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THE DEFOCUSING ENERGY-CRITICAL KLEIN–GORDON–HARTREE EQUATION

BY

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Abstract. We study the scattering theory for the defocusing energy-critical Klein–Gordon equation with a cubic convolution $u_{tt} - \Delta u + u + (|x|^{-4} * |u|^2)u = 0$ in spatial dimension $d \ge 5$. We utilize the strategy of Ibrahim et al. (2011) derived from concentration compactness ideas to show that the proof of the global well-posedness and scattering can be reduced to disproving the existence of a soliton-like solution. Employing the technique of Pausader (2010), we consider a virial-type identity in the direction orthogonal to the momentum vector to exclude such a solution.

1. Introduction. This paper is devoted to the Cauchy problem for the defocusing energy-critical Klein–Gordon–Hartree equation

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ d \ge 5, \\ u(0,x) = u_0(x), & u_t(0,x) = u_1, \end{cases} \tag{1.1}$$

where $f(u) = (V(x) * |u|^2)u$ with $V(x) = |x|^{-4}$. Here u is a real-valued function defined in $\mathbb{R}^{d+1}$, the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^d$, $V(x)$ is called the potential, and $*$ denotes the spatial convolution in $\mathbb{R}^d$.

Formally, the solution u of (1.1) conserves the energy,

$$\begin{aligned} E(u(t), \dot{u}(t)) &= \frac{1}{2} \int_{\mathbb{R}^d} (|\dot{u}(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2)\, dx \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^4}\, dx\, dy \\ &= E(u_0, u_1), \end{aligned}$$

and the momentum,

$$P(u)(t) = \int_{\mathbb{R}^d} u_t(t,x) \nabla u(t,x)\, dx = P(u)(0).$$

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For the equation (1.1) with nonlinearity $f(u) = \mu(|x|^{-\gamma} * |u|^2)u$, $\mu = \pm 1$, using the ideas of Strauss [30], [31] and Pecher [29], Mochizuki [24] showed that if $d \geq 3$ and $2 \leq \gamma < \min(d, 4)$, then global well-posedness and scattering results with small data hold in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For general initial data, we refer to [23] where we develop a complete scattering theory in the energy space for (1.1) with a subcritical nonlinearity (i.e. $2 < \gamma < \min(d, 4)$) for both the defocusing ($\mu = 1$) and focusing ($\mu = -1$) cases in spatial dimension $d \geq 3$. In this paper, we will focus on the energy-critical case, i.e. $\gamma = 4$ and $d \geq 5$. We refer also to Miao–Zhang [21] where the low regularity for the cubic convolution defocusing Klein–Gordon–Hartree equation is discussed.

Before stating our main results, we recall the scattering theory for the classical Klein–Gordon equation, i.e. (1.1) with nonlinearity $f(u) = \mu|u|^{p-1}u$. For $\mu = 1$ and

$$1 + \frac{4}{d} < p < 1 + \frac{4\gamma_d}{d-2}, \qquad \gamma_d = \begin{cases} 1, & 3 \leq d \leq 9, \\ \dfrac{d}{d+1}, & d \geq 10, \end{cases}$$

Brenner [5] established scattering results in the energy space in dimension $d \geq 10$. Thereafter, Ginibre and Velo [7] exploited the Birman–Solomyak space $\ell^m(L^q, I, B)$ of [3] and delicate estimates to improve the results in [5], which covered all subcritical cases. Finally, K. Nakanishi [25] obtained scattering results for the energy-critical case by the strategy of induction on energy [6] and a new Morawetz-type estimate. And recently, S. Ibrahim, N. Masmoudi, and K. Nakanishi [10, 11] utilized the concentration compactness ideas to give the scattering threshold for the focusing (i.e. $\mu = -1$) nonlinear Klein–Gordon equation. We remark that their method also works for the defocusing case. We will utilize their argument to study the scattering theory for the defocusing energy-critical Klein–Gordon–Hartree equation.

On the other hand, the scattering theory for the Hartree equation

$$i\dot{u} = -\Delta u + (|x|^{-\gamma} * |u|^2)u$$

has also been studied by many authors (see [9, 15–19]). For the energy-subcritical case, i.e. $\gamma < 4$, Ginibre and Velo [9] obtained the asymptotic completeness in the energy space $H^1(\mathbb{R}^d)$ by deriving the associated Morawetz inequality and a useful Birman–Solomyak-type estimate. Nakanishi [26] improved the results by using a new Morawetz estimate. For the energy-critical case ($\gamma = 4$ and $d \geq 5$), Miao, Xu, and Zhao [16] took advantage of a new kind of a localized Morawetz estimate to rule out the possibility of energy concentration at the origin and established scattering results in the energy space for radial data. We also refer to [17–19] for general data and also for the mass-critical case.

Compared with the classical Klein–Gordon equation with the local nonlinearity $f(u) = |u|^{p-1}u$, the nonlinearity $f(u) = (V(\cdot) * |u|^2)u$ is nonlocal, which brings many difficulties. The main difficulty is the absence of Lorentz invariance which could be used to control the momentum efficiently. We will overcome this difficulty by considering a virial-type identity in the direction orthogonal to the momentum vector, following the technique of [28].

Now we introduce the definition of a strong solution for (1.1).

DEFINITION 1.1 (Solution). A function $u : I \times \mathbb{R}^d \to \mathbb{R}$ on a nonempty time interval $0 \in I$ is a *strong solution* to (1.1) if for any compact $J \subset I$, $(u, u_t) \in C_t^0(J; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ and

$$u \in W(J), \qquad W(J) := L_t^{\frac{2(d+1)}{d-1}}(J; B^{1/2}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)),$$

and for each $t \in I$, $(u(t), \dot{u}(t))$ satisfies the following Duhamel formula:

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds,$$

where

$$V_0(t) = \begin{pmatrix} \dot{K}(t), K(t) \\ \ddot{K}(t), \dot{K}(t) \end{pmatrix}, \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = (1 - \Delta)^{1/2}.$$

The interval I is called the *lifespan* of u. Moreover, if the solution u cannot be extended to any strictly larger interval, then u is a *maximal-lifespan solution*. We say that u is a *global solution* if $I = \mathbb{R}$.

REMARK 1.2. From Remark 2.5 below, we find that the solution u lies in the space $W(I)$ locally in time. Also, the finiteness of the norm on the maximal lifespan implies that the solution is global and scatters in both time directions, by a standard argument. In view of this, we define

$$S_I(u) = \|u\|_{ST(I)} = \|u\|_{[W](I)} \tag{1.2}$$

to be the *scattering size* of u.

Our main result is the following global well-posedness and scattering result in the energy space.

THEOREM 1.3. *Assume that* $d \geq 5$, *and* $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. *Then there exists a unique global solution* $u(t)$ *of* (1.1) *which scatters in the sense that there exist solutions* $v_\pm$ *of the free Klein–Gordon equation*

$$\ddot{v} - \Delta v + v = 0$$

with $(v_\pm(0), \dot{v}_\pm(0)) \in H^1 \times L^2$ *such that*

$$\|(u(t), \dot{u}(t)) - (v_\pm(t), \dot{v}_\pm(t))\|_{H^1 \times L^2} \to 0 \quad \text{as } t \to \pm\infty.$$

Let us outline the proof of Theorem 1.3: we define

$$\Lambda(E) = \sup\{\|u\|_{ST(I)} \mid E(u, u_t) \leq E\}$$

where the supremum is taken over all strong solutions u of (1.1) on any interval I with energy not greater than E, and set

$$E_{\max} = \sup\{E \mid \Lambda(E) < \infty\}.$$

The small data scattering (Theorem 2.4 below) tells us that $E_{\max} > 0$. Our goal next is to prove that $E_{\max} = \infty$. We show that if $E_{\max} < \infty$, then there exists a nonlinear solution of (1.1) with energy exactly $E_{\max}$. Moreover, this solution has some strong compactness properties. This is completed in Section 4 where we utilize the profile decomposition established in [10], and a strategy introduced by Kenig and Merle [13]. We consider a virial-type identity in the direction orthogonal to the momentum vector following the technique of [28] to obtain a contradiction. We refer to Section 5 for more details.

The paper is organized as follows. In Section 2, we deal with the local theory for equation (1.1). In Section 3, we give the linear and nonlinear profile decomposition and show some properties of the profile. In Section 4, we show that nonscattering entails the existence of a critical solution. Finally, in Section 5, we preclude the critical solution, which completes the proof of Theorem 1.3.

2. Preliminaries

2.1. Notation. First, we give some notation which will be used throughout this paper. We always assume the spatial dimension $d \geq 5$ and let $2^* = 2d/(d-2)$. For any $1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^d)$, and by r' the conjugate exponent defined by $1/r + 1/r' = 1$. For any $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ the usual Sobolev space. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\widehat{\psi} \subseteq \{\xi \mid 1/2 \leq |\xi| \leq 2\}$ and $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. Define ψ_0 by $\widehat{\psi_0} = 1 - \sum_{j\geq 1} \widehat{\psi}(2^{-j}\xi)$. Thus $\operatorname{supp}\widehat{\psi_0} \subseteq \{\xi \mid |\xi| \leq 2\}$ and $\widehat{\psi_0} = 1$ for $|\xi| \leq 1$. We denote by Δ_j and $\mathcal{P}_0$ the convolution operators whose symbols are respectively $\widehat{\psi}(\xi/2^j)$ and $\widehat{\psi_0}(\xi)$. For $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, the inhomogeneous Besov space $B^s_{r,2}(\mathbb{R}^d)$ is defined by

$$B^s_{r,2}(\mathbb{R}^d) = \left\{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|\mathcal{P}_0 u\|^2_{L^r} + \left\|2^{js}\|\Delta_j u\|_{L^r}\right\|^2_{l^2_{j\in\mathbb{N}}} < \infty\right\}.$$

For details on Besov spaces, we refer to [2]. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $\mathcal{C}(I; X)$ the space of strongly continuous functions from I to X, and by $L^q(I; X)$ the space of strongly measurable functions from I to X with $\|u(\cdot); X\| \in L^q(I)$. Given d, we define, for $2 \leq$

$r \le \infty$,

$$\delta(r) = d\left(\frac{1}{2} - \frac{1}{r}\right).$$

Sometimes we abbreviate $\delta(r)$, $\delta(r_i)$ to δ, δ_i respectively. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2. We let L^p_* denote the weak L^p space.

