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NEW INFINITE FAMILIES OF RAMANUJAN-TYPE CONGRUENCES MODULO 9 FOR OVERPARTITION PAIRS

BY

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Abstract. Let $\overline{pp}(n)$ denote the number of overpartition pairs of n. Bringmann and Lovejoy (2008) proved that for $n \geq 0$, $\overline{pp}(3n+2) \equiv 0 \pmod 3$. They also proved that there are infinitely many Ramanujan-type congruences modulo every power of odd primes for $\overline{pp}(n)$. Recently, Chen and Lin (2012) established some Ramanujan-type identities and explicit congruences for $\overline{pp}(n)$. Furthermore, they also constructed infinite families of congruences for $\overline{pp}(n)$ modulo 3 and 5, and two congruence relations modulo 9. In this paper, we prove several new infinite families of congruences modulo 9 for $\overline{pp}(n)$. For example, we find that for all integers $k, n \geq 0$, $\overline{pp}(2^{6k}(48n+20)) \equiv \overline{pp}(2^{6k}(384n+32)) \equiv \overline{pp}(2^{3k}(48n+36)) \equiv 0 \pmod 9$.

1. Introduction. The objective of this paper is to derive some new infinite families of congruences modulo 9 for the number of overpartition pairs.

We start with an overview of the terminology and notation of partitions and q-series. Recall that a *partition* of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. An *overpartition* [CoL] of n is a partition in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. As noted in [CoL], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_2}{f_1^2}; \tag{1.1}$$

here and in the rest of this paper, f_k is defined by

$$f_k = \prod_{n=1}^{\infty}(1-q^{kn}), \qquad |q| < 1, \tag{1.2}$$

where k is a positive integer. Some congruences for the function $\overline{p}(n)$ have been established; see, for example, Chen and Xia [CX], Fortin, Jacob and Mathieu [FJM], Hirschhorn and Sellers [HS], Kim [K1], Lovejoy and Osburn [LO], Mahlburg [M], and Xia and Yao [XY].

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An *overpartition pair* π of n is a pair of overpartitions (λ, μ) where the sum of all of the parts is n. It has recently become clear that overpartition pairs play an important role in the theory of q-series and partitions (see [BL, L, LM]). Let $\overline{pp}(n)$ denote the number of overpartition pairs of n. By convention, we agree that $\overline{pp}(0) = 1$. By (1.1), it is easy to see that the generating function for $\overline{pp}(n)$ is

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{f_2^2}{f_1^4}. \tag{1.3}$$

Recently, various arithmetic properties for overpartition pairs have been revealed. Modulo powers of 2, employing the theory of q-series, Keister, Sellers and Vary [KSV] proved that for any positive integer n,

$$\overline{pp}(n) \equiv \begin{cases} 4 \pmod 8 & \text{if } n = m^2 \text{ or } 2m^2 \text{ for some integer } m, \\ 0 \pmod 8 & \text{otherwise.} \end{cases} \tag{1.4}$$

Chen and Lin [CL] established the following Ramanujan-type congruence:

$$\overline{pp}(8n+7) \equiv 0 \pmod{64}. \tag{1.5}$$

Kim [K2] proved (1.4) by using a purely combinatorial argument. He also showed that

$$\overline{pp}(10672200n + 624855) \equiv 0 \pmod{2^8} \tag{1.6}$$

and, for almost all positive integers n,

$$\overline{pp}(n) \equiv 0 \pmod{2^8}. \tag{1.7}$$

For the case of modulus 3, Bringmann and Lovejoy [BL] proved that for any integer $n \geq 0$,

$$\overline{pp}(3n+2) \equiv 0 \pmod 3. \tag{1.8}$$

They also introduced a rank for overpartition pairs, and gave a combinatorial explanation for this congruence. Moreover, employing the theory of Klein forms, Bringmann and Lovejoy proved that there exist infinitely many nonnested arithmetic progressions $An + B$ satisfying

$$\overline{pp}(An+B) \equiv 0 \pmod{p^k} \tag{1.9}$$

for any odd prime p and any positive integer k. However, the theory of Klein forms used to derive congruence (1.9) is not constructive and it does not give explicit arithmetic progressions $An + B$ in the statement. So it is still desirable to establish explicit congruences for $\overline{pp}(n)$.

