp. in-splittings*) *in direction* $1 \le i \le d$, *i.e. if* X *is given in terms of* $(A_1, \dots, A_d)$ *then* X' *can be represented in terms of* $(A'_1, \dots, A'_d)$ *with*

$$\begin{array}{lll} A_i = D \cdot E & \xrightarrow{\mathrm{os}_i} & A'_i = E \cdot D \\ A_k \ \ (k \neq i) & & A'_k = (D^\mathsf{T} \cdot A_k \cdot D) \ominus (E \cdot A_k \cdot E^\mathsf{T}) \end{array}$$

(*resp.*

$$\begin{array}{lll} A_i = E \cdot C & \xrightarrow{\mathrm{is}_i} & A'_i = C \cdot E \\ A_k \ \ (k \neq i) & & A'_k = (E^\mathsf{T} \cdot A_k \cdot E) \ominus (C \cdot A_k \cdot C^\mathsf{T})). \end{array}$$

Proof. We consider only the case of l_1^+, the other cases being similar.

The alphabet of X' is $\mathcal{A}' := \{[a, h(b)] \in \mathcal{A} \times \widetilde{\mathcal{A}} \mid a\,b \in \mathcal{B}_{2\times 1^{d-1}}(X)\}$. Define a surjective map $\Delta : \mathcal{A}' \to \mathcal{A}$, $[a, \widetilde{b}] \mapsto a$, which gives rise to the division matrix D. The entries of the $0/1$-matrix E are:

$$E_{[a,h(b)],c} = \begin{cases} 1, & h(c) = h(b) \wedge a\,c \in \mathcal{B}_{2\times 1^{d-1}}(X), \\ 0, & \text{otherwise.} \end{cases}$$

An easy calculation shows $A_1 = D \cdot E$.

Let $A'_1 := E \cdot D$, $A'_k := (D^\mathsf{T} \cdot A_k \cdot D) \ominus (E \cdot A_k \cdot E^\mathsf{T})$ for all $2 \le k \le d$ and let $X' := \{(x'_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \in \mathcal{A}'^{\mathbb{Z}^d} \mid \forall \vec{\jmath} \in \mathbb{Z}^d, 1 \le k \le d : \ (A'_k)_{x'_{\vec{\jmath}}, x'_{\vec{\jmath}+\vec{e}_k}} = 1\}$.

We have to show that $l_1^+(X) = X'$. Suppose $x \in X$, i.e. $(A_k)_{x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_k}} = 1$ for all $\vec{\jmath} \in \mathbb{Z}^d$ and $1 \le k \le d$, and look at $l_1^+(x)$: for all $\vec{\jmath} \in \mathbb{Z}^d$,

$$\begin{aligned} (A'_1)_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], [x_{\vec{\jmath}+\vec{e}_1}, h(x_{\vec{\jmath}+2\vec{e}_1})]} &= (E \cdot D)_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], [x_{\vec{\jmath}+\vec{e}_1}, h(x_{\vec{\jmath}+2\vec{e}_1})]} \\ &= E_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], \Delta([x_{\vec{\jmath}+\vec{e}_1}, h(x_{\vec{\jmath}+2\vec{e}_1})])} \\ &= E_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], x_{\vec{\jmath}+\vec{e}_1}} = 1, \end{aligned}$$

and for $2 \le k \le d$,

$$\begin{aligned} &(A'_k)_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], [x_{\vec{\jmath}+\vec{e}_k}, h(x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})]} \\ &\quad = ((D^\mathsf{T} \cdot A_k \cdot D) \ominus (E \cdot A_k \cdot E^\mathsf{T}))_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], [x_{\vec{\jmath}+\vec{e}_k}, h(x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})]} \\ &\quad = \min\Big\{ (A_k)_{\Delta([x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})]), \Delta([x_{\vec{\jmath}+\vec{e}_k}, h(x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})])}, \\ &\qquad\qquad \sum_{a,b\in\mathcal{A}} E_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})], a} \cdot (A_k)_{a,b} \cdot E_{[x_{\vec{\jmath}+\vec{e}_k}, h(x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})], b} \Big\} \\ &\quad \ge \min\{(A_k)_{x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_k}}, (A_k)_{x_{\vec{\jmath}+\vec{e}_1}, x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k}}\} = 1. \end{aligned}$$

This calculation immediately yields $l_1^+(X) \subseteq X'$ but it even proves the reverse inclusion: If $x \notin X$ there is a $\vec{\jmath} \in \mathbb{Z}^d$ such that either $(A_1)_{x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_1}} = 0$, which would force $(A_1')_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})],[x_{\vec{\jmath}+\vec{e}_1}, h(x_{\vec{\jmath}+2\vec{e}_1})]} = 0$ and thus contradicts $l_1^+(x) \in X'$, or there is $2 \le k \le d$ such that $(A_k)_{x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_k}} = 0$, which in turn violates $(A_k')_{[x_{\vec{\jmath}}, h(x_{\vec{\jmath}+\vec{e}_1})],[x_{\vec{\jmath}+\vec{e}_k}, h(x_{\vec{\jmath}+\vec{e}_1+\vec{e}_k})]} = 1$ by the last part of the above calculation. Therefore $l_1^+(X) = X'$ as desired. ∎