2.2. Strichartz estimate. In this section, we consider the Cauchy problem for equation (1.1),

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, \\ u(0) = u_0, \quad \dot{u}(0) = u_1. \end{cases} \tag{2.1}$$

The integral equation for the Cauchy problem (2.1) can be written as

$$u(t) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))\,ds,$$

or

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t)\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s)\begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds,$$

where

$$K(t) = \frac{\sin(t\omega)}{\omega}, \qquad V_0(t) = \begin{pmatrix} \dot{K}(t), K(t) \\ \ddot{K}(t), \dot{K}(t) \end{pmatrix}, \qquad \omega = (1-\Delta)^{1/2}.$$

Let $U(t) = e^{it\omega}$. Then

$$\dot{K}(t) = \frac{U(t) + U(-t)}{2}, \qquad K(t) = \frac{U(t) - U(-t)}{2i\omega}.$$

Now we recall the following dispersive estimate for the operator $U(t) = e^{it\omega}$.

LEMMA 2.1 ([5,7]). *Let $2 \le r \le \infty$ and $0 \le \theta \le 1$. Then*

$$\|e^{i\omega t} f\|_{B_{r,2}^{-(d+1+\theta)(1/2-1/r)/2}} \le \mu(t)\|f\|_{B_{r',2}^{(d+1+\theta)(1/2-1/r)/2}},$$

where

$$\mu(t) = C\min\{|t|^{-(d-1-\theta)(1/2-1/r)_+}, |t|^{-(d-1+\theta)(1/2-1/r)}\}, \quad a_+ := \max\{a, 0\}.$$

Combining the above lemma, the abstract duality and an interpolation argument (see [8, 12]) yields the following Strichartz estimates.

LEMMA 2.2 ([5,7,20]). *Let $0 \le \theta_i \le 1$, $\rho_i \in \mathbb{R}$, $2 \le q_i, r_i \le \infty$, $i = 1, 2$. Assume that $(\theta_i, d, q_i, r_i) \ne (0, 3, 2, \infty)$ satisfy the following admissibility* *conditions:*

$$\begin{cases} 0 \le \dfrac{2}{q_i} \le \min\Big\{(d-1+\theta_i)\Big(\dfrac{1}{2}-\dfrac{1}{r_i}\Big), 1\Big\}, \qquad i=1,2, \\ \rho_1 + (d+\theta_1)\Big(\dfrac{1}{2}-\dfrac{1}{r_1}\Big) - \dfrac{1}{q_1} = \mu, \\ \rho_2 + (d+\theta_2)\Big(\dfrac{1}{2}-\dfrac{1}{r_2}\Big) - \dfrac{1}{q_2} = 1-\mu. \end{cases}$$

Then, for $f \in H^\mu$, we have

$$\begin{aligned} \|U(\cdot)f\|_{L^{q_1}(\mathbb{R};B^{\rho_1}_{r_1,2})} &\le C\|f\|_{H^\mu}, \\ \|K * f\|_{L^{q_1}(I;B^{\rho_1}_{r_1,2})} &\le C\|f\|_{L^{q_2'}(I;B^{-\rho_2}_{r_2',2})}, \\ \|K_R * f\|_{L^{q_1}(I;B^{\rho_1}_{r_1,2})} &\le C\|f\|_{L^{q_2'}(I;B^{-\rho_2}_{r_2',2})}, \end{aligned}$$

where the subscript R stands for "retarded", and

$$K * f = \int_{\mathbb{R}} K(t-s)f(u(s))\,ds, \qquad K_R * f = \int_0^t K(t-s)f(u(s))\,ds.$$

In addition to the W-norm defined in (1.2), we also need the following space:

$$[W]^*(I) = L_t^{\frac{2(d+1)}{d+3}}(I; B^{1/2}_{\frac{2(d+1)}{d+3},2}(\mathbb{R}^d)).$$

Now we give a nonlinear estimate which will be applied to show small data scattering.

LEMMA 2.3. *We have*

$$\begin{aligned} \|(V(\cdot) * |u|^2)v\|_{[W]^*(I)} &+ \|(V(\cdot) * (uv))u\|_{[W]^*(I)} \\ &\le C\|v\|_{[W](I)}\|u\|^{\frac{2(d-3)}{d-1}}_{L_t^\infty(I;\dot H_x^1)}\|u\|^{\frac{4}{d-1}}_{[W](I)} \\ &\quad + C\|u\|^{1+\frac{2}{d-1}}_{[W](I)}\|u\|^{\frac{d-3}{d-1}}_{L_t^\infty(I;\dot H_x^1)}\|v\|^{\frac{d-3}{d-1}}_{L_t^\infty(I;\dot H_x^1)}\|v\|^{\frac{2}{d-1}}_{[W](I)}. \end{aligned} \tag{2.2}$$

In particular,

$$\|(V(\cdot) * |u|^2)u\|_{[W]^*(I)} \le C\|u\|^{1+\frac{4}{d-1}}_{[W](I)}\|u\|^{\frac{2(d-3)}{d-1}}_{L^\infty(I;H^1)}.$$

Proof. We only need to estimate $\|(V(\cdot) * |u|^2)v\|_{[W]^*(I)}$, since estimating $\|(V(\cdot) * (uv))u\|_{[W]^*(I)}$ is similar. From the Sobolev embedding $W^{s,p}(\mathbb{R}^d) \hookrightarrow B^s_{p,2}(\mathbb{R}^d)$, $p \le 2$, and $B^s_{q,2}(\mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}^d)$, $q \ge 2$, the fractional Leibniz rule [14], and the Hölder and Young inequalities, we have

$$\begin{aligned}
&\|(V*|u|^2)v\|_{L^{q'}(I;B^{1/2}_{r',2})}\\
&\lesssim \|V\|_{L^p_*}\|v\|_{L^q(I;B^{1/2}_{r,2})}\|u\|^2_{L^k(I;L^s)}+\|V\|_{L^p_*}\|u\|_{L^q(I;B^{1/2}_{r,2})}\|u\|_{L^k(I;L^s)}\|v\|_{L^k(I;L^s)},
\end{aligned}$$

where the exponents satisfy

$$\begin{cases} \dfrac{d}{p} = 2\delta(r)+2\delta(s), \\ \dfrac{2}{q}+\dfrac{2}{k}=1. \end{cases}$$

Since $V(x)=|x|^{-4}\in L^{d/4}_*$, if we take the admissible pair $q=r=\frac{2(d+1)}{d-1}$ and $\delta(s)=1+1/k$ (then $\delta(r)=d/(d+1)$, $k=d+1$), then

$$\begin{aligned}
\|(V*|u|^2)v\|_{L^{q'}(I;B^{1/2}_{q',2})} &\lesssim \|v\|_{[W](I)}\|u\|^2_{L^k(I;L^s)}\\
&\quad+\|u\|_{[W](I)}\|u\|_{L^k(I;L^s)}\|v\|_{L^k(I;L^s)}.
\end{aligned}\tag{2.3}$$

The Hölder inequality and the Sobolev embedding theorem yield

$$\|v\|_{L^k(I;L^s)}\le \|v\|^{\frac{d-3}{d-1}}_{L^\infty_t L^{2^*}_x}\|v\|^{\frac{2}{d-1}}_{L_t^{\frac{2(d+1)}{d-1}}L_x^{\frac{2d(d+1)}{d^2-2d-1}}}\lesssim \|v\|^{\frac{d-3}{d-1}}_{L^\infty_t\dot H^1_x}\|v\|^{\frac{2}{d-1}}_{[W](I)}.\tag{2.4}$$

Plugging (2.4) into (2.3), we get

$$\begin{aligned}
\|(V*|u|^2)v\|_{L^{q'}(I;B^{1/2}_{q',2})} &\lesssim \|v\|_{[W](I)}\|u\|^{\frac{2(d-3)}{d-1}}_{L^\infty_t\dot H^1_x}\|u\|^{\frac{4}{d-1}}_{[W](I)}\\
&\quad+\|u\|^{1+\frac{2}{d-1}}_{[W](I)}\|u\|^{\frac{d-3}{d-1}}_{L^\infty_t\dot H^1_x}\|v\|^{\frac{d-3}{d-1}}_{L^\infty_t\dot H^1_x}\|v\|^{\frac{2}{d-1}}_{[W](I)}. \blacksquare
\end{aligned}$$

Now, we can state the local well-posedness for (1.1) with large initial data and small data scattering in the energy space $H^1\times L^2$.

THEOREM 2.4 (Small data scattering). *Assume that $d\ge 5$ and (u_0,u_1) is in $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$. There exists a small constant $\delta=\delta(E)$ such that if $\|(u_0,u_1)\|_{H^1\times L^2}\le E$ and I is an interval such that*

$$\|\dot K(t)u_0+K(t)u_1\|_{W(I)}\le\delta,$$

then there exists a unique strong solution u to (1.1) in $I\times\mathbb{R}^d$, with u in $C(I;H^1)\cap C^1(I;L^2)$ and

$$\|u\|_{W(I)}\le 2C\delta.$$

Let $(T_-(u_0,u_1),T_+(u_0,u_1))$ be the maximal time interval on which u is well-defined.

REMARK 2.5. (1) There exists $\tilde\delta$ such that if $\|(u_0,u_1)\|_{H^1\times L^2}\le\tilde\delta$, the conclusion of Theorem 2.4 applies to any interval I. Indeed, by Strichartz estimates, $\|\dot K(t)u_0+K(t)u_1\|_{W(I)}\le C\tilde\delta$ and the claim follows.

(2) Given $(u_0, u_1) \in H^1 \times L^2$, there exists $(0 \in)\ I$ such that the hypothesis of Theorem 2.4 is satisfied on I. This is clear because, by Strichartz estimates, $\|\dot{K}(t)u_0 + K(t)u_1\|_{W(\mathbb{R})} < \infty$.

We conclude this subsection by recalling the following standard finite blow-up criterion.

Lemma 2.6 (Standard finite blow-up criterion). *If $T_+(u_0, u_1) < \infty$, then*

$$\|u\|_{W([0,T_+(u_0,u_1)))} = \infty.$$

A corresponding result holds for $T_-(u_0, u_1)$.

The proof is similar to the one of [13, Lemma 2.11].

2.3. Perturbation lemma. In this part, we give a perturbation result for solutions of (1.1) with a global space-time estimate. First we recall some notation of [10].