Chen and Lin [CL] deduced a Ramanujan-type identity

$$\sum_{n=0}^{\infty} \overline{pp}(3n+2) q^n = 12 \frac{f_2^6 f_3^6}{f_1^{14}}, \tag{1.10}$$

which yields (1.8). Moreover, they also established some infinite families of congruences modulo 3 for $\overline{pp}(n)$. For example, for all integers $\alpha \geq 1$ and $n \geq 0$,

$$\overline{pp}(9^\alpha(3n+1)) \equiv \overline{pp}(9^\alpha(3n+2)) \equiv 0 \pmod 3; \tag{1.11}$$

for all integers $n \geq 0$,

$$\overline{pp}(20n+11) \equiv \overline{pp}(20n+15) \equiv \overline{pp}(20n+19) \equiv 0 \pmod 5; \tag{1.12}$$

and for all integers $\alpha \geq 1$ and $n \geq 0$,

$$\overline{pp}(5^\alpha(5n+2)) \equiv \overline{pp}(5^\alpha(5n+3)) \equiv 0 \pmod 5. \tag{1.13}$$

Chen and Lin [CL] also established some strange congruences modulo 5 and 9 for $\overline{pp}(n)$, for example,

$$\overline{pp}(5 \times 29^k) \equiv 3(k+1) \pmod 5, \tag{1.14}$$

$$\overline{pp}(2 \times 13^k) \equiv 3(k+1) \pmod 9. \tag{1.15}$$

In this paper, we prove some new infinite families of Ramanujan-type congruences modulo 9 for $\overline{pp}(n)$ by established generating functions for $\overline{pp}(An+B)$ modulo 9 for some A and B. We list our main results in the following theorems.

From this point on, all congruences for $\overline{pp}(n)$ are to the modulus 9.

THEOREM 1.1. *For all integers $k \geq 1$ and $n \geq 0$,*

$$\overline{pp}(2^k(12n+9)) \equiv \begin{cases} \overline{pp}(12n+9) & \text{if } k \equiv 1 \pmod 3, \\ 0 & \text{if } k \equiv 2 \pmod 3, \\ -\overline{pp}(12n+9) & \text{if } k \equiv 0 \pmod 3. \end{cases} \tag{1.16}$$

Using Theorem 1.1 and the facts that $\overline{pp}(21) \equiv 0$ and $\overline{pp}(45) \equiv 6$, we obtain the following corollary.

COROLLARY 1.2. *For any integer $k \geq 0$,*

$$\overline{pp}(21 \cdot 2^{k+1}) \equiv \overline{pp}(21) \equiv 0, \tag{1.17}$$

$$\overline{pp}(45 \cdot 2^{3k+1}) \equiv \overline{pp}(45) \equiv 6, \tag{1.18}$$

$$\overline{pp}(45 \cdot 2^{3k+2}) \equiv 0, \tag{1.19}$$

$$\overline{pp}(45 \cdot 2^{3k+3}) \equiv -\overline{pp}(45) \equiv 3. \tag{1.20}$$

THEOREM 1.3. *For all integers $i, j, k, n \geq 0$,*

$$\overline{pp}(2^{6k}(6n+2)) \equiv \overline{pp}(6n+2), \tag{1.21}$$

$$\overline{pp}(2^{6k}(48n+20)) \equiv 0, \tag{1.22}$$

$$\overline{pp}(2^{6k}(384n+32)) \equiv 0, \tag{1.23}$$

$$\overline{pp}(2^{6k}(96n+8)) \equiv -\overline{pp}(2^{6j}(24n+2)), \tag{1.24}$$

$$\overline{pp}(2^{6k}(192n+80)) \equiv \overline{pp}(12n+5), \tag{1.25}$$

$$\overline{pp}(2^{6k}(192n+176)) \equiv \overline{pp}(2^{6j}(48n+44)) \equiv -\overline{pp}(12n+11), \tag{1.26}$$

$$\overline{pp}(2^{6k}(384n+224)) \equiv \overline{pp}(2^{6j}(96n+56)) \equiv \overline{pp}(2^{6i}(24n+14)), \tag{1.27}$$

$$\overline{pp}(2^{6k}(384n+320)) \equiv -\overline{pp}(6n+5). \tag{1.28}$$

Employing (1.21) and the facts $\overline{pp}(8) \equiv 6$, $\overline{pp}(14) \equiv 3$, $\overline{pp}(20) \equiv 0$, we can deduce the following corollary.

COROLLARY 1.4. *For any integer $k \geq 0$,*

$$\overline{pp}(2^{6k+3}) \equiv 6, \tag{1.29}$$

$$\overline{pp}(7 \cdot 2^{6k+1}) \equiv 3, \tag{1.30}$$

$$\overline{pp}(5 \cdot 2^{6k+2}) \equiv 0. \tag{1.31}$$

This paper is organized as follows. In Section 2, we present several lemmas, which are then used in Sections 3 and 4 to prove Theorems 1.1 and 1.3.

2. Preliminaries. By the binomial theorem, it is easy to prove the following one.

LEMMA 2.1. *Let k be a positive integer. Then*

$$f_1^{3k} \equiv f_3^k \pmod 3, \tag{2.1}$$

$$f_3^{3k} \equiv f_1^{9k} \pmod 9. \tag{2.2}$$

The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [B, p. 40].