COROLLARY 4.11. *The 2-higher block maps $r_i^+ : X \to X'$, $(x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mapsto ([x_{\vec{\jmath}}, x_{\vec{\jmath}+\vec{e}_i}])_{\vec{\jmath}\in\mathbb{Z}^d}$ (and $r_i^- : X \to X'$, $(x_{\vec{\jmath}})_{\vec{\jmath}\in\mathbb{Z}^d} \mapsto ([x_{\vec{\jmath}-\vec{e}_i}, x_{\vec{\jmath}}])_{\vec{\jmath}\in\mathbb{Z}^d}$) induce complete out-splittings (resp. in-splittings) in direction $1 \le i \le d$, i.e. if X is given in terms of $(A_1, \ldots, A_d)$ then X' can be represented in terms of $(A_1', \ldots, A_d')$ with*

$$\begin{aligned} &A_i = D \cdot C \quad \xrightarrow{\mathrm{cs}_i} \quad A_i' = C \cdot D \\ &A_k \ \ (k \ne i) \qquad\quad A_k' = (D^{\mathsf{T}} \cdot A_k \cdot D) \ominus (C \cdot A_k \cdot C^{\mathsf{T}}). \end{aligned}$$

Proof. Again we consider r_1^+ only. Using $h = \mathrm{Id}_{\mathcal{A}}$ in Lemma 4.10 one gets $\mathcal{A}' = \{[a,b] \in \mathcal{A}^2 \mid a\,b \in \mathcal{B}_{2\times 1^{d-1}}(X)\}$ and

$$E_{[a,b],c} = E_{[a,h(b)],c} = \left\{ \begin{matrix} 1, & c = b \wedge a\,b \in \mathcal{B}_{2\times 1^{d-1}}(X) \\ 0, & \text{otherwise} \end{matrix} \right\} = \delta_{\Gamma([a,b]),c}$$

with $\Gamma : \mathcal{A}' \to \mathcal{A}$, $[a,b] \mapsto b$. Therefore E is an amalgamation matrix associated to the surjective map Γ and the out-splitting obtained is in fact already a complete splitting. ∎

THEOREM 4.12. *Every in-/out-splitting induces a topological conjugacy between the corresponding higher-dimensional SFTs. Every topological conjugacy can be decomposed into a finite chain of in-/out-splittings followed by in-/out-amalgamations.*

Proof. The first statement is already in Lemma 4.8. For the second statement use a slight generalization of a result by Aso (Theorem 2.1 in [1]):

> *Every topological conjugacy ϕ between two $\mathbb{Z}^d$-shift spaces X, Y can be factorized into a finite number of bipartite codes as follows*:

$$\begin{array}{cccccccccc} (A_1,\ldots,A_d): & X & \xrightarrow{\lambda_1} & X_1 & \xrightarrow{\lambda_2} & X_2 & \cdots & X_{n-1} & \xrightarrow{\lambda_n} & X_n \\ & {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle (H_1)_\infty} & & \big\downarrow{\scriptstyle (H_2)_\infty} & & \big\downarrow{\scriptstyle (H_{n-1})_\infty} & & \big\downarrow{\scriptstyle (H_n)_\infty} \\ (B_1,\ldots,B_d): & Y & \xrightarrow[\varrho_1]{} & Y_1 & \xrightarrow[\varrho_2]{} & Y_2 & \cdots & Y_{n-1} & \xrightarrow[\varrho_n]{} & Y_n \end{array}$$

Here the bipartite codes λ_k $(1 \le k \le n)$ are of type l_i^+ or l_i^- and the ϱ_k are of type r_i^+ or r_i^-. The map $(H_n)_\infty$ is a symbolic conjugacy which only acts on the alphabets by renaming symbols but has no effect on the set of transition matrices.

Using Lemma 4.10 and Corollary 4.11 the diagram translates into a finite chain of matrix transformations connecting $(A_1, \ldots, A_d)$ to $(B_1, \ldots, B_d)$ starting with a finite sequence of in- and out-splittings followed by a finite sequence of in- and out-amalgamations. ∎

PROPOSITION 4.13. *The order of two d-dimensional out-splittings acting on different directions can be exchanged in a canonical way, i.e. if there is an out-splitting in direction i followed by another out-splitting in direction j one can construct—with a fixed procedure—an out-splitting in direction j followed by an out-splitting in direction i such that the corresponding diagram commutes. The same is possible for two in-splittings acting on different directions.*

Proof. Suppose there is an out-splitting in direction 1 given by D_1, E_1 followed by an out-splitting in direction 2 given by D_2, E_2:

$$\begin{array}{lclcl}
A_1 = D_1 \cdot E_1 & \xrightarrow{\text{os}_1} & A_1' = E_1 \cdot D_1 & & A_1'' \\
A_2 & & A_2' = D_2 \cdot E_2 & \xrightarrow{\text{os}_2} & A_2'' = E_2 \cdot D_2 \\
A_k \ (k \neq 1,2) & & A_k' & & A_k''
\end{array}$$

such that $(A_1, \ldots, A_d)$, $(A_1', \ldots, A_d')$ and $(A_1'', \ldots, A_d'')$ are strictly essential sets of square 0/1-matrices of size $l \times l$, $m \times m$ and $n \times n$ respectively. We denote by $\Delta_1 : \{1, \ldots, m\} \to \{1, \ldots, l\}$ and $\Delta_2 : \{1, \ldots, n\} \to \{1, \ldots, m\}$ the surjective maps given by D_1 and D_2 respectively.