With any real-valued function $u(t,x)$, we associate the complex-valued function $\vec{u}(t,x)$ by

$$\vec{u} = \langle\nabla\rangle u - i\dot{u}, \qquad u = \operatorname{Re}\langle\nabla\rangle^{-1}\vec{u}.$$

Then the free and nonlinear Klein–Gordon equations can be given by

$$\begin{cases} (\Box + 1)u = 0 \Leftrightarrow (i\partial_t + \langle\nabla\rangle)\vec{u} = 0, \\ (\Box + 1)u = -f(u) \Leftrightarrow (i\partial_t + \langle\nabla\rangle)\vec{u} = -f(\langle\nabla\rangle^{-1}\operatorname{Re}\vec{u}), \end{cases}$$

and the energy can be written as

$$\begin{aligned} \tilde{E}(\vec{u}) = E(u, \dot{u}) = \frac{1}{2}\int_{\mathbb{R}^d} (|\dot{u}|^2 + |\nabla u|^2 + |u|^2)\, dx \\ + \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\, dx\, dy. \end{aligned}$$

Lemma 2.7. *Let I be a time interval, $t_0 \in I$, and $\vec{u}, \vec{w} \in C(I; L^2(\mathbb{R}^d))$ satisfy*

$$\begin{aligned} (i\partial_t + \langle\nabla\rangle)\vec{u} &= -f(u) + \mathrm{eq}(u), \\ (i\partial_t + \langle\nabla\rangle)\vec{w} &= -f(w) + \mathrm{eq}(w). \end{aligned}$$

for some functions $\mathrm{eq}(u), \mathrm{eq}(w)$. *Assume that for some constants $M, E > 0$, we have*

$$\|w\|_{ST(I)} \le M, \qquad \|\vec{u}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} + \|\vec{w}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \le E.$$

Let $t_0 \in I$, and let $(u(t_0), u_t(t_0))$ be close to $(w(t_0), w_t(t_0))$ in the sense that

$$\|(u(t_0) - w(t_0), u_t(t_0) - w_t(t_0))\|_{H^1\times L^2} \le E'.$$

Let $\vec{\gamma}_0 = e^{i\langle\nabla\rangle(t-t_0)}(\vec{u}-\vec{w})(t_0)$ *and assume also that we have the smallness condition*

$$\|\gamma_0\|_{ST(I)} + \|(\mathrm{eq}(u),\mathrm{eq}(w))\|_{ST^*(I)} \le \epsilon, \tag{2.5}$$

where $0<\epsilon<\epsilon_1=\epsilon_1(M,E)$ *is a small constant and*

$$ST^*(I) = [W]^*(I) + L_t^1(I; L_x^2(\mathbb{R}^d)).$$

Then

$$\|u-w\|_{ST(I)} \le C(M,E)\epsilon$$

and

$$\|u\|_{ST(I)} \le C(M,E,E').$$

Proof. Since $\|w\|_{ST(I)} \le M$, there exists a partition of I to the right of t_0:

$$t_0 < t_1 < \cdots < t_N, \qquad I_j = (t_j, t_{j+1}), \qquad I\cap(t_0,\infty) = (t_0, t_N),$$

such that $N \le C(L,\delta)$ and for any $j=0,1,\ldots,N-1$, we have

$$\|w\|_{ST(I_j)} \le \delta \ll 1. \tag{2.6}$$

The estimate to the left of t_0 is analogous; we omit it.

Let

$$\gamma(t) = u(t)-w(t), \qquad \vec{\gamma}_j(t) = e^{i\langle\nabla\rangle(t-t_j)}\vec{\gamma}(t_j), \qquad 0\le j\le N-1.$$

Then γ satisfies the following difference equation:

$$\begin{cases} (i\partial_t + \langle\nabla\rangle)\vec{\gamma} = (V*|w|^2)\gamma + 2[V*(\gamma w)] + 2[V*(\gamma w)]\gamma \\ \qquad\qquad + (V*|\gamma|^2)w + (V*|\gamma|^2)\gamma + \mathrm{eq}(u) - \mathrm{eq}(w), \\ \vec{\gamma}(t_j) = \vec{\gamma}_j(t_j), \end{cases}$$

which implies that

$$\vec{\gamma}(t) = \vec{\gamma}_j(t) - i\int_{t_j}^{t} e^{i\langle\nabla\rangle(t-s)}\big((V*|w|^2)\gamma + 2[V*(\gamma w)]w + 2[V*(\gamma w)]\gamma$$
$$+ (V*|\gamma|^2)w + (V*|\gamma|^2)\gamma + \mathrm{eq}(u)-\mathrm{eq}(w)\big)\,ds,$$

$$\vec{\gamma}_{j+1}(t) = \vec{\gamma}_j(t) - i\int_{t_j}^{t_{j+1}} e^{i\langle\nabla\rangle(t-s)}\big((V*|w|^2)\gamma + 2[V*(\gamma w)]w + 2[V*(\gamma w)]\gamma$$
$$+ (V*|\gamma|^2)w + (V*|\gamma|^2)\gamma + \mathrm{eq}(u)-\mathrm{eq}(w)\big)\,ds.$$

By Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
(2.7)\qquad & \|\gamma-\gamma_j\|_{ST(I_j)}+\|\gamma_{j+1}-\gamma_j\|_{ST(\mathbb{R})} \\
&\lesssim \|(V*|w|^2)\gamma+2[V*(\gamma w)]w+2[V*(\gamma w)]\gamma+(V*|\gamma|^2)w+(V*|\gamma|^2)\gamma\|_{[W]^*(I_j)} \\
&\quad+\|(\mathrm{eq}(u),\mathrm{eq}(w))\|_{ST^*(I_j)} \\
&\lesssim \|\gamma\|_{[W](I_j)}\|w\|^{\frac{2(d-3)}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|w\|^{\frac{4}{d-1}}_{[W](I_j)} \\
&\quad+\|w\|^{1+\frac{2}{d-1}}_{[W](I_j)}\|w\|^{\frac{d-3}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|\gamma\|^{\frac{d-3}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|\gamma\|^{\frac{2}{d-1}}_{[W](I_j)} \\
&\quad+\|w\|_{[W](I_j)}\|\gamma\|^{\frac{2(d-3)}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|\gamma\|^{\frac{4}{d-1}}_{[W](I_j)} \\
&\quad+\|\gamma\|^{1+\frac{2}{d-1}}_{[W](I_j)}\|\gamma\|^{\frac{d-3}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|w\|^{\frac{d-3}{d-1}}_{L_t^\infty(I_j;\dot H_x^1)}\|w\|^{\frac{2}{d-1}}_{[W](I_j)} \\
&\quad+\|\gamma\|^{1+\frac{4}{d-1}}_{[W](I_j)}\|\gamma\|^{\frac{2(d-3)}{d-1}}_{L^\infty(I_j;H^1)}+\|(\mathrm{eq}(u),\mathrm{eq}(w))\|_{ST^*(I_j)}.
\end{aligned}
$$

Therefore, assuming that

$$
(2.8)\qquad \|\gamma\|_{ST(I_j)}\le\delta\ll 1,\qquad \forall j=0,1,\dots,N-1,
$$

by (2.6) and (2.7) we have

$$
\|\gamma\|_{ST(I_j)}+\|\gamma_{j+1}\|_{ST(t_{j+1},t_N)}\le C\|\gamma_j\|_{ST(t_j,t_N)}+\epsilon
$$

for some absolute constant $C>0$. By (2.5) and iteration on j, we obtain

$$
\|\gamma\|_{ST(I)}\le(2C)^N\epsilon\le\delta/2,
$$

if we choose ϵ_1 sufficiently small. Hence the assumption (2.8) is justified by continuity in t and induction on j. Then repeating the estimate (2.7) once again, we can get the ST-norm estimate on γ, which implies the Strichartz estimates on u. ∎

3. Profile decomposition. In this section, we first recall the linear profile decomposition of the sequence of H^1-bounded solutions of (1.1) which was established in [10]. Then we utilize it to give the orthogonal analysis of the nonlinear energy and the nonlinear profile decomposition which will be used to construct the critical element and obtain its compactness properties.

3.1. Linear profile decomposition. First, we give some notation as introduced in [10]. For any triple $(t_n^j,x_n^j,h_n^j)\in\mathbb{R}\times\mathbb{R}^d\times(0,\infty)$ with arbitrary n and j, let τ_n^j, T_n^j, and $\langle\nabla\rangle_n^j$ respectively denote the scaled time shift, the unitary operator, and the self-adjoint operator in $L^2(\mathbb{R}^d)$, defined by

$$
\tau_n^j=-\frac{t_n^j}{h_n^j},\qquad T_n^j\varphi(x)=(h_n^j)^{-d/2}\varphi\Big(\frac{x-x_n^j}{h_n^j}\Big),\qquad \langle\nabla\rangle_n^j=\sqrt{-\Delta+(h_n^j)^2}.
$$

We denote the set of Fourier multipliers by

$$\mathcal{MC} = \Big\{\mu = \mathcal{F}^{-1}\tilde{\mu}\mathcal{F} \;\Big|\; \tilde{\mu} \in C(\mathbb{R}^d),\ \exists \lim_{|x|\to\infty} \tilde{\mu}(x) \in \mathbb{R}\Big\}.$$

Now we can state the linear profile decomposition:

LEMMA 3.1 (Linear profile decomposition, [10]). *Let $\vec{v}_n(t) = e^{i\langle\nabla\rangle t}\vec{v}_n(0)$ be a sequence of free Klein–Gordon solutions with uniformly bounded L^2_x norm. Then after replacing it with some subsequence, there exist $K \in \{0, 1, \dots, \infty\}$ and, for each integer $j \in [0, K)$, $\varphi^j \in L^2(\mathbb{R}^d)$ and $\{(t^j_n, x^j_n, h^j_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}\times\mathbb{R}^d\times(0,1]$ satisfying the following. Define $\vec{v}^j_n$ and $\vec{\omega}^k_n$ for each $j < k \le K$ by*

$$\vec{v}_n(t,x) = \sum_{j=0}^{k-1} \vec{v}^j_n(t,x) + \vec{\omega}^k_n(t,x),$$

$$\vec{v}^j_n(t,x) = e^{i\langle\nabla\rangle(t-t^j_n)} T^j_n \varphi^j(x) = T^j_n\big(e^{i\langle\nabla\rangle^j_n \frac{t-t^j_n}{h^j_n}} \varphi^j\big).$$

Then

$$\lim_{k\to K} \overline{\lim_{n\to\infty}} \|\vec{\omega}^k_n\|_{L^\infty(\mathbb{R}; B^{-d/2}_{\infty,\infty}(\mathbb{R}^d))} = 0, \tag{3.1}$$

and for any $\mu \in \mathcal{MC}$, any $l < j < k \le K$, and any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \langle \mu\vec{v}^l_n, \mu\vec{v}^j_n\rangle^2_{L^2_x} = 0 = \lim_{n\to\infty} \langle \mu\vec{v}^j_n, \mu\vec{\omega}^k_n\rangle^2_{L^2_x}, \tag{3.2}$$

$$\lim_{n\to\infty} \left(\left|\frac{h^l_n}{h^j_n}\right| + \left|\frac{h^j_n}{h^l_n}\right| + \frac{|t^j_n - t^k_n| + |x^j_n - x^k_n|}{h^l_n} \right) = \infty. \tag{3.3}$$

Moreover, each sequence $\{h^j_n\}_{n\in\mathbb{N}}$ either tends to 0 or is identically 1 for all n.