LEMMA 2.2. *The following 2-dissections are true:*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.3}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}. \tag{2.4}$$

The following lemma gives generating functions for $\overline{pp}(3n)$ and $\overline{pp}(3n+2)$ modulo 9.

LEMMA 2.3. *We have*

$$\sum_{n=0}^{\infty} \overline{pp}(3n) q^n \equiv \frac{f_1^{20}}{f_2^{10}} + 7q \frac{f_2^{14}}{f_1^4}, \tag{2.5}$$

$$\sum_{n=0}^{\infty} \overline{pp}(3n+2) q^n \equiv 3 f_1^4 f_2^6. \tag{2.6}$$

Proof. Fortin, Jacob and Mathieu [FJM] and Hirschhorn and Sellers [HS] proved that

$$\sum_{n=0}^{\infty} \overline{p}(3n)q^n = \frac{f_2^4 f_3^6}{f_1^8 f_6^3}, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \overline{p}(3n+1)q^n = 2\frac{f_2^3 f_3^3}{f_1^7}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} \overline{p}(3n+2)q^n = 4\frac{f_2^2 f_6^3}{f_1^6}. \tag{2.9}$$

In view of (1.1), (1.3) and (2.7)–(2.9), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}(n)q^n &= \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2 \\ &= \frac{f_6^8 f_9^{12}}{f_3^{16} f_{18}^6} + 4q\frac{f_6^7 f_9^9}{f_3^{15} f_{18}^3} + 12q^2\frac{f_6^6 f_9^6}{f_3^{14}} \\ &\quad + 16q^3\frac{f_6^5 f_9^3 f_{18}^3}{f_3^{13}} + 16q^4\frac{f_6^4 f_{18}^6}{f_3^{12}},\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}(3n)q^n = \frac{f_2^8 f_3^{12}}{f_1^{16} f_6^6} + 16q\frac{f_2^5 f_3^3 f_6^3}{f_1^{13}}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} \overline{pp}(3n+1)q^n = 4\frac{f_2^7 f_3^9}{f_1^{15} f_6^3} + 16q\frac{f_2^4 f_6^6}{f_1^{12}}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} \overline{pp}(3n+2)q^n = 12\frac{f_2^6 f_3^6}{f_1^{14}}. \tag{2.12}$$

Note that (2.12) has been established by Chen and Lin [CL]. Lemma 2.3 follows from (2.1), (2.2), (2.10) and (2.12). ∎

Next, we deduce generating functions for $\overline{pp}(6n+i)$ $(i = 2, 3, 5)$ modulo 9.

Lemma 2.4. *We have*

$$\sum_{n=0}^{\infty} \overline{pp}(6n+2)q^n \equiv 3\frac{f_1^4 f_2^{10}}{f_4^4}, \tag{2.13}$$

$$\sum_{n=0}^{\infty} \overline{pp}(6n+3)q^n \equiv 7\frac{f_2^{38}}{f_1^{16} f_4^{12}} + 7\frac{f_2^{14}}{f_4^4} + 8q\frac{f_2^{14} f_4^4}{f_1^8} + 2q^2\frac{f_4^{20}}{f_2^{10}} \tag{2.14}$$

and

$$\sum_{n=0}^{\infty}\overline{pp}(6n+5)q^n \equiv 6\frac{f_1^8f_4^4}{f_2^2}. \tag{2.15}$$

Proof. Substituting (2.3) and (2.4) into (2.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(3n)q^n &\equiv \frac{1}{f_2^{10}}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)^5+7qf_2^{14}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\\ &\equiv \frac{f_4^{50}}{f_2^{20}f_8^{20}}+7q\frac{f_4^{38}}{f_2^{16}f_8^{12}}+7q\frac{f_4^{14}}{f_8^4}+7q^2\frac{f_4^{26}}{f_2^{12}f_8^4}\\ &\quad+q^2f_2^4f_4^2f_8^4+8q^3\frac{f_4^{14}f_8^4}{f_2^8}+2q^4\frac{f_4^2f_8^{12}}{f_2^4}+2q^5\frac{f_8^{20}}{f_4^{10}},\end{aligned} \tag{2.16}$$

which yields (2.14). Substituting (2.4) into (2.6), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(3n+2) &\equiv 12f_1^4f_2^6 \equiv 3\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)f_2^6\\ &\equiv 3\frac{f_2^4f_4^{10}}{f_8^4}+6q\frac{f_2^8f_8^4}{f_4^2}.\end{aligned} \tag{2.17}$$

Congruences (2.13) and (2.15) follow from (2.17). ∎

Finally, we give generating functions for $\overline{pp}(12n+i)$ ($i=2,5,8,9,11$) modulo 9.