Let

$$\widetilde{\mathcal{A}} := \left\{ \begin{bmatrix} \{\Delta_1(i) \mid 1 \le i \le m \wedge (E_2)_{j,i} = 1\} \\ \Delta_1(\Delta_2(j)) \end{bmatrix} \,\middle|\, 1 \le j \le n \right\}$$

be a new alphabet of cardinality $l \le |\widetilde{\mathcal{A}}| \le n$. We can define surjective maps

$$\widetilde{\Delta}_1 : \{1, \ldots, n\} \to \widetilde{\mathcal{A}}, \qquad j \mapsto \begin{bmatrix} \{\Delta_1(i) \mid 1 \le i \le m \wedge (E_2)_{j,i} = 1\} \\ \Delta_1(\Delta_2(j)) \end{bmatrix},$$

and

$$\widetilde{\Delta}_2 : \widetilde{\mathcal{A}} \to \{1, \ldots, l\}, \qquad \begin{bmatrix} S \\ i \end{bmatrix} \mapsto i.$$

Those can be thought of as projections from the split alphabets onto the unsplit ones.

These maps give rise to division matrices $\widetilde{D}_1$ of size $|\widetilde{\mathcal{A}}| \times n$ and $\widetilde{D}_2$ of size $l \times |\widetilde{\mathcal{A}}|$ via $(\widetilde{D}_k)_{i,j} := \delta_{i, \widetilde{\Delta}_k(j)}$ $(k = 1, 2)$. Moreover let $\widetilde{E}_2$ be the 0/1-matrix of size $|\widetilde{\mathcal{A}}| \times l$ such that

$$(\widetilde{E}_2)_{\alpha,j} := \begin{cases} 1, & \alpha = \begin{bmatrix} S \\ i \end{bmatrix} \in \widetilde{\mathcal{A}} \wedge j \in S, \\ 0, & \text{otherwise}. \end{cases}$$

Now $A_2 = \widetilde{D}_2 \cdot \widetilde{E}_2$ by construction and this decomposition induces an out-splitting in direction 2 in the obvious way:

$$\begin{array}{lcl} A_1 & & \widetilde{A}_1 = (\widetilde{D}_2^{\intercal} \cdot A_1 \cdot \widetilde{D}_2) \ominus (\widetilde{E}_2 \cdot A_1 \cdot \widetilde{E}_2^{\intercal}) \\ A_2 = \widetilde{D}_2 \cdot \widetilde{E}_2 & \xrightarrow{\text{os}_2} & \widetilde{A}_2 = \widetilde{E}_2 \cdot \widetilde{D}_2 \\ A_k \ (k \neq 1,2) & & \widetilde{A}_k = (\widetilde{D}_2^{\intercal} \cdot A_k \cdot \widetilde{D}_2) \ominus (\widetilde{E}_2 \cdot A_k \cdot \widetilde{E}_2^{\intercal}). \end{array}$$

Using $\widetilde{A}_1$ we define a 0/1-matrix $\widetilde{E}_1$ of size $n \times |\widetilde{\mathcal{A}}|$ by

$$(\widetilde{E}_1)_{i,\alpha} := \begin{cases} 1, & (E_1)_{\Delta_2(i),\widetilde{\Delta}_2(\alpha)} = 1 \wedge (\widetilde{A}_1)_{\widetilde{\Delta}_1(i),\alpha} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $\widetilde{A}_1 = \widetilde{D}_1 \cdot \widetilde{E}_1$ and in fact performing the induced out-splitting in direction 1 we end up with our original set of matrices $A_1'', \dots, A_d''$. Thus we have reversed the order of the two splittings.

Notice that the matrices $\widetilde{A}_1, \dots, \widetilde{A}_d$ are only determined up to simultaneous row and column permutations (reflecting the freedom of choosing an order on the alphabet $\widetilde{\mathcal{A}}$). The given matrices $A_1, \dots, A_d$ and $A_1'', \dots, A_d''$ however are not affected by this.

The proof for other pairs of directions is exactly the same. To get the statement about two in-splittings just transpose all matrices—in-splittings become out-splittings—and use the result on out-splittings. ∎

We can use Proposition 4.13 to partially rearrange the order of a given finite sequence of splittings and amalgamations. However, in general it is not possible to interchange an in-splitting and an out-splitting, as can be seen in the following example.