REMARK 3.2. We call $\{\vec{v}^j_n\}_{n\in\mathbb{N}}$ a *free concentrating wave* for each j, and $\vec{w}^k_n$ the *remainder*. From (3.2), we have the following asymptotic orthogonality:

$$\lim_{n\to\infty} \Big(\|\mu\vec{v}_n(t)\|^2_{L^2} - \sum_{j=0}^{k-1} \|\mu\vec{v}^j_n(t)\|^2_{L^2} - \|\mu\vec{\omega}^k_n(t)\|^2_{L^2} \Big) = 0, \quad \forall \mu \in \mathcal{MC}. \tag{3.4}$$

Next we begin the orthogonal analysis of the nonlinear energy. It follows from Mikhlin's theorem that for $1 < p < \infty$,

$$\|[|\nabla| - \langle\nabla\rangle_n]\varphi\|_p \lesssim h_n \|\langle\nabla/h_n\rangle^{-1}\varphi\|_p, \tag{3.5}$$

$$\|[|\nabla|^{-1} - \langle\nabla\rangle^{-1}_n]\varphi\|_p \lesssim \|\langle\nabla/h_n\rangle^{-2}|\nabla|^{-1}\varphi\|_p, \tag{3.6}$$

uniformly for $0 < h_n \le 1$.

LEMMA 3.3. *Let $\vec{v}_n$ be a sequence of free Klein–Gordon solutions satisfying $\vec{v}_n(0) \in L^2_x$. Let $\vec{v}_n = \sum_{j=0}^{k-1} \vec{v}^j_n + \vec{\omega}^k_n$ be the linear profile decomposition* *given by Lemma* 3.1. *If* $\overline{\lim}_{n\to\infty}\tilde{E}(\vec{v}_n(0))<\infty$, *then* $\vec{v}_n^j(0)\in L_x^2$ *for large* n, *and*

$$\lim_{k\to K}\overline{\lim_{n\to\infty}}\Big|\tilde{E}(\vec{v}_n(0))-\sum_{j=0}^{k-1}\tilde{E}(\vec{v}_n^j(0))-\tilde{E}(\vec{\omega}_n^k(0))\Big|=0. \tag{3.7}$$

Moreover, for all $j<k$,

$$0\le \lim_{n\to\infty}\tilde{E}(\vec{v}_n^j(0))\le \overline{\lim_{n\to\infty}}\,\tilde{E}(\vec{v}_n^j(0))\le \overline{\lim_{n\to\infty}}\,\tilde{E}(\vec{v}_n(0)), \tag{3.8}$$

where the last inequality becomes an equality only if $K=1$ *and* $\vec{\omega}_n^1\to 0$ *in* $L_t^\infty L_x^2$.

Proof. First, we claim that

$$\|u\|_{L_x^{2^*}}\lesssim \|u\|_{H^1}^{(d-2)/d}\|u\|_{B^{1-d/2}_{\infty,\infty}}^{2/d},\qquad 2^*=\frac{2d}{d-2}. \tag{3.9}$$

In fact, on the one hand, by the Hölder and Bernstein equalities, we have

$$\|P_{\le 1}u\|_{L_x^{2^*}}\lesssim \|P_{\le 1}u\|_{L_x^2}^{(d-2)/d}\|P_{\le 1}u\|_{L_x^\infty}^{2/d}\lesssim \|u\|_{H^1}^{(d-2)/d}\|u\|_{B^{1-d/2}_{\infty,\infty}}^{2/d}.$$

On the other hand, from sharp interpolation [1], we know that

$$\|P_{>1}u\|_{L_x^{2^*}}\lesssim \|P_{>1}u\|_{L_x^2}^{(d-2)/d}\|P_{>1}u\|_{\dot{B}^{1-d/2}_{\infty,\infty}}^{2/d}\lesssim \|u\|_{H^1}^{(d-2)/d}\|u\|_{B^{1-d/2}_{\infty,\infty}}^{2/d},$$

which gives the claim.

Thus, by (3.9) and (3.1), we obtain

$$\lim_{k\to K}\overline{\lim_{n\to\infty}}\,\|\omega_n^k\|_{L_x^{2^*}}\le \lim_{k\to K}\overline{\lim_{n\to\infty}}\,\|\omega_n^k\|_{H^1}^{(d-2)/d}\|\omega_n^k\|_{B^{1-d/2}_{\infty,\infty}}^{2/d}=0,$$

where $\omega_n^k=\operatorname{Re}\langle\nabla\rangle^{-1}\vec{\omega}_n^k$. This implies that if there exists $i\in\{1,2,3,4\}$ such that $u_i=\omega_n^k$, then by the Hölder and the Hardy–Littlewood–Sobolev inequalities, we get

$$\lim_{k\to K}\overline{\lim_{n\to\infty}}\,\|(V(x)*(u_1u_2))(u_3u_4)\|_{L_x^1}\le \lim_{k\to K}\overline{\lim_{n\to\infty}}\prod_{i=1}^4\|u_i\|_{L_x^{2^*}}=0.$$

This together with (3.4) reduces our task to proving

$$\lim_{k\to K}\overline{\lim_{n\to\infty}}\Big|F\Big(\sum_{j<k}v_n^j(0)\Big)-\sum_{j<k}F(v_n^j(0))\Big|=0$$

where $F(u)=\|(V(x)*|u|^2)|u|^2\|_{L_x^1}$.

Moreover, using the decay of $e^{it\langle\nabla\rangle}$ in $\mathcal{S}\to L_x^{2^*}$ uniform with respect to n and the Sobolev embedding $\dot{H}^1(\mathbb{R}^d)\subset L^{2^*}(\mathbb{R}^d)$, we have

$$\|v_n^j\|_{L_x^{2^*}}\le \|\langle\nabla\rangle^{-1}e^{-i\langle\nabla\rangle_n^j\tau_n^j}\varphi^j(x)\|_{L_x^{2^*}}\to 0\quad \text{as } n\to\infty.$$

Thus, we can discard those j where $\tau_n^j=-t_n^j/h_n^j\to\infty$.

Hence, up to a subsequence, we may assume that $\tau_n^j \to \tau_\infty^j \in \mathbb{R}$ for all j. Let

$$\psi^j := \operatorname{Re} e^{-i\langle\nabla\rangle_\infty^j \tau_\infty^j}\varphi^j \in L_x^2(\mathbb{R}^d).$$

We have

$$\begin{aligned}
&\Big|F\Big(\sum_{j<k} v_n^j(0)\Big) - \sum_{j<k} F(v_n^j(0))\Big| \\
&\quad\le \Big|F\Big(\sum_{j<k} v_n^j(0)\Big) - F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big)\Big| \\
&\qquad + \Big|\sum_{j<k} F(v_n^j(0)) - \sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j)\Big| \\
&\qquad + \Big|F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big) - \sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j)\Big|.
\end{aligned} \tag{3.10}$$

By the continuity of the operator $e^{it\langle\nabla\rangle}$ in t in H^1, we have

$$v_n^j(0) - \langle\nabla\rangle^{-1}T_n^j\psi^j \to 0 \quad \text{in } H^1(\mathbb{R}^d) \text{ as } n\to\infty.$$

This together with the nonlinear estimate

$$\|(V(\cdot) * (g_1g_2))g_3g_4\|_{L_x^1} \lesssim \prod_{j=1}^4 \|g_j\|_{L_x^{2^*}} \tag{3.11}$$

shows that as $n\to\infty$,

$$\Big|F\Big(\sum_{j<k} v_n^j(0)\Big) - F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big)\Big| \to 0,$$

$$\Big|\sum_{j<k} F(v_n^j(0)) - \sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j)\Big| \to 0.$$

Now we consider the last term on the right hand side of (3.10). Let

$$\hat\psi^j = \begin{cases} |\nabla|^{-1}\psi^j & \text{if } h_n^j \to 0,\\ \langle\nabla\rangle^{-1}\psi^j & \text{if } h_n^j \equiv 1.\end{cases}$$

Then $\hat\psi^j \in L_x^{2^*}$ and

$$\begin{aligned}
&\Big|F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big) - \sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j)\Big| \\
&\quad\lesssim \Big|F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big) - F\Big(\sum_{j<k} h_n^jT_n^j\hat\psi^j\Big)\Big| \\
&\qquad + \Big|\sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j) - \sum_{j<k} F(h_n^jT_n^j\hat\psi^j)\Big| \\
&\qquad + \Big|F\Big(\sum_{j<k} h_n^jT_n^j\hat\psi^j\Big) - \sum_{j<k} F(h_n^jT_n^j\hat\psi^j)\Big|.
\end{aligned}$$

By (3.5), one has

$$
\begin{aligned}
\|\langle\nabla\rangle^{-1}T_n^j\psi^j - h_n^jT_n^j\hat{\psi}^j\|_{L_x^{2^*}} &= \begin{cases} \|\langle\nabla\rangle^{-1}T_n^j\psi^j - h_n^jT_n^j|\nabla|^{-1}\psi^j\|_{L_x^{2^*}} & \text{if } h_n^j \to 0, \\ \|\langle\nabla\rangle^{-1}T_n^j\psi^j - h_n^jT_n^j\langle\nabla\rangle^{-1}\psi^j\|_{L_x^{2^*}} & \text{if } h_n^j \equiv 1 \end{cases} \\
&= \begin{cases} \|(\langle\nabla\rangle_n^j)^{-1}\psi^j - |\nabla|^{-1}\psi^j\|_{L_x^{2^*}} & \text{if } h_n^j \to 0, \\ 0 & \text{if } h_n^j \equiv 1 \end{cases} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Combining this with (3.11), we find that as $n \to \infty$,

$$
\begin{aligned}
\Big|F\Big(\sum_{j<k}\langle\nabla\rangle^{-1}T_n^j\psi^j\Big) - F\Big(\sum_{j<k} h_n^jT_n^j\hat{\psi}^j\Big)\Big| &\to 0, \\
\Big|\sum_{j<k} F(\langle\nabla\rangle^{-1}T_n^j\psi^j) - \sum_{j<k} F(h_n^jT_n^j\hat{\psi}^j)\Big| &\to 0.
\end{aligned}
$$

Thus it suffices to show that as $n \to \infty$,

$$
\Big|F\Big(\sum_{j<k} h_n^jT_n^j\hat{\psi}^j\Big) - \sum_{j<k} F(h_n^jT_n^j\hat{\psi}^j)\Big| \to 0. \tag{3.12}
$$

Now we define $\hat{\psi}_{n,R}^j$ for any $R \gg 1$ by

$$
\hat{\psi}_{n,R}^j(x) = \chi_R(x)\hat{\psi}^j \prod\{(1-\chi_{h_n^{j,l}R})(x - x_n^{j,l}) \mid 1 \le l < k,\ h_n^lR < h_n^j\},
$$

where $(h_n^{j,l}, x_n^{j,l}) = (h_n^l, x_n^j - x_n^l)/h_n^j$, and $\chi_R(x) = \chi(x/R)$ with $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Then $\hat{\psi}_{n,R}^j \to \chi_R\hat{\psi}^j$ in $L_x^{2^*}$ as $n \to \infty$, since either $h_n^{j,l} \to 0$ or $|x_n^{j,l}| \to \infty$ by (3.3). Moreover, $\chi_R\hat{\psi}^j \to \hat{\psi}^j$ in $L_x^{2^*}$ as $R \to \infty$.