Lemma 2.5. *We have*

$$\sum_{n=0}^{\infty}\overline{pp}(12n+2)q^n \equiv 3\frac{f_1^8f_2^6}{f_4^4}, \tag{2.18}$$

$$\sum_{n=0}^{\infty}\overline{pp}(12n+5)q^n \equiv 6\frac{f_2^{24}}{f_1^6f_4^8}+6qf_1^2f_4^8, \tag{2.19}$$

$$\sum_{n=0}^{\infty}\overline{pp}(12n+8)q^n \equiv 6\frac{f_1^{12}f_4^4}{f_2^6}, \tag{2.20}$$

$$\sum_{n=0}^{\infty}\overline{pp}(12n+9)q^n \equiv 3\frac{f_2^{32}}{f_1^{14}f_4^8}+3q\frac{f_2^8f_4^8}{f_1^6}, \tag{2.21}$$

$$\sum_{n=0}^{\infty}\overline{pp}(12n+11)q^n \equiv 6\frac{f_2^{12}}{f_1^2}. \tag{2.22}$$

Proof. Substituting (2.4) into (2.13) and (2.15), we obtain

$$\sum_{n=0}^{\infty}\overline{pp}(6n+2)q^n \equiv 3\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)\frac{f_2^{10}}{f_4^4} \equiv 3\frac{f_2^8f_4^6}{f_8^4}+6q\frac{f_2^{12}f_8^4}{f_4^6} \tag{2.23}$$

and

$$\sum_{n=0}^{\infty}\overline{pp}(6n+5)q^n \equiv 6\bigg(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\bigg)^2\frac{f_4^4}{f_2^2} \tag{2.24}$$
$$\equiv 6\frac{f_4^{24}}{f_2^6f_8^8}+6q\frac{f_4^{12}}{f_2^2}+6q^2f_2^2f_8^8.$$

Congruences (2.18)–(2.20) and (2.22) follow from (2.23) and (2.24).

Substituting (2.3) into (2.14), we get

$$\sum_{n=0}^{\infty}\overline{pp}(6n+3)q^n \equiv 7\frac{f_2^{38}}{f_4^{12}}\bigg(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\bigg)^4+7\frac{f_2^{14}}{f_4^4}$$
$$+8qf_2^{14}f_4^4\bigg(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\bigg)^2+2q^2\frac{f_4^{20}}{f_2^{10}}$$
$$\equiv 7\frac{f_2^{14}}{f_4^4}+7\frac{f_4^{44}}{f_2^{18}f_8^{16}}+3q\frac{f_4^{32}}{f_2^{14}f_8^8}+3q^3\frac{f_4^8f_8^8}{f_2^6}+q^4\frac{f_8^{16}}{f_2^2f_4^4},$$

which yields (2.21). ■

3. Proof of Theorem 1.1. To begin, we give generating functions for $\overline{pp}(2^kn)$ and $\overline{pp}(2^k(12n+9))$ modulo 9.

LEMMA 3.1. *We have*

$$\sum_{n=0}^{\infty}\overline{pp}(3\cdot 2n)q^n \equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}}+7q\frac{f_2^{26}}{f_1^{12}f_4^4}+qf_1^4f_2^2f_4^4+2q^2\frac{f_2^2f_4^{12}}{f_1^4}. \tag{3.1}$$

Moreover, for any positive integer k, if

$$\sum_{n=0}^{\infty}\overline{pp}(3\cdot 2^kn)q^n \equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}}+aq\frac{f_2^{26}}{f_1^{12}f_4^4}+bqf_1^4f_2^2f_4^4+cq^2\frac{f_2^2f_4^{12}}{f_1^4}, \tag{3.2}$$

where a, b and c are integers, then

$$\sum_{n=0}^{\infty}\overline{pp}(3\cdot 2^{k+1}n)q^n \equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}}+(7+3a+c)q\frac{f_2^{26}}{f_1^{12}f_4^4} \tag{3.3}$$
$$-4bqf_1^4f_2^2f_4^4+(2+a)q^2\frac{f_2^2f_4^{12}}{f_1^4},$$

$$\sum_{n=0}^{\infty}\overline{pp}(2^k(12n+9))q^n \equiv (6+a+4c)\frac{f_2^{32}}{f_1^{14}f_4^8}+(6+7a+c)q\frac{f_2^8f_4^8}{f_1^6}. \tag{3.4}$$