Example 4.14. Starting with the 2-dimensional full shift on two symbols given as $A_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, we first perform an out-splitting in direction 1. This yields

$$A_1' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A_2' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

After another in-splitting in direction 2 we end up with

$$A_1'' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad A_2'' = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Reversing the order of these two splittings would imply the existence of two square 0/1-matrices $\widetilde{A}_1$, $\widetilde{A}_2$ that are direction-2 in-splittings of A_1, A_2 and at the same time direction-1 out-amalgamates of A_1'', A_2''. Note that the only possibility to out-amalgamate A_1'' is to collapse columns 2 and 3. This would give a pair $\widetilde{A}_1$, $\widetilde{A}_2$ of size 5×5; but even the direction-2 complete (in-)splitting of A_1, A_2 yields only matrices of size 4×4. Thus an appropriate intermediate pair $\widetilde{A}_1$, $\widetilde{A}_2$ does not exist and we cannot interchange our initial splittings.

As every complete splitting is at the same time an out-splitting (as well as an in-splitting) we get the following result.

Corollary 4.15. *A complete splitting in direction i followed by a complete splitting in direction j results in the same set of transition matrices (up to simultaneous row and column permutations) as a complete splitting in direction j followed by a complete splitting in direction i.*

Proof. This is an immediate consequence of Proposition 4.13 as complete splittings may be interpreted as special out-splittings. ∎

One can replace an in-/out-splitting with a complete splitting followed by an in-/out-amalgamation. This on the one hand justifies the notion of complete splitting and on the other hand proves the possibility of obtaining every matrix presentation by making some amalgamations of a sufficiently high block presentation.

Proposition 4.16. *Given any d-dimensional in-/out-splitting in direction i from $(A_1, \dots, A_d)$ to $(A'_1, \dots, A'_d)$ there is an in-/out-amalgamation from the direction-i complete splitting of $(A_1, \dots, A_d)$ to $(A'_1, \dots, A'_d)$. This amalgamation is given as the inverse of a splitting of all those symbols of the alphabet not modified in the initial splitting.*

Proof. Denote the initial out-splitting as

$$\begin{aligned} &A_i = D_1 \cdot E_1 \xrightarrow{\mathrm{os}_i} A'_i = E_1 \cdot D_1 \\ &A_j \quad (j \neq i) \qquad\quad A'_j = (D_1^{\intercal} \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^{\intercal}). \end{aligned}$$

According to the last statement in Remark 2.4, E_1 can be decomposed into a product $E_1 = D_2 \cdot C_2$ of another division matrix D_2 and an amalgamation matrix C_2 such that $A_i = D_1 \cdot E_1 = D_1 \cdot (D_2 \cdot C_2) = (D_1 \cdot D_2) \cdot C_2$ with $D_1 \cdot D_2$ a division matrix.

The complete splitting of A_i is given as $\widetilde{A}_i = C_2 \cdot (D_1 \cdot D_2) = E_2 \cdot D_2$ with $E_2 := C_2 \cdot D_1$. Thus the out-amalgamation $\widetilde{A}_i = E_2 \cdot D_2 \xrightarrow{\mathrm{oa}_i} D_2 \cdot E_2 = D_2 \cdot (C_2 \cdot D_1) = E_1 \cdot D_1 = A'_i$ reproduces A'_i as desired.

It remains to show that $\widetilde{A}_j := (D_2^{\intercal} \cdot D_1^{\intercal} \cdot A_j \cdot D_1 \cdot D_2) \ominus (C_2 \cdot A_j \cdot C_2^{\intercal}) \xrightarrow{\mathrm{oa}_i} A'_j$, which is equivalent to checking the following matrix equation for each entry:

$$(D_2^{\intercal} \cdot D_1^{\intercal} \cdot A_j \cdot D_1 \cdot D_2) \ominus (C_2 \cdot A_j \cdot C_2^{\intercal}) = (D_2^{\intercal} \cdot A'_j \cdot D_2) \ominus (E_2 \cdot A'_j \cdot E_2^{\intercal}). \tag{4.3}$$

We start by transforming the right-hand side of (4.3) using Lemma 4.2:

$$\begin{aligned}
&(4.4)\quad (D_2^\intercal \cdot A'_j \cdot D_2) \ominus (E_2 \cdot A'_j \cdot E_2^\intercal)\\
&= (D_2^\intercal \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^\intercal)) \cdot D_2) \ominus (E_2 \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^\intercal)) \cdot E_2^\intercal)\\
&= (D_2^\intercal \cdot D_1^\intercal \cdot A_j \cdot D_1 \cdot D_2) \ominus (D_2^\intercal \cdot E_1 \cdot A_j \cdot E_1^\intercal \cdot D_2) \ominus (E_2 \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^\intercal)) \cdot E_2^\intercal).
\end{aligned}$$

So for all pairs of indices a, b with $(D_2^\intercal \cdot D_1^\intercal \cdot A_j \cdot D_1 \cdot D_2)_{a,b} = 0$ both sides in (4.3) are zero.