Hence we may replace $\hat{\psi}^j$ by $\hat{\psi}_{n,R}^j$ in (3.12). Since $\{\operatorname{supp}_{(t,x)} h_n^jT_n^j\hat{\psi}_{n,R}^j\}$ are mutually disjoint for large n, it follows that for large n,

$$
\Big|\sum_{j<k} h_n^jT_n^j\hat{\psi}_{n,R}^j\Big|^2 = \sum_{j<k} |h_n^jT_n^j\hat{\psi}_{n,R}^j|^2.
$$

Then

$$
\begin{aligned}
&\Big|F\Big(\sum_{j<k} h_n^jT_n^j\hat{\psi}_{n,R}^j\Big) - \sum_{j<k} F(h_n^jT_n^j\hat{\psi}_{n,R}^j)\Big| \\
&\qquad \le \sum_{j\neq l} \Big\|(V(\cdot) * |h_n^jT_n^j\hat{\psi}_{n,R}^j|^2)|h_n^lT_n^l\hat{\psi}_{n,R}^l|^2\Big\|_{L_x^1(\mathbb{R}^d)} \\
&\qquad = \sum_{j\neq l} (h_n^{j,l})^{2-d}\Big\|(V(\cdot) * |\hat{\psi}_{n,R}^j|^2)\Big|\hat{\psi}_{n,R}^l\Big(\frac{x - x_n^{j,l}}{h_n^{j,l}}\Big)\Big|^2\Big\|_{L_x^1(\mathbb{R}^d)} \\
&\qquad \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

by Lebesgue's dominated convergence theorem, since either $h_n^{j,l} \to 0$ or $|x_n^{j,l}| \to \infty$ by (3.3). This concludes the proof of Lemma 3.3. ■

3.2. Nonlinear profile decomposition. Having established the linear profile decomposition of a sequence of initial data in the last subsection, we now show the nonlinear profile decomposition of a sequence of solutions of (1.1) with the same initial data in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ by following the argument in [10].

First we construct a nonlinear profile corresponding to a free concentrating wave. Let $\vec{v}_n$ be a free concentrating wave for a sequence (t_n, x_n, h_n) in $\mathbb{R} \times \mathbb{R}^d \times (0, 1]$,

$$\begin{cases} (i\partial_t + \langle\nabla\rangle)\vec{v}_n = 0, \\ \vec{v}_n(t_n) = T_n\phi(x), \quad \phi \in L^2(\mathbb{R}^d). \end{cases}$$

Then by Lemma 3.1, we have a sequence of free concentrating waves $\vec{v}_n^j(t, x)$ with $\vec{v}_n^j(t_n^j) = T_n^j\varphi^j$, $\varphi^j \in L^2(\mathbb{R}^d)$ for $j = 0, 1, \ldots, k-1$, such that

$$\begin{aligned} \vec{v}_n(t, x) &= \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{\omega}_n^k(t, x) = \sum_{j=0}^{k-1} e^{i\langle\nabla\rangle(t-t_n^j)} T_n^j \varphi^j(x) + \vec{\omega}_n^k(t, x) \\ &= \sum_{j=0}^{k-1} T_n^j e^{i(\frac{t-t_n^j}{h_n^j})\langle\nabla\rangle_n^j} \varphi^j + \vec{\omega}_n^k(t, x). \end{aligned}$$

Now for any concentrating wave $\vec{v}_n^j$, we undo the group action T_n^j to look for the linear profile $\vec{V}^j$. Let

$$\vec{v}_n^j(t, x) = T_n^j \vec{V}_n^j((t - t_n^j)/h_n^j).$$

Then

$$\vec{V}_n^j(t, x) = e^{it\langle\nabla\rangle_n^j}\varphi^j.$$

Now let $\vec{u}_n^j$ be the nonlinear solution with the same initial data $\vec{v}_n^j(0)$,

$$\begin{cases} (i\partial_t + \langle\nabla\rangle)\vec{u}_n^j = -f(\mathrm{Re}\langle\nabla\rangle^{-1}\vec{u}_n^j), \\ \vec{u}_n^j(0) = \vec{v}_n^j(0) = T_n^j \vec{V}_n^j(\tau_n^j), \end{cases}$$

where $\tau_n^j = -t_n^j/h_n^j$. In order to look for the nonlinear profile $\vec{U}_\infty^j$ associated with the free concentrating wave $\vec{v}_n^j$, we also need to undo the group action. Define

$$\vec{u}_n^j(t, x) = T_n^j \vec{U}_n^j((t - t_n^j)/h_n^j);$$

then $\vec{U}_n^j$ satisfies the rescaled equation

$$\begin{cases} (i\partial_t + \langle\nabla\rangle_n^j)\vec{U}_n^j = -f(\mathrm{Re}\,(\langle\nabla\rangle_n^j)^{-1}\vec{U}_n^j), \\ \vec{U}_n^j(\tau_n^j) = \vec{V}_n^j(\tau_n^j). \end{cases}$$

Up to a subsequence, we may assume that there exist $h^j_\infty \in \{0,1\}$ and $\tau^j_\infty \in [-\infty,\infty]$ for every j, such that as $n \to \infty$,

$$h^j_n \to h^j_\infty \quad \text{and} \quad \tau^j_n \to \tau^j_\infty.$$

Then the limit equations are given by

$$\vec{V}^j_\infty = e^{it\langle\nabla\rangle^j_\infty}\varphi^j, \qquad \begin{cases} (i\partial_t + \langle\nabla\rangle^j_\infty)\vec{U}^j_\infty = -f(\hat{U}^j_\infty), \\ \vec{U}^j_\infty(\tau^j_\infty) = \vec{V}^j_\infty(\tau^j_\infty), \end{cases}$$

where

$$\hat{U}^j_\infty := \mathrm{Re}\,((\langle\nabla\rangle^j_\infty)^{-1}\vec{U}^j_\infty = \begin{cases} \mathrm{Re}\,\langle\nabla\rangle^{-1}\vec{U}^j_\infty & \text{if } h^j_\infty = 1, \\ \mathrm{Re}\,|\nabla|^{-1}\vec{U}^j_\infty & \text{if } h^j_\infty = 0. \end{cases}$$

We remark that by using the standard iteration with the Strichartz estimate, we can obtain the unique existence of a local solution $\vec{U}^j_\infty$ around $t = \tau^j_\infty$ in all cases, including $h^j_\infty = 0$ and $\tau^j_\infty = \pm\infty$ (the latter corresponding to the existence of the wave operators). We define $\vec{U}^j_\infty$ on the maximal existence interval to be the nonlinear profile associated with the free concentrating wave $(\vec{v}^j_n; t^j_n, x^j_n, h^j_n)$.

The nonlinear concentrating wave $\vec{u}^j_{(n)}$ associated with $\vec{v}^j_n$ is defined by

$$\vec{u}^j_{(n)}(t,x) := T^j_n\vec{U}^j_\infty((t-t^j_n)/h^j_n).$$

It is easy to see that $u^j_{(n)}$ solves (1.1) when $h^j_\infty = 1$. If $h^j_\infty = 0$, then $u^j_{(n)}$ solves

$$\begin{cases} (\partial_{tt} - \Delta + 1)u^j_{(n)} = (i\partial_t + \langle\nabla\rangle)\vec{u}^j_{(n)} = (\langle\nabla\rangle - |\nabla|)\vec{u}^j_{(n)} - f(|\nabla|^{-1}\langle\nabla\rangle u^j_{(n)}), \\ \vec{u}^j_{(n)}(0) = T^j_n\vec{U}^j_\infty(\tau^j_n). \end{cases}$$

The existence time interval of $u^j_{(n)}$ may be finite and even go to 0, but at least we have

$$\begin{aligned} \|\vec{u}^j_n(0) - \vec{u}^j_{(n)}(0)\|_{L^2_x} &= \|T^j_n\vec{V}^j_n(\tau^j_n) - T^j_n\vec{U}^j_\infty(\tau^j_n)\|_{L^2_x} \\ &\le \|\vec{V}^j_n(\tau^j_n) - \vec{V}^j_\infty(\tau^j_n)\|_{L^2_x} \\ &\quad + \|\vec{V}^j_\infty(\tau^j_n) - \vec{U}^j_\infty(\tau^j_n)\|_{L^2_x} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.13}$$

Let u_n be a sequence of (local) solutions of (1.1) around $t = 0$, and let v_n be the sequence of the free solutions with the same initial data. We consider the linear profile decomposition of $\{\vec{v}_n\}$ given by Lemma 3.1,

$$\vec{v}_n = \sum_{j=0}^{k-1}\vec{v}^j_n + \vec{\omega}^k_n, \qquad \vec{v}^j_n = e^{i\langle\nabla\rangle(t-t^j_n)}T^j_n\varphi^j.$$

DEFINITION 3.4 (Nonlinear profile decomposition). Let $\{\vec{v}^j_n\}_{n\in\mathbb{N}}$ be the free concentrating waves, and $\{\vec{u}^j_{(n)}\}_{n\in\mathbb{N}}$ be the sequence of the nonlinear concentrating waves associated with $\{\vec{v}_n^j\}_{n\in\mathbb{N}}$. Then we define the nonlinear profile decomposition of u_n by

$$\vec{u}_{(n)}^{<k} := \sum_{j=0}^{k-1} \vec{u}_{(n)}^j = \sum_{j=0}^{k-1} T_n^j \vec{U}_\infty^j \left(\frac{t - t_n^j}{h_n^j} \right). \tag{3.14}$$

We will show that $\vec{u}_{(n)}^{<k} + \vec{\omega}_n^k$ is a good approximation for $\vec{u}_n$, provided that each nonlinear profile has finite global Strichartz norm.