Proof. It is easy to see that congruence (3.1) follows from (2.16). Suppose that

$$\sum_{n=0}^{\infty}\overline{pp}(3\cdot 2^kn)q^n \equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}}+aq\frac{f_2^{26}}{f_1^{12}f_4^4}+bqf_1^4f_2^2f_4^4+cq^2\frac{f_2^2f_4^{12}}{f_1^4}, \tag{3.5}$$

where a, b and c are integers. Substituting (2.3) and (2.4) into (3.5), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(3\cdot 2^k n)q^n &\equiv \frac{f_2^{50}}{f_4^{20}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^5+aq\frac{f_2^{26}}{f_4^4}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^3\\ &\quad+bqf_2^2f_4^4\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)\\ &\quad+cq^2f_2^2f_4^{12}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\\ &\equiv \frac{f_4^{50}}{f_2^{20}f_8^{20}}+(2+a)q\frac{f_4^{38}}{f_2^{16}f_8^{12}}+bq\frac{f_4^{14}}{f_8^4}-4bq^2f_2^4f_4^2f_8^4\\ &\quad+(7+3a+c)q^2\frac{f_4^{26}}{f_2^{12}f_8^4}+(1+3a+4c)q^3\frac{f_4^{14}f_8^4}{f_2^8}\\ &\quad+(2+a)q^4\frac{f_4^2f_8^{12}}{f_2^4}+7q^5\frac{f_8^{20}}{f_4^{10}},\end{aligned}$$

which implies that (3.3) is true and

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(2^k(6n+3))q^n &\equiv (2+a)\frac{f_2^{38}}{f_1^{16}f_4^{12}}+b\frac{f_2^{14}}{f_4^4}\\ &\quad+(1+3a+4c)q\frac{f_2^{14}f_4^4}{f_1^8}+7q^2\frac{f_4^{20}}{f_2^{10}}.\end{aligned}\tag{3.6}$$

Substituting now (2.3) into (3.6), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(2^k(6n+3))q^n &\equiv (2+a)\frac{f_2^{38}}{f_4^{12}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^4+b\frac{f_2^{14}}{f_4^4}\\ &\quad+(1+3a+4c)qf_2^{14}f_4^4\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2+7q^2\frac{f_4^{20}}{f_2^{10}}\\ &\equiv (2+a)\frac{f_4^{44}}{f_2^{18}f_8^{16}}+b\frac{f_2^{14}}{f_4^4}+(6+a+4c)q\frac{f_4^{32}}{f_2^{14}f_8^8}\\ &\quad+(3a+5c)q^2\frac{f_4^{20}}{f_2^{10}}+(6+7a+c)q^3\frac{f_4^8f_8^8}{f_2^6}\\ &\quad+(8+4a)q^4\frac{f_8^{16}}{f_2^2f_4^4},\end{aligned}$$

which yields (3.4). ∎

Lemma 3.2. *For all integers $k\ge 1$ and $n\ge 0$,*

$$\overline{pp}(2^{k+3}(12n+9))\equiv\overline{pp}(2^k(12n+9)).\tag{3.7}$$

Proof. By Lemma 3.1 and mathematical induction, we can prove that for all integers $k \geq 1$ and $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{pp}(3 \cdot 2^k n) q^n \equiv \frac{f_2^{50}}{f_1^{20} f_4^{20}} + aq \frac{f_2^{26}}{f_1^{12} f_4^4} + bq f_1^4 f_2^2 f_4^4 + cq^2 \frac{f_2^2 f_4^{12}}{f_1^4}, \tag{3.8}$$

where a, b and c are integers related to k.

Substituting $k \mapsto k+1$, $a \mapsto 7 + 3a + c$, $b \mapsto -4b$ and $c \mapsto 2 + a$ in Lemma 3.1, and employing (3.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{pp}(3 \cdot 2^{k+2} n) q^n &\equiv \frac{f_2^{50}}{f_1^{20} f_4^{20}} + (30 + 10a + 3c) q \frac{f_2^{26}}{f_1^{12} f_4^4} \\ &\quad + 16bq f_1^4 f_2^2 f_4^4 + (9 + 3a + c) q^2 \frac{f_2^2 f_4^{12}}{f_1^4} \\ &\equiv \frac{f_2^{50}}{f_1^{20} f_4^{20}} + (3 + a + 3c) q \frac{f_2^{26}}{f_1^{12} f_4^4} \\ &\quad + 7bq f_1^4 f_2^2 f_4^4 + (3a + c) q^2 \frac{f_2^2 f_4^{12}}{f_1^4} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{pp}(2^{k+1}(12n+9)) q^n &\equiv (21 + 7a + c) \frac{f_2^{32}}{f_1^{14} f_4^8} + (57 + 22a + 7c) q \frac{f_2^8 f_4^8}{f_1^6} \\ &\equiv (3 + 7a + c) \frac{f_2^{32}}{f_1^{14} f_4^8} + (3 + 4a + 7c) q \frac{f_2^8 f_4^8}{f_1^6}. \end{aligned}$$