Now suppose $(D_2^\intercal \cdot D_1^\intercal \cdot A_j \cdot D_1 \cdot D_2)_{a,b} = 1$ but $(C_2 \cdot A_j \cdot C_2^\intercal)_{a,b} = (A_j)_{\Gamma_2(a),\Gamma_2(b)} = 0$, i.e. the left-hand side in (4.3) is again zero. Then

$$\begin{aligned}
&(E_2 \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^\intercal)) \cdot E_2^\intercal)_{a,b}\\
&\quad = (C_2 \cdot D_1 \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (D_2 \cdot C_2 \cdot A_j \cdot C_2^\intercal \cdot D_2^\intercal)) \cdot D_1^\intercal \cdot C_2^\intercal)_{a,b}\\
&\quad = \sum_{k,l} \delta_{\Gamma_2(a),\Delta_1(k)} \min\{(A_j)_{\Delta_1(k),\Delta_1(l)}; (D_2 \cdot C_2 \cdot A_j \cdot C_2^\intercal \cdot D_2^\intercal)_{k,l}\} \delta_{\Delta_1(l),\Gamma_2(b)}\\
&\quad = 0
\end{aligned}$$

since $(A_j)_{\Delta_1(k),\Delta_1(l)} = 0$ for all $\Delta_1(k) = \Gamma_2(a)$ and $\Delta_1(l) = \Gamma_2(b)$ and at least one of the Kronecker symbols are zero for all $\Delta_1(k) \neq \Gamma_2(a)$ or $\Delta_1(l) \neq \Gamma_2(b)$. So the right-hand side in (4.4) and consequently in (4.3) is also zero.

Finally, suppose $(D_2^\intercal \cdot D_1^\intercal \cdot A_j \cdot D_1 \cdot D_2)_{a,b} = 1$ and $(C_2 \cdot A_j \cdot C_2^\intercal)_{a,b} = (A_j)_{\Gamma_2(a),\Gamma_2(b)} = 1$, i.e. the left-hand side in (4.3) is one. For the right-hand side consider the equivalent three terms from (4.4):

The first term equals $(D_2^\intercal \cdot D_1^\intercal \cdot A_j \cdot D_1 \cdot D_2)_{a,b}$ and is thus 1. The second term is

$$\begin{aligned}
(D_2^\intercal \cdot E_1 \cdot A_j \cdot E_1^\intercal \cdot D_2)_{a,b} &= (D_2^\intercal \cdot D_2 \cdot C_2 \cdot A_j \cdot C_2^\intercal \cdot D_2^\intercal \cdot D_2)_{a,b}\\
&= \sum_{k,l} \delta_{\Delta_2(a),\Delta_2(k)} (A_j)_{\Gamma_2(k),\Gamma_2(l)} \delta_{\Delta_2(l),\Delta_2(b)} \geq (A_j)_{\Gamma_2(a),\Gamma_2(b)} = 1
\end{aligned}$$

and the third term is again

$$\begin{aligned}
&(E_2 \cdot ((D_1^\intercal \cdot A_j \cdot D_1) \ominus (E_1 \cdot A_j \cdot E_1^\intercal)) \cdot E_2^\intercal)_{a,b}\\
&\quad = \sum_{k,l} \delta_{\Gamma_2(a),\Delta_1(k)} \min\{(A_j)_{\Delta_1(k),\Delta_1(l)}; (D_2 \cdot C_2 \cdot A_j \cdot C_2^\intercal \cdot D_2^\intercal)_{k,l}\} \delta_{\Delta_1(l),\Gamma_2(b)}\\
&\quad = \sum_{\substack{k \in \Delta_1^{-1}(\Gamma_2(a))\\ l \in \Delta_1^{-1}(\Gamma_2(b))}} \Big\{(A_j)_{\Gamma_2(a),\Gamma_2(b)}; \sum_{c,d} \delta_{k,\Delta_2(c)} (A_j)_{\Gamma_2(c),\Gamma_2(d)} \delta_{\Delta_2(d),l}\Big\}\\
&\quad = \sum_{\substack{k \in \Delta_1^{-1}(\Gamma_2(a))\\ l \in \Delta_1^{-1}(\Gamma_2(b))}} \min\Big\{1; \sum_{\substack{c \in \Delta_2^{-1}(k)\\ d \in \Delta_2^{-1}(l)}} (A_j)_{\Gamma_2(c),\Gamma_2(d)}\Big\}.
\end{aligned}$$

The last expression is zero if and only if for all fixed $k \in \Delta_1^{-1}(\Gamma_2(a)) \neq \emptyset$ and all fixed $l \in \Delta_1^{-1}(\Gamma_2(b)) \neq \emptyset$ we have

$$\sum_{\substack{c \in \Delta_2^{-1}(k) \\ d \in \Delta_2^{-1}(l)}} (A_j)_{\Gamma_2(c),\Gamma_2(d)} = 0.$$

This is true if and only if

$$\sum_{\substack{c \in \Delta_2^{-1}(\Delta_1^{-1}(\Gamma_2(a))) \\ d \in \Delta_2^{-1}(\Delta_1^{-1}(\Gamma_2(b)))}} (A_j)_{\Gamma_2(c),\Gamma_2(d)} = 0,$$

which contradicts $\widetilde{A}_j$ being strictly essential.