Next we introduce some Strichartz norms. Let $ST(I)$ and $ST^*(I)$ be the function spaces on $I \times \mathbb{R}^d$ defined as above,

$$\begin{aligned} ST(I) &= [W](I) = L_t^{\frac{2(d+1)}{d-1}}(I; B^{1/2}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)), \\ ST^*(I) &= [W]^*(I) + L_t^1(I; L^2(\mathbb{R}^d)). \end{aligned}$$

The Strichartz norm for the nonlinear profile $\hat{U}_\infty^j$ depends on the scaling h_∞^j:

$$ST_\infty^j(I) := \begin{cases} ST(I) & \text{if } h_\infty^j = 1, \\ L_t^q(I; \dot{B}^{1/2}_{q,2}) \left(q = \frac{2(d+1)}{d-1}\right) & \text{if } h_\infty^j = 0. \end{cases}$$

The following two lemmas derive from Lemma 3.1 and the perturbation lemma. The first lemma concerns orthogonality in the Strichartz norms.

Lemma 3.5. *Assume that in the nonlinear profile decomposition* (3.14), *we have*

$$\|\hat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})} + \|\vec{U}_\infty^j\|_{L_t^\infty L_x^2(\mathbb{R})} < \infty, \qquad \forall j < k.$$

Then, for any finite interval I and $j < k$, we have

$$\overline{\lim_{n\to\infty}} \|u_{(n)}^j\|_{ST(I)} \lesssim \|\hat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})}, \tag{3.15}$$

$$\overline{\lim_{n\to\infty}} \|u_{(n)}^{<k}\|_{ST(I)}^2 \lesssim \overline{\lim_{n\to\infty}} \sum_{j=0}^{k-1} \|u_{(n)}^j\|_{ST(\mathbb{R})}^2, \tag{3.16}$$

where the implicit constants do not depend on the interval I or j. Moreover,

$$\lim_{n\to\infty} \left\| f(u_{(n)}^{<k}) - \sum_{j=0}^{k-1} f((\langle\nabla\rangle_\infty^j)^{-1} \langle\nabla\rangle u_{(n)}^j) \right\|_{ST^*(I)} = 0, \tag{3.17}$$

where $f(u) = (V(x) * |u|^2)u$.

Proof. One can refer to [10] for the proof of (3.15) and (3.16). Now we turn to (3.17). By the definition of $u_{(n)}^j$ and $\hat{U}_\infty^j$, we know that

$$
\begin{aligned}
u^j_{(n)}(x,t) &= \mathrm{Re}\,\langle\nabla\rangle^{-1}\vec{u}^j_{(n)}(t,x) = \mathrm{Re}\,\langle\nabla\rangle^{-1}T^j_n\vec{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big)\\
&= h^j_nT^j_n\frac{\langle\nabla\rangle^j_\infty}{\langle\nabla\rangle^j_n}\hat{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big).
\end{aligned}
$$

Let $u^{<k}_{\langle n\rangle}(t,x) = \sum_{j<k} u^j_{\langle n\rangle}(x,t)$, where

$$
u^j_{\langle n\rangle}(x,t) := \frac{\langle\nabla\rangle}{\langle\nabla\rangle^j_\infty}u^j_{(n)} = h^j_nT^j_n\hat{U}^j_\infty\Big(\frac{t-t^j_n}{h^j_n}\Big).
$$

Then

$$
\begin{aligned}
\Big\|f(u^{<k}_{(n)}) - \sum_{j<k} f((\langle\nabla\rangle^j_\infty)^{-1}\langle\nabla\rangle u^j_{(n)})\Big\|_{ST^*(I)} &= \Big\|f(u^{<k}_{(n)}) - \sum_{j<k} f(u^j_{\langle n\rangle})\Big\|_{ST^*(I)}\\
&\le \|f(u^{<k}_{(n)}) - f(u^{<k}_{\langle n\rangle})\|_{ST^*(I)} \qquad (3.18)\\
&\quad + \Big\|f(u^{<k}_{\langle n\rangle}) - \sum_{j<k} f(u^j_{\langle n\rangle})\Big\|_{ST^*(I)}. \qquad (3.19)
\end{aligned}
$$

First, we estimate (3.18). Let $[G](I) = L^{3(d+1)}_t(I; L_x^{\frac{6d(d+1)}{3d^2-3d-8}})$. It follows from (2.2) and the Hölder inequality that ($q = 2(d+1)/(d-1)$)

$$
\begin{aligned}
(3.18) &\le \|f(u^{<k}_{(n)}) - f(u^{<k}_{\langle n\rangle})\|_{L^{q'}(I;\dot{B}^{1/2}_{q',2})} + \|f(u^{<k}_{(n)}) - f(u^{<k}_{\langle n\rangle})\|_{L^{q'}_{t,x}(I\times\mathbb{R}^d)}\\
&\lesssim \|u^{<k}_{(n)} - u^{<k}_{\langle n\rangle}\|^{\frac{2}{d-1}}_{ST^j_\infty(I)}(\|(u^{<k}_{(n)}, u^{<k}_{\langle n\rangle})\|_{L^\infty_t\dot{H}^1_x\cap ST^j_\infty(I)})^{3-\frac{2}{d-1}}\\
&\quad + |I|^{1/2}\|u^{<k}_{(n)} - u^{<k}_{\langle n\rangle}\|_{[G](I)}(\|(u^{<k}_{(n)}, u^{<k}_{\langle n\rangle})\|_{[G](I)})^2\\
&\lesssim \Big(\sum_{j<k,\, h^j_\infty=0} \|\langle\nabla/h^j_n\rangle^{-2}\hat{U}^j_\infty\|_{ST^j_\infty(I)}\Big)^{\frac{2}{d-1}}(\|(u^{<k}_{(n)}, u^{<k}_{\langle n\rangle})\|_{L^\infty_t\dot{H}^1\cap ST^j_\infty(I)})^{3-\frac{2}{d-1}}\\
&\quad + |I|^{1/2}\sum_{j<k,\, h^j_\infty=0}\|\langle\nabla/h^j_n\rangle^{-2}\hat{U}^j_\infty\|_{[G](I)}(\|(u^{<k}_{(n)}, u^{<k}_{\langle n\rangle})\|_{[G](I)})^2\\
&\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

where we utilize (3.6) in the second last inequality and the fact that $\hat{U}^j_\infty \in L^\infty_t\dot{H}^1_x\cap ST^j_\infty(I) \subset [G](I)$.

Next we estimate (3.19). For $R \gg 1$, we define

$$
\hat{U}^j_{n,R}(t,x) = \chi_R(t,x)\hat{U}^j_\infty(t,x)\prod_{l<k}\{(1-\chi_{h^{j,l}_nR})(t-t^{j,l}_n, x-x^{j,l}_n) \mid h^{j,l}_n < R^{-1}\},
$$

where $(h^{j,l}_n, t^{j,l}_n, x^{j,l}_n) = (h^l_n, t^j_n - t^l_n, x^j_n - x^l_n)/h^j_n$, and $\chi_R(t,x) = \chi(t/R, x/R)$ with $\chi \in C^\infty_c(\mathbb{R}^{d+1})$ satisfying $\chi(t,x) = 1$ for $|(t,x)| \le 1$ and $\chi(t,x) = 0$ for $|(t,x)| \ge 2$. Then, noting that either $h^{j,l}_n \to 0$ or $|t^{j,l}_n| + |x^{j,l}_n| \to \infty$ by (3.3), we obtain $\hat{U}^j_{n,R} \to \chi_R \hat{U}^j_\infty$ in $ST^j_\infty(\mathbb{R})$ and $[G](\mathbb{R})$ as $n \to \infty$. Furthermore, $\chi_R \hat{U}^j_\infty \to \hat{U}^j_\infty$ in the same spaces.

Therefore, we may replace $u^j_{\langle n\rangle}$ by

$$u^j_{\langle n\rangle,R} := h^j_n T^j_n \hat{U}^j_{n,R}((t - t^j_n)/h^j_n).$$

By the support property of $u^j_{\langle n\rangle,R}$ we have, for large n,

$$\Big(\sum_{j<k} u^j_{\langle n\rangle,R}\Big)^2 = \sum_{j<k} |u^j_{\langle n\rangle,R}|^2.$$

Thus, we obtain

$$\begin{aligned}
&\Big\| f(u^{<k}_{\langle n\rangle,R}) - \sum_{j<k} f(u^j_{\langle n\rangle,R})\Big\|_{ST^*(I)} \\
&\quad \le \sum_{j\neq l} \|(V(\cdot) * |u^j_{\langle n\rangle,R}|^2) u^l_{\langle n\rangle,R}\|_{ST^*(I)} \\
&\quad = \sum_{j\neq l} (h^{j,l}_n)^{1-d/2} \Big\| (V(\cdot) * |\hat{U}^j_{\langle n\rangle,R}|^2)(t,x) \hat{U}^l_{\langle n\rangle,R}\Big(\frac{t - t^{j,l}_n}{h^{j,l}_n}, \frac{x - x^{j,l}_n}{h^{j,l}_n}\Big)\Big\|_{ST^*(I)} \\
&\quad \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

by Lebesgue's dominated convergence theorem, since either $h^{j,l}_n \to 0$ or $|t^{j,l}_n| + |x^{j,l}_n| \to \infty$ by (3.3). This concludes the proof of Lemma 3.5. ■

After these preliminaries, we now show that $\vec{u}^{<k}_{(n)} + \vec{\omega}^k_n$ is a good approximation for $\vec{u}_n$ provided that each nonlinear profile has finite global Strichartz norm.