Taking $k \mapsto k+2$, $a \mapsto 3 + a + 3c$, $b \mapsto 7b$ and $c \mapsto 3a + c$ in Lemma 3.1, and using (3.9), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{pp}(3 \cdot 2^{k+3} n) q^n &\equiv \frac{f_2^{50}}{f_1^{20} f_4^{20}} + (16 + 6a + 10c) q \frac{f_2^{26}}{f_1^{12} f_4^4} \\ &\quad - 28bq f_1^4 f_2^2 f_4^4 + (5 + a + 3c) q^2 \frac{f_2^2 f_4^{12}}{f_1^4} \\ &\equiv \frac{f_2^{50}}{f_1^{20} f_4^{20}} + (7 + 6a + c) q \frac{f_2^{26}}{f_1^{12} f_4^4} \\ &\quad - bq f_1^4 f_2^2 f_4^4 + (5 + a + 3c) q^2 \frac{f_2^2 f_4^{12}}{f_1^4} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{pp}(2^{k+2}(12n+9)) q^n &\equiv (9 + 13a + 7c) \frac{f_2^{32}}{f_1^{14} f_4^8} + (27 + 10a + 22c) q \frac{f_2^8 f_4^8}{f_1^6} \\ &\equiv (4a + 7c) \frac{f_2^{32}}{f_1^{14} f_4^8} + (a + 4c) q \frac{f_2^8 f_4^8}{f_1^6}. \end{aligned}$$

Letting $k \mapsto k+3$, $a \mapsto 7+6a+c$, $b \mapsto -b$ and $c \mapsto 5+a+3c$ in Lemma 3.1, and utilizing (3.10), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}(3\cdot 2^{k+4}n)q^n &\equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + (33+19a+6c)q\frac{f_2^{26}}{f_1^{12}f_4^4} \\ &\quad + 4bqf_1^4f_2^2f_4^4 + (9+6a+c)q^2\frac{f_2^2f_4^{12}}{f_1^4} \\ &\equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + (6+a+6c)q\frac{f_2^{26}}{f_1^{12}f_4^4} \\ &\quad + 4bqf_1^4f_2^2f_4^4 + (6a+c)q^2\frac{f_2^2f_4^{12}}{f_1^4}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}(2^{k+3}(12n+9))q^n &\equiv (33+10a+13c)\frac{f_2^{32}}{f_1^{14}f_4^8} \\ &\quad + (60+43a+10c)q\frac{f_2^8f_4^8}{f_1^6} \\ &\equiv (6+a+4c)\frac{f_2^{32}}{f_1^{14}f_4^8} + (6+7a+c)q\frac{f_2^8f_4^8}{f_1^6}.\end{aligned} \tag{3.11}$$

Congruence (3.7) follows from (3.4) and (3.11). ∎

Proof of Theorem 1.1. By Lemma 3.2, we see that for any positive integer k,

$$\overline{pp}(2^k(12n+9)) \equiv \begin{cases} \overline{pp}(24n+18) & \text{if } k \equiv 1 \pmod 3, \\ \overline{pp}(48n+36) & \text{if } k \equiv 2 \pmod 3, \\ \overline{pp}(96n+72) & \text{if } k \equiv 0 \pmod 3. \end{cases} \tag{3.12}$$

Letting $k=1$, $a=7$, $b=1$ and $c=2$ in Lemma 3.1, and using (3.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}(12n)q^n &\equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + 30q\frac{f_2^{26}}{f_1^{12}f_4^4} - 4qf_1^4f_2^2f_4^4 + 9q^2\frac{f_2^2f_4^{12}}{f_1^4} \\ &\equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + 3q\frac{f_2^{26}}{f_1^{12}f_4^4} - 4qf_1^4f_2^2f_4^4,\end{aligned} \tag{3.13}$$

$$\sum_{n=0}^{\infty} \overline{pp}(24n+18)q^n \equiv 21\frac{f_2^{32}}{f_1^{14}f_4^8} + 57q\frac{f_2^8f_4^8}{f_1^6} \equiv 3\frac{f_2^{32}}{f_1^{14}f_4^8} + 3q\frac{f_2^8f_4^8}{f_1^6}. \tag{3.14}$$

Taking $k=2$, $a=3$, $b=-4$ and $c=0$ in Lemma 3.1, and employing (3.13), we see that

$$\sum_{n=0}^{\infty} \overline{pp}(24n)q^n \equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + 16q\frac{f_2^{26}}{f_1^{12}f_4^{4}} + 16qf_1^4f_2^2f_4^4 + 5q^2\frac{f_2^2f_4^{12}}{f_1^4}$$
$$\equiv \frac{f_2^{50}}{f_1^{20}f_4^{20}} + 7q\frac{f_2^{26}}{f_1^{12}f_4^{4}} + 7qf_1^4f_2^2f_4^4 + 5q^2\frac{f_2^2f_4^{12}}{f_1^4} \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}(48n+36)q^n \equiv 9\frac{f_2^{32}}{f_1^{14}f_4^{8}} + 27q\frac{f_2^8f_4^8}{f_1^6} \equiv 0,$$

which implies that for all integers $n \geq 0$,

$$\overline{pp}(48n+36) \equiv 0. \tag{3.16}$$

Setting $k = 3$, $a = 7$, $b = 7$ and $c = 5$ in Lemma 3.1, and utilizing (3.15), we deduce that

$$\sum_{n=0}^{\infty} \overline{pp}(96n+72)q^n \equiv 33\frac{f_2^{32}}{f_1^{14}f_4^{8}} + 60q\frac{f_2^8f_4^8}{f_1^6} \equiv 6\frac{f_2^{32}}{f_1^{14}f_4^{8}} + 6q\frac{f_2^8f_4^8}{f_1^6}. \tag{3.17}$$