The proof for an initial in-splitting is similar. ∎

In one dimension every ESSE $A = R \cdot S,\ B = S \cdot R$ can be seen as a composition of an out-splitting followed by an in-amalgamation:

$$A = R \cdot S = (D \cdot C) \cdot S = D \cdot (C \cdot S) \xrightarrow{\text{os}} C \cdot S \cdot D \xrightarrow{\text{ia}} S \cdot D \cdot C = S \cdot R = B.$$

(Likewise there is a decomposition of this ESSE into an in-splitting followed by an out-amalgamation.) In general such a factorization is not possible in higher dimensions, even if we restrict R and S to be 0/1-matrices. So we have to use elementary splittings and amalgamations as given in Definition 4.3 instead of the R-S-formalism.

Example 4.17. Let

$$A_1 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Define

$$R := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A_1 = R \cdot S \quad \text{and} \quad A_1'' = S \cdot R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Factoring

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = D \cdot C$$

one can decompose this one-dimensional ESSE into an out-splitting followed by an in-amalgamation:

$$A_1 = D \cdot (C \cdot S) \xrightarrow{\mathrm{os}_1} (C \cdot S) \cdot D = A_1' = C \cdot (S \cdot D) \xrightarrow{\mathrm{ia}_1} (S \cdot D) \cdot C = A_1''.$$

In the second direction the out-splitting gives

$$A_2' = (D^\intercal \cdot A_2 \cdot D) \ominus ((C \cdot S) \cdot A_2 \cdot (C \cdot S)^\intercal) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

But the matrix A_2' cannot be amalgamated to a 0/1-matrix A_2'' of size 4×4 using C and $E = S \cdot D$ because the equation

$$\begin{aligned} A_2' &\overset{!}{=} ((S \cdot D)^\intercal \cdot A_2'' \cdot (S \cdot D)) \ominus (C \cdot A_2'' \cdot C^\intercal) \\ &= \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & a''_{1,3} + a''_{1,4} \\ * & * & * & * & a''_{1,3} + a''_{1,4} \\ * & * & a''_{3,1} + a''_{4,1} & a''_{3,1} + a''_{4,1} & * \end{pmatrix} \ominus \begin{pmatrix} a''_{1,1} & a''_{1,4} & a''_{1,2} & a''_{1,3} & a''_{1,1} \\ a''_{4,1} & a''_{4,4} & a''_{4,2} & a''_{4,3} & a''_{4,1} \\ a''_{2,1} & a''_{2,4} & a''_{2,2} & a''_{2,3} & a''_{2,1} \\ a''_{3,1} & a''_{3,4} & a''_{3,2} & a''_{3,3} & a''_{3,1} \\ a''_{1,1} & a''_{1,4} & a''_{1,2} & a''_{1,3} & a''_{1,1} \end{pmatrix} \end{aligned}$$

forces $a''_{i,j} = 1$ for all $i + j \leq 5$ (use the position of ones in A_2'). Thus the entries at coordinates $(3,5)$, $(4,5)$, $(5,3)$ and $(5,4)$ in A_2' would be 1, a contradiction.

5. Strictly essential presentations of the full 2-shift and the golden-mean shift in two dimensions. As we have seen in the proof of our main theorem 4.12, every strictly essential presentation of a d-dimensional SFT can be generated by successive amalgamations of some higher block presentation. As an application of our matrix formalism the author has implemented a fast computer algorithm to produce all possible amalgamations of a given pair of 0/1-matrices. The algorithm can handle matrices of size up to around 20 in reasonable time.

In the following two tables we give a complete list of all strictly essential presentations of the 2-dimensional full shift on 2 symbols (Table 1) and the 2-dimensional golden-mean shift (Table 2) that can be obtained by amalgamations of the L-block presentation of the full shift (the alphabet consists of the eight $\begin{smallmatrix} a & \\ b & c \end{smallmatrix}$-patterns with $a, b, c \in \{0, 1\}$), respectively the $(2,2)$-higher block presentation of the golden-mean shift (the alphabet consists of all allowed $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$-patterns with $a, b, c, d \in \{0, 1\}$ and no adjacent 1's).

Table 1. Complete list of the strictly essential matrix presentations of the 2-dimensional full 2-shift that are obtained as amalgamations of its L-block presentation

Starting with the two matrices of size 8×8 representing the L-block presentation of the full 2-shift in two dimensions we get the following matrix amalgamations of size 7×7 down to 2×2.