LEMMA 3.6. *Let u_n be a sequence of local solutions of* (1.1) *around $t = 0$ satisfying $\overline{\lim}_{n\to\infty} E(u_n, \dot{u}_n) < \infty$. Assume that in the nonlinear profile decomposition* (3.14), *for any j we have*

$$\|\hat{U}^j_\infty\|_{ST^j_\infty(\mathbb{R})} + \|\vec{U}^j_\infty\|_{L^\infty_t L^2_x(\mathbb{R})} < \infty. \tag{3.20}$$

Then, for large n, u_n is bounded in the Strichartz and the energy norms, that is,

$$\overline{\lim_{n\to\infty}} (\|u_n\|_{ST(\mathbb{R})} + \|\vec{u}_n\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^d)}) < \infty.$$

Proof. We only need to verify the conditions of Lemma 2.7. For this purpose, we use the fact that

$$(i\partial_t + \langle\nabla\rangle)(\vec{u}^{<k}_{(n)} + \vec{\omega}^k_n) = -f(u^{<k}_{(n)} + \omega^k_n) + \mathrm{eq}(u^{<k}_{(n)}, \omega^k_n),$$

where

$$\begin{aligned}\mathrm{eq}(u^{<k}_{(n)}, \omega^k_n) = &\sum_{j<k}(\langle\nabla\rangle - \langle\nabla\rangle^j_\infty)\vec{u}^j_{(n)} + f(u^{<k}_{(n)}) - \sum_{j<k} f(u^j_{\langle n\rangle}) \\ &+ f(u^{<k}_{(n)} + \omega^k_n) - f(u^{<k}_{(n)}),\end{aligned}$$

and $u^j_{\langle n\rangle} = (\langle\nabla\rangle^j_\infty)^{-1}\langle\nabla\rangle u^j_{(n)}$ is as before.

First, by the definition of the nonlinear concentrating wave $u^j_{(n)}$ and (3.13), we have

$$\|(\vec{u}^{<k}_{(n)}(0) + \vec{w}^k_n(0)) - \vec{u}_n(0)\|_{L^2_x} \leq \sum_{j=0}^{k-1} \|\vec{u}^j_{(n)}(0) - \vec{u}^j_n(0)\|_{L^2_x} \to 0$$

as $n \to \infty$.

Next, by the linear profile decomposition in Lemma 3.1, we get

$$\begin{aligned}\|\vec{u}_n(0)\|^2_{L^2} = \|\vec{v}_n(0)\|^2_{L^2} &\geq \sum_{j=0}^{k-1} \|\vec{v}^j_n(0)\|^2_{L^2} + o_n(1) \\ &= \sum_{j=0}^{k-1} \|\vec{u}^j_{(n)}(0)\|^2_{L^2} + o_n(1).\end{aligned} \tag{3.21}$$

Thus, using the small data scattering (Lemma 2.4), we find that except for a finite set $J \subset \mathbb{N}$, the energy of $u^j_{(n)}$ with $j \notin J$ is smaller than the iteration threshold. Hence

$$\|u^j_{(n)}\|_{ST(\mathbb{R})} \lesssim \|\vec{u}^j_{(n)}(0)\|_{L^2_x}, \qquad j \notin J.$$

This together with (3.15), (3.16), (3.20), and (3.21) shows that for any finite interval I,

$$\begin{aligned}\sup_k \overline{\lim_{n\to\infty}} \|u^{<k}_{(n)}\|^2_{ST(I)} &\lesssim \sum_{j\in J} \|u^j_{(n)}\|^2_{ST(\mathbb{R})} + \sum_{j\notin J} \|u^j_{(n)}\|^2_{ST(\mathbb{R})} \\ &\lesssim \sum_{j\in J} \|\hat{U}^j_\infty\|^2_{ST^j_\infty(\mathbb{R})} + \overline{\lim_{n\to\infty}} \|\vec{u}_n(0)\|^2_{L^2} < \infty.\end{aligned}$$

Combining this with the Strichartz estimate for ω^k_n, we get

$$\sup_k \overline{\lim_{n\to\infty}} \|u^{<k}_{(n)} + \omega^k_n\|_{ST(I)} < \infty.$$

By Lemmas 3.1 and 3.5, we have

$$\|f(u^{<k}_{(n)} + \omega^k_n) - f(u^{<k}_{(n)})\|_{ST^*(I)} \to 0,$$

$$\Big\|f(u^{<k}_{(n)}) - \sum_{j=0}^{k-1} f(u^j_{(n)})\Big\|_{ST^*(I)} \to 0,$$

as $n \to \infty$. On the other hand, the linear part in $\mathrm{eq}(u_{(n)}^{<k}, \omega_n^k)$ vanishes when $h_\infty^j = 1$, and is controlled when $h_\infty^j = 0$ by

$$\begin{aligned}
\|(\langle\nabla\rangle - |\nabla|)\vec{u}_{(n)}^j\|_{L_t^1(I;L_x^2)} &\lesssim |I|\, \|\langle\nabla\rangle^{-1}\vec{u}_{(n)}^j\|_{L_t^\infty(\mathbb{R};L_x^2)} \\
&\simeq |I|\, \|\langle\nabla/h_n^j\rangle^{-1}\vec{U}_\infty^j\|_{L_t^\infty(\mathbb{R};L_x^2)} \\
&\lesssim |I|\big(\|P_{\le(h_n^j)^{1/2}}\vec{U}_\infty^j\|_{L_t^\infty(\mathbb{R};L_x^2)} + (h_n^j)^{1/2}\|P_{>(h_n^j)^{1/2}}\vec{U}_\infty^j\|_{L_t^\infty(\mathbb{R};L_x^2)}\big) \to 0
\end{aligned}$$

as $n \to \infty$. Thus, $\|\mathrm{eq}(u_{(n)}^{<k}, \omega_n^k)\|_{ST^*(I)} \to 0$ as $n \to \infty$.

Therefore, for k sufficiently close to K and n large enough, the true solution u_n and the approximate solution $u_{(n)}^{<k} + \omega_n^k$ satisfy all the assumptions of the perturbation Lemma 2.7. Hence we obtain the desired result. ∎

4. Concentration compactness. By the profile decomposition in the previous section and perturbation theory, we show in this section that if the scattering result does not hold, then there must exist a minimal energy solution with some good compactness properties. This is the object of the following proposition.

PROPOSITION 4.1. *Suppose that $E_{\max} < \infty$. Then there exists a global solution u_c of* (1.1) *satisfying*

$$E(u_c) = E_{\max}, \qquad \|u_c\|_{ST(\mathbb{R})} = \infty.$$

Moreover, there exists $c : \mathbb{R}^+ \to \mathbb{R}^d$ such that $K = \{(u_c, \dot{u}_c)(t, x - c(t)) \mid t \in \mathbb{R}^+\}$ is precompact in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Moreover, one can assume that c is C^1 and

$$|\dot{c}(t)| \lesssim_{u_c} 1$$

uniformly in t.

Proof. The proof of [10] can be adopted verbatim, but we give a sketch for completeness. By the definition of $E_{\max}$, we can choose a sequence $\{u_n\}$ such that

$$E(u_n, \dot{u}_n) \to E_{\max} \quad \text{and} \quad \|u_n\|_{ST(I_n)} \to \infty \qquad \text{as } n \to \infty. \tag{4.1}$$

Now we consider the linear and nonlinear profile decompositions of u_n, using Lemma 3.1,

$$\begin{aligned}
&e^{it\langle\nabla\rangle}\vec{u}_n(0) = \sum_{j=0}^{k-1}\vec{v}_n^j + \vec{\omega}_n^k, \qquad \vec{v}_n^j = e^{i\langle\nabla\rangle(t-t_n^j)}T_n^j\varphi^j(x), \\
&u_{(n)}^{<k} = \sum_{j=0}^{k-1}\vec{u}_{(n)}^j, \qquad \vec{u}_{(n)}^j(t,x) = T_n^j\vec{U}_\infty^j((t-t_n^j)/h_n^j), \\
&\|\vec{v}_n^j(0) - \vec{u}_{(n)}^j(0)\|_{L_x^2} \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Lemma 3.6 precludes that all the nonlinear profiles $\vec{U}^j_\infty$ have finite global Strichartz norm. On the other hand, every solution of (1.1) with energy less than $E_{\max}$ has global finite Strichartz norm by the definition of $E_{\max}$. Hence by (3.7), we deduce that there is only one profile, i.e. $K = 1$, and so for large n,

$$\tilde{E}(\vec{u}^0_{(n)}) = E_{\max}, \quad \|\hat{U}^0_\infty\|_{ST^0_\infty(\mathbb{R})} = \infty, \quad \lim_{n\to\infty} \|\vec{\omega}^1_n\|_{L^\infty_t L^2_x} = 0.$$

If $h^0_n \to 0$, then $\hat{U}^0_\infty = \operatorname{Re}|\nabla|^{-1}\vec{U}^0_\infty$ solves the $\dot{H}^1$-critical wave-Hartree equation

$$\partial_{tt}u - \Delta u + (|x|^{-4} * |u|^2)u = 0$$

and satisfies

$$E(\hat{U}^0_\infty(\tau^0_\infty)) = E_{\max} < \infty, \quad \|\hat{U}^0_\infty\|_{L^q_t(\mathbb{R};\dot{B}^{1/2}_{q,2})} = \infty, \quad q = \frac{2(d+1)}{d-1}.$$

But Miao–Zhang–Zheng [22] have proven that there is no such solution. Hence $h^0_n = 1$, and so there exist a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ and $\phi \in L^2(\mathbb{R}^d)$ such that along some subsequence,

$$\|\vec{u}_n(0,x) - e^{-it_n\langle\nabla\rangle}\phi(x - x_n)\|_{L^2_x} \to 0 \quad n \to \infty. \tag{4.2}$$

Now we show that $\hat{U}^0_\infty = \langle\nabla\rangle^{-1}\vec{U}^j_\infty$ is a global solution. Assume not; then we can choose a sequence $t_n \in \mathbb{R}$ which approaches the maximal existence time. Since $\hat{U}^0_\infty(t + t_n)$ satisfies (4.1), applying the above argument to it, we infer by (4.2) that there are $\psi \in L^2$ and another sequence $(t'_n, x'_n) \in \mathbb{R}\times\mathbb{R}^d$ such that

$$\|\vec{U}^0_\infty(t_n) - e^{-it'_n\langle\nabla\rangle}\psi(x - x'_n)\|_{L^2_x} \to 0 \tag{4.3}$$

as $n \to \infty$. Let $\vec{v} := e^{it\langle\nabla\rangle}\psi$. For any $\varepsilon > 0$, there exist $\delta > 0$ such that, with $I = [-\delta, \delta]$,

$$\|\langle\nabla\rangle^{-1}\vec{v}(t - t'_n)\|_{ST(I)} \le \varepsilon,$$

which together with (4.3) shows that for sufficiently large n,

$$\|\langle\nabla\rangle^{-1}e^{it\langle\nabla\rangle}\vec{U}^0_\infty(t_n)\|_{ST(I)} \le \varepsilon.$$

If ε is small enough, this implies that the solution $\vec{U}^0_\infty$ exists on $[t_n - \delta, t_n + \delta]$ for large n by small data theory (Lemma 2.4). This contradicts the choice of t_n. Thus $\hat{U}^0_\infty$ is a global solution and it is just the desired critical element u_c. Moreover, since (1.1) is symmetric in t, we may assume that

$$\|u_c\|_{ST(0,\infty)} = \infty.$$

We call such a u a *forward critical element.*

One can refer to [23] for the choice of $c(t)$. This concludes the proof of Proposition 4.1. ∎

As a consequence of the above proposition and the Hardy–Littlewood–Sobolev inequality, we have

COROLLARY 4.2. *Let u be a forward critical element, and denote*

$$E_{R,c} = \int\limits_{|x-c|\geq R} (|u|^2 + |\nabla u|^2 + |\dot u|^2)\,dx + \iint\limits_{\substack{|x-c|\geq R\\ y\in\mathbb{R}^d}} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy.$$

Then for any $\eta > 0$, there exists $R(\eta) > 0$ such that

$$E_{R(\eta),c(t)} \leq \eta E(u,\dot u) \quad \textit{for any } t > 0.$$

The next corollary concludes this section.