By means of (2.21), (3.14) and (3.17), we find that for any integer $n \geq 0$,

$$\overline{pp}(96n+72) \equiv -\overline{pp}(24n+18) \equiv -\overline{pp}(12n+9). \tag{3.18}$$

Congruence (1.16) follows from (3.12), (3.16) and (3.18). ∎

4. Proof of Theorem 1.3. Substituting (2.4) into (2.18) and (2.20), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}(12n+2)q^n \equiv 3\left(\frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2}\right)^2\frac{f_2^6}{f_4^4}$$
$$\equiv 3\frac{f_2^2f_4^{16}}{f_8^8} + 3qf_2^6f_4^4 + 3q^2\frac{f_2^{10}f_8^8}{f_4^8}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} \overline{pp}(12n+8)q^n \equiv 6\left(\frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2}\right)^3\frac{f_4^4}{f_2^6} \equiv 6\frac{f_4^{34}}{f_2^{12}f_8^{12}} + 3q^3\frac{f_8^{12}}{f_4^2}. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\sum_{n=0}^{\infty} \overline{pp}(24n+2)q^n \equiv 3\frac{f_1^2f_2^{16}}{f_4^8} + 3q\frac{f_1^{10}f_4^8}{f_2^8}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} \overline{pp}(24n+8)q^n \equiv 6\frac{f_2^{34}}{f_1^{12}f_4^{12}}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} \overline{pp}(24n+14)q^n \equiv 3f_1^6f_2^4 \tag{4.5}$$

and

$$\sum_{n=0}^{\infty}\overline{pp}(24n+20)q^n \equiv 3q\frac{f_4^{12}}{f_2^2}. \tag{4.6}$$

Substituting (2.3) into (4.4), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(24n+8)q^n &\equiv 6\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^3\frac{f_2^{34}}{f_4^{12}}\\ &\equiv 6\frac{f_4^{30}}{f_2^8f_8^{12}}+6q^3\frac{f_2^4f_8^{12}}{f_4^6}.\end{aligned} \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$\sum_{n=0}^{\infty}\overline{pp}(48n+8)q^n \equiv 6\frac{f_2^{30}}{f_1^8f_4^{12}}, \tag{4.8}$$

$$\sum_{n=0}^{\infty}\overline{pp}(48n+20)q^n \equiv 0, \tag{4.9}$$

$$\sum_{n=0}^{\infty}\overline{pp}(48n+32)q^n \equiv 6q\frac{f_1^4f_4^{12}}{f_2^6}, \tag{4.10}$$

$$\sum_{n=0}^{\infty}\overline{pp}(48n+44)q^n \equiv 3\frac{f_2^{12}}{f_1^2}. \tag{4.11}$$

It follows from (4.9) that for $n \geq 0$,

$$\overline{pp}(48n+20) \equiv 0. \tag{4.12}$$

Substituting (2.3) and (2.4) into (4.8) and (4.10), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(48n+8)q^n &\equiv 6\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\frac{f_2^{30}}{f_4^{12}}\\ &\equiv 6\frac{f_2^2f_4^{16}}{f_8^8}+3qf_2^6f_4^4+6q^2\frac{f_2^{10}f_8^8}{f_4^8},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pp}(48n+32)q^n &\equiv 6q\frac{f_4^{12}}{f_2^6}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)\\ &\equiv 6q\frac{f_4^{22}}{f_2^8f_8^4}+3q^2\frac{f_4^{10}f_8^4}{f_2^4}.\end{aligned} \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\sum_{n=0}^{\infty}\overline{pp}(96n+8)q^n \equiv 6\frac{f_1^2f_2^{16}}{f_4^8}+6q\frac{f_1^{10}f_4^8}{f_2^8}, \tag{4.15}$$

$$\sum_{n=0}^{\infty} \overline{pp}(96n+32)q^n \equiv 3q\frac{f_2^{10}f_4^4}{f_1^4}, \tag{4.16}$$

$$\sum_{n=0}^{\infty} \overline{pp}(96n+56)q^n \equiv 3f_1^6f_2^4, \tag{4.17}$$

$$\sum_{n=0}^{\infty} \overline{pp}(96n+80)q^n \equiv 6\frac{f_2^{22}}{f_1^8f_4^4}. \tag{4.18}$$