$$\begin{pmatrix} 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&1&1&1&1&1\\ 0&1&1&0&1&1&0\\ 0&1&1&0&1&1&0\\ 1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 0&1&1&0&1&1&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0 \end{pmatrix} \qquad \begin{pmatrix} 1&0&1&1&0&1&0\\ 0&1&1&0&1&0&1\\ 0&1&1&0&1&0&1\\ 1&0&1&1&0&1&0\\ 1&0&1&1&0&1&0\\ 0&1&1&0&1&0&1\\ 1&0&1&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0&0\\ 0&1&0&1&0&1&1\\ 0&1&0&1&0&1&1\\ 1&0&1&0&1&0&0\\ 1&0&1&0&1&0&0\\ 0&1&0&1&0&1&1\\ 1&0&1&0&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&1&1&1&1\\ 0&1&1&0&1&0\\ 1&1&1&1&1&1\\ 1&0&0&1&0&1\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1\\ 0&1&1&0&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0\\ 0&1&1&0&1&0\\ 0&1&1&0&1&0 \end{pmatrix} \qquad \begin{pmatrix} 1&0&1&1&0&1\\ 0&1&1&1&1&0\\ 1&0&1&1&0&1\\ 0&1&1&1&1&0\\ 1&0&1&1&0&1\\ 0&1&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&0\\ 0&1&1&0&0&1\\ 1&0&0&1&1&0\\ 0&1&1&0&0&1\\ 1&0&0&1&1&0\\ 0&1&1&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&1&1&1&1\\ 0&1&0&1&1&1\\ 1&1&1&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&0&0\\ 1&0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0\\ 0&1&0&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&0&0\\ 0&1&0&1&1&1\\ 1&0&1&0&0&0 \end{pmatrix} \qquad \begin{pmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&1&1&1&1\\ 0&1&1&0&1&0\\ 0&1&1&0&1&0\\ 1&0&0&1&0&1\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1\\ 0&1&1&0&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0\\ 1&1&1&1&1&1\\ 0&1&1&0&1&0 \end{pmatrix} \qquad \begin{pmatrix} 1&1&1&1&1&1\\ 0&1&0&1&1&0\\ 1&0&1&0&0&1\\ 1&0&1&0&0&1\\ 1&0&1&0&0&1\\ 0&1&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&1\\ 1&1&1&1&1&1\\ 0&1&0&1&1&0\\ 0&1&0&1&1&0\\ 1&0&1&0&0&1\\ 0&1&0&1&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&1&1&0&1\\ 0&1&1&0&1&0\\ 0&1&1&0&1&0\\ 1&0&1&1&0&1\\ 1&0&1&1&0&1\\ 0&1&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0\\ 0&1&0&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&1&1 \end{pmatrix} \qquad \begin{pmatrix} 1&1&1&1&0\\ 0&1&1&1&1\\ 1&1&1&1&0\\ 0&1&1&1&1\\ 1&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1\\ 0&1&1&0&0\\ 1&0&0&1&1\\ 0&1&1&0&0\\ 1&0&0&1&1 \end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&0&0&1&1\\0&1&1&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1&1\\0&1&1&0&0\\1&0&0&1&1\\0&1&1&0&0\\0&1&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1&1&1\\0&1&0&1&1\\1&1&1&1&1\\0&1&0&1&1\\1&0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0&0\\0&1&0&1&1\\0&1&0&1&1\\1&0&1&0&0\\1&1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&0&0\\0&1&1&1&1\\0&1&1&1&1\\1&1&1&0&0\\1&1&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\1&1&0&0&1\\1&1&0&0&1\\1&0&1&1&0\\1&0&1&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1&1&1\\0&1&1&1&1\\0&1&1&1&1\\1&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1&1\\0&1&1&1&1\\1&0&0&0&0\\0&1&1&1&1\\1&0&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1&0\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&0&1\\1&1&0&1\\1&0&1&1\end{pmatrix}\qquad\begin{pmatrix}1&1&1&1\\0&1&1&1\\1&1&1&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1\\0&1&1&1\\1&0&0&0\\0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1\\1&1&1&1\\0&1&1&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1\\0&1&1&1\\1&0&0&0\\1&0&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix},\begin{pmatrix}1&1&1\\0&1&1\\1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

Table 2. Complete list of the strictly essential matrix presentations of the 2-dimensional golden-mean shift that are obtained as amalgamations of its $(2,2)$-higher block presentation

Starting with the two matrices of size 7×7 representing the $(2,2)$-higher block presentation of the golden-mean shift in two dimensions we get the following matrix amalgamations of size 6×6 down to 2×2.