COROLLARY 4.3. *Let u be a nonlinear strong solution of* (1.1) *such that the set K defined in Proposition* 4.1 *is precompact in $H^1 \times L^2$, and $E(u,\dot u) \neq 0$. Then there exists a constant $\beta = \beta(\tau) > 0$ such that, for all time $t > 0$, we have*

$$\int\limits_t^{t+\tau} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x_2-y_2|^2}{|x-y|^6}|u(s,x)|^2|u(s,y)|^2\,dx\,dy\,ds \geq \beta,$$

where x_2 denotes the second component of $x \in \mathbb{R}^d$. In particular,

$$\int\limits_0^t \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x_2-y_2|^2}{|x-y|^6}|u(t,x)|^2|u(t,y)|^2\,dx\,dy\,ds \gtrsim t.$$

Proof. One can refer to [23] for the detailed proof. ∎

5. Extinction of the critical element. In this section, we utilize the technique in [28] to prove that the critical solution constructed in Section 4 does not exist, thus ensuring that $E_{\max} = \infty$. This implies Theorem 1.3.

PROPOSITION 5.1. *Assume that $d \geq 5$. Then $E_{\max} = \infty$.*

Proof. We use a virial-type estimate in a direction orthogonal to the momentum vector. Up to relabeling the coordinates, we might assume that $\mathrm{Mom}(u)$ is parallel to the first coordinate. Thus

$$\int\limits_{\mathbb{R}^d} u_t(t,x)\partial_j u(t,x)\,dx = 0, \qquad \forall j \geq 2.$$

Let $\phi_R(x) = \phi(x/R)$, where $\phi(x)$ is a nonnegative smooth radial function such that $\operatorname{supp}\phi \subseteq B(0,2)$ and $\phi \equiv 1$ in $B(0,1)$. We define the virial action

$$I(t) = \int\limits_{\mathbb{R}^d} z_2\phi_R(z)\partial_2 u(t,x)u_t(t,x)\,dx,$$

where $z = x - c(t)$ and z_2 denotes the second component of $z \in \mathbb{R}^d$. Integrating by parts we get, by (1.1),

$$\begin{aligned}
\partial_t I(t) = & \int_{\mathbb{R}^d} \partial_t(z_2\phi_R(z))\partial_2 u(t,x)u_t(t,x)\,dx + \tfrac{1}{2}\int_{\mathbb{R}^d} z_2\phi_R(z)\partial_2(u_t(x,t))^2\,dx \\
& + \int_{\mathbb{R}^d} z_2\phi_R(z)\partial_2 u(t,x)\big(\Delta u - u - (V(\cdot) * |u|^2)u\big)\,dx \\
= & \ \tfrac{1}{2}\int_{\mathbb{R}^d} \big(-|u_t|^2 + |u|^2 + |\nabla u|^2 + (V(\cdot) * |u|^2)|u|^2\big)\,dx - \int_{\mathbb{R}^d} |\partial_2 u|^2\,dx \\
& + \dot{z}_2 \int_{\mathbb{R}^d} u_t\partial_2 u\,dx - 2\int_{\mathbb{R}^d} z_2\phi_R(z)|u|^2\bigg(\frac{x_2}{|x|^6} * |u|^2\bigg)\,dx \\
& + \int_{|z|\geq R} \mathcal{O}_1(u)\,dx,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{O}_1(u) = & \frac{1}{2}\bigg[\frac{z_2}{R}\phi_R' - (1-\phi_R(x))\bigg]\big[-|u_t|^2 + |u|^2 + |\nabla u|^2 + (V(\cdot) * |u|^2)|u|^2\big] \\
& - (c'(t)\cdot\nabla\phi_R)\frac{z_2}{R}\partial_2 u u_t - c_2'(t)(1-\phi_R(z))\partial_2 u u_t - (\nabla\phi_R\cdot\nabla u)z_2\partial_2 u
\end{aligned}$$

is supported on the set $|z| \geq R$ and satisfies

$$\bigg|\int_{|z|\geq R} \mathcal{O}_1(u)\,dx\bigg| \lesssim \int_{|z|\geq R} (|u|^2 + |\nabla u|^2 + |\dot{u}|^2)\,dx.$$

Moreover, we define the equirepartition of energy action

$$J(t) = \int_{\mathbb{R}^d} \phi_R(z)u(t,x)u_t(t,x)\,dx.$$

Then

$$\partial_t J(t) = \int_{\mathbb{R}^d} \big(|u_t|^2 - |u|^2 - |\nabla u|^2 - (V(\cdot) * |u|^2)|u|^2\big)\,dx + \int_{|z|\geq R} \mathcal{O}_2(u)\,dx,$$

where

$$\begin{aligned}
\mathcal{O}_2(u) = & (1-\phi_R(z))\big[|u_t|^2 - |u|^2 - |\nabla u|^2 - (V(\cdot) * |u|^2)|u|^2\big] \\
& + (c'(t)\cdot\nabla\phi_R)\frac{uu_t}{R} - \frac{u}{R}\nabla\phi_R\cdot\nabla u
\end{aligned}$$

has the same properties as $\mathcal{O}_1(u)$.

Considering $A(t) = I(t) + \frac{1}{2}J(t)$, we get

$$|A(t)| \lesssim RE(u,\dot{u}) \quad \text{for all time } t, \tag{5.1}$$

and

$$\begin{aligned}\partial_t A(t) = &- \int\limits_{\mathbb{R}^d} |\partial_2 u|^2 \, dx \\ &- 2 \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \phi_R(x - c(t))(x_2 - c_2(t)) \frac{x_2 - y_2}{|x - y|^6} |u(t,x)|^2 |u(t,y)|^2 \, dx \, dy \\ &- \int\limits_{|z| \geq R} \big(\mathcal{O}_1(u) + \tfrac{1}{2}\mathcal{O}_2(u)\big) \, dx.\end{aligned}$$

And so by symmetrization, $\partial_t A(t)$ can be rewritten as

$$\begin{aligned}-\partial_t A(t) = &\int\limits_{\mathbb{R}^d} |\partial_2 u|^2 \, dx + \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^6} |u(t,x)|^2 |u(t,y)|^2 \, dx \, dy \\ &+ I_2 + \int\limits_{|z| \geq R} (\mathcal{O}_1(u) + \mathcal{O}_2(u)) \, dx,\end{aligned}$$

where

$$\begin{aligned}I_2 = \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} &\big[(x_2 - c_2(t))\phi_R(x - c(t)) - (y_2 - c_2(t))\phi_R(y - c(t)) - (x_2 - y_2)\big] \\ &\times \frac{x_2 - y_2}{|x - y|^6} |u(t,x)|^2 |u(t,y)|^2 \, dx \, dy.\end{aligned}$$

We will show that I_2 constitutes only a small fraction of $E(u, u_t)$. First, by Corollary 4.2, we know that if R is sufficiently large depending on u and η, then

$$E_{R,c(t)}(u, u_t) \leq \eta E(u, u_t).$$

Let χ denote a smooth cutoff to the region $|x - c(t)| \geq R/2$ such that $\nabla\chi$ is bounded by R^{-1} and supported where $|x - c(t)| \sim R$. In the region where $|x - c(t)| \sim |y - c(t)|$, we have

$$|x - c(t)| \sim |y - c(t)| \gtrsim R,$$

since otherwise I_2 vanishes. Moreover, noting that

$$|(x_2 - c_2(t))\phi(x - c(t)) - (y_2 - c_2(t))\phi(y - c(t))| \lesssim |x - y|,$$

we use the Hardy–Littlewood–Sobolev inequality and the Sobolev embedding theorem to control the contribution to I_2 from this regime by

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\chi u(t,x)|^2 |\chi u(t,y)|^2}{|x - y|^4} \, dx \, dy \lesssim \|\nabla(\chi u)\|_2^4 \lesssim \eta^2.$$

In the region where $|x - c(t)| \ll |y - c(t)|$, we use the fact that

$$|x - c(t)| \ll |y - c(t)| \sim |x - y| \quad \text{and} \quad |y - c(t)| \gtrsim R$$

to estimate the contribution from this regime by

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^4}|\chi u(t,y)|^2|u(t,x)|^2\,dx\,dy \lesssim \|\nabla(\chi u)\|_{L_x^2}^2\|\nabla u\|_{L_x^2}^2 \lesssim \eta.$$

The last line follows from the same computation as in the first case. Finally, since the remaining region $|y-c(t)| \ll |x-c(t)|$ can be estimated in the same way, we conclude that

$$I_2 \lesssim \eta.$$

Choosing η sufficiently small depending on u, and R sufficiently large depending on u and η, we obtain

$$-\partial_t A(t) \geq \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x_2-y_2|^2}{|x-y|^6}|u(t,x)|^2|u(t,y)|^2\,dx\,dy - \eta E(u,u_t). \tag{5.2}$$

If $E_{\max} < \infty$, then integrating (5.2) from 0 to $T > 0$ and using Corollary 4.3, we find that there exists $\alpha = \alpha(1,u) > 0$ such that

$$\int_0^T \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x_2-y_2|^2}{|x-y|^6}|u(s,x)|^2|u(s,y)|^2\,dx\,dy\,ds \geq \alpha T$$

for all $T > 1$. Thus $-A(t) \gtrsim T$ for large T, which contradicts (5.1). Hence $E_{\max} = \infty$, which concludes the proof of Proposition 5.1. ∎

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