Now, from (4.3) and (4.15), we find that for any nonnegative integer n,

$$\overline{pp}(96n+8) \equiv -\overline{pp}(24n+2). \tag{4.19}$$

Substituting (2.3) into (4.16) and (4.18), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}(96n+32)q^n &\equiv 3qf_2^{10}f_4^4\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\\ &\equiv 3q\frac{f_4^{18}}{f_2^4f_8^4}+3q^2f_4^6f_8^4,\\ \sum_{n=0}^{\infty} \overline{pp}(96n+80)q^n &\equiv 6\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\frac{f_2^{22}}{f_4^4}\\ &\equiv 6\frac{f_4^{24}}{f_2^6f_8^8}+3q\frac{f_4^{12}}{f_2^2}+6q^2f_2^2f_8^8,\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}(192n+32)q^n \equiv 3qf_2^6f_4^4, \tag{4.20}$$

$$\sum_{n=0}^{\infty} \overline{pp}(192n+80)q^n \equiv 6\frac{f_2^{24}}{f_1^6f_4^8}+6qf_1^2f_4^8, \tag{4.21}$$

$$\sum_{n=0}^{\infty} \overline{pp}(192n+128)q^n \equiv 3\frac{f_2^{18}}{f_1^4f_4^4}, \tag{4.22}$$

$$\sum_{n=0}^{\infty} \overline{pp}(192n+176)q^n \equiv 3\frac{f_2^{12}}{f_1^2}. \tag{4.23}$$

By (2.19) and (4.21), we see that for any nonnegative integer n,

$$\overline{pp}(192n+80) \equiv \overline{pp}(12n+5). \tag{4.24}$$

In view of (2.22), (4.11) and (4.23), we deduce that for any nonnegative integer n,

$$\overline{pp}(192n+176) \equiv \overline{pp}(48n+44) \equiv -\overline{pp}(12n+11). \tag{4.25}$$

Substituting (2.3) into (4.22), we get

$$\sum_{n=0}^{\infty}\overline{pp}(192n+128)q^n \equiv 3\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\frac{f_2^{18}}{f_4^4} \equiv 3\frac{f_2^4f_4^{10}}{f_8^4}+3q\frac{f_2^8f_8^4}{f_4^2}. \tag{4.26}$$

Employing (4.20) and (4.26), we see that

$$\sum_{n=0}^{\infty}\overline{pp}(384n+32)q^n \equiv 0, \tag{4.27}$$

$$\sum_{n=0}^{\infty}\overline{pp}(384n+128)q^n \equiv 3\frac{f_1^4f_2^{10}}{f_4^4}, \tag{4.28}$$

$$\sum_{n=0}^{\infty}\overline{pp}(384n+224)q^n \equiv 3f_1^6f_2^4, \tag{4.29}$$

$$\sum_{n=0}^{\infty}\overline{pp}(384n+320)q^n \equiv 3\frac{f_1^8f_4^4}{f_2^2}. \tag{4.30}$$

It follows from (4.27) that for $n \geq 0$,

$$\overline{pp}(384n+32) \equiv 0. \tag{4.31}$$

By (2.13), (2.15), (4.5) and (4.17), (4.28)–(4.30), we find that for any integer $n \geq 0$,

$$\overline{pp}(384n+128) \equiv \overline{pp}(6n+2), \tag{4.32}$$

$$\overline{pp}(384n+224) \equiv \overline{pp}(96n+56) \equiv \overline{pp}(24n+14), \tag{4.33}$$

$$\overline{pp}(384n+320) \equiv -\overline{pp}(6n+5). \tag{4.34}$$

Now, we prove congruence (1.21) by induction. Obviously, (1.21) holds when $k=0$. From (4.32), we see that it is true when $k=1$. Assume that it holds when $k=m$, that is,

$$\overline{pp}(2^{6m}(6n+2)) \equiv \overline{pp}(6n+2). \tag{4.35}$$

Replacing n by $64^m n+(64^m-1)/3$ in (4.32), we deduce that

$$\overline{pp}(2^{6m+6}(6n+2)) \equiv \overline{pp}(2^{6m}(6n+2)). \tag{4.36}$$

Thus, (1.21) holds when $k=m+1$ after combining (4.35) and (4.36).

Replacing n by $8n+3$ in (1.21) and employing (4.12), we obtain (1.22). Replacing n by $64n+5$ in (1.21) and using (4.27), we deduce (1.23). Congruence (1.24) can be obtained by replacing n by $4n$ and by $16n+1$ in (1.21), and then using (4.19). Replacing n by $32n+13$ in (1.21) and employing (4.24), we get (1.25). Replacing n by $8n+7$ and by $32n+29$ in (1.21), and then utilizing (4.25), we deduce (1.26). Replacing n by $64n+37$, by $16n+9$

and by $4n+2$ in (1.21), and then using (4.33), we obtain (1.27). Replacing n by $64n+53$ in (1.21) and employing (4.34), we get (1.28). ∎

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