$$\begin{pmatrix}1&0&0&1&1&0&0\\1&0&0&1&1&0&0\\1&0&0&1&1&0&0\\0&1&0&0&0&0&1\\0&0&1&0&0&1&0\\0&1&0&0&0&0&1\\0&0&1&0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&1&0&1&0&0\\1&0&1&0&1&0&0\\0&1&0&0&0&0&1\\1&0&1&0&1&0&0\\0&0&0&1&0&1&0\\0&1&0&0&0&0&1\\0&0&0&1&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1&0&0\\1&1&0&1&0&1\\1&1&0&1&0&0\\0&0&1&0&1&0\\0&1&0&0&0&1\\0&0&1&0&1&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\\1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&0&1&0&1\\0&0&1&0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&0&0&1&1&0\\1&0&1&1&1&1\\1&0&0&1&1&0\\0&1&0&0&0&0\\0&0&1&0&0&1\\0&1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0&1&0\\1&0&1&1&1&1\\0&1&0&0&0&0\\1&0&1&0&1&0\\0&0&0&1&0&1\\0&1&0&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&1&1&0\\1&0&0&1&1&0\\1&1&0&1&1&1\\0&1&0&0&0&1\\0&0&1&0&0&0\\0&0&1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0&1&0\\1&0&1&0&1&0\\0&1&0&0&0&1\\1&0&1&0&1&0\\0&0&1&1&0&0\\0&0&1&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1&1&0\\1&0&0&1&0&0\\1&0&0&1&0&0\\0&0&1&0&0&1\\0&0&1&0&0&1\\0&1&0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0&0\\1&0&1&1&0&0\\0&1&0&0&1&0\\1&0&0&0&0&1\\1&0&0&0&0&1\\0&1&0&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1&1&0\\1&0&0&1&0&0\\1&0&0&1&0&0\\0&0&1&0&0&1\\0&0&1&0&0&1\\0&1&0&0&1&0\end{pmatrix},\begin{pmatrix}1&1&0&1&1&0\\0&0&1&0&0&1\\1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&1&1&0\end{pmatrix}\qquad\begin{pmatrix}1&0&0&1&1\\1&0&1&1&1\\1&1&0&1&1\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0&1\\1&0&1&1&1\\0&1&0&0&0\\1&0&1&0&1\\0&0&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1&0\\1&1&0&1&1\\1&1&0&1&1\\0&0&1&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\1&0&1&1&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1&0\\1&1&0&1&1\\1&1&0&1&1\\0&0&1&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&0&1&1&1\\1&1&0&1&1\\1&1&0&0&0\\0&0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1&1\\1&0&0&1&0\\1&1&0&1&1\\0&0&1&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&1&0&1&1\\0&0&1&0&0\\1&1&0&1&1\\1&1&0&0&0\\0&0&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1&1\\1&0&0&1&0\\1&1&0&1&1\\0&0&1&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\1&0&1&1&0\\0&1&0&0&1\\1&0&1&0&0\\1&0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1&0\\1&0&1&1&1\\1&0&0&1&0\\0&0&1&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&1&0\\0&0&1&0&1\\1&1&0&0&0\\1&1&0&0&0\\0&1&0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1&0\\1&0&1&1&1\\1&0&0&1&0\\0&0&1&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\1&0&1&1&1\\0&1&0&0&0\\1&0&0&0&1\\0&1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&1&1\\1&0&0&1&0\\1&0&0&0&1\\0&1&1&0&0\\0&1&1&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1&1\\1&0&0&0&0\\0&0&0&1&1\\1&0&0&0&0\\0&1&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1\\1&1&0&1\\1&1&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&1&0&0\\0&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&1\\1&0&1&1\\1&1&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&0&1&1\\0&1&0&0\\1&0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1&0&1\\1&0&1&1\\1&1&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&1&0&1\\0&0&1&0\\1&1&0&1\\1&1&0&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1&1\\1&0&1&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1\\1&0&0&0\\1&1&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&1&0\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1&1\\1&0&1\\1&1&0\end{pmatrix},\begin{pmatrix}1&0&1\\1&0&1\\0&1&0\end{pmatrix}\qquad\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

REMARK 5.1. Here two matrix presentations are regarded the same if they differ only by a renumbering of states and/or by simultaneous transposition of both matrices (our examples are symmetric with respect to transposition). Thus the displayed set of matrix pairs in Tables 1 and 2 contains exactly one representative for each class of presentations with respect to the following equivalence relation:

$$\begin{aligned}(A,B)\sim(A',B') \;:\Leftrightarrow\; &\exists P\in\mathcal{P}:(A'=P\cdot A\cdot P^{-1}\;\wedge\; B'=P\cdot B\cdot P^{-1})\\ &\vee(A'=P\cdot A^{\mathsf{T}}\cdot P^{-1}\;\wedge\; B'=P\cdot B^{\mathsf{T}}\cdot P^{-1})\end{aligned}$$

where $(A,B),(A',B')$ are strictly essential 0/1-matrix presentations and $\mathcal{P}$ is the set of permutation matrices.

The set of strictly essential presentations of fixed matrix size that are amalgamations of some higher block presentation grows with increasing block size. This can be seen in our example of the 2-dimensional golden-mean shift. Its L-block presentation corresponds to the fourth pair of 5×5 matrices in Table 2. But amalgamating this presentation only gives two of the five classes of matrix size 4×4 (the third and the fourth pair of size 4×4 in Table 2). Similarly the set of all amalgamations of the $(2,2)$-higher block presentation of the full shift on two symbols yet contains many additional presentations. There are 1, 1, 19, 82, 366, 1368, 3815 etc. equivalence classes of size $2,3,4,5,6,7$ and 8 respectively.

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Centro de Modelamiento Matemático
Universidad de Chile (CNRS UMI 2807)
Av. Blanco Encalada 2120
Santiago de Chile, Chile
E-mail: mschraudner@dim.uchile.cl